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Abstract: The aim of this research was to analyze the level of mathematical understanding based on the mathematical connections made by a group of Mexican High School students when solving mathematical tasks related to the concepts of equation and linear function. The study employed the thinking aloud method to collect data, whereby students verbalized their thought processes while solving three mathematical tasks, followed by answering some final questions. The collected data were analyzed using a simplified version of thematic analysis based on a preliminary framework adopted in this research. The findings revealed that students demonstrated different levels of mathematical understanding depending on the types of mathematical connections they made. Moreover, these outcomes facilitated the refinement of the preliminary framework, providing a more precise set of indicators for each tier of mathematical understanding. These indicators can be valuable resources for educators and researchers, offering guidance in developing instructional strategies aimed at promoting higher levels of mathematical understanding within the classroom.

Keywords: Mathematical Understanding, Mathematical Connections, Thinking Aloud, Thematic Analysis, Linear Equations and Functions.

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Introduction

Mathematical understanding is a fundamental goal in Mathematics Education (National Council of Teacher of Mathematics [NCTM], 2013; Campo-Meneses et al., 2023), as it enables students to apply mathematical knowledge, solve problems, and engage in critical thinking. This important because mathematics is a science based on the interrelationships among its concepts, procedures, theorems, and meanings (Businskas, 2008; Tanışlı & Kalkan, 2018; García-García & Dolores-Flores, 2018). It is a sequential and cumulative science that builds upon previous knowledge to construct new knowledge (Bingölbali & Coşkun, 2016). Boaler and Staples (2012) demonstrated the positive impact of a curriculum emphasizing mathematical connections on students' mathematical reasoning abilities.

Research reveals that studying the mathematical connections that make a student is possible to infer his or her level of understanding (Bingölbali & Coskun, 2016; García-García & Dolores-Flores, 2021a). Mathematical understanding has been studied from different theoretical approaches (APOS theory, onto-semiotic approach [OSA], Pirie and Kieren's model, etc.), but there are few studies that propose a framework to study the mathematical understanding of a student based on the mathematical connections that he or she makes.

The significance of studying mathematical understanding through the lens of mathematical connections lies in its potential to inform instructional practices and curriculum design. When students develop strong connections among mathematical concepts, they are better equipped to transfer their knowledge to new situations, apply mathematical principles in real-world contexts, and solve complex problems. By understanding the nature of mathematical connections, educators can design learning experiences that explicitly foster and promote these connections, thereby enhancing students' overall mathematical understanding.

In Campo-Meneses and García-García (2021), we introduced a framework designed to assess students' mathematical understanding by examining the mathematical connections they establish while tackling tasks related to exponential and logarithmic functions. Among other results, we found that students have difficulties getting high levels of mathematical understanding. However, it's important to note that this framework has a limitation —it exclusively focuses on evaluating students' mathematical understanding when they work on tasks involving exponential and logarithmic functions—. For this reason, in this research, we seek to extend its use when students solve tasks involving concepts of function and equation linear. This led us to set the following aim: *analyze the level of mathematical understanding based on the mathematical connections made by a group of Mexican High School students when solving mathematical tasks related to the concepts of equation and linear function*.

To accomplish this aim, our approach involves several key steps. Firstly, we provide a concise review of the relevant literature, which helps establish the significance of our research. Following this, we articulate the research question, outline the preliminary framework, delineate our methodology, present the findings, engage in a comprehensive discussion, draw conclusive remarks, and finally, provide a list of references.

Literature Review

Mathematical Connections

According to Garbín (2005), the act of making mathematical connections enables persons to establish relationships between problems, particularly in terms of mathematical language and semiotic representation registers. This ability facilitates the discovery of coherent solutions. Furthermore, as Koestler et al. (2013) pointed out, students who can make mathematical connections are better positioned to delve deeper into problem-solving, thus understanding mathematics as a unified body of knowledge. Consequently, the NCTM (2014) advocated for instructional practices aimed at encouraging students to build connections among various mathematical ideas, representations, and problem-solving strategies. They underscored the importance of educators explicitly addressing these connections and fostering mathematical coherence within the classroom environment.

Mathematical connections have emerged as a prevalent focus in curricula across various countries, including the United States of North America, South Africa, Mexico, Australia, Turkey, Spain, Colombia, and many others (García, 2018; García-García, 2019). This widespread emphasis on mathematical connections has consequently elevated their significance within the realm of Mathematics Education research. A review of the existing literature in this field reveals several distinct approaches to studying mathematical connections, which include:

- Studies exploring mathematical connections that appear when students or teachers solve mathematical tasks (Bingölbali and Coşkun, 2016; García-García & Dolores-Flores, 2018; Campo-Meneses & García-García, 2020).

- Research that identifies the mathematical connections utilized by teachers or students when working with various representations (Moon et al., 2013; García-García & Dolores-Flores, 2021a).
- Researches examining the mathematical connections that emerge when teachers or students solve application or modeling problems (Gainsburg, 2008; Özgen, 2013; García-García & Dolores-Flores, 2021b).
- Studies focusing on the beliefs associated with mathematical connections (Ji-Eun, 2012; Soltani et al., 2013).
- Research exploring the mathematical connections promoted within the curriculum (García-García et al., 2022; Campo-Meneses et al., 2023; Cruz-Acevedo & García-García, 2023).
- Studies that establish connections between mathematical connections and other theories (Rodríguez-Nieto, Font, Borji & Rodríguez-Vásquez, 2021; Rodríguez-Nieto, Rodríguez-Vásquez, Font & Morales-Carballo, 2021; Campo-Meneses & García-García, 2021).

Previous literature has underscored the significance of mathematical connections in research. It has also highlighted the challenges both students and teachers encounter in establishing these mathematical connections (Mhlolo, 2012; Özgen, 2013; Moon et al., 2013; Radmehr & Drake, 2017; Dolores & García-García, 2017). Additionally, it has been identified that in various tasks, including graphical ones, students tend to persistently rely on algebraic representations (Dawkins & Mendoza, 2014; Hong & Thomas, 2015). Notably, the literature indicates that even high-achieving math students often exhibit a preference for algebraic techniques when solving problems or working with graphical tasks (Hong & Thomas, 2015; Dawkins & Mendoza, 2014). Nevertheless, Dawkins and Mendoza (2014) have suggested that the inappropriate use of algebraic techniques can limit students' comprehension and their ability to transition effectively between different representation formats.

It is significant that efforts are being made to promote the use of mathematical connections in the classroom. For instance, Boaler and Staples (2012) conducted a longitudinal study that investigated the effects of a mathematics curriculum emphasizing mathematical connections. Their research revealed that students who received instruction focused on these connections demonstrated significant improvements in their mathematical reasoning abilities when compared to students in traditional classrooms. This study provides empirical evidence supporting the notion that nurturing mathematical connections can enhance students' understanding of mathematics.

The reviewed research has described various types of mathematical connections that emerge in the work of teachers and students. They propose different typologies of mathematical connections categorized into two significant groups: intra-mathematical and extra-mathematical connections. Consequently, in recent years, several frameworks for studying mathematical connections have been proposed, including those by Evitts (2004), Businskas (2008), Eli et al. (2011), García-García and Dolores-Flores (2018, 2021a, 2021b), and Rodríguez-Nieto et al. (2020), who have shared some of their findings. However, the research has also reported that the mathematical connections used can vary depending on the specific population under study.

The Relationship Between Mathematical Connections and Mathematical Understanding

The relationship between mathematical connections and mathematical understanding is intricate and profound. Mathematical connections serve as the scaffolding upon which mathematical understanding is built. They foster coherence, support knowledge transfer, enhance fluency, and promote metacognitive development. Recognizing and harnessing the power of these connections is essential for educators and researchers striving to advance mathematical learning and pedagogy.

Consistent with the previous idea, Mason et al. (2010) argued that making connections extends beyond procedural knowledge and enhances conceptual understanding, problem-solving, and critical thinking. Likewise, research has highlighted the importance of teacher knowledge and pedagogical practices in facilitating mathematical connections. Teachers need a strong understanding of mathematical connections themselves to effectively support students in making those connections (Ruthven, 2017). Educators can encourage mathematical connections by incorporating instructional methods that emphasize the relationships between mathematical concepts and promote the application of knowledge in new contexts. In doing so, we can support students in constructing a strong foundation in mathematics and equip them with the skills necessary for success in the classroom.

