The Mathematics Enthusiast

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Part 1: Mathematical Alchemy

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Abstract: In this set of four articles, we invite Calculus teachers to encounter some of the ideas that shaped and motivated the innovation of the subject, presented in story form. We find that in traditional treatments of these topics, the dramatic tension between ways of thinking, representations, and phenomena in the world can be lost in a sea of calculations, procedures, and partially-understood formal proofs. However, the fundamental ideas and insights can be identified in nascent form in students' ways-of-thinking about Calculus, and in their struggles to connect Calculus with their prior mathematical experiences.

We are honored to contribute these Stories of Calculus to this number of *The Mathematics Enthusiast*, honoring the life and thinking of David Tall. David's work with technology in Calculus learning brought him into close contact with Jim Kaput, the SimCalc Projects, and the Kaput Center. He and Jim discussed ideas that came to represent his "three worlds" of mathematics, a story he tells in part in his contribution to the volume, *The SimCalc Vision and Contributions* (Hegedus & Roschelle, 2013). His ideas, and his sense of both the challenges of learning Calculus and the potential transformations offered by digital and executable representations, are coherent with the approach we are advocating in these articles.

Keywords: Calculus teaching and learning, mathematical foundations, stories of ideas, indivisibles, Cartesian graphs of functions, infinitesimals.

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> Nature has been defined as a 'principle of motion and change,' and it is the subject of our inquiry. We must therefore see that we understand the meaning of 'motion'; for if it were unknown, the meaning of 'nature' too would be unknown.
>
> …Now motion is supposed to belong to the class of things which are *continuous*; and the *infinite* presents itself first in the continuous—that is how it comes about that 'infinite' is often used in definitions of the continuous ('what is infinitely divisible is continuous').
>
> – Aristotle, *Physics*, Book III, Ch 1, emphasis added

Introduction

This set of articles is written for Calculus teachers, and for educators who are interested in Calculus as a subject area whose development was produced through a dynamic interplay among powerful ideas and perspectives, radically expanding the reach of science as quantitative inquiry into phenomena in the natural world.

Our goal is to *animate* these ideas, capturing the drama of their interaction in fictional characters based loosely on historical figures who carried these ideas forward. The result is not a historical account; it is a collection of narrative vignettes: stories that illustrate the generative tensions among ideas from different branches of mathematics. We aim to present the *Bildung* of the ideas of Calculus through accessible stories of development for students and teachers of the subject.

Our rationale is that idea development at the foundations of Calculus occurs across *ontogenetic*, *sociogenetic* and *phylogenetic* scales. The sociogenetic development of the subject exhibited accidental particularities of history, but its unfolding expressed underlying necessities in the structures of disciplinary ideas. Moreover, at this level the social aftereffects of the innovations of Calculus have been enormous. Virtually all aspects of modern technological life bear their imprint. These dynamics also echo at the individual (ontogenetic) and evolutionary (phylogenetic) scales.

At the level of the student, learning Calculus provokes a set of crises, both for the conceptual systems and the symbol systems that learners have developed over their career with school mathematics. These crises relate to ideas about the infinitely large and infinitely small, in the context of modeling phenomena of change, variation, and accumulation. Moreover, these crises are fundamental: they were felt to be so by Aristotle, as witnessed by his quote in the epigraph, in which phenomena of change and motion are described as the essence of Nature; in turn, they are seen as essentially dependent on concepts of infinity and continuity. And at the *evolutionary* level, the crises provoked in accommodating the ideas of Calculus reach down to roots in the fundamental nature of humans' use of mathematical objects and symbolic actions.

In the classroom, these dynamics can also be realized at an interpersonal scale, by provoking conversations among the ideas that students raise about the conceptual problems of Calculus. Even more: these historically-inspired Stories of Ideas may serve the teacher as a guide to how to orchestrate discussions in class that can re-animate their generative interaction. The precise storyline will likely be different—just as our stories are different from history—but the dramatic interactions may be familiar.

Beginning in this article, we will *re-present* dynamics at the foundations of Calculus, interleaving commentary with episodes from larger stories of ideas. Broadly, we begin with mathematical *intuition* and the roots of mathematical concepts; later, we discuss *symbols* and *symbolic action*; and finally, we discuss processes of *systematization* and *formalization*. These ideas will be foregrounded by stories of the generative tensions at the roots of Calculus across three branches of mathematics and their perspectives on *continuity*. In a geometric perspective, the concept of the *indivisible* arises as a means to navigate the divide between the polygonal and the smooth. Next, in a modeling and data analytic perspective, novel graphical representations mediate discrete data and analytic models and connect with the geometric tradition. And finally, in an algebraic/arithmetic perspective, the construct of the *infinitesimal* provides an accessible and reliable means of operating coherently with paradoxical expressions that combine infinitely large and infinitely small quantities.

Dramatis Personae and Key Sources

In developing our stories of ideas, we have identified main characters, whose perspectives found expression in several historical mathematicians. These are:

Sereno.	Inspired by Archimedes.
Guillermo.	Inspired by Leibniz.
Nicolas.	Inspired by Oresme.
Peter.	Inspired by Fermat
Leonardo.	Inspired by Euler.
Juan.	Inspired by Wallis.
Carlos Federico.	Inspired by Gauss.
Agustín.	Inspired by Cauchy.
Ricardo.	Inspired by Dedekind.
Jorge.	Inspired by Cantor.

Felix.	Inspired by Klein.
Teodoro.	Inspired by Weierstrass.
Jaime.	Inspired by Pierpont.

In addition to these characters, several historical mathematicians enter peripherally into the narrative; for these figures, we have maintained their historical names.

Before introducing the first of our characters, we want to note four works in particular that have supported and inspired our thinking as we conceived of these stories. First is *The Historical Development of the Calculus* by C. H. Edwards. This book is a treasure-trove of the history of ideas of Calculus. Moreover, Edwards captures key elements of the spirit of inquiry that drove major discoveries in the field. Second is Merlin Donald's *A Mind So Rare: The Evolution of Human Consciousness*. Donald offers wise guidance into the evolution—biological and cultural—of forms of thought, and into the sources of human inspiration and insight—environmental, social, and cognitive. The final two texts are works written by two of the historical mathematicians that have inspired characters in our stories. In his *Introduction to Analysis of the Infinite* (two volumes), Leonard Euler gives insights into his way of operating with series, which enabled him to extract results like a magician—results that cut to the hidden structure of mathematics. And, for Archimedes, a series of fortunate historical accidents enabled the rediscovery of his letter to Eratosthenes, in which he describes his "Mechanical Method" for generating fantastic geometric results. The history of this document (which is also a marvelous story), along with the reconstruction and translation of the text, appears in *The Archimedes Codex*, by Reviel Netz and William Noel.

Intuition and the sources of mathematical concepts

Calculus is a complex of ideas that captures basic intuitions of human beings. The cognitive capacity of people makes it inevitable that when we look around us, in addition to perceiving numerosity and shape, we can also perceive phenomena of variation and accumulation. Now, thanks to the symbolic capacity we possess, we can translate these experiences into symbolic models. Numerosity is translated into number, into arithmetic; form into geometry. And yet these translations carry with them aspects of their sources as a kind of sensory heritage. Geometric figures *inherit* properties of the material forms that gave them origin. For phenomena of variation, the same holds: we have diverse experiences of the variation in temperature, in the intensity of light, or in the speed of a car driving along a road. In short, we have enough experiences to make the phenomena of variation and accumulation familiar to us. Well, Calculus is the mathematical (symbolic) structure that captures all these experiences, inheriting shades of their human meanings. And the stories of how this symbolic structure arose (and arises for every new learner) are tinged with these inherited experiences. For the sake of brevity, in this article we have focused on simple examples that make tangible the symbolic nature of the mathematical "objects" involved in the narrative.

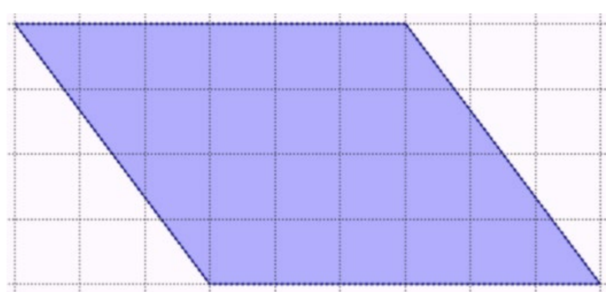
Sereno and Dissection

We would like to begin by introducing our friend Sereno and asking him to describe an approach to geometric arguments, which can sometimes appear as a game, and sometimes as a kind of *geometric alchemy*. It is called the “method of dissection,” and it is a way of converting one shape or figure into another. To prove that two shapes have the same area, we find a way to cut one of the shapes into pieces, which can then be assembled to *build* the other, or *cover it* without 'gaps' or ‘overlaps.'

To get a feel for dissection and the kind of thinking it promotes, Sereno asks us first to try to convert a parallelogram into a rectangle. Consider the shape in Figure 1. Do you see a way to cut this parallelogram into two pieces, which can then be rearranged to form a rectangle? (Note: there is more than one way to do this!).

