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David Tall and Advanced Mathematical Thinking

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Abstract: I discuss David Tall's influence on advanced mathematical thinking.
Keywords: Advanced Mathematical Thinking; David Tall

For the past 45 years, David Tall has been a larger than life figure in educational research in advanced mathematical thinking. David was the first mathematics education researcher to use the phrase 'advanced mathematical thinking'. His paper with Shlomo Vinner on concept image (