Over the past decade, research has emphasized the significance of mathematical connections in supporting mathematical understanding. As Maier and Steinbring (1998) point out, understanding is a unique and individual process of constructing meaning and knowledge, and it varies among each student. Additionally, Schlöglmann (2007) highlights the role of the teacher and the interaction between students, underscoring the social component in the development of mathematical understanding.

Within the field of mathematics education, researchers have increasingly recognized the significance of promotion meaningful connections among mathematical concepts, procedures, and representations to enhance students' overall understanding of mathematics. In this context, research has emphasized that making mathematical connections in the classroom contributes to perceiving mathematics as an integrated discipline (Evitts, 2004; Mwakapenda, 2008; Jaijan & Loipha, 2012). In essence, this approach enables students to articulate knowledge from diverse mathematical domains and identify their interrelations. When this is achieved, students enhance their mathematical understanding (Mhlolo, 2012; Eli, Mohr-Schroeder, & Lee, 2011) and become better equipped to tackle real-world application problems. As proposed by Jones (2015), the development of mathematical understanding holds paramount importance in an educational setting. Instructional strategies such as employing open-ended tasks, problem-solving methodologies, and fostering mathematical discourse have been demonstrated to facilitate mathematical connections and promote mathematical understanding (Boaler, 2016).

On the other hand, in the literature, we identify that while understanding has been studied from theoretical perspectives like APOS, OSA, Pirie and Kieren's model, there has been limited research on mathematical understanding based on the mathematical connections that a student can make. Notably, recent efforts in this regard include the work of Rodríguez-Nieto et al. (2021), which examines the quality of mathematical connections, and the framework proposed by Campo-Meneses and García-García (2021) to investigate students' mathematical understanding while engaging with tasks related to exponential and logarithmic functions.

Research on mathematical connections confirms that, from the student's standpoint, making mathematical connections contributes to the development of mathematical understanding. Meanwhile, from the perspectives of teachers and researchers, analyzing the mathematical connections forged by a learner enables the assessment of their level of mathematical understanding (Campo-Meneses and García-García, 2021). Despite these common assertions in various research studies, we did not find any studies that specifically address understanding from only the perspective of mathematical connections. Hence, we aim to make a valuable contribution to this area of research.

Studies About Equation and Function Linear

Research closely related to the aim of this study was conducted by Hatisaru (2022), who explored the types of mathematical connections created in the classroom during the teaching of functions. However, we have not come across any research that specifically identifies the level of understanding among students when solving tasks involving equations and linear functions, despite the teaching experience indicate that students often confuse these concepts in the classroom. This lack of research is significant considering that understanding is highlighted as a curricular goal in various countries (Stylianides and Stylianides, 2007).

Furthermore, we believe that "mathematical understanding and mathematical connections are strongly linked" (García-García and Dolores-Flores, 2021a, p. 4). This viewpoint aligns with authors such as Berry and Nyman (2003), who argue that making mathematical connections between different ideas serves as a crucial indicator of understanding. Similarly, we concur with García-García and Dolores-Flores (2021a), who suggest that students who possess a certain level of mathematical understanding are better equipped to make mathematical connections.

According to Tanışlı and Kalkan (2018), linear functions and slope represent fundamental concepts, forming the bedrock for more advanced mathematical ideas and possessing relevance for understanding various real-life phenomena. Their study revealed that over half of Turkish students exhibited low levels of achievement in both of these concepts. Moreover, an analysis of conceptual understanding unveiled that the majority of students grappled with determining the slope of a line. Similar findings were reported by Dolores-Flores et al. (2019), who identified challenges among pre-university students when it came to solving tasks related to rate of change. Furthermore, they identified that students most frequently relied on procedural mathematical connections when solving these tasks.

Other research interests have focused on studying the reasoning behind covariation between quantities that leads to a functional relationship (Carlson et al., 2002; Thompson & Carlson, 2017; Kafetzopoulos & Psycharis, 2022). Additionally, there have been efforts to analyze textbooks from various countries to examine how the concept of functions is presented (Kaur, Wong & Govindani, 2016; Hong & Choi, 2018). In this sense, Kaur et al. (2016) reported that both Singaporean and Dutch textbooks provide opportunities for students to connect mathematical concepts to real-life situations, engage in self-assessment, and reflect on their learning. The Singaporean textbooks present topics in a structured and systematic manner, while the Dutch textbooks present mathematical concepts in an intuitive manner, intertwined with real-life contexts.

Similarly, Hong and Choi (2018) revealed differences in the sequencing of topics, exercise problems, and solved examples between Korean and American textbooks. They also found that both countries' textbooks incorporate multiple algorithms for graphing and writing linear functions in both solved examples and exercises. While the standards in both countries emphasize understanding linear functions, Korean textbooks tend to emphasize a more procedural and symbolic treatment of the function concept, while North American textbooks focus more on interpreting the meanings of linear functions.

The concept of a linear equation is a fundamental topic in algebra because, as pointed out by Otten et al. (2019), a significant component of learning algebra involves solving algebraic equations, often utilizing the balance model. In Mexico, the formal introduction of this concept typically begins in Junior High School, extending through High School and University. Nonetheless, research indicates the possibility of introducing equations as early as the Elementary Level (Brizuela and Schliemann, 2004; Hewitt, 2012). Regarding this, Hewitt (2012) reported that children in Year 5 (aged 9 to 10) demonstrated remarkable confidence when working with complex linear equations expressed in formal notation.

Both linear equations and linear functions serve as valuable tools for modeling various mathematical and real-world scenarios. However, their utilization often poses challenges for students, resulting in difficulties with comprehension and application.

Research Question

Based on the literature review, it is evident that there is limited research examining the mathematical understanding of students in conjunction with the mathematical connections they make. Consequently, we propose to investigate these relationships by addressing the following research questions: *What levels of mathematical understanding do Mexican High School students achieve when making mathematical connections to solve tasks involving equations and linear functions?*

To answer these questions, we adopted as preliminary framework the proposed by Campo-Meneses and García-García (2021). Subsequently, we designed tasks aligned with this framework. Finally, we refined the preliminary framework based on empirical results obtained from the study. So, this research makes two primary theoretical contributions. Firstly, we extend the use of the previously mentioned framework. Secondly, we introduce a new typology of mathematical connection that has not been documented in the existing literature.

Framework

Mathematical understanding can be approached from various frameworks, and one such perspective, as considered in our research question, is through the lens of mathematical connections. In this context, Lambdin (2003) suggests that "understanding is represented by an increasingly connected and complex web of mathematical knowledge" (p. 5). Therefore, in this section, we aim to define several key concepts that will serve as a guide for our research and aid us in refining our preliminary framework (See Table 1 at the end of the framework section).

Mathematical Connections

Several authors have used the term 'mathematical connections' to describe student actions or activities, although their exact definitions for this construct have varied. For example, Eli et al. (2011) define mathematical connections as links wherein students utilize prior knowledge to facilitate their understanding of the relationships between two or more mathematical concepts, strands, or representations within a mental network. Similarly, Foster and Lee (2021) describe it as the relationship constructed by an individual between a mathematical object and another mathematical or non-mathematical object. Likewise, Kleden et al. (2021) argue that "the ability to make connections is a way to create understanding through linking mathematical concepts with previously learned concepts, other fields of science, and the environment" (p. 262).

However, for the aim of this research, we chosen to adopt García-García and Dolores-Flores' (2018) definition of mathematical connections as a true relation make by a student between "two or more ideas, concepts, definitions, theorems, procedures, representations, and meanings with each other, with other disciplines or with real life" (p. 229). Mathematical connections become evident when students solve specific tasks and demonstrate their ability to recognize and justify them through their written answers. Therefore, it is crucial to analyze both their written and verbal answers to identify the mathematical connections used by students.

In investigating mathematical connections, previous research has identified various types (Businskas, 2008; Eli et al., 2011; García-García and Dolores-Flores, 2018, 2021a, 2021b; Rodríguez-Nieto, Rodríguez-Vásquez, & Font, 2020) that provide insights into their use. To study mathematical connections, researchers have identified typologies that have enabled us to identify the mathematical connections demonstrated by students while solving the proposed mathematical tasks. These types are described below.