Figure 1

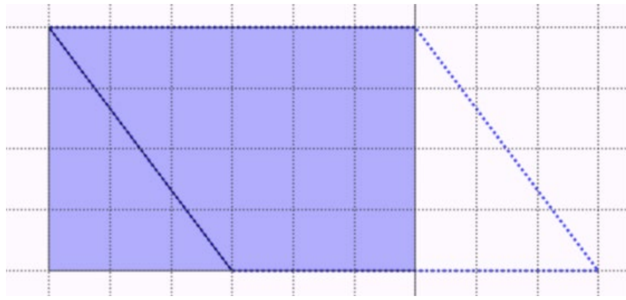
A Parallelogram to Dissect



The diagram below shows one possible solution. From the figure, can you see what cut we made, and how we put the pieces back together? Sereno explains: "Choose one of the sides of the parallelogram as the 'base' (Here, we’ve chosen the ‘top’ side.) Then, starting from a vertex of that base (here, the right-hand one), cut the parallelogram in a direction perpendicular to the base. This cut will produce a (right) triangle. Next, slide this triangle to the other side of the figure. Its hypotenuse is *parallel* to the other side, so the pieces will fit perfectly.

Figure 2

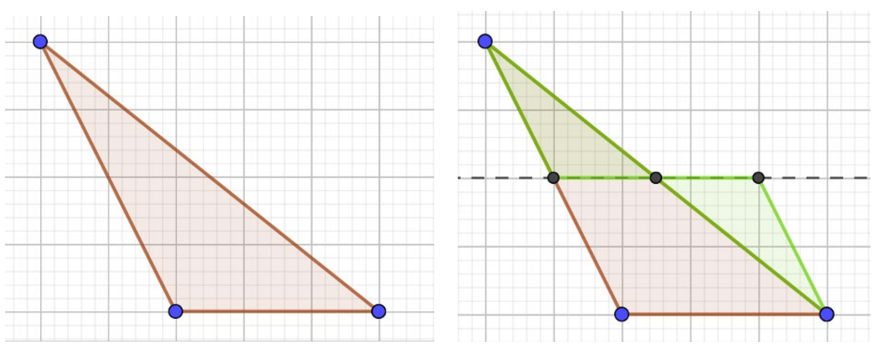
One Possible Dissection and Rearrangement into a Rectangle



What can dissection tell us about the area of triangles? The authors of many geometry textbooks ask you to think of triangles in terms of *half*-parallelograms. Here is one possible dissection:

Figure 3

Dissecting a Triangle into a Parallelogram



Cut the triangle at exactly half its height. Then, rotate the triangular 'tip' that results, 180 degrees about its bottom-right vertex. (Note: we could have used any of the triangle's sides as its "base" and cut at half the corresponding height.) Do you see why the rotated section of the triangle "matches" the bottom, and why the result is in fact a parallelogram? Now, look at the parallelogram we created. From the prior dissection, we know that this parallelogram's area is equal to its base multiplied by its height. But its height is half the height of the original triangle.

Commentary

The Method of Dissection captures some important themes about the nature of symbolic mathematical thinking and practice. It reflects an attitude to representations that draws on physical, embodied experience of the world but idealizes aspects of that experience. Dissection draws on embodied and sensory experience, and it invites interactive and enactive work with geometric shapes that foster 'involved' perspectives. Moreover, it creates a new system of reversible operations that broadens the ways that we can conceptualize geometric objects. And it invites a form of systematizing that can be seen in other areas when new objects, relations, and operations are introduced.

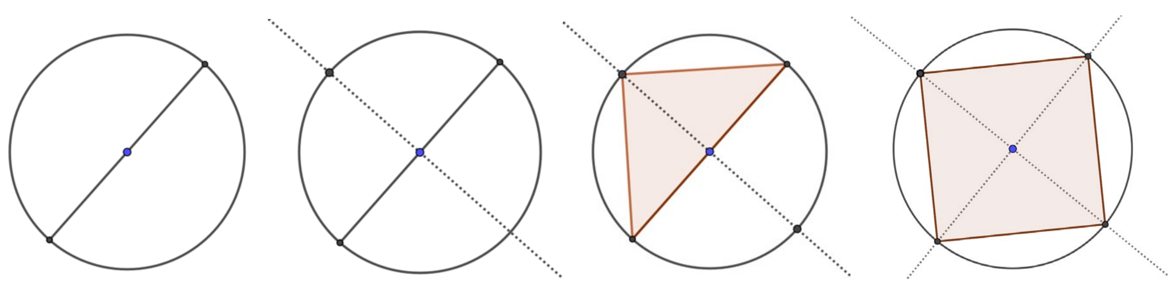
Sereno and an Infinite Process

We have a feeling that dissection seems somehow *limited*: after all, we only allow straight cuts! We ask Sereno, "With dissection it seems we only relate polygons to polygons? Or is it possible somehow to equate polygons with figures whose 'sides' are curvy?" Sereno smiles. He says that if we allow ourselves to consider making an *unlimited* or *undefined* number of cuts, we can relate curved and polygonal figures.

Let's consider simple curvy shape: a circle. Let's begin by identifying a diameter, as shown below. On either side of the diameter, we can raise a perpendicular from the center, till it meets the circle. Along with the two points defining the diameter, this creates an isosceles triangle on either side of the diameter. These will be our two first cuts out of the circle. Together, they make a square, which is inscribed in the circle.

Figure 4

First Steps: Cutting a Square Out of the Circle

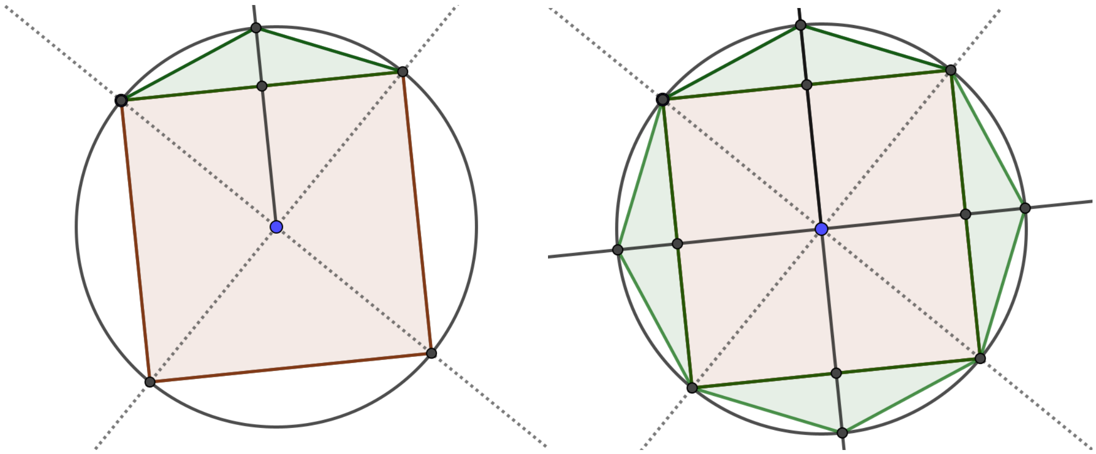


Now, inspired by that first pair of cuts, let's define a *process* that we can repeat indefinitely – without end – and that will cut a series of polygons out of the remaining parts of the circle.

At the second step, for each of the four sides of the square, we're going to cut out another isosceles triangle from the part of the circle 'above' that square's side. For one side, this is shown in green below. To find this triangle, we raise a perpendicular from the midpoint of the side of the square. The point at which it meets the circle will be our isosceles triangle's third vertex.

Figure 5

Progressing to an Octagon by Cutting Out Isosceles Triangles on Each Side

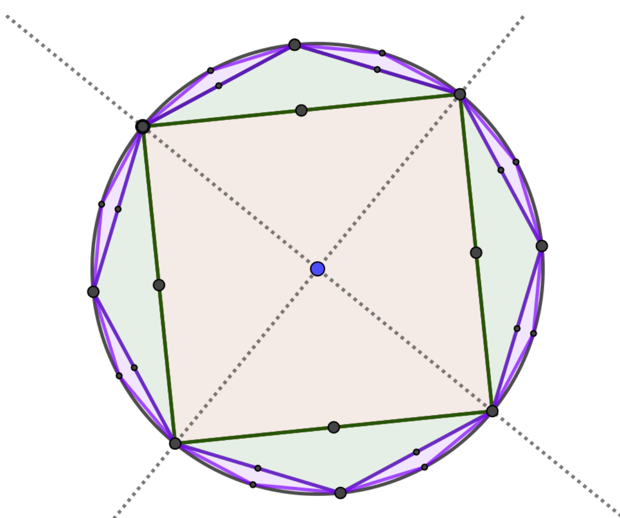


In the diagram above, you can see that the four resulting triangles combine with our original square to form a regular *octagon*, again inscribed in the circle.

Now, for each side of this octagon, there is another piece of area, bordered by the octagon's side and an arc of the circle. As before, we can cut an isosceles triangle out of each of *these* regions. At this third step, the 8 triangles come together with the octagon to form a regular 16-gon.

Figure 6

Third Step: The 16-Gon



This process can continue indefinitely: at each step, the isosceles triangle cut described above can be accomplished; it consumes more and more of the circular area and at the n^{th} step, and the total set of pieces up to that point forms a $2^{(n+1)}$-gon.

Note that at each step, we are cutting more and more pieces, of smaller and smaller sizes. At the n^{th} step, we have $2^{(n+1)}$ pieces. If we imagine our infinite process *completing*, we would have an infinite number of pieces–the sizes of each of which quickly become extremely small. Because we are cutting up the area of a finite object (the circle), we know that the 'infinite sum' of the areas of these pieces must add to a finite number.

