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Part 2: Sailor on the Seas of Infinity

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Abstract: Mathematics is the result of a long process of research with the instruments of rationality that each age makes available to its members. Euclidean geometry crystallizes an impulse of deductive rationality. But this impulse has not always been the dominant one. At other times, such as the one we are going to explain in this part, a form of inductive reasoning has been given primacy. We will see the development of ideas that today constitute the foundations of the Calculus and that indicated the future task of its deductive consolidation. Our protagonist, the bold sailor, was identified as one who *calculated without apparent inductive effort, just as eagles hover in the air*.

Keywords: Logarithmic areas, infinitesimal, infinity, symbolic apparatus, inductive thinking.

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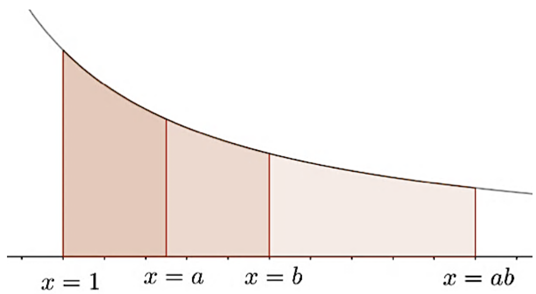
Leonardo

The young Leonardo followed the steps of Guillermo. He had learned that integrating the function y=1/x might unlock secrets of logarithms. He knew that the area under that function from x=1 to x=ab was the same as the sum of areas between x=1 and x=a plus x=1 and x=b:

Figure 1

Logarithmic Areas



If we write L(a) the area between x=1 and x=a, then what Leonardo knew was that the function L satisfies: *L(ab)=L(a)+L(b)*. In other words: L *behaves as a logarithm function, converting products into sums*.

The question Leonardo then asked himself was: what is the *base* of this particular logarithmic function? To answer this question Leonardo proceeded as follows, using the new infinitesimal idea: Integrate the function L between 1 and 1+w, where w is an infinitesimal. The graph, *if we could see it with the naked eye*, would look like this:

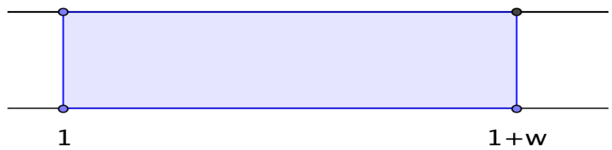
Figure 2

Microscopic Vision

But 1+w is infinitely close to 1, so the function y=1/x between 1 and 1+w actually looks like this —thanks to the mathematical microscope, which he possessed:

Figure 3

Infinitesimal Area



Then L(1+w) corresponds to the area between 1 and 1+w. Then, we arrive at a very important conclusion: L(1+w)=w, the area of the infinitesimal rectangle shown above.

Leonardo chose *e* to denote the base of these logarithms. By applying the exponential to both sides he got:

$$w = L(1+w)$$

$$e^{w} = 1 + w$$

Since w is an infinitesimal, he represented it as 1/N, where N is an infinitely large integer. Consequently, by raising both sides to the power of N:

$$e^{\frac{1}{N}} = 1 + \frac{1}{N}$$

$$e = (1 + \frac{1}{N})^{N}$$

Then Leonardo developed the expression $(1 + \frac{1}{N})^{N}$ by means of Newton's binomial formula:

$$e = \sum_{k=0}^{N} \frac{N}{k} \binom{N}{k} \left(\frac{1}{N^k}\right) = \sum_{k=0}^{N} \frac{N(N-1)(N-2)\ldots(N-k+1)}{N^k} \frac{1}{k!}$$

Each factor $\frac{N-r}{N} = 1 - \frac{r}{N}$, with r is a finite number. Since N is infinitely large, $\frac{r}{N}$ is an *infinitesimal*! Remembering Guillermo´s lessons about working with infinitesimals, Leonardo wrote:

$$\frac{N-r}{N} = 1 - \frac{r}{N} = 1$$

Again, because $\frac{r}{N}$ is an infinitesimal compared to 1. He thus achieved a wonderful result:

$$e = \sum_{0}^{\infty} \frac{1}{k!} = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

This was an *analytical* way of proceeding: a way of proceeding in which Leonardo became an excellent master. The partial sums (of this infinite sum) are rapidly approaching the value of *e*:

Table 1

Increasing Approximation

N = 2	$\sum_{k=0}^{N} \frac{1}{k!} = 1 + 1 + \frac{1}{2}$	$e \approx 2.5$
N = 4	$\sum_{k=0}^{N} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$	$e \approx 2.70833$
N = 8	$\sum_{k=0}^{N} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320}$	$e \approx 2.71827$

Going a bit further…(as Leonardo liked it): ***e=2.7182818284590452353…***

Guided by Leonardo's master hand, we enlarge the numerical field allowing the presence of infinitesimals and infinite numbers with which we have more room to maneuver —in the large and in the small—to operate on algebraic expressions. In the end, given the 'replacement rules' of Guillermo, these infinitesimals and infinite numbers *disappear*: The results remain in terms of finite numbers.

