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Part 4: Epilogue

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Abstract: In closing our set of of articles, we reflect back on the nature of mathematical thinking and learning with representations, especially the computational, executable representations that are enabled by modern dynamic mathematics environments. These aspects of the development of mathematical ideas have profound implications for our approaches to the teaching and learning of Calculus. Our narrative approach has dramatized the radically fruitful and generative nature of the period during which the ideas of Calculus were stabilized (especially in the stories of Guillermo/Leibniz and Leonardo/Euler). An analogy between the development of the field and the development of individual learners, should caution us not to cut short the vivid imaginative life of these ideas in our students through a premature push to formalism and analytic rigor; rather, to encourage students to systematize their thinking in domains of abstraction.

Keywords: Representations, Co-Action, Concrete, Abstract, Domain of Abstraction

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Introduction

In the pages of this set of articles, we have attempted to realize a fantasy: imagining personal encounters with some of the mathematical ideas that were pivotal in the construction of Calculus (and more broadly, Analysis)—as a new discipline within mathematics. Our goal in pursuing this fantasy has been to invite teachers of the subject to explore approaches that animate these ideas and the potential drama of the generative tension between them, as opposed to traditional teaching methods.

Mathematical Objects and Representations: Concrete and Abstract

Since mathematical objects are *ideal*, conceptual objects, in order to refer to them we must resort to symbolic representations. Our intuitions, which connect our embodied understandings to our prior representational experiences with mathematical objects, can motivate us to engage in representational action and spur us in initial directions; but those initial intentions will evaporate if we fail to represent our ideas. This reminds us of some lines of the poet Ossip Mandelstam: *I have forgotten the word I wanted to pronounce and my thought, incorporeal, returns to the world of shadows*. So it is with those mathematical intuitions that we fail to capture through representation.

If we speak of real numbers, it is convenient to have them represented, for example, by their decimal expansions. In this way, we will be able to apply arithmetic operations to them and gradually develop a feeling that we are working with concrete objects. What we have just said illustrates a more general cognitive feature: as we work with an abstraction, the feeling that we are working with something intangible disappears, and after a certain time, we feel that we are working in a more immediate way, with something more "concrete" (cf Wilensky, 1991). That is to say, concepts are not in themselves *abstract* or *concrete*. Rather, this designation reflects the nature of our developing relationship with them: they are abstract *to us* or increasingly concrete *to us*. Some authors have spoken of *the ascent from the abstract to the concrete* precisely to point out this fact. We always go from the unknown, from the abstract, towards the known, the concrete. The process is an *ascent,* because it is about reaching a more solid level of understanding. It is in that sense that we have ascended towards the clarity of the concrete. This ascent reflects a development in familiarity, a process of constructing connections, and an appropriation of the representation(s) in question.

Representations are critical to the process, as they structure interactions and direct attention. The history of number systems tells us that complex numbers were originally called *imaginary* because an appropriate representation of them had not yet been developed. When it was possible to represent them as points on a plane, and when geometric interpretations of the arithmetic operations associated with them were developed, they came to be called *complex* numbers. That is, they acquired "citizenship" in the mathematical world, and they became *concrete* for mathematicians. Something similar can be said of the other number systems, where 'alien' names (e.g., 'negative' numbers, 'fractions' [interpreted as 'broken' or 'discordant, fractious'], or 'irrational' numbers) were replaced by more 'familiar' and positive ones (e.g., 'integers' [i.e., integrated], 'rational' numbers, or 'real' numbers) as the process of ascending to the concrete became more available.

Today, when the ideas of symbolic representation have become clearer, we say that a mathematical object is not independent of its representations, and we add that, as new representations appear—notably digital, executable representations—they shed new light on our understanding of objects. We thus note that no system of representation (or systems, in the plural) completely exhausts the understanding of the mathematical object in question. The object is always unfolding, always under construction. The work of innumerable generations leaves its mark through the systems of representation that are being elaborated.