The idea is puzzling, but we can imagine even more ordinary dissection processes that lead to similar conundrums. For example, let's imagine that we cut an everyday object—say, a loaf of bread—in a single direction/dimension, using slices of equal width. Suppose our bread is 40 cm long. If we cut it into 2 pieces, each one will be 20 cm long. With 10 pieces, each one will be 4 cm long. With 100, each one will be 0.4 cm long. As the number of slices increases, each time the width of each slice becomes smaller and smaller.

Now, if we continue this process of cutting, until we can no longer perceive the thickness of the pieces, we begin to conceive some surprising results. For example, we can imagine three-dimensional objects constructed from sheets, each of which *looks* to be two-dimensional (i.e., having zero thickness). Changing dimensions, we could have two-dimensional figures built up from sets of sub-figures that appeared to be mere line segments. Or, we can imagine one-dimensional segments or paths divided into collections of objects that appear to be simple zero-dimensional points.

Sereno now tells us that the style of thinking behind the Dissection Method brings about a "delicious paradox." He says, "Suppose we have completed the process of cutting up our loaf of bread into an infinite number of infinitely small slices. Now, like in a Dissection proof, we start building up another shape by taking slices and stacking them. Suppose we stack and stack, piling up a million slices. My question to you is, how high is our pile?"

A strange puzzle. We answer, "Even after a million, it cannot be very high! Maybe the thickness of a hair? Or one page of this book?" Sereno smiles. "Suppose it is so. Then, after a thousand million slices, the stack will be as tall as a very thick volume. And after ten million million slices, as tall as any building!"

“Then,” we say, “it must have no thickness at all!” Again, Sereno smiles, “Then, we can pile up as many slices as we want, and still our pile will have no height.”

Sereno mentions that Aristotle has hinted at the paradoxes of infinity and how they become more striking when the infinities of division (slicing in our case), and of addition (piling up slices) are considered together. (Our epigraph is one example of Aristotle’s commentary on this point.) Sereno tells us about a disagreement between his companions Zeno and Democritus around this issue.

“Zeno constructed the following absurd scenario: a 50-meter race between the hero Achilles and a tortoise. Suppose Achilles gives the tortoise a head start of, say, 10 meters, and we call this point T_0. Then, Zeno asked: When and where will Achilles catch up to the tortoise?

“Zeno proposed that Achilles can *never* catch the tortoise. Here is the logic: Achilles will take some time to run 10 meters - that is, to arrive at the tortoise’s starting point (T_0). During that time, the tortoise will also have advanced some distance; call this point T_1. So, Achilles is still behind. In the next phase of the race, Achilles will take a while to reach point T_1; During that time, the tortoise will again have moved forward a little, to a point we’ll call T_2. Achilles is still behind. But this process goes on endlessly! Every time Achilles advances to the next point T_N, the tortoise has run forward to a point $T_{(N+1)}$. So, Zeno says, ‘Achilles *never* will be able to reach the tortoise.’

“After listening to Zeno, Democritus went off to think. Several days later, he returned, saying that this story gave him a key to the secrets of the microscopic fabric of the universe. The strength of Zeno's logic had impressed him; thus, he felt there must have been some problem with his assumptions. This led Democritus to a new idea: that each aspect of the universe—time, space, all matter, and each quantity—has a minimum part, an ‘indivisible,’ which cannot be subdivided. In the case of matter, he called this indivisible an ‘atom.’

“With this concept in hand, Democritus explained to Zeno how to resolve his paradox with Achilles. If we grant that there an ‘indivisible’ element of space, then when we follow the process of Zeno’s logic, at some iteration the tortoise will not be able to advance from its point T_N even the minimum distance, during the time it takes Achilles to arrive from $T_{(N-1)}$ to T_N. Achilles and the tortoise will then be together at T_N. And from that point onward, Achilles will be in the lead.

“In our bread-slicing example,” Sereno says, “the indivisible unit limits the subdividing process. We cannot slice thinner than the indivisible width, and when we reach this limit, the loaf will be divided into an enormously large but still finite number of slices. To our human eyes, these slices will appear to have no thickness, but the paradox of piling up slices will be resolved, as there will be a finite number of these."

Commentary

This episode raises both the promise and the perils raised by indefinite processes of subdividing. On the "promise" side, the dissection of the circle gives a way of thinking about the circle that is rich and definite enough to allow us to begin to calculate. In fact, calculating areas from this sequence of polygons can also give us remarkably rapidly-converging estimates for the ratio π. Also, while the focus of the dissection approach was on areas, the success of the approach might lead us to contemplate other aspects of the sequence of polygons that become increasingly circle-like. Perimeter and circumference are a natural choice for exploration, but even less quantitative aspects like 'axes of symmetry' are interesting candidates as well. At each step, in the sequence above, the inscribed polygon has double the number of axes of symmetry of its predecessor. In step 1, we go from the diameter (2 axes of symmetry) to the square (4 axes). In step 2, from the square (4) to the octagon (8); and so on. The circle has infinite symmetries, and in fact, for every point on the circle, it is symmetric over the axis through its center and that point. This might lead one to think about how the points on the circumference might be represented by their "coordinates" in terms of numbered vertices of a $2^{(n+1)}$-gon for sufficiently large n. Such thinking will begin to resemble thinking about binary-decimal expansions; and the question of coordinates of points when the 'expansion' can be extended infinitely.

On the "perils" side, Zeno's story of Achilles and the Tortoise shows that infinite processes can generate paradoxes. In one sense, it can be powerful to conceptualize every point on the circle as a vertex of an inscribed 2^n-gon, with the caveat that *n* may perhaps need to be infinite. In another sense, certain questions about this infinity-gon will have apparently absurd answers: for example, what is its side length? Democritus's idea of indivisibles may provide a way of making progress, allowing us to gain some experience and familiarity with the kinds of operations, comparisons, and calculations that we can do when we contemplate infinite processes.

Sereno, using indivisibles to weigh a parabola

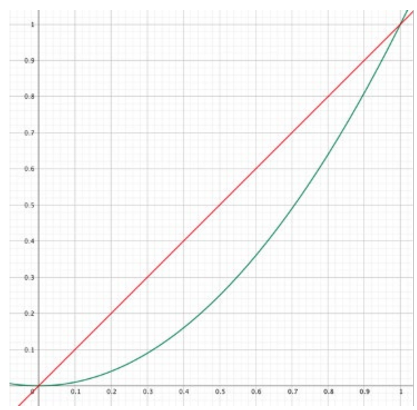
Sereno says, "Now let me show you how I can combine the idea of Indivisibles with procedures from the Method of Dissection in a new 'Mechanical Method,' which allows me to compare many shapes in new ways.

"This method requires acts of imagination to conceive of geometric shapes both as material bodies that have mass, and as ideal Platonic objects that can be dissected at will. Moreover, the art of the method involves identifying shapes to be compared, for which it is useful to attend to the suggestions of algebra and the system of coordinates, which later episodes will describe.

"To see the power of this method, consider the figure below. We want to find the area between the diagonal line (drawn in red) and the parabola (drawn in green). This area is called a *section* of the parabola.

Figure 7

A Section of a Parabola, Drawn on a Modern Coordinate Grid

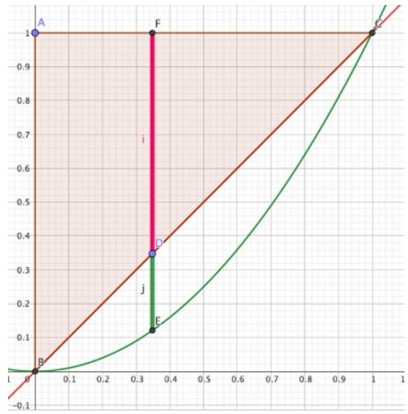


"In the language of coordinates, the line is described by the equation, $y=x$, while the parabola is described by the equation, $y=x^2$. I am using coordinates for the purpose of communication; many of you know the parabola best as the graph of this algebraic relation, rather than as a section of a cone. Nevertheless, the figures are geometric ones, not the graphs of functions.

Next, to bring some more familiar shapes into play, let's add the horizontal line segment from (0,1) to the intersection point of the two graphs, (1,1), part of the line with equation y=1. The region between this segment, the line $y=x$ and the y-axis defines a right triangle. This large triangle (of area ½ u^2) is shaded in orange in the figure below.

Figure 8

Considering a Way to Slice the Section and an Adjacent Triangle

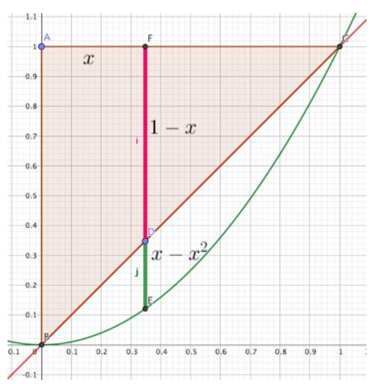


Our focus will be to compare this triangular region with the region between the line and the parabola, using a dissection approach, cutting both region into many, many vertical slices. One pair of the slices is shown above (the triangular slices in red; the parabolic slice in green).

Consider how the coordinate system gives the vertical measures of these strips. The red slice extends from the segment (y=1) to the line (y=x). Therefore, at the horizontal coordinate x, it has measure (1-x). The green slice extends from the line (y=x) to the parabola ($y=x^2$). At horizontal coordinate x, it thus has vertical measure ($x-x^2$).

Figure 9

Vertical Measures of the Slices

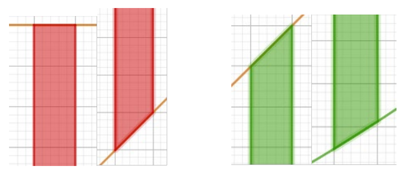


An algebraic connection here is suggestive: multiplying the red measure (1-x) by a factor of x would give us the green result ($x-x^2$). It is not yet clear how to make use of this, but it seems like a key part of the puzzle.