Procedural Mathematical Connection

This mathematical connection occurs when a student employs rules, algorithms, formulas, or even graphs as tools to arrive at a solution for a given mathematical task, providing a justification for their application (Businskas, 2008; García-García & Dolores-Flores, 2021a). For example, when a student uses the graph of a linear function to determine the solution of a corresponding linear equation and provides reasoning for their approach, they are using this mathematical connection.

Different Representations Mathematical Connection

According to Businskas (2008), this mathematical connection can be categorized into two types: alternate representations and equivalent representations. Alternate representations occur when a student make a conversion (in the sense of Duval, 2017) from one semiotic register to another of a mathematical concept. In other words, the student performs a transformation between different representations, such as algebraic-geometric or algebraic-graphical, among others. For example, when a student constructs a graph of a line associated with a linear function based on its given algebraic representation, they are employing this mathematical connection. On the other hand, equivalent representations occur when a student make a treatment (in the sense of Duval, 2017) from one representation to another within the same semiotic register, such as algebraic-algebraic, graphic-graphic, etc. For instance, a student who starts from an equation in the form $ax + b = cx + d$, performs operations, and transforms it into another form, $ax = b$ (to find the value of x), all within the same algebraic register, is using this mathematical connection by making equivalent representations.

Feature Mathematical Connection

This type of mathematical connection occurs when a student identifies an invariant attribute or quality that discriminates a mathematical concept from others (Eli et al., 2011; García-García and Dolores-Flores, 2018, 2021a, 2021b). It also appears when the student describes the properties of mathematical concepts that make them different or similar to others (García-García and Dolores-Flores, 2018, 2021a, 2021b). This may include verbal or written descriptions of the components of a representation (algebraic, geometric, graphical, etc.) or even characteristics of a theorem. For instance, a student who indicates that a linear equation of the form $ax + b = c$ consists of an unknown *x* with an exponent of 1, and that has constants like a, b or c, is engaging in this mathematical connection.

Reversibility Mathematical Connection

This type of mathematical connection comes into play for certain mathematical concepts that possess a specific characteristic. It emerges when a student is able to move from concept A to concept B, and then inverse the process by starting from B to return to A (García-García & Dolores-Flores, 2018, 2021b). For instance, a student who indicate that he linear function $f(x) = 2x + 1$ is the inverse of $f(x) = \frac{x-1}{2}$ is making this mathematical connection.

Meaning Mathematical Connection

This type of mathematical connection emerges when a student assigns sense to a mathematical concept or provides a personal definition that he or she has constructed for it. This mathematical connection is different from the feature because a student who make it does not describe properties or qualities but rather explains what for a student is the mathematical concept itself and may include its context of use (García-García & Dolores-Flores, 2021a, 2021b). For instance, when a student explains that a linear function can be utilized to estimate the costs of a product that maintains a constant price, regardless of the quantity of products sold, is making in this mathematical connection as they are elaborating on its contextual use.

Part-whole Mathematical Connection

This mathematical connection is exhibit when a student establishes a logical relationship between mathematical concepts. In other words, it occurs when a student recognizes that A (a general case) includes B (a particular case), or conversely, that B is a component of A (Businskas, 2008). For example, when a student argues that a linear equation like $2x + 4 = 8$ is a particular case of an equation in the form $ax + b = c$, is making this type of mathematical connection.

Inter-conceptual Mathematical Connection

Is used when a student establishes a relationship between two or more different mathematical concepts. This relationship is forged because the interconnected concepts are essential for solving specific mathematical tasks. A student who links a linear equation to a linear function, which each possess their own mathematical properties, as well as associated concepts like slope and the domain of a function, among others, is utilizing the inter-conceptual mathematical connection.

The intra-mathematical connections previously defined are the basis for the preliminary framework (Table 1) to study mathematical understanding from them. On the other hand, we propose the inter-conceptual mathematical connection based on the mathematical concepts involved, such as equations and linear functions, as well as associated mathematical ideas like arithmetic progression, general term of an arithmetic progression, domain of a function, image set of a function, inverse function, variables, slope, etc. We introduce the prefix inter to emphasize the connection between different mathematical concepts. Furthermore, we hypothesize that beyond mathematical tasks, students may need to link mathematical concepts to concepts from other disciplines or real-life situations, which we term extra-conceptual mathematical connections.

A Tool for Analyzing Mathematical Understanding through Mathematical Connections

Mathematical understanding hinges on a student's capacity to make mathematical connections (Hiebert & Carpenter, 1992; Good et al., 1992; Silver et al., 2009). When evaluating someone's grasp of a mathematical concept, the quality and robustness of these connections within their internal representation serve as reliable indicators (Boaler, 2002; Berry & Nyman, 2003; Patterson & Norwood, 2004; Barmby, Harries, Higgins & Suggate, 2009). Hiebert and Carpenter (1992) liken mathematical understanding to an internal network of representations encompassing mathematical ideas, procedures, and facts, characterized by two connection types. The first type is based on similarities and differences between different external representations or within a single form of representation, and the second type refers to hierarchical relationships, such as those between specific cases and general ones. Therefore, the strongest connection that is built is the mathematical understanding achieved. Consistent with this perspective, Barmby et al. (2009) suggest that understanding develops through connections made between different internal representations, in addition to being able to reason about them.

On the contrary, Eli et al. (2011) argue that constructing and understanding mathematical concepts, ideas, facts, or procedures necessitate linking prior knowledge with new information. This viewpoint aligns with Mousley's (2004) assertion that fostering mathematical connections is a crucial classroom activity for building understanding, with mathematical meanings emerging from the process of making such connections (Noss et al., 1997). Furthermore, Cai and Ding (2015) add that some characteristics of mathematical understanding are as follows: it is both a process of understanding (or knowing) and a result of the act of understanding (sometimes referred to as knowledge); it is both the act of making mathematical connections and a result of making those connections; it is a dynamic and continuous process; understanding can have different levels and types; and the goal is to achieve a deep understanding of mathematics. These authors directly emphasize the relationship between mathematical understanding and mathematical connections.

Lambdin (2003) indicated that while students expand their personal web of connections, their understanding evolves and becomes increasingly complex. Along these lines, he contends that learning with understanding offers several advantages, including serving as motivation, enhancing comprehension, aiding memory retention, facilitating knowledge transfer, influencing attitudes and beliefs, and nurturing the development of independent learners.

On the other hand, Tanışlı and Kalkan (2018) emphasize that making connections between mathematical concepts, operations and relationships is an important indicator of conceptual understanding. According to these authors, understanding and reasoning play pivotal roles in enhancing learning outcomes, alleviating students' difficulties, and preventing misconceptions. In this respect, García-García and Dolores-Flores (2021a) consider that misconceptions avoid students for making mathematical connections and, as a consequence, they show errors in their procedures and conceptual difficulties in their arguments.

Tanışlı and Kalkan (2018) assume that "understanding does not mean memorizing formulas that allow for performing practical operations or solving problems correctly by using formulas" (p. 1223). Instead, Campo-Meneses and García-García (2021) propose that mathematical understanding manifests when students can apply mathematical concepts in various contexts. For this reason, they accept that a student understands a mathematical concept when he performs practices from which objects emerge (problems, definitions, propositions, procedures, and arguments). Moreover, the student should be able to make the central[1] mathematical connections among these elements, allowing them to consistently solve assigned

[1] This mathematical connection is the most important and it is directly associated with the concept or concepts involved in the task, in such a way that if the student fails to do it, they will not be able consistently solve all the proposed tasks. In addition, it has the characteristic that it is used as a justification for other connections made by students (García-García & Dolores-Flores, 2018). In the case of the present research, the central mathematical connection is the inter-conceptual connection because in all the tasks it appears as a necessary connection to solve them completely, in addition to the fact that when doing so, students will be able to demonstrate the use of other mathematical connections such as meaning, different representations, feature, part-whole, procedural, etc.

tasks (demonstrating competent usage) and justify these central connections. This research aligns with Campo-Meneses and García-García's (2021) perspective on mathematical understanding.