This invites a reflection that goes beyond what Leonardo has just shown us: *It is the actions of human beings* that force us to expand the numerical domains.

Continuing, *Leonardo knew the formula:* $\cos x + i \sin x = e^{ix}$. Then, assuming his work above extended to imaginary arguments…

$$\cos x + i \sin x = e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} + \frac{ix^3}{3!} - \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \cdots$$

Separating the real part and the imaginary part we have:

$$\cos x + i \sin x = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Consequently,

$$\cos(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right); \quad \sin(x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

And finally, remembering Guillermo´s basic rules, he arrived at:

$$sin(dx) = dx, \ \ cos(dx) = 1$$

The deep intention of his work was to fully develop the *symbolic apparatus* of his master, Guillermo, and to realize its fully analytical operative field. By working with infinitely small and infinitely large quantities, he facilitated algebraic manipulations, and this is most remarkable, *the results are expressed in terms of finite quantities* (numbers).

Leonardo's contributions are monumental. But if we had to choose just *one* of his marvelous discoveries, we would choose the following:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Now, we will try to explain it. When Leonardo approached the study of this infinite sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

It was known that its numerical value was less than 2. But what was the *exact value*? He put his marvelous inductive thinking into play. Leonardo knew that a polynomial P(x), of second degree, with two non-zero roots a, b has the following structure:

$$P(x) = \left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{b}\right) = \frac{x^2}{ab} - \left(\frac{1}{a} + \frac{1}{b}\right)x + 1$$

A polynomial of degree 4 with non-zero double roots ±a, ±b can be written as:

$$P(x) = \left(1 - \frac{x}{a}\right)\left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{b}\right)\left(1 + \frac{x}{b}\right) = \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{x^2}{b^2}\right)$$

$$P(x) = \frac{x^4}{a^2b^2} - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)x^2 + 1$$

Note that the coefficient of x^2 contains the sum of the squares of the double roots.

If the polynomial has *six* non-zero roots, ±a, ±b, ±c, then:

$$P(x) = \left[\frac{x^4}{a^2b^2} - \left(\frac{1}{a^2} + \frac{1}{b^2}\right)x^2 + 1\right]\left(1 - \frac{x^2}{c^2}\right), \text{ or:}$$

$$P(x) = -\frac{x^6}{a^2b^2c^2} + \left(\frac{1}{a^2b^2} + \frac{1}{b^2c^2} + \frac{1}{a^2c^2}\right)x^4 - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)x^2 + 1$$

Here again, the coefficients of x^2 contains the sum of the squares of the double roots.

Now the *inductive* reasoning, one of Leonardo's specialties when it comes to algebraic (analytical) expressions: If a polynomial has non-zero roots $\pm a_1, \pm a_2, \pm a_3, \ldots, \pm a_n$, then the coefficient of x^2 is given by**:** $-\sum_{i=1}^{n} \frac{1}{(a_i)^2}$

Leonardo's heuristic genius is present: *the passage from the finite to the infinite*.

Now he claims that if there exists a *polynomial* with an *infinite* number of non-zero roots a_i, its quadratic coefficient will be: $-\sum_{i=1}^{\infty} \frac{1}{(a_i)^2}$;

This *polynomial* "*almost-exists*." Leonardo affirms that it *does* exist, and that it is given by the expression (in power series) of the function:

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

The *roots* of this *polynomial* are: $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots, \pm n\pi \ldots$. Considering that the quadratic term of this polynomial has coefficient: $-\frac{1}{3!}$, then:

$$-\frac{-1}{3!} = \frac{1}{6} = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2},$$

Moving the factor of π^2 to the left-hand side, he has, explicitly:

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

This was the result that made Leonardo famous, a result of enormous value for *inductive* thinking. Leonardo was a master in the art of reading in the face of the finite, that which was valid in the domain of infinity.