Now let's note an important issue that faced Sereno here, and how the idea of Indivisibles helped him to overcome it. If the slices described above are relatively wide, they are certainly *not* rectangular, and they have different shapes at their edges. For the red slice, the upper edge is horizontal, and the lower edge is diagonal. For the green slice, the upper edge is diagonal and the lower edge is a small portion of the parabola.

Figure 10

Upper and Lower Edges of the Red Slice (Left)...and of the Green Slice (Right)

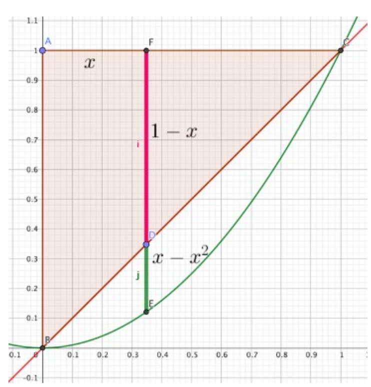


However, consider what happens if we follow our procedure of slicing to the limit created by the spatial *indivisible*. When we reach the indivisible minimal width, the length of the slice cannot vary across that width: it is bound to take on a single value. For a member of our current digital culture, this situation is familiar from the context of screen resolution in a pixel-based display with no anti-aliasing. If we are analyzing the representation of a shape, any given pixel will either be part of the shape or exterior to it. Another way of viewing the situation is to note that the bottom edge of the slice has to be horizontal, as it is only a single "pixel" wide.

Now, we can return to considering the vertical dimensions of infinitesimal slices, knowing that they are rectangular. We imagined slicing the two shapes, and we found an almost tantalizing comparison between the two expressions: (1-x) and x(1-x). Moreover, there actually was another "x" in our setup: the distance between our slices and the y-axis.

Figure 11

Considering the Algebraic Similarity in the Measures



This algebraic similarity provokes us to look for phenomena in the world that we could represent by multiplying these those quantities, (x and 1-x)), which appear in the red triangular region and making it comparable to the slice from the green region. This was a moment of brilliant insight on Sereno's part. After first having used indivisibles to imagine rectangular slices, Sereno now thinks about the slices as physical objects (slightly thickened so they have mass, and made of the same, uniformly dense substance), which can be placed on a two-armed balance.

Sereno, incidentally, has had a long interest in balance, weights, and torque. He has studied properties of levers, balances, centers of mass and so forth. To understand Sereno's solution to the problem of the parabolic segment, three facts suffice: First, when placed on a balance, the moment of force (or "torque") exerted by a body is its mass times its distance from the center fulcrum. Second, the force of a body suspended along the arm of a balance is equal to the force exerted by an equal mass concentrated at the center of mass. And third, the center of mass of a triangle is the intersection of its medians (its "circumcenter").

Now, let's imagine that the two sides (positive and negative) of the line y=1 are the two arms of a balance, with fulcrum at x=0.

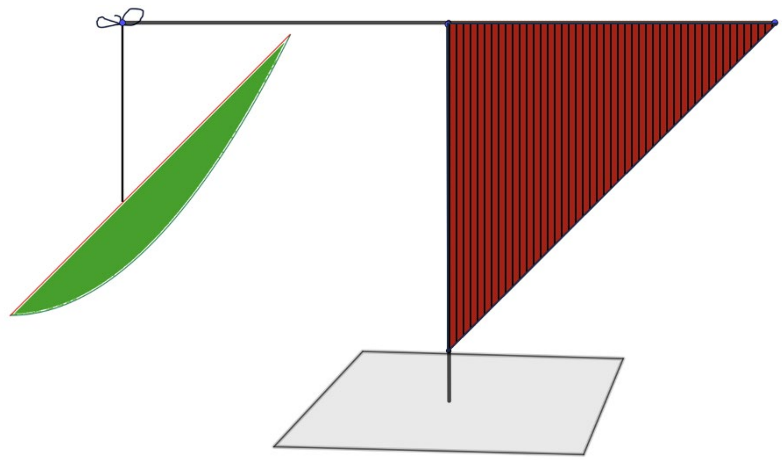
In our setup, we are cutting the two shapes into strips, all with the same indivisible width. (Let's call this width w.) Therefore, the red strip from the triangle has area (or "mass") $(1-x)$w. Furthermore, it is situated on the right arm of the scale, at a distance of x from the fulcrum. So it exerts a moment of force of magnitude $\boldsymbol{x(1-x)}$**w**.

Now consider the green strip from the parabolic section. It has area (or "mass") $(x-x^2)$w. In order for it to exert the same moment of force as the red stripe, we need to place it at a distance of 1 from the fulcrum, on the opposite side (i.e., at x=-1). If we do that, its moment of force will also have magnitude $\mathbf{(x-x^2)w}$.

If we do this for *every one* of the slices: hanging the red ones on the balance in place and the green ones at one unit to the left, all at x = -1, we build up the figure below, where the whole parabolic segment is hung at x = -1 (shown here suspended by a piece of string). Since our equal-moment-of-force relationship applies to ***each*** pair of slices from the two regions as x varies between 0 and 1, we know that the two resulting shapes are perfectly balanced!

Figure 12

Balancing the Parabola and the Triangle



Commentary

In inventing the Mechanical Method, Sereno has taken a step further, guided by the concreteness and familiarity that he has developed in working with geometric shapes. He has been able to expand a notion of "equivalence" of area, under Dissection, to a notion of "balance," using the metaphor of an equal arm balance. Now unequal quantities can be "balanced" by placing them along the two arms at points where they would exert the same moment of force. Not only does this new relation radically expand the reach of the thinking developed in the dissection method, but it also opens the way for new kinds of interaction with geometric shapes that will cultivate a new level of familiarity with them, generating a new kind of concreteness.

Nicolas: Modeling, and an Abuse of Geometric Space

Next, we introduce Nicolas. He has been listening to Sereno with respect and wants to add his point of view. "As I and my colleagues have encountered and processed the ideas of Sereno and others, we have felt a deep appreciation for their sublimity, but we also sense that they are somehow separated from our world. In our time, we are seeing more and more examples in nature, where analyzing measurable quantities seems to reveal the reasons for phenomena. There are people who dedicate their lives to *observation* of phenomena in the world and recording data. The word 'observation' itself reveals the connection to religious observance—care, diligence, and dedication to something grander than oneself (cf Daston & Lunbeck, 2019). We began with observation of celestial phenomena, but as our ability to measure time has grown, we have turned more attention to worldly phenomena. Motion has a particular interest, including the *intensive* *quality* of speed. Galileo has proposed to study the fall of bodies, slowing the phenomenon down through his use of ramps, and using the beat of music, the flow of 'water clocks' and other new devices, to mark time precisely.

"I have found a way to make a visual record of the tables of observation data we have compiled. I wanted an approach that, like the track of an animal, would let us recover a phenomenon after it has passed. For a long time, I looked at the diagrams of the Greeks, and I thought about how to make their statuesque figures come alive. But a moving diagram effaces itself, as each successive moment of a figure violates the place of its predecessor, as is the nature of bodies in space. Nevertheless, I recognized we could take inspiration from geometry and *express our data in spatial form,* by imagining any measure as the length of a segment. This was a metaphorical leap, since we have to violate the nature of weights, or sizes, or durations to see them as lengths; however, the transformation has yielded possibilities.

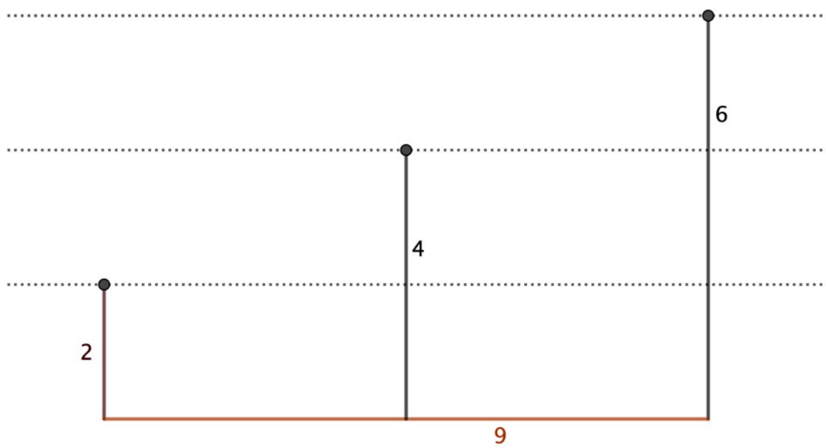
"With this approach, I can represent any measure in relation to others of the same kind. If I were to stop at this point, my idea would be incomplete and ineffective—I would only achieve an imprecise way of depicting the values from my table. However, I next realized that if I drew the segments of the successive observations moving from left to right, in my mind's eye I could imagine the phenomenon changing *between* my readings.

"The drawings I am making would, I know, be blasphemy to the Greeks. If oriented vertically, a segment's length means the magnitude of a reading. But horizontally, a segment's length means a duration of time. What an absurd geometric space I have made! What could a diagonal line here mean? Nothing, surely!