Furthermore, we share the viewpoints of Cai and Ding (2015) and Hiebert, Carpenter, Fennema, Fuson, Wearne, et al. (1997), which acknowledge that mathematical understanding is a dynamic process that can vary among students and encompass different levels. While mathematical understanding and mathematical connections are closely intertwined, we recognize that mathematical understanding represents a more intricate cognitive process and learning goal compared to mathematical connections; they are not synonymous. However, the ability to make mathematical connections acts as a crucial bridge, fundamental to students achieving mathematical understanding. According to Hiebert, Carpenter, Fennema et al. (1997), two cognitive processes, reflection, and communication, are fundamental in students' efforts to achieve a deep understanding. This research embraces the notion that these same processes are indispensable in the process of forging mathematical connections.

We have adopted the framework introduced by Campo-Meneses and García-García (2021) with the aim of extending its applicability to study mathematical understanding through the examination of students' mathematical connections (Table 1). We have expanded this framework to investigate an inverse function—our belief being that it could prove valuable for the study of various functions and their inverses—alongside associated concepts. Initially, we view this as a preliminary framework suitable for the examination of linear equations, functions, and the inverse of the linear function. However, it is important to note that this framework will be refined based on the empirical data we gather in the course of this research.

Table 1

Preliminary framework to assess levels of mathematical understanding based on the mathematical connections made by students

Level	Description
0	The student does not solve the proposed tasks and does not make any mathematical connections. In their answers, conceptual difficulties and errors in their procedures are appear.
1	The student solves part of some tasks and makes less elaborate mathematical connections such as feature or procedural.

2	In addition to making some of the mathematical connections made in the previous level, the student also makes other mathematical connections such as different representations, meaning, and part-whole. The student solves some proposed tasks, but not all.
3	Furthermore, beyond successfully establishing the majority of the mathematical connections reached in the earlier level, the student also demonstrates the capacity to form more intricate connections, specifically those related to reversibility mathematical connection (when feasible within the mathematical concepts involved in the tasks) and the inter-conceptual mathematical connection (in some tasks). Likewise, the student consistently solves all the proposed tasks.
4	The student consistently solves all the given tasks while also demonstrating a strong ability to make most of the required mathematical connections. Particularly, in this research the central mathematical connection is the inter-conceptual connections that appears in the arguments when the student solves each task. In other words, the student uses the inter-conceptual connection as an argument to make the other mathematical connections that appear in the process of solving the proposed tasks.

Adapted from Campo-Meneses and García-García (2021).

Method

This research adopts a qualitative approach and employs the thinking aloud method to collect data. According to Kucan and Beck (1997), this method serves as both an inquiry and instructional tool for enhancing reading comprehension. It provides insight into students' thought processes as they articulate their thoughts while reading, solving tasks, or responding to questions. We consider this method adequate because it allows the students to say aloud everything they were thinking while solving the proposed tasks, in addition to the fact that the researcher did not influence the answers given by the students. The entire process was video recorded for subsequent analysis.

Context and Participants

Data collection took place at a high school in southern Mexico. This school predominantly admits students who have faced rejection from other institutions at the same educational level, leading to a relatively high dropout rate during the six semesters that they study at this school. This school is situated in an urban area characterized by high levels of poverty and violence, the school serves a challenging environment.

Eight Mexican student volunteers, consisting of four women and four men aged between 16 and 17, participated in this study. These students had previously covered subjects such as Algebra, Geometry, and Trigonometry. At the time of data collection, they were engaged in the study of Analytical Geometry, where they had already explored various aspects of linear equations and preliminary concepts related to linear functions. According to their teacher, all these students had regular and high academic performance in the course.

Data Collection Instrument

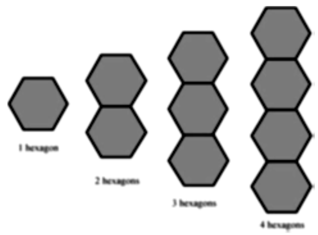
For data collection, a questionnaire including three mathematical tasks and final questions (as outlined in Table 2) was employed. These tasks were adapted from two previous studies (Ronda, 2009; Wilkie and Ayalon, 2018) and were modified to align with the specific focus of this research. These tasks were selected due to their appropriateness for the participants in this study and their potential to elicit a wide range of mathematical connections, diverse problem-solving strategies, and the use of various representations.

The first task was adapted from Wilkie and Ayalon (2018). This considers a geometric pattern and involves concepts such as perimeter, general term of an arithmetic progression, linear equation, and its solution, mainly. The second and third tasks were adapted from Ronda (2009). In this sense, the second task considers the variation in the height of the water in a container over time. The task involved concepts such as linear equation and its solution, linear function, graph of a function, initial value, etc. In the end, the third task considered the behavior of two inverse functions whose values are given in two tables of values. The first function, in addition, is represented algebraically; while for the second table of values the student is asked to find the function that describes this behavior, in addition to other additional questions. The task involved concepts such as linear function, inverse function, numerical values of a function, slope, and linear equation.

Table 2

Data Collection Tasks

Task	Mathematical Connections Expected

1. For the geometric pattern below, there is a chain of regular hexagons (meaning all 6 sides are equal) whose sides are 1 centimeter:  Observe that: For 1 hexagon, the perimeter is 6 cm. For 3 hexagons, the perimeter is 14 cm. a) For 2 hexagons, what is the perimeter? _____________. b) For 7 hexagons, what is the perimeter? _____________. c) If the figure has a perimeter of 82 cm, how many regular hexagons are there in the figure? Explain how you arrive to that result. d) Describe the process to calculate the perimeter of 100 hexagons, without knowing the perimeter of 99 hexagons. e) Write a formula to describe the perimeter of any number of hexagons in the chain (it is not necessary to simplify it). You can use: $p(n) =$ f) Explain why you think your formula is correct.	Procedural Part-whole Different representations Feature Inter-conceptual
2. Imagine that water is flowing through a faucet into a container. The following equations indicate the height of the water (w) in the container according to the number of minutes (t) when the faucet was open for 10 minutes. $w = t + 8$ for the first four minutes ($t = 0$ to 4). $w = 3t$ for the remaining six minutes ($t = 4$ to 10). Where: *w* refers to the water level (height) measured in centimeters. *t* refers to the number of minutes.	Procedural Meaning Feature Different representations Inter-conceptual

<table>
<tr>
<td>Note: Remember that 3t means multiplying 3 by the value of t.
Use the information above to answer the following questions.
a) Based on the information provided, do you think that the height of the water in the container increases at the same rate during the 10 minutes? Justify your answer.
b) From the given information, do you think that the container already contained water before the faucet was turned on? Justify your answer.
c) How many minutes have passed if the water reaches a height of 10 cm in the container? What if he had a height of 24 cm? Justify your answer.
d) How many minutes will have passed if the water has a height of 22.5 cm in the container? Explain your answer.
e) Draw a picture and construct a graph that describes the situation. Justify your constructions.</td>
<td></td>
</tr>
<tr>
<td>3. The relationship between x and y in Table 1 is $y = 2x + 1$. In Table 2, the values of x and y in Table 1 were swapped or interchanged. Please write the equation which shows the relationship between x and y in Table 2.
Table 1
<table><tr><th>x</th><th>y</th></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>3</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>9</td></tr></table>$y = 2x + 1$
Table 2
<table><tr><th>x</th><th>y</th></tr><tr><td>1</td><td>0</td></tr><tr><td>3</td><td>1</td></tr><tr><td>5</td><td>2</td></tr><tr><td>7</td><td>3</td></tr><tr><td>9</td><td>4</td></tr></table>___________
a) Explain in detail how you got the requested equation. Why is the formula you found correct?
b) Find other y-values for $x = 11,13,15$ and 17 for the function you found previously.
c) Graph the values of table 1 and 2 on the same Cartesian plane. What characteristics do both graphs have? What are they like to each other?</td>
<td>Reversibility

Procedural

Meaning

Feature

Different representations

Inter-conceptual</td>
</tr>
</table>

d) For what value of *x* is it true that the value of *y* is 30 in table 2? Explain your answer thoroughly.	
Final questions: a) Did you identify any similarities or differences between the three tasks worked on? What mathematical concepts are involved? b) What is a linear equation? c) What is a linear function? d) Will the equation and the linear function have any relationship? Explain and give examples.	Reversibility Meaning Feature Different representations Inter-conceptual.