Dynamic Representations and Developing Ideas about Infinitesimals

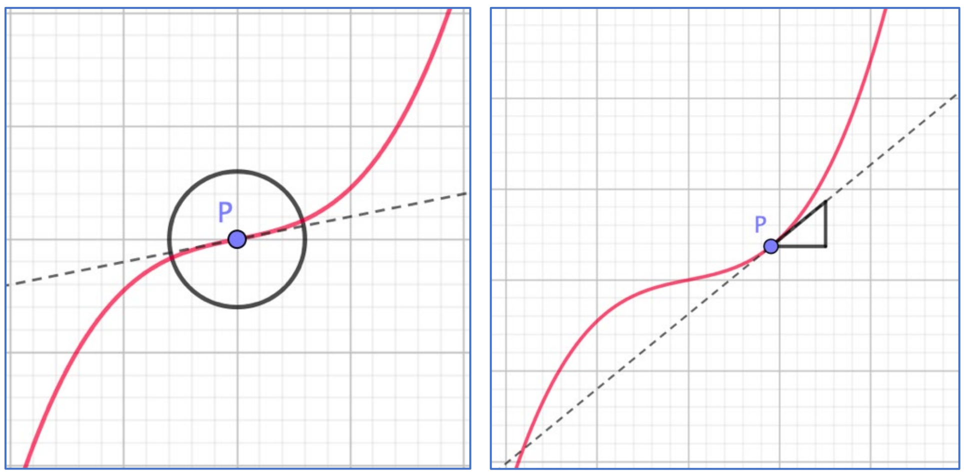
We have noted that mathematical objects cannot be approached except through the "mediated" path offered by representations of them. This is phrased in a 'tragic' way—as a loss, or as a lack of 'immediacy.' However, it is also true that representations offer powerful affordances. Any representational system that we engage with offers feedback properties that can make expressing ideas a generative and dialogic process. Even in static media, the process of representing an idea can be a transformative one. E. M. Forster (Forster, 1927) captured the generativity of representing in *Aspects of the Novel,* writing, "How can I know what I think until I see what I say?"

In dynamic and/or executable representations, this generative and dialogic relation occurs not only at the moment of construction (or 'writing'), but also to some extend at the moment of use (or 'reading'). Such representations offer relations with the human reader or writer that we have described as "co-action" (Moreno-Armella & Brady, 2018).

Let us see how this can happen with the ideas of Calculus. First, in the context of differential calculus, let us look at the geometric concept of the tangent line. Through its etymology, the word 'tangent' evokes the sense of *touch*, and it suggests an experience that underlies the concept, which could be brought into the Cartesian coordinate system through the innovations of Nicolas and Guillermo. Digital media, such as dynamic geometry software, can bring interactive and sensory dimensions to this mathematics, and this can amplify the expressivity of the infinitesimal innovation.

Figure 1

Focusing on a Movable Point of the Graph, We Construct a Tiny Tangent Segment

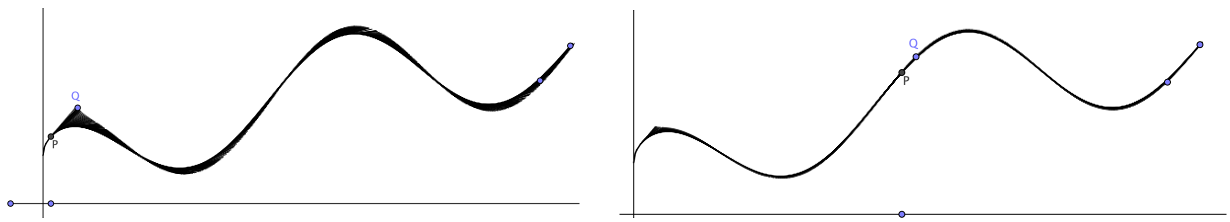


For example, we can create a small segment of the tangent line to a movable trace point P on the graph of a function. This is the hypotenuse of the "characteristic triangle," whose legs represent the ideas of "dx" and "dy." In the dynamic geometry medium, we can zoom – either symmetrically or in one dimension, as appropriate for the conceptual analysis we are doing (cf Tall, 2003; and continuous graphs as those that can be 'pulled flat,' Tall, 2013). In this setting, we can express infinitesimals as small with respect to any scale, and we can adjust the zoom to *as small a scale as needed* for the analysis (Tall, 2001; 2004).

Suppose we ask our environment to leave a 'trace' of our tiny tangent segment, as we slide P along the graph (as if the tangent segment had been coated in ink or paint). As we move P, it leaves a trace that practically coincides with the graph of the function, as shown in the left-hand graph of the figure below.

Figure 2

The "Footprints" of the Tiny Tangent Segment



Reducing the drawn scale of the characteristic triangle's quantities dx and dy enables us to reason about them as infinitesimals. With a smaller dx and dy, we repeat the interaction, and the 'footprints' of the tangent segment follow the graph even more closely. This experience gives a visual and synoptic meaning to the concept that the tangent line "best approximates the curve near P." Practically, the small segment traces the same graph as the original function. Of course, as long as we use visible dx and dy infinitesimals, deviations from the graph are detectable. But we can see these deviations vanishing as they are reduced.