"And yet, there are some glimmers of sense even in this utter nonsense. My British colleagues have been theorizing about motion, and they have focused on two kinds, based on the nature of the 'intensive quality' of speed. In their first kind of motion, speed remains constant. And in their second kind, speed *grows constantly* – incrementally – increasing by a given magnitude in every regular interval of time. Now, suppose a body begins with a speed of 2 feet per second, it increases its speed 'uniformly' and ends 9 seconds later with a speed of 6 feet per second. The picture below would capture this situation in my new geometry:

Figure 13

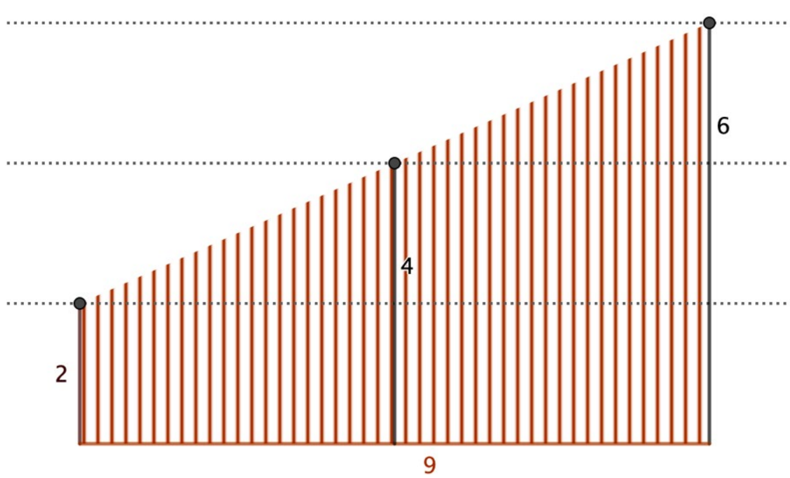
A Way of Representing Time-Series Velocity Measurements



"Although we cannot fathom the rapidity of observation that would allow us to capture the corresponding speed measurements, we can imagine that our vertical magnitude must have grown *continuously* over the 9 seconds. For example, at precisely halfway through (4.5 seconds), it would have reached a height of 4 (feet per second), as shown. And we can imagine a moment halfway between each of these observations, and so on, leading us to fill the whole time interval with measurement segments. It would start to look like the picture below:"

Figure 14

Imagining Very Rapid Data Collection



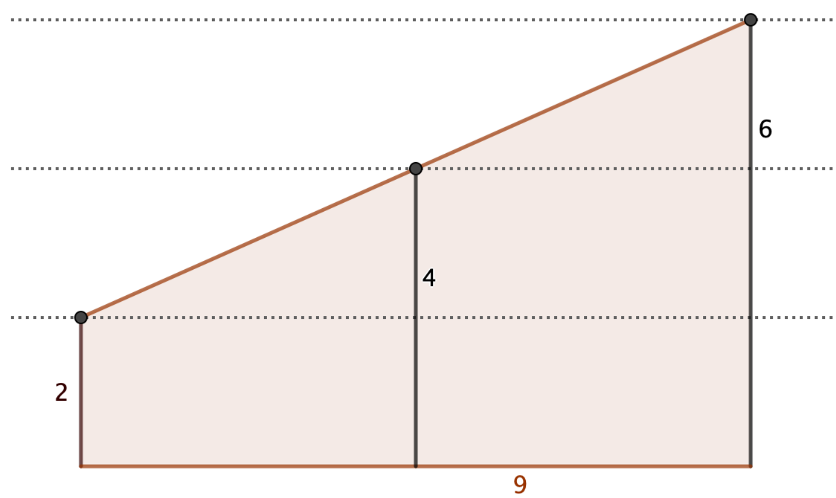
(A hint of surprise comes over us as we recognize something in this picture that is familiar from Sereno's story. But Nicholas seems to be taking the ideas in a different direction.)

He continues: “Now, lest you think that this is just a pretty picture, look how my bold abuse of the Euclidean plane yields fruits in calculations. If our moving body kept its initial speed (following the English idea of ‘uniform motion’), then the body would move 2x9 = 18 feet. Numerically, this is the area of the rectangle cut by the lowest dotted line in the figure above. Or if the body had moved at its final speed (the picture of its motion following the highest dotted line in the figure), it would have covered 6x9=54 feet, again the area of the rectangle that the picture would describe.

“What about in our *uniformly-changing* case? We imagine that all of our infinite measurement segments together trace out the **trapezoid** shown in orange below:

Figure 15

Considering the Area of the Shape Created in the Data



“The area of this trapezoid is the mean of its two heights times its width. ½(2+6)*9=36. This agrees with the English Merton Rule, or their ‘mean speed theorem.’

“Further, the geometry of dissection suggests other findings, which give meaning to different algebraic transformations of the equation in the Merton Rule. For example, our motion would cover the same distance as a uniform motion at the sum of the initial and final speeds, for half the duration. (To see this, cut the trapezoid at the measure at 4.5 seconds, then rotate the right half around the upper vertex of that measure segment, to create a rectangle of height 8 and width 4.5. This new picture represents the conjectured motion.) Or, note that our motion would also cover the same distance as a motion at the initial speed for the first half of the time, followed by a motion at the final speed for the second half of the time. (To see this, cut along the trace of the initial measure, to the midpoint measure, making a right triangle. Then, rotate that triangle 180 degrees about its upper-right vertex, to create the two desired rectangles.)"

Commentary

Nicolas's innovation shows the power of a representation that mixes modes of thinking. The problem that motivated it was from data analysis and modeling. The visualization enabled pattern-recognition, inference, and new ways of thinking about phenomena. But the graph was more than a picture: it had enough resemblance to geometric figures to invoke the very different kinds of thinking that had matured in *that* setting. Even in the very narrow field of velocity graphs of constant-velocity and linear-velocity motion, the new representation suggests new ways of reasoning across algebraic and geometric modes. Nevertheless, Nicolas's new spatial representation has paradoxical aspects, and its relation to the space we know from experience is complicated. Importantly, these paradoxes can be viewed as *strengths* of the representation as well as weaknesses. They are strengths when they enable new ways of mapping to the world, as we will see with our next episode.

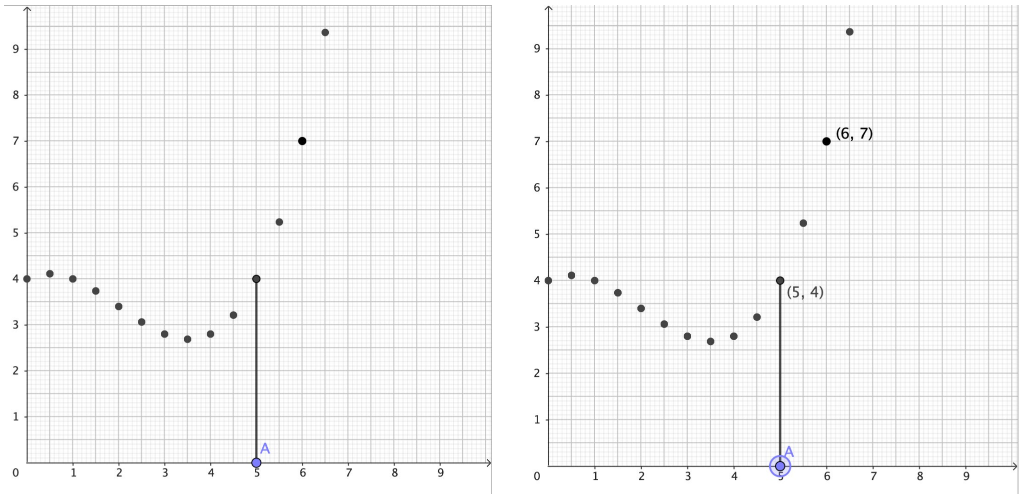
An infinite-sided irregular polygon

Nicholas's colleague Peter has something to add. He says, "Nicholas's idea may have been inspired by animating tables of empirical observations, but it is also a wonderful device for theoretical explorations. Nicolas's examples with the Merton Rule are just the beginning. We can speculate about and simulate any "function" that might relate one quantity to the other with his device. If we imagine that a phenomenon in the world might be described by *any* such expression relating two quantities, then we can make a drawing or *graph* of the relationship, in a space like Nicholas's, by 'remembering' only the topmost vertex of the segment that reflects the value of the resultant quantity. Each point of this graph will capture both "co-ordinate" values of the reading that it corresponds to.

"Something interesting happens if we use one of these Graphs to describe a body's *position* (measured from a spatial reference point), and the *time* it was observed in that position (measured from a temporal reference moment). In this case, we can actually use geometric properties of the graph to capture qualities of that motion—in particular, the intensive quality of speed, which Nicholas took as his starting point.