We included what we refer to as "final questions" in Table 2 because we anticipated the possibility that some students might not clarify the meaning of the equation and linear function in their arguments while solving the given tasks. Additionally, there was the potential for these students to encounter difficulties in making the inter-conceptual connection in all the tasks. That is, we tried to anticipate questions for those participants who were unable to make the central mathematical connection.

Data Analysis

Data analysis was made using simplified stages of the thematic analysis proposed by Braun and Clarke (2006). This method was considered adequate for the aim of this research because it is useful to carry out an analysis based on theory and even for few data. In addition, it allows analyzing written and verbal productions together. In this sense, for the interests of this research, the following steps were undertaken:

Phase 1: Familiarization with the Data

At this stage, a transcript was made of all the arguments and justifications offered by the students in the video recordings when they solved the proposed tasks. In addition, the students' written productions were simultaneously observed to identify the relationship between what each student writes and what each student argued verbally.

Phase 2: Generate Initial Themes

For this research, a "theme" was conceptualized as a form of mathematical connection. Therefore, to achieve this phase, both the verbal and written productions of the students were reviewed to identify "true relationships between two or more ideas, concepts, definitions, theorems, procedures, representations and meanings among themselves", that is, the first mathematical connections that the students made when solving the proposed tasks. For example, when Alexandra explained "to find the perimeter of a chain made up of one hundred regular hexagons, I multiply one hundred by four and add two, the result is four hundred two centimeters" it is initially emerged as a procedural mathematical connection, whereas when she argued "to find the perimeter of a chain formed by a number n of regular hexagons, the formula $p(n) = 4n + 2$ (she writes the expression) is used" allowed us to identify the part-whole mathematical connection.

Phase 3: Review and Definition of Themes

In this phase, the initial themes recognized in the previous phase were reviewed and contrasted with the written productions of the students. Once we had gained confidence in the pattern of meaning conveyed by the students in their procedures and arguments, we were able to objectively identify the mathematical connections established by them. Theme definitions were grounded in the students' productions and the relationships they constructed. For example, in the case of Alexandra we were able to identify that when she explained that “to find the perimeter of a chain formed by a number n of regular hexagons, the formula $p(n) = 4n + 2$ is used" she made the part-whole mathematical connection because the whole was the proposed expression ($p(n) = 4n + 2$) and the part was when she validated that formula with particular cases, for example, when $n = 2, 7, 20$ and 100.

Phase 4: Report Elaboration

The fourth phase of our analysis involved the preparation of the research report. This report was crafted with due consideration of the preliminary framework, specifically focusing on the students' levels of mathematical understanding. To achieve this, we opted to provide illustrative examples representing each identified level, supported by the mathematical connections made by the students. Additionally, we recognized the significance of highlighting certain difficulties and errors that surfaced in the participants' productions. This served the dual purpose of better justifying their assigned level of understanding and offering an overview of their mathematical reasoning processes.

Results

Through the analysis of the data, we successfully identified the anticipated mathematical connections outlined in our framework. By evaluating the quality of these mathematical connections evident in the students' arguments (comprising both verbal explanations and written productions), we were able to categorize them into various levels of understanding, considering their performance across all the assigned tasks (see Table 3). It is important to note that none achieved level 4; however, the other three levels of mathematical understanding were identified. Furthermore, these results played a fundamental role in refining the preliminary framework employed in this research (See Table 1), whose applicability extends to other mathematical concepts. This aspect will be explored further in the discussion and conclusion section.

Table 3

Level of Mathematical Understanding of Participants

Level	Students' Name[2]
3	Alexandra
2	Emir Monserrat
1	Christian Rafael Mario
0	Luis Maria

Mathematical Understanding Level 0: The Case of Luis

Luis' written and verbal productions allowed us to classify his level of mathematical understanding of the linear equation and function as level 0. This assessment is grounded in the inconsistencies evident in the relationships he attempted to establish across all the tasks. He exhibited difficulties in his arguments and errors in his procedures, suggesting that Luis struggled to grasp the underlying mathematical concepts. Instead, he relied on common sense to attempt the task solutions, albeit with limited success.

When faced with Task 1, Luis initially perceived the questions as straightforward and concluded that no formulas were necessary to address them. Consequently, he attempted to calculate the perimeter of a figure consisting of two hexagons by manually counting their sides. However, when tasked with finding the perimeter of a figure composed of 100 hexagons, he

[2] We use pseudonyms to protect the identity of the participants.

faltered due to his inability to create a drawing of the figure and count its sides accurately. Similarly, when asked for the algebraic expression to compute the perimeter of any figure, he wrote $p = nl$ (where n represents the number of sides, and l signifies the side length). This answer likely stemmed from his Geometry and Trigonometry course, where he learned this formula for calculating the perimeter of a regular polygon with *n* sides of length l. Nevertheless, the expected answer for this task was the expression $p(n) = 4n + 2$ or a variation thereof, where *n* indicates the position of each figure, and $p(n)$ denotes the perimeter of the chain formed by *n* regular hexagons, each with sides measuring 1 centimeter.

In Task 2, Luis confronted difficulty in understanding the problem and the questions inquired. He contemplated the time variation more in line with common sense, concluding that the water level in the container would not remain constant. However, he failed to consider the provided functions that calculated the water's behavior in the container. Consequently, Luis tried to grasp the meaning of the "8" in the expression $w = t + 8$, where *w* represents the water height in centimeters, and *t* signifies the time measured in minutes. Although he believed that the container already contained water when he opened the tap, he couldn't provide a satisfactory explanation. This misunderstanding led to another error: he assumed that for the water in the container to reach a height of 10 cm, precisely 10 minutes must elapse. This conclusion was based only on the range of *t* changing from 0 to 10 as indicated in the problem, without utilizing the provided functions to analyze how *t* and *w* changed during that time.

In Task 3, which included a table of values and the algebraic representation of a function ($y = 2x + 1$), with the task of finding the algebraic representation for the function associated with a second table of values, where the *x* and *y* values were exchanged, Luis incorrectly wrote $y = 1x + 2$ (which he drew in the cartesian plane). He did not demonstrate whether this expression corresponded to the values presented in the second table. This answer highlighted Luis's struggles in understanding the concept of linear functions, inverses of linear functions, and related ideas such as slope, *y*-intercept, numerical function values, and linear equations.

Mathematical Understanding Level 1: The Case of Christian

Christian has been classified at mathematical understanding level 1 due to his performance in solving some of the assigned tasks, which sets him apart from Luis. Additionally, Christian demonstrated the ability to make procedural, feature, and different representation connections, albeit the latter was only evident when he created a graph from the values provided in a table of values (see Table 4). This particular skill appears to be closely linked to his classroom instruction. However, Christian encountered difficulties when attempting to correlate the second table of values (presented in task 3) with the requested function (algebraic representation) in the task.

Table 4

Mathematical Connections Made by Christian

Task	Mathematical Connection Identified
1	Procedural
2	Procedural
3	Different representations
Final questions	Feature

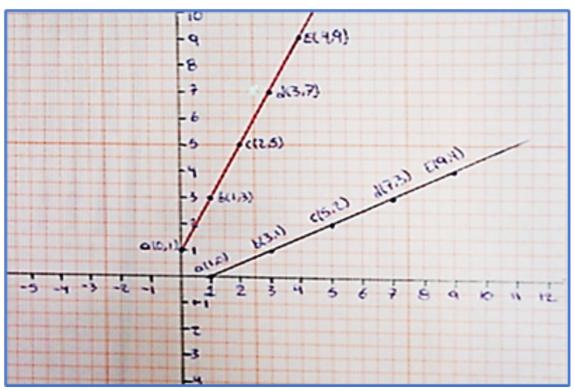
In Task 1, Christian exhibited the ability to make only the procedural mathematical connection (refer to Table 4). He utilized the figures provided in the task as tools to calculate the perimeter of chains composed of two or seven hexagons. However, when tasked with determining the number of hexagons in an 82-centimeter chain, he simply divided this value by six. This indicated a lack of understanding regarding the pattern followed by the figures presented in the problem. While Christian correctly recognized that hexagons have six sides, he failed to comprehend the pattern these figures followed, namely the concept of an arithmetic progression. Therefore, he mistakenly formulated the expression $p = 6l$ (where l represents the figure number) to calculate the perimeter of any given figure.