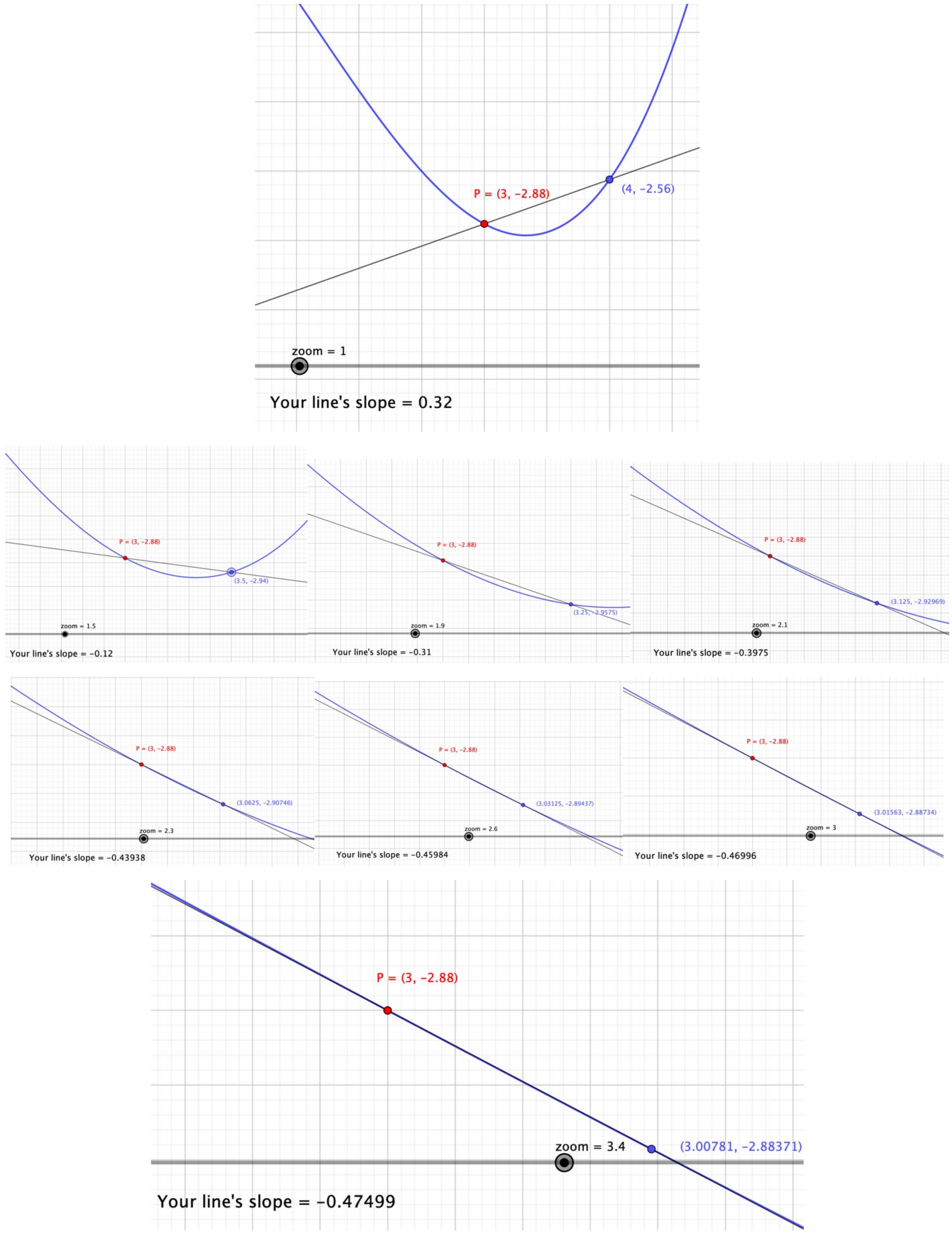
This simulation can also be understood as follows: from any given point P on the curve, adding the tangent line segment is an operation that is equivalent to integrating the derivative; by doing this, the original function is recovered. This gives us another expression of the first half of the Fundamental Theorem of Calculus!