Figure 16

How the New Representation Suggests Measuring Coordinates across Readings

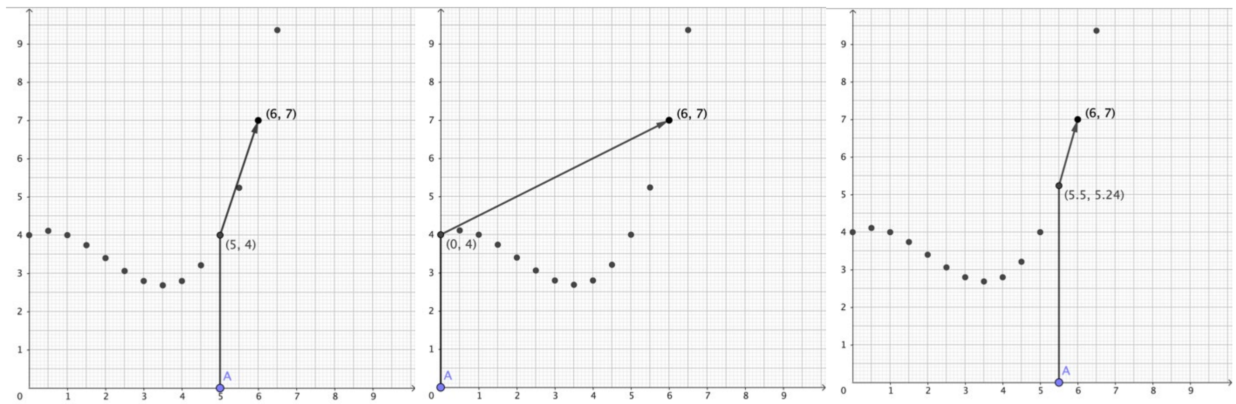


"For example, in the picture above, we can see that when the body moves from a position of 4 feet at 5 seconds to a position of 7 feet at 6 seconds, its position has changed 3 feet in the span of 1 second. In other words, it has moved 3 feet per second on average, during that one-second interval. That is *faster* than the average speed over the whole 6-second interval ending at the 6-second point, during which the body moved 2 feet in 6 seconds, or 1/3 feet per second. And it is *slower* than the average speed during the interval between 5 ½ seconds and 6 seconds, when its average speed was approximately 1.76 feet / .5 seconds or just over 3 ½ feet per second.

"The pictures below show these three intervals. I have highlighted the readings in question by connecting them with a segment. This reminds me of Nicolas's trapezoidal graph of constantly changing speed. In our setting, these trapezoids reflect constantly changing *position*—that is, a fixed speed. In fact, algebraically, the ratio of the change in distance to the change in time is the *average* speed. Notice that when the segment connecting two points of the graph is more steeply inclined, this indicates a greater average speed over the corresponding interval of time. The segments I have drawn are not proper geometric objects (Nicolas was right that a diagonal segment has no meaning), but if, like Nicolas did, we imagine it as indicating a sequence of readings from a hypothetical alternative motion, that motion would have fixed velocity and cover the same distance in the same time. The segment thus measures the *average* speed of the actual motion.

Figure 17

The New Representation Reveals Average Speed or Average Rate of Change between Measures



"Between adjacent readings, we also might imagine our connecting segments as embodying a motion like the one that the body actually enacted. If we could take readings of the distance more and more frequently, we would gain a clearer and clearer picture of the graph. But for any given data collection frequency, we are ignorant of the phenomenon between readings, and so we are left with no better alternative than to *assume* that the change was uniform. This means we sketch the graph as a many-sided, irregular polygon. However, we might imagine in our mind's eye that, if we had infinite readings, our graph would become a smooth curve instead of a jagged polygonal path."

Commentary

Nicolas and Peter began to explore whether instincts from geometry could be useful in the context of the new representation system of "Cartesian" graphs. Guided by the units of measure in their data, they identified ways to interpret areas and slopes, always beginning from the Data Analysis and Modeling perspective, in which they used their imagination to extend their image of the motion of a body from a limited set of readings that they had taken.

From this perspective, the jaggedness of a graph is a measure of inevitable gaps in our knowledge between readings. Fantasizing about infinite knowledge brings this polygonal path to its limit in a smooth curve. That curve is the projected end of a process of data collection, and like the infinite processes Sereno encountered, it is a paradoxical object. In gaining that smooth-curve representation of a motion, we lose the ability to calculate the actual speed over any interval. However, we can resort to the "average speed" calculations that we made when we lacked knowledge. And then we can "recall" making this calculation over any interval we like. We observe that as our knowledge increased, our estimate of the speed stabilized, and we are looking for a way to talk about that stabilization process.

In the final step of Peter's logic, we note that this stable value of the speed over a shorter and shorter interval corresponds in many ways to the geometric notion of a tangent line (as the line that best approximates a curve locally).

In a traditional Calculus course, we often approach the tangent-line problem from the perspective of analyzing the functions with which we *model* motion—functions defined at all real values and given by algebraic expressions. This reverses the problem, placing us in a position of absolute knowledge of the function's value and asking how we define the tangent line (speaking geometrically) or instantaneous rate of change (speaking arithmetically). The data-analytic perspective, in contrast, naturalizes the geometric conception of the graph as a polygonal path with an enormous number of sides. In that conception the tangent line *is* the graph, except at measured data points, where the value is known but the tangent is not well defined! The paradox is that in the limiting case, as the 'vertices' of the polygonal path become increasingly ubiquitous, the tangent line goes from being undefined at many-many places, to being *defined* everywhere.

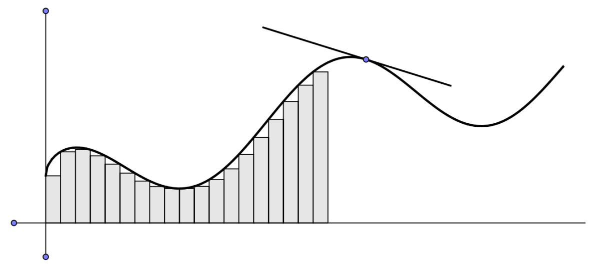
Guillermo: Infinitesimals and a notation that acts as a guide for thought

Using the Cartesian representation to study functions defined algebraically (versus through periodic measurements) raises complementary perspectives. For the area problem, progress was made in calculating the areas under polynomial graphs, advancing the theoretical foundation for modeling phenomena of change with functions.

Here, Guillermo signals he would like to share an episode. "It struck me as important to find a way of describing both the area under the curve (quadrature) problem and the tangent line problem with one general symbolic language. This seemed very strange to many colleagues, as the two problems did not seem to share any connection:

Figure 18

Two Kinds of Measurements We Can Take on the New Representation: They Seem Very Different



“However, both problems involve thinking that we saw in the Data Analysis context as moving from polygonal paths to smooth curves, by thinking about infinite processes that constructed a sequence of better and better approximations. Our algebraic language failed us in similar ways in both settings, and it seemed reasonable that a single innovation might address both failings.

“In both settings we aimed to take measures of polygonal paths, measures which collapsed in the limiting curve. For quadrature, it was the area of an infinitely thin rectangle. For the tangent line, it was a ratio of two infinitely small changes.

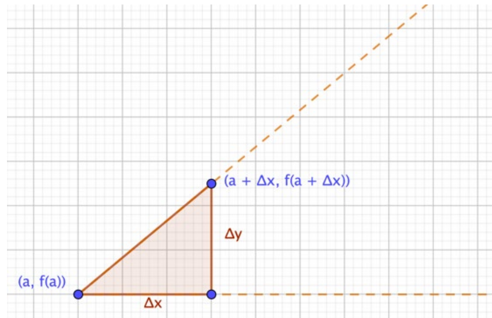
“It seemed to me that we needed a tool for thinking. Nicolas noted that a diagonal segment has no sense as a Euclidean geometric object in the Cartesian plane, but it provides tools for thinking about patterns of change that are key to the application of geometric perspectives to the problems of Calculus. In a similar way, it seemed to me that we needed a tool that would enable us to talk about the values that “stabilized” when we took smaller and smaller measures.

“In fact, in designing the new notational tool, it occurred to me that thinking about a specific ‘size’ for these measures may be a trap of thinking. In practice, we think of these components of change as being “as small as they need to be” for the current problem. They therefore should be a kind of adjustable conceptual framework that we set up around the function we are studying.

“Fortunately, the operations we are interested in describing, guiding, inspiring, and understanding are tolerant of the very small. For area, we have seen paradoxes when we do not speak with precision about slicing figures and adding the slices, but for any fixed (tiny) size of slice, we will produce a fixed (large) number of slices. Adding these areas is a coherent proposition, and if for very small widths, the function itself looks like a rectangle, slicing it further will not change the result. For the tangent, we draw many versions of a ‘characteristic triangle,’ like the one below, where our attention is on the *ratio* between its vertical and horizontal 'legs.' If the value of this ratio stabilizes, it does not matter how long the legs are – different lengths will produce *similar* triangles, and the same ratio.

Figure 19

Drawing a 'Characteristic Triangle' with Infinitesimal Legs



"The central issue is that while we can imagine a smallest necessary size for any given situation, there is no size that is 'small enough' for *all* situations. However, as tools for thought, rather than as physical entities, these objects have no limits on how small they can be. While Physicists talk of the Planck length as a minimum unit - the smallest unit of conceivable measure (an epistemological minimum) or as the smallest possible feature of the universe (an ontological minimum). In this sense, it is related to the "indivisible" mentioned above. But in designing a mathematical tool for *guiding thought* (not for measuring), we are not limited by the Planck length. Mathematics offers an "augmented reality" that has reference to the world but is exempt from its limitations.

"Our challenge, then, is to build a framework that permits us to speak (and think) coherently about these tiny measurements, not as numbers of the type we are familiar with, which have a determinate size, but rather as tools that specify the scale of our work, giving us a means of investigating the stabilization phenomena of both the area and the tangent line problems."

We have heard Guillermo describe his mathematical contribution of a notation system as a "medium"—that would "guide the mind" in the way that exceptional intuitions had guided the work of prior thinkers. That is, Guillermo aimed to develop these *external instruments of thought*, which could *democratize* as well as *systematize* the new field of Calculus. He also recognized how the notation needed to combine concepts from geometry and algebra, saying its guidance should operate like "the lines drawn in geometry and the formulas…laid down for the learner in arithmetic" (qtd. in Edwards, 2012, p. 232).