In Task 2, Christian claimed that the container filled with water at a constant rate from $t = 0$ to $t = 10$ minutes. This misconception occurred because he did not consider the information provided by the task, specifically the functions describing the water's height as a function of time. He justified his reasoning by contemplating the pressure or force with which the water entered the container, without factoring in the piecewise functions provided. Consequently, Christian overlooked the initial value given in the expression $w = t + 8$ (which models the container's filling during the initial four minutes), despite acknowledging the presence of an initial amount of water in the container. This difficulty in interpreting the piecewise function led him to conclude that if the water in the container reached a height of 10 cm, approximately 3 minutes had elapsed because 3 multiplied by 3 equals 9 (a calculation seemingly based on another part of the function: $w = 3t$), which closely approximated the desired 10 cm. On the other hand, if the water reached a height of 24 cm, Christian believed that 8 minutes had occurred, deducing this because 3 multiplied by 8 is 24.

In Task 3, Christian struggled to grasp the concept of inverse functions, hindering his ability to make a mathematical connection of the reversibility type. Consequently, he could only generate a graph from the values presented in Tables 1 and 2 (see Figure 1). Christian's proficiency in plotting points on the Cartesian plane, acquired during his Analytical Geometry course, enabled this task accomplishment. However, he did not conceptually grasp the essence of the task. He argued that "both lines are equal and pass through the same points" (possibly referring to what both are straight lines). While he made this assertion, he failed to identify other characteristics, such as similarities and differences, between these mathematical concepts. Consequently, he could not make the inter-conceptual mathematical connection in Task 3, nor in the other two tasks proposed.

Figure 1

Graphics Built by Christian to Solve Part of the Task 3



Mathematical Understanding Level 2: The Case of Emir

Emir's written answers and arguments led us to classify his level of mathematical understanding as level 2. This assessment is attributed to Emir's ability to make the reversibility mathematical connection between a linear function and its inverse in Task 3 (refer to Table 5). However, he did not recognize the mathematical relationship between a linear equation and a linear function, including their characteristics, properties, and meanings. Emir, therefore, failed to make the inter-conceptual mathematical connection in all three proposed tasks. Furthermore, he did not consistently employ this mathematical connection to solve the assigned tasks and substantiate his responses. Emir's answers imply that he perceives these concepts (equation and function) as synonymous or equivalent.

Table 5

Mathematical Connections Made by Emir

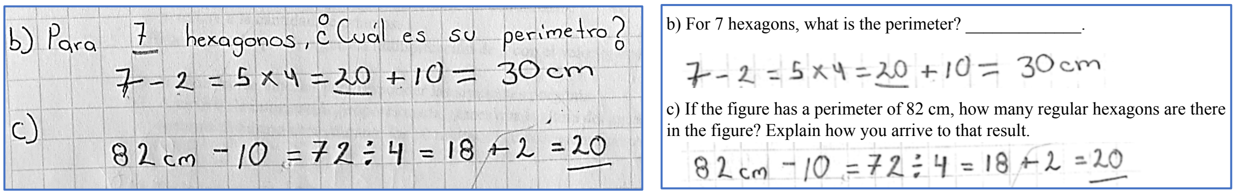
Task	Mathematical Connection Identified
1	Procedural Part-whole
2	None
3	Procedural Different representations Reversibility
Final questions	Feature

Similar to Christian, Emir also demonstrated a procedural mathematical connection in task 1, specifically for sections *a, b*, and *c*, as depicted in Table 2 and Figure 2. To achieve this, Emir recognized that in determining the perimeter of chains containing 2 and 7 hexagons, he could simply subtract 2 from the total number of hexagons (denoted as '*n*'), multiply the result by 4, and then add 10. Consequently, when faced with the challenge of finding the number of hexagons in a chain with a perimeter of 82 cm (as presented in section c of task 1), he applied the same rule he had previously inferred, although in a reverse manner. While Emir grasped the underlying concepts of these procedures, it is worth noting that his mathematical syntax may not have been entirely precise, as illustrated in Figure 2.

Emir's deduced rule not only enabled him to perform tasks similar to Christian but also facilitated the making of a part-whole mathematical connection, which is pivotal for sections *d, e,* and *f* of task 1, as outlined in Table 2. In this context, Emir arrived at a mathematical model for calculating the perimeter of any figure based on the parameter 'n', representing the number of hexagons comprising the chain. His formula took the form of $p = 4(n - 2) + 10$, which can be seen as another representation compared to the conventional $p = 4n + 2$ representation. This modified mathematical model, $p = 4(n - 2) + 10$, served as a valuable tool for Emir to confirm its accuracy. As an illustration, when 'n' equaled 100 hexagons ($n = 100$), he accurately calculated the perimeter of the hexagon chain to be 402 cm by substituting the value of 'n' and executing the necessary operations.

Figure 2

Procedure Performed by Emir to Solve Sections b and c of Task 1



However, it's worth noting that while Emir consistently succeeded in solving task 1, his arguments did not explicitly highlight his recognition of the relationship between the linear function used in section e and the linear equation in section d as a critical element for validating his result in section *f*. Emir missed the opportunity to make an inter-conceptual mathematical connection, for instance, by providing examples or creating a graphical representation to illustrate these relationships. Such efforts could have helped him recognize how these concepts relate to other associated mathematical ideas.

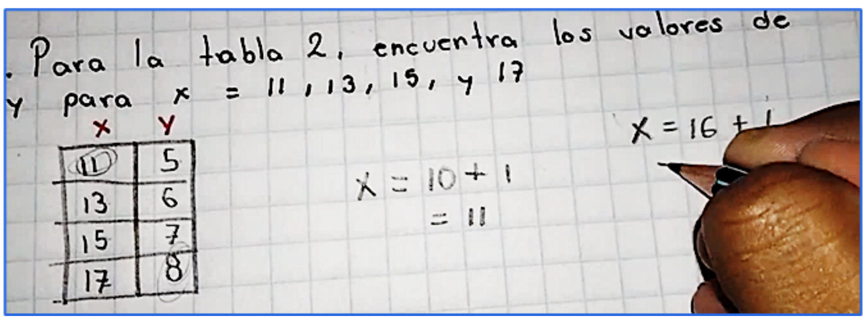
In task 2, Emir encountered challenges in grasping the use of the provided functions within the context of the tasks. For instance, when addressing the problem, he noted that the rate at which the water's height in the container increases wasn't uniform. However, his justification for this was only based on the time variation, the significance of the piecewise function provided in the task. This difficulty in understanding the concept of piecewise function led to an incorrect reply where he concluded that "if the expression $w = t + 8$ allows us to calculate the height of the water in the container from $t = 0$ to $t = 4$ minutes and the expression $w = 3t$ allows us to calculate the height of the water in the container from $t = 4$ minutes to $t = 10$ minutes (Table 5), then, for the water to reach a height of 10 cm or 24 cm in a container, 5 and 12 minutes must pass, respectively". Essentially, he failed to establish the correct equation to answer the proposed question.

It appears that Emir mistakenly divided the value of height (w) by two, indicating a misunderstanding of the relationship between the change in time (t, in minutes) and the change in height (w, in centimeters). Similar to his experience in task 1, Emir struggled to make an inter-conceptual mathematical connection in task 2. His arguments did not demonstrate recognition of the mathematical relationship between the various mathematical concepts presented in the given task.

In task 3, Emir successfully obtained the inverse function of $y = 2x + 1$ as $x = 2y + 1$, and he efficiently validated his results by calculating values from Table 2 (Figure 3). This accomplishment demonstrated Emir's ability to make reversibility and different representations mathematical connections. The different representations mathematical connection became evident as he utilized the expression $x = 2y + 1$ to verify his proposed values for y when x equaled 11, 13, 15, and 17, respectively. Furthermore, Emir employed a table of values to document his numerical computations, which served as a crucial component of his argument to substantiate his findings (as seen in Figure 3). So, Emir demonstrated exceptional versatility as he effortlessly navigated through various mathematical registers, including algebraic, numerical, tabular, and graphical representations. This showcased his high level of proficiency in effectively handling diverse representation registers.

Figure 3

Some of Representations Used by Emir to Solve Part of Task 3



To summarize, unlike Christian, who only partially solved the given tasks, Emir consistently tackled tasks 1 and 3, successfully making mathematical connections, including procedural, part-whole, reversibility, and different representations (as shown in Table 5). However, Emir encountered difficulties in solving task 2, and in his explanations, the use of inter-conceptual connections was not clearly evident across all three tasks, even though he mentioned in the final questions that "all three tasks involve equations and graphs". These results lead us to infer that Emir's level of mathematical understanding falls within level 2.