As we interact with the digital representation, sliding the segment over the graph by dragging the point P, the simulation evokes the embodied experience of tracing our index finger over a surface to explore its texture. Since the Cartesian graph is paradoxical as a geometric space (as we have seen above), it is remarkable to be able to associate a tactile experience to this representation. By means of the interface, we are reaching into a space where our body cannot exist. This 'texture' of a graph suggests a qualitative means of access to the condition of differentiability—or in the case of the blancmange function, of *non*-differentiability. The dynamic representation's almost *haptic* possibilities for interaction open the potential for reflecting in new ways on a function and on the phenomena it models.

Zooming itself can provide visual support and build intuition for the concept that with infinitesimal change dx, the secant line between (a,f(a)) and (a+dx, f(a+dx)) approximates the tangent line to f(x) at x=a. Below, we zoom to investigate the situation at x=3 for a particular function f(x), beginning with dx=1 and then exploring $dx=2^{-n}$ for n=1,2,…,7.

Figure 3

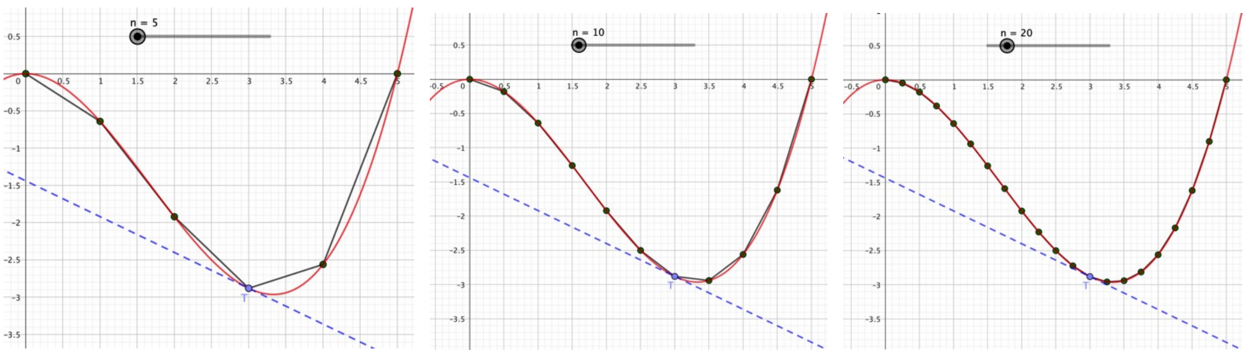
Secant Segments over Increasingly Small Sections of the Curve



And at the same time, at the global scale, this operation can be monitored for the way in which it comes to approximate f(x) as a many-sided polygonal path.

Figure 4

Visualizing a Function's Graph as a Many-Sided Polygonal Path

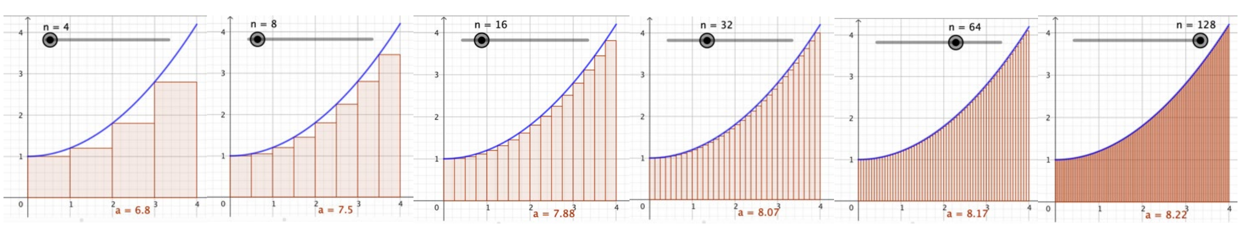


The emergence and dissemination of dynamic, executable representations in the past 50 years creates another chapter in the development of Calculus which, as we have seen, has depended on such innovations. This highlights the disciplinary and pedagogical importance of the rich relations between mathematical objects, our understandings of them, and the symbolic representation systems in which these objects *live*.

Now consider how the dynamic environment can support meaning-making inquiry and discussion around the topic of area-under-the-curve.