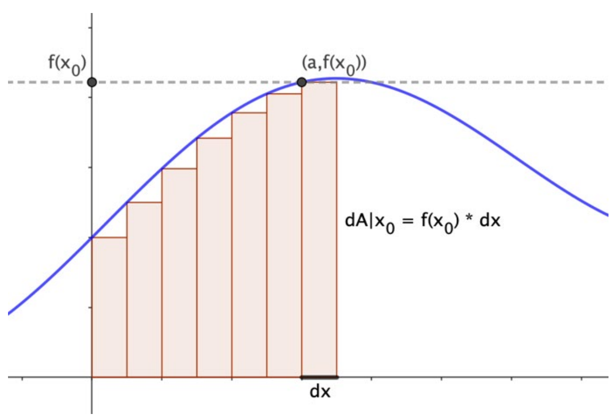
Guillermo continues: "To do this, I developed the concept of an *infinitesimal*. If a main quantity in our investigation is indicated by a letter (e.g., x, y, or A), then an *infinitesimal change* in that quantity can be indicated by adding a 'd' prefix (e.g., dx, dy, or dA). This innovation provides a notation to capture 'in mid-stream' the iterative processes we are studying, at whatever scale of change we are studying: the *d_* notation indicates reducing the scale 'sufficiently for the precision of the current analysis.' It thus allows one to draw diagrams and write equations that are not conceptually paradoxical but that allow the mind to contemplate the *eventual* (stabilizing) behavior of an infinite process.

"For calculating area under a curve f(x), we are interested in adding the areas of a series of rectangles; a representative rectangle at x_0 will have width dx and with height $f(x_0)$. Adding this rectangle to the accumulating area, then, causes the change:

Figure 20

The Change in Area Produced by Adding the Area of a Rectangle with Infinitesimal Width

$$dA|x_0 = f(x_0) * dx$$



…and we are interested in whether the sum of all of these rectangles' areas over the interval of interest 'eventually stabilizes' to a number that represents the area under the curve.

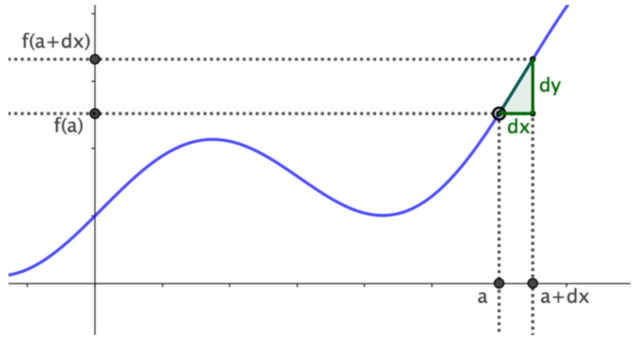
"For the tangent line to a function f(x) at x=a, we need for the average rate of change on the interval (a, a + dx) to 'stabilize' to a value that we will call the instantaneous rate of change at a. Using the d_ notation, we are interested in determining whether the ratio…

$$\frac{dy}{dx} \quad \text{or} \quad \frac{f(a \pm dx) - f(a)}{dx}$$

…'eventually stabilizes.'

Figure 21

Infinitesimal Triangle Capturing and Stabilizing to Show an Imagined 'Instantaneous Rate'



"In each case, the use of infinitesimals 'freezes' an infinite dynamic process we are imagining, which allows us to work with indefinite propositions and trace them to root causes. Simplifying this calculation at *any* point x=a raises the possibility of considering $\frac{dy}{dx}$ as a *function*. The result, $\frac{dy}{dx}$ or f'(x), can then be considered as a function in its own right.

"Infinitesimals provide a 'work around' for two challenging problems of reasoning about 'approaching,' namely (1) *initially* the function may not be approaching f(a); the sum may not approximate the area under the curve well; and the difference quotient may not be stable; but (2) if we wait until x actually reaches a, the phenomenon of interest has disappeared (for area, if dx=0, all our rectangles have zero are; and for the tangent line, at x=a, the difference quotient is undefined).

"One dimension of the power of this innovation in notation derives from a human ability, honed over evolutionary time. As humans, *we can imagine the final result of a dynamic process when it is not yet finished*. Our species has honed this ability in hunting, for example, to anticipate the location of prey and to coordinate attacks. Infinitesimals extend this ability in a subtle way, since we have seen that *we can infer properties of the* ***eventual*** *result of an* ***infinite, iterative*** *process when it is not yet finished.* Another perspective is: the infinitesimal is a tool for seeing the infinite iterative process as a finite dynamic process. Through the notation 'a + dx' we imagine the approach of x to a as ***movement toward*** a, governed by the relevant infinite iterative process, and arrested when the 'eventual' behaviors of the quantities of interest become clear. The infinitesimal truncates the infinity, but only after its true character has revealed itself.

"We can see these *infinitesimals* as *convenient fictions*, in case we do not want to accept them yet as genuine numbers. If we reflect on the history of ideas, we'll see that this is not an unusual way to begin. Even the passage from natural numbers to negative integers met with great resistance—what was a negative quantity? Today, we can recognize not only the value of negative integers for the number system, but also their role in modeling the world: we say that -10,000 represents a debt. Similar stories tell us that the expansion of the numerical system from the natural to the irrational was the result of overcoming resistance to "quantities" that had no representation in the material world. When ways were found to interpret these numbers in the daily life of societies, these resistances were overcome and, in this way, they entered the symbolic world for their further mathematical development and integration into the growing systems of numbers. On being admitted to those systems, numbers are enriched. For example, the number 7, which we perhaps conceived as representing numerosity in the world (e.g., 7 apples), acquired the status of a prime number in the augmented reality of our mathematical imagination. Alongside 7, and π, we also have dx, dy and other wonders to expand our calculation capabilities.

Commentary: Symbol Systems offer Interactions, beyond Depictions

A primary function of a symbol is to take the place of something *absent* – or something *inaccessible*. Merlin Donald traces the sources of symbolization to our shared biology that which has been *sculpted* (Donald, 2001) by evolution to respond to symbols. An example of this is our responsiveness to motion—in particular, we have a heightened ability to notice motion, and we are good at anticipating the destination of a moving entity in our field of vision) (Llinás, 2002).

This suggests that the primary relation we have with reality which we aim to replicate in symbol systems is an *interactive* relation. (That is, we do not focus on describing 'what is' in the world, so much as we aim to replicate how we interact with that world). In turn, this shows us that our symbol systems are not so much created as mimetic, descriptive mirrors of the world; rather we construct them 'in the image' of the world primarily in the sense that the world offers *possibilities for interaction* (or inspires fantasies for alternative forms of interaction). We should thus look at symbol systems (whether literary works or mathematical works) as *interpellating* 'readers,' calling them to assume an interactive relation with the system and offering the possibility that this relation can generate discoveries about the nature of the 'virtual world.'

Our symbolic representation systems—in both language and mathematics—have thus presented themselves as executable 'programs' to be 'run' - or as virtual or augmented realities to be actively tested and explored. The ability to execute a high-fidelity 'run' of a novel or a proof has been a cultivated skill in our cultures. And such work has a great deal of generative possibility, in that it places one not only in interaction with the author, but also, through the author, in interaction with the symbol system itself. In this way, in both literary and mathematical 'reading,' the reader can encounter insights not dreamed of by the author. Unfortunately, this 'execution' ability has been a rather rare and elite achievement—even more so in mathematics than in literature, given that mathematics deals with a symbol system constructed by a smaller community for a more narrow set of purposes. However, with dynamic executable representations, mathematics has gained a remarkable advantage—the ability to offer *co-action* with a construction and the underlying symbol system to a reader with a much lower threshold of simulation ability. Dynamic mathematics environments like GeoGebra and Cabri provide an interface that enables an important proportion of the interactive agency that characterizes powerful reading.

This use of a symbol system for discovery is a kind of symbolic play – enacting operations on symbols that obey *rules* understood to preserve the validity of the result (its tether to the referent). These rules ensure that transformations we enact are trustworthy without requiring constant checking back to the field of reference to ensure that the symbolic link is not broken.

Guillermo: Calculating with Infinitesimals

"Exactly!" says Guillermo. "The greatest value of the infinitesimal notation is the way it guides calculation. There are two guiding principles – one a way of conceptualizing a function that we have seen before; the other a rule for simplifying expressions that include a mixture of infinitesimals and traditional Real numbers.

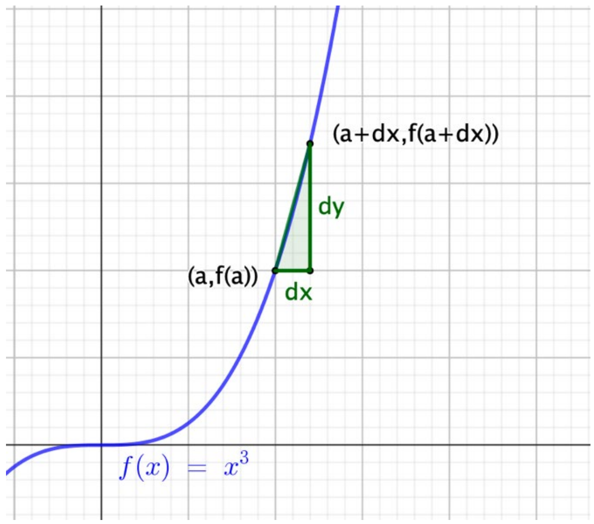
> **Principle 1 (Conceptualizing function graphs)**: Any function graph can be conceptualized as a polygonal path with an infinitely large number of sides, each of which is a segment of infinitesimal length.
> **Principle 2 (Rule for simplifying)**: Given terms A and μ, where μ is infinitesimal in comparison with A, the sum $A+\mu$ can be replaced by A.