It has been written of Leonardo that he exercised his algebraic, analytical, skills as naturally as an eagle exercises its abilities in flight. His boldness was such that he himself sought over the years various confirmations of his procedures to solve this problem. One day he managed to see that his solution led smoothly to the answer that years earlier an islander, Juan, had discovered. It was one way the ubiquitous number π could be expressed. Let's see this expression, achieved by Juan, which gave him great prestige among his contemporaries. His result:

$$\frac{\pi}{2} = \left(\frac{2*2}{1*3}\right)\left(\frac{4*4}{3*5}\right)\left(\frac{6*6}{5*7}\right)\left(\frac{8*8}{7*9}\right)\dots$$

That is, Juan managed to find an expression for the number π, as an *infinite product*! Now, what did Leonardo see in his own result that allowed him to link it to Juan's infinite product?

The key that opens the window from which Leonardo contemplates Juan, is in the expression of Leonardo's infinite polynomial, when it is factored. Recall that its roots come in pairs: nπ and -nπ (n=1, 2, 3, …), and recall that Leonardo thought about the expression in terms of product of the terms ($1-\frac{x}{r}$), for each root, r. Since our roots come in pairs, we have $\left(1-\frac{x}{r}\right) * \left(1+\frac{x}{r}\right) = \left(1-\frac{x^2}{r^2}\right)$.

So, with roots of $\pm n\pi, P(x) = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\dots = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$

If $x = \frac{\pi}{2}$, then sin (π/2) =1, and we have:

$$\frac{\sin\left(\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}.$$

And, substituting $x = \frac{\pi}{2}$ into the general term:

$$\mathrm{P(x)} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$$

…we get:

$$\left(1-\frac{(\frac{\pi}{2})^2}{n^2\pi^2}\right)=(1 - \frac{1}{2^2n^2}) = \left(\frac{(4n^2-1)}{4n^2}\right) = \left(\frac{(2n-1)(2n+1)}{(2n)^2}\right).$$

Thus:

$$\frac{2}{\pi}=\prod_{n=1}^{\infty}\left(\frac{(2n-1)(2n+1)}{(2n)^2}\right)$$

Now, inverting each term:

$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\left(\frac{(2n)^2}{(2n-1)(2n+1)}\right)=\left(\frac{2*2}{1\cdot 3}\right)\left(\frac{4*4}{3\cdot 5}\right)\left(\frac{6*6}{5\cdot 7}\right)\left(\frac{8*8}{7\cdot 9}\right)\dots$$

This is the result obtained by Juan, reproduced by Leonardo through an extremely complex and audacious procedure motivated by analogy and intuition. He obtained it as a consequence of his own inductive way of thinking.

The study of this type of relations highlights the appearance of infinity as *an object* and as *a process*, one of the most characteristic features of the mathematical way of thinking. An infinite process such as the infinite sum or the infinite product, results in a mathematical object. Processes and objects, in mathematics, can constitute a single, or *hybrid* idea, and represent the same thing. This is very different from in real life: the process of making a table is not the table itself. The process *is not* the object in this case.

Leonardo kept a piece of paper on which it was written:

> It will suffice to make use of them [infinitesimals and infinitely large quantities] as a tool that has advantages for the purposes of calculating, just as the algebraist works with imaginary roots to great advantage. We do so because in them [in the infinitesimals] there is at hand a tool for calculating, as is clearly verified in each case by the method we have already presented.

We do not know if these lines were original to him, but there is no doubt that he made them his own as a profound approach for dealing with infinity.

There is in all of this a *tension between the finite and the infinite*, both as processes and as the results (*objects*) of such processes. The infinite, so elusive, seems always to disappear. But it always reappears with another face. Here, with Leonardo´s face…

Finally, although his mathematical audacity led him into previously unexplored territories, Leonardo never ceased to think that mathematics bore the deep imprint of nature. However, his mathematical elaborations seemed to move away from their physical referents and with that, their legitimacy, which until then had been fundamental, almost vanished in front of the eyes of his contemporaries.

These concerns reached the doors of the academies, which offered the scientific community a prize for anyone who could explain in a clear and precise manner what is called *the infinite* in mathematics. The die was cast. It was now a matter of finding a way to overcome the dissonance between Leonardo's inductive methods and the need for a foundation that would trace a *safe* path.

Searching for models in mathematical culture, eyes turned their gaze to the *deductive* methods of Euclid's *Elements*. Agustín's eyes tried to meet the challenge. A new time was opening to a new century.

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