Mathematical Understanding Level 3: The Case of Alexandra

The results clearly highlight Alexandra's remarkable ability to consistently solve all the tasks presented, particularly those that directly involve equations and functions. She demonstrated proficiency in representing and recognizing patterns, as well as a sound understanding, at least procedurally, of when to apply a linear equation versus a linear function. In accordance with the framework employed in this research, it is evident that Alexandra has achieved level 3 of mathematical understanding. This level signifies her aptitude for making a wide range of mathematical connections, with a notable emphasis on the more intricate ones such as reversibility and inter-conceptual connections. These advanced mathematical connections require a deep conceptual knowledge of the mathematical concepts involved in the tasks, which Alexandra has clearly demonstrated (as indicated in Table 6).

Table 6

Mathematical Connections Made by Alexandra

Task	Mathematical Connection Identified
1	Procedural Part-whole Feature
2	Procedural Feature Meaning Different representations
3	Procedural Feature Different representations Reversibility
Final questions	Inter-conceptual Different representations Part-whole Meaning

In contrast to Emir's approach, Alexandra exhibited a higher level of mathematical understanding and made additional mathematical connections in the tasks.

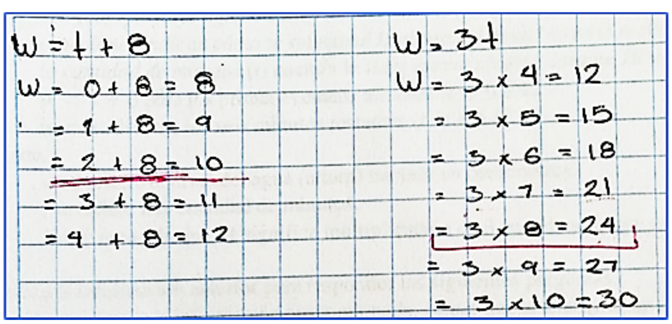
In task 1, Alexandra not only makes the procedural mathematical connection but also made the feature type mathematical connection. This connection emerged as she observed that "the perimeter of the figure increases progressively by 4 centimeters as the number of hexagons in the chain increases," leading her to derive the formula $p = 4n + 2$, where 'p' represents the perimeter and 'n' the number associated with the figure. This insight enabled her to establish the part-whole mathematical connection. While her procedural mathematical connection resembled Emir's, her ability to recognize and utilize patterns and features added depth to her understanding.

In task 2, Alexandra outperformed Emir by consistently solving the task and making various mathematical connections, including procedural, feature, meaning, and different representations. For instance, the feature mathematical connection emerged as she discerned a pattern in the height of the water in the container relative to time. She noted that "the rate of increase in the height of the water in the bowl during the first 4 minutes is one centimeter (using the formula $w = t + 8$), while from 4 to 10 minutes the increase was at a rate of three centimeters (utilizing the formula $w = 3t$ as provided in the problem)". To further support her findings, she calculated values for 'w' based on discrete 't' values (as shown in Figure 4).

Alexandra's ability to make the feature mathematical connection also facilitated her showing of the meaning mathematical connection. She recognized that the '8' in the expression $w = t + 8$ (from task 2) indicated the initial height of the water when the container started filling. She argued that if time is zero ($t = 0$), then this value represented the height of the water in the container when the water faucet was first opened. Additionally, she used the meaning mathematical connection when she defined a linear equation as "an equality where one or more unknowns appear raised to the power one" during the final questions.

Figure 4

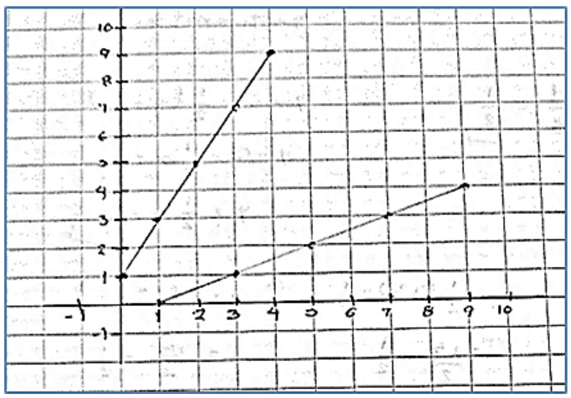
Computations Performed by Alexandra to Make the Feature Mathematical Connection



In task 3, Alexandra not only continued to demonstrate her proficiency in various mathematical connections but also expanded her understanding by making additional connections, including the reversibility and inter-conceptual connections (final questions). Alexandra's ability to make the reversibility type mathematical connection was evident as she recognized the inverse relationship between the two functions graphically and in the table of values. She pointed out that they were inverse functions and validated this by noting that their slopes were reciprocal. Her graphical observation of the graphs moving apart further supported her argument (Figure 5). To solidify her case, she even calculated the slope and the equation of the line (as shown in Figure 6) using principles from Analytical Geometry.

Figure 5

Graph of Inverse Linear Functions Made by Alexandra

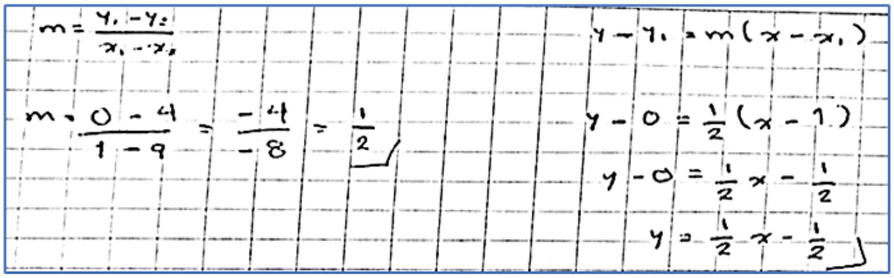


Additionally, Alexandra skillfully employed various mathematical connections throughout task 3. She utilized the procedural mathematical connection when calculating the slope of the line associated with the linear function based on the values from table 2. The connection of different representations came into play as when performing a treatment of the equation of the line in the register algebraic. She also used the part-whole mathematical connection when recognizing and expressing the general representation of the equation of a line, along with the specific case she worked with. Alexandra not only executed these actions but also provided clear justifications for each step in her process. These mathematical connections appeared at other moments when Alexandra solved task 3. At another time, she made the mathematical connections of feature type when indicating that "the points that are associated or form part of a straight line are collinear".

Moreover, she made the inter-conceptual mathematical connection (Table 6) by recognizing that "a linear function graphically represents a straight line and that it helps to find the value of the unknown of a linear equation associated with it". This answer showcased her recognition of the mathematical relationship between linear equations and linear functions, as well as associated concepts like the slope of a straight line.

Figure 6

Procedures Made by Alexandra to Solve Part of Task 3



Alexandra was not classified as level 4 of mathematical understanding. Despite her use of reversibility and inter-conceptual mathematical connections, the latter only becomes evident in the final questions, rather than being explicitly present within the three proposed tasks. Consequently, Alexandra did not employ the inter-conceptual mathematical connection as a primary argument for making other mathematical connections during her process of solving the proposed tasks. This made it difficult to identify the central mathematical connection that Alexandra used as an argument to make other mathematical connections.

Discussion and Conclusion

The research question that was asked is: What levels of mathematical understanding do Mexican High School students achieve when making mathematical connections to solve tasks involving equations and linear functions? To attend this inquiry, we employed an instrument comprising three tasks accompanied by a set of final questions (refer to Table 2), complemented by the use of the thinking aloud method for data collection. The classification of students' levels of mathematical understanding was made possible by identifying various mathematical connections evident in their answers, as per the expectations outlined in our framework. Consequently, our findings reveal that among the eight participants, all of whom were enrolled in the same Analytical Geometry course, instructed by the same teacher, and demonstrating varying degrees of performance according to their teacher's assessment, they exhibited distinct levels of mathematical understanding when solving the proposed tasks. One student achieved level 3, two reached level 2, three achieved level 1, and two were classified at level 0.