Figure 5

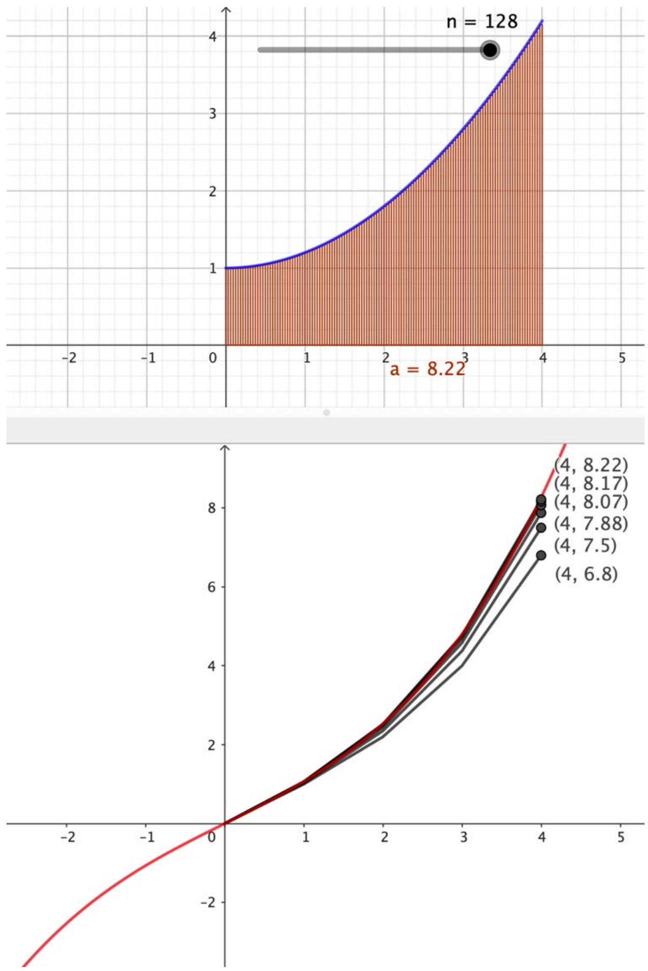
Multiplying the Number of Rectangles Inscribed under the Curve



Comparing the values of the left-hand sums at x=1, 2, 3, and 4, with the number of rectangles increasing (beginning from rectangle one per integer interval, doubling to 2, 4, 8,16, and 32) we can watch the area under the curve function approach the theoretical function, shown below in red, with the value of approximating functions shown for x=4.

Figure 6

How Increasingly Fine Area Decompositions Lead to Increasingly Precise Estimates of the Accumulated-Area Function



The formalism of integration can be daunting for students, involving sigma notation and the idea of an infinite limit operating on the subdivisions over which the series is calculated. Yet the visual representation is more straightforward, and when we substitute "large N" for "infinitely-large N," the drawings enable connections with Sereno's work with parabolas and other shapes using the notion of the "indivisible" quantity.

Systematizing and Symbolic Generativity

As we contemplate the changes in thinking and learning about Calculus when humans partner (co-act) with dynamic representations, we are led to reflect on the nature of thinking in the presence of symbolic systems, more generally. As suggested in the "ascent from the abstract to the concrete," operating in a symbolic realm allows us to build new instincts, making the symbolic objects increasingly familiar and concrete. We begin to *live* in the symbolic, to develop new intuitions there. Our work and our understanding also begin to have a *systematic* character.

There are at least two, symmetric, ways that fluent use of a symbol system can reflect back on our understanding of the world. In the first, a concept or an intuition arises in the reference field (world), which is not expressible or which is invalid or meaningless in the symbolic field. Before Calculus (both in the *sociogenic* sense, historically, and in the *ontogenic* sense, in a student's learning), the construct of instantaneous velocity is an example. We have vivid personal experiences of speed, and it makes intuitive sense to think about speed at a moment. Yet our existing symbol system, algebra, fails to extend our expression of *velocity over an interval* to provide *velocity at an instant*. Here, the intuition, educated and focused by experience and play with the symbolic system and its connection with the field of reference, can be a guide for innovation in the symbolic system by cultivating the conception of infinitesimals and toward an intuition for the idea of limits.

There is a second way that systematization of a symbol system can affect our understanding of the world. This occurs when symbolic activity goes *beyond* its foundation in the experiential world, to develop objects that are consistent with the symbolic system but that are difficult even to conceptualize in the realm of experience, let alone to encounter there "in the wild." In this case, the systematizing effect of the symbolic realm shows its productive potential. The new objects conform to the intuitively-grounded symbolic description – this description is then increasingly taken as the definition of the intuitive construct, allowing it to be extended beyond experience. An example of this in the conceptual field of Calculus occurs as, increasingly, the definite integral operation is taken as a *definition* of area, allowing new regions to be assigned areas.