"As implied in Principle 2, the introduction of infinitesimals brings in a hierarchy of infinitesimals. If μ is infinitesimal with respect to A, $A*\mu$ is also infinitesimal with respect to A, and μ^2 is infinitesimal with respect to μ. There are also *infinities*, Ω, such that A is infinitesimal with respect to Ω. One example of an infinity is $\frac{1}{\mu}$. But for the calculations we will consider, only infinitesimals will be needed.

"Let's consider an example. Take a function such as $f(x)=x^3$. Consider the problem of finding the slope of the tangent line at any point x=a.

Figure 22

Infinitesimal Triangle Used for Studying $y=x^3$



$$\frac{dy}{dx}\Big|_{x=a} = \frac{f(a+dx)-f(a)}{dx}$$

$$= \frac{(a+dx)^3 - a^3}{dx}$$

$$= \frac{a^3 + 3a^2dx + 3adx^2 + dx^3 - a^3}{dx}$$

$$= \frac{3a^2dx + 3adx^2 + dx^3}{dx}$$

$$= 3a^2 + 3adx + dx^2 \quad \text{(since dx} \neq 0)$$

Now, following the rule of Guillermo's Principle #2, the terms containing (dx) and (dx^2) are infinitely small in comparison with the term $3a^2$. Thus, the average slope over the interval (a, a+dx) stabilizes to the value $3a^2$.

$$\frac{dy}{dx}\Big|_{x=a} = \quad 3a^2$$

The notation acts as an even stronger guide for thought in showing the relation between the area and tangent problems: the operations of *differentiation* and *integration* are inverses. Earlier, we noted that the slope of the tangent at a point can be the basis for a new function, the derivative. The mapping of a function to its derivative is called 'differentiation.' Also, the operation of calculating the area under the graph of f(x) between a fixed reference value x=a and a variable value x=b, also generates a new function, the definite integral. This mapping is called 'integration.'

The infinitesimal notation helps to suggest that these two mappings are inverses. Namely, integrating the derivative function about the reference point x=a yields the difference between the function and its value at x=a. And differentiating the integral function yields the original function.

To see the first part of the inverse relation: consider calculating the area under the curve of the derivative function.

The notation suggests, by 'canceling': $\frac{dy}{dx}|_{x=a}$* dx = $dy|_{x=a}$

The left-hand side of this is the area of a rectangle in a calculation of the area under the graph of the derivative function.

Figure 23

The Name for the Height of the Infinitesimally-Wide Rectangle under f'(x), in Guillermo's Notation

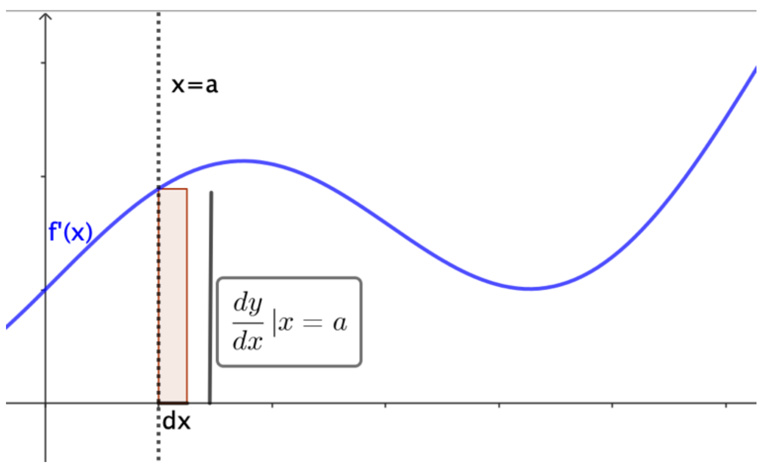


Figure 24

Guillermo's Notation Suggests a Mapping Between the Two Graphs

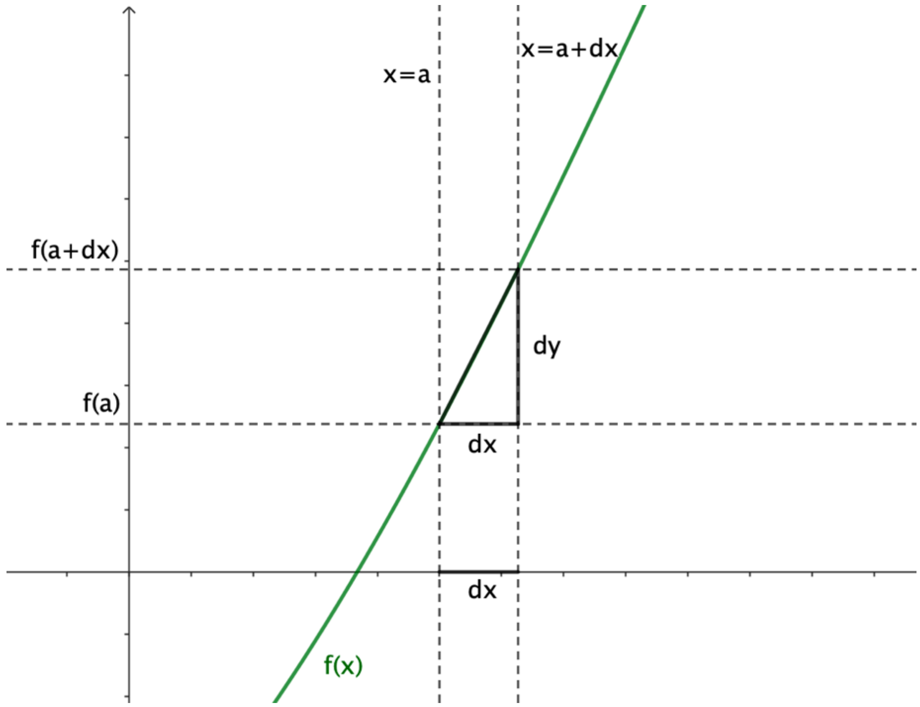


Figure 23.

The right-hand side represents the change in the original function triggered by a change, dx, in the independent variable. Moreover, $dy|_{x=a}$ = f(a+dx) - f(a). If we repeated this identity for each of the rectangles of the area calculation, we would have

Area under f'(x) between x=a and x=b

EQUALS

f(a+dx) - f(a) + f(a+2dx) - f(a + dx) + … + f(b) - f(b-dx)

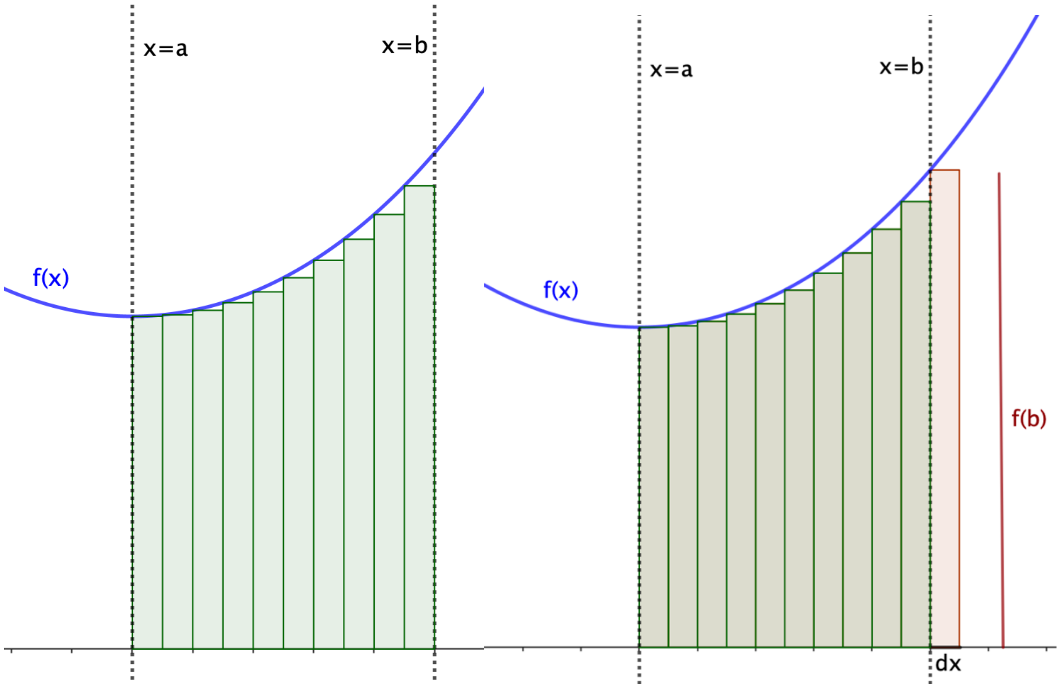
This is a telescoping sum, equaling **f(b) – f(a),** as desired.

Now, to see the *second* part of the inverse relation, we want to show that differentiating the integral function gives the original function. We want to evaluate the rate of change in the integral function (the area under the graph of f between a and x) at x=b. Let's begin with the differential below, and consider how the Area function is changing at x=b.

$$dA|_{x=b}$$

Figure 25

How the Area Changes with One New Infinitesimally-Wide Rectangle.



The change in the area function is the area of the final rectangle. That rectangle has height f(b) and width dx.

$$dA|_{x=b} = \text{f(b)dx}$$

And so, dividing by dx, we get the desired rate of change:

$$\frac{dA}{dx}|_{x=b} = \text{f(b)}$$

In other words, the derivative of the integral function is the original function.

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