It's crucial to note that a student hypothetically achieving level 4 of mathematical understanding would ideally make the same (or most) mathematical connections as Alexandra did. Moreover, they would explicitly incorporate the inter-conceptual mathematical connection into their arguments. The presence of this inter-conceptual mathematical connection could have been identified in task 1 (see Table 2) if the hypothetical student recognized that the task encompassed the notion of an arithmetic progression while also relating to concepts such as the perimeter of a regular polygon, linear equations, linear functions, and other fundamental characteristics distinguishing these concepts. Similarly, in task 2, the hypothetical student might recognize the concept of rate of change, piecewise function, linear equations, linear functions, initial values of a function, and employ them cohesively to solve the task. In task 3, the hypothetical student could identify concepts like slope, rate of change, inverse functions, and linear equations. The inter-conceptual connection would serve as the central mathematical linkage, allowing the hypothetical student to make other mathematical connections. Therefore, when evaluating a student's level of mathematical understanding concerning specific mathematical concepts, it is essential to design tasks encompassing various contexts to uncover the most complete mathematical understanding possible.

In this study, we acknowledge that using the "thinking aloud" method as a data collection technique may have limitations in identifying the central mathematical connection within participants' arguments. Therefore, future research should explore participants' arguments more extensively through methods such as task-based interviews.

Furthermore, our findings suggest that the framework proposed by Campo-Meneses and García-García (2021) can be extended to encompass mathematical concepts like linear equations and linear functions. However, it will be important to develop more research to extend its use to different mathematical concepts. In general, we believe that this framework could prove valuable for studying any function and its inverse. Overall, our results align with those reported by Campo-Meneses and García-García (2021), indicating that students face challenges in achieving high levels of mathematical understanding. To strengthen these findings, future research should involve a larger number of participants and incorporate diverse methods to gain a more comprehensive understanding of this issue.

Based on our results, we have refined the preliminary framework by incorporating explicit indicators for assessing a student's level of mathematical understanding based on their ability to make various mathematical connections (Table 7). We believe this refined framework will be practical when conducting interviews to recognize the use of a central mathematical connection within students' arguments while they solve proposed tasks. This refinement addresses a limitation in this research due to the data collection method employed. Therefore, future research should consider replicating this study using different data collection methods to obtain a more comprehensive view of participants' levels of mathematical understanding when addressing tasks related to linear equations and linear functions.

Table 7

Framework to Assess Levels of Mathematical Understanding Based on Mathematical Connections

Level	Mathematical Connections Expected	Indicators
0	None	▪ A student provided mathematically inconsistent answers in all of the given tasks. ▪ Some of the student's arguments were based on common sense. ▪ The student used the data provided by the task inconsistently. ▪ Difficulties are identified in the students' arguments. ▪ Errors in the student's procedures are also identified.
1	Feature. Procedural.	▪ A student successfully completes some of the assigned tasks but not all. ▪ In certain instances, a student relies on procedural steps guided by formulas or memorized rules. ▪ A student recognizes certain characteristics or properties of the mathematical concepts involved in some tasks. ▪ In a few tasks, a student encounters challenges and makes errors, such as partially grasping the meaning of the data or executing incorrect procedures.

		▪ At this level, a student might make additional mathematical connections beyond those typically expected but they may not be crucial for consistently solving the assigned tasks.
2	Feature. Procedural. Part-whole. Different representations.	▪ A student successfully completes some of the proposed tasks but not all. ▪ The mathematical connections used at level 1 continue to be applied in this level. ▪ In contrast to level 1, at this level, a student demonstrates an understanding of logical relationships between mathematical concepts and/or properties relevant to certain tasks. ▪ A student makes the mathematical connection of different representations to solve some of the proposed tasks. ▪ Beyond the more common mathematical connections expected at this level, a student might utilize additional ones, although they may not be essential for solving all the assigned tasks. ▪ The absence of the mathematical connections of reversibility (depending on the mathematical concepts involved) or the inter-conceptual connection at this level can pose challenges for a student in solving all the tasks.
3	Feature. Procedural. Part-whole. Different representations.	▪ At this level, a student consistently solves all tasks. ▪ Some or even all the mathematical connections used in level 2 continue to be used in this level. ▪ In contrast to level 2, a student adeptly employs the mathematical connection of different

	Meaning. Reversibility. Inter-conceptual.	representations across more than two representation registers. ▪ A student demonstrates the ability to attribute meaning to the mathematical concepts involved in the tasks and their numerical solutions within the task's context. ▪ While a student occasionally makes the mathematical connections of reversibility or inter-conceptual in certain proposed tasks, it may not be discernible as the central mathematical connection within their arguments.
4	Feature. Procedural. Part-whole. Different representations. Meaning. Reversibility. Inter-conceptual. Central mathematical connection.	▪ A student consistently solves all tasks proficiently. ▪ Some or even all of the mathematical connections employed in level 3 are consistently applied at this level. ▪ A student adeptly identifies both similarities and differences among the proposed tasks. ▪ Through their written and verbal productions, a student showcases a profound understanding of mathematical structures and the relationships among the concepts involved, demonstrating a genuine capability for doing mathematics. ▪ In a student's arguments, a central mathematical connection is clearly recognized. This connection serves as the foundation for justifying their application of other mathematical connections in all tasks.

We believe that further research is essential to continue refining the indicators recognized in Table 7. Additionally, we speculate that if a student were to attain level 4 of mathematical understanding, he or she might be capable of making extra-mathematical connections. These connections emerge when a student establishes a true relationship between mathematical concepts or ideas and concepts from other disciplines, sciences, or real-life situations, using them consistently to solve problems. However, this hypothesis should be validated or refuted in future research.

Our findings also indicate that students' arguments reveal their difficulties and errors in the solving process of the proposed tasks. These challenges hinder their ability to make mathematical connections, as also reported by García (2018). Moreover, we identified that students who achieved levels 0 and 1 of mathematical understanding often struggle to grasp the concept of a function expressed in a table of discrete values or through its algebraic representation. They also encounter difficulties in formulating linear equations to determine unknown values within certain aspects of the proposed tasks. Assigning meaning to the result within the task's context is another challenge expressed by students at these levels of mathematical understanding.

Regarding the identified mathematical connections, our results align with the existing literature that shares a similar approach (Businskas, 2008; García-García & Dolores-Flores, 2018, 2021a, 2021b; Rodríguez-Nieto, Font et al., 2021; Rodríguez-Nieto, Rodríguez-Vásquez, et al., 2021; Campo-Meneses & García-García, 2020, 2021). Specifically, we identified mathematical connections such as feature, procedural, part-whole, different representations, meaning, and reversibility in this research. Also, we identified at least in one of Alexandra's task resolutions, the inter-conceptual mathematical connection. Future research could provide more empirical evidence of how students use this mathematical connection and how it could serve as the central mathematical connection in their arguments, facilitating the establishing of other mathematical connections.

Our results partially align with other research indicating that students encounter difficulties in making the mathematical reversibility connection, even when dealing with mathematical concepts other than linear equations and functions (Weber, 2002; Campo-Meneses & García-García, 2020, 2021; García-García & Dolores-Flores, 2021a). Out of the eight participants, only three were able to make this mathematical connection. We believe that, both this mathematical connection and the inter-conceptual one, are among the most challenging for students to make. Therefore, future research could focus on designing instructional processes that promote the utilization of these connections by students when solving tasks involving various mathematical concepts. The indicators listed in Table 7 could serve as valuable tools for achieving this objective, fostering mathematical understanding in students through the establishment of mathematical connections.

Finally, regarding linear equations, our results align with those of Jupri, Drijvers and Heuvel-Panhuizen (2014), who found that students struggle with arithmetic skills, understanding algebraic expressions, and grasping the concept of variables. Furthermore, students often make errors when solving mathematical tasks involving linear equations due to difficulties in interpreting and understanding the sentences presented, as also reported by Adu, Assuah and Asiedu-Addo (2015). These challenges also manifest when students tackle tasks related to the concept of linear functions. Our findings suggest that students may face difficulties in interpreting the algebraic representation of piecewise function, assuming that it should only be expressed using an algebraic representation, as also reported by Borke (2021). Among the eight participants, only one (Alexandra) employed the concept of slope to determine the required linear function. In this regard, we concur with Tanışlı and Kalkan (2018), who highlighted the need for students to enhance their utilization and interpretation of functions and slopes while considering multiple representations.

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