These two ways of operating are 'dual' to one another. In the first, intuition inexpressible in the symbol system works to motivate innovation to extend the expressivity of the system. In the second, the system is able to express more than can be grasped by human intuition, and the human works with the system to extend the reach of their intuition.

This dual relation allows "boostrapping" to develop a conceptual field; this process develops well in what we have called "domains of abstraction" (Moreno-Armella & Sriraman, 2010). Where intuition and an emerging symbol system are configured so as to allow the bootstrapping relation above, rapid organic development can occur. At the sociogenic scale, in formative and developmental periods, as in the time of Guillermo, Leonardo, and Agustín, we observe mathematical thinking that is audacious and creative, using intuition to extend symbol systems, and attending to the suggestions of symbol systems to illuminate new intuitions for thinking about the world.

The "story" approach we have followed in these articles recognizes that analogues of these historically significant developments can unfold in the discourse among a classroom group of students struggling to make sense of Calculus. On the ontogenic scale, the flashes of intuition that students experience and their gradual appropriation of new symbol systems can recapitulate some of the dynamics (but not the precise sequence or tone) of history.

Our understanding the potential for links between classroom learning and the cultural development of Calculus suggests that the most fruitful forms of engagement to emulate may involve struggles to expand both intuition and symbolic expression. In them, learners expand their symbolic systems in order to be able to create mathematics that captures new ideas and phenomena in the world; and at the same time, they expand their intuitions about those phenomena—coming more deeply to understand the world when guided by symbolic action. In the history of Calculus, these times involved ambitious, audacious thought: new conjectures that gave rise to local efforts at systematization, as seen in Guillermo and Leonardo. Such times contrast with the drive toward rigor and formalism that is characteristic of the later nineteenth and early twentieth centuries in the history of Calculus. Yet, if introductory Calculus as traditionally taught today draws from *any* historical period, it is in fact this latter period that dominates, with aims of formal rigor and goals to connect Calculus with Analysis.

But formalization of a symbolic system involves "looking at" it as an instrument, potentially to the expense of "looking through" it as a means for gaining new understandings of the world. Surely, for students of Calculus, our goal should be supporting them in encountering and appropriating powerful ideas and building a conceptual system, which need not (yet) be a formally airtight construction.

This proposition is complicated by the history of nineteenth century mathematics. As mentioned above, the philosophical and epistemological rupture of the 1830s with non-Euclidean geometries had a two-leveled effect. The primary effect—a healthful one in our view—was a *mathematical* crisis, a recognition that geometry (and mathematics in general) can offer only *models* of the world, rather than direct access to reality. The secondary effect—a pernicious one in our view—was a *psychological and pedagogical* crisis, a loss of confidence in human intuition and in visual/geometric reasoning as a guide for mathematical inquiry. David Henderson (1996) reflected on the logical extension of this trend into the twentieth century, identifying it as a cause of alienation among mathematicians and as a barrier to diversity of thinking and participation in mathematics.

In the context of Calculus, we argue that emphasizing formalism can rob mathematical ideas of their best "stories." And an effort to convince students that the formalism (e.g., of epsilons and deltas) is *necessary* can easily become an attempt to shake students' confidence in their intuitions (cf. De Villiers, 2012, for parallel effects of proof in geometry instruction). This can have tragic effects, converting philosophical adventures into tautological exercises in the name of rigor. Returning to René Thom (1971; 1973), we note that the challenge in teaching Calculus is to cultivate students' development of systems of *meaning*, rather than to train them to produce rigorous, yet empty, arguments. This pedagogical assertion must struggle against the impulses of university professors of mathematics, whose professional lives give them ample evidence of the value of rigor, when it enters as a guide for intuition and vibrant understanding. We hope that the approach of telling "stories" of the ideas of Calculus can revive in them the memory of the emotional and intellectual power of coming-to-understand, so that they may appreciate the prior need for students to construct and develop meaning—both the meaning *of* mathematical constructs and the meanings that can be built *with* them.

Conclusion

Mathematics education is a field of research at the intersection of several disciplines. Of course, the list includes mathematics, but it also includes cognitive theories, the theory of representations (static and dynamic), and the epistemology of mathematics. This list is not exhaustive. The social and cultural dimensions of mathematical knowledge and practice make us think that the progress of knowledge in this, our discipline, requires a deeper understanding of the conditions that make knowledge possible. Therefore, we are continually obligated to return to the study of the same foundational ideas through which we came to understand the objects of mathematics and the nature of mathematics. This explains the character and purpose of the present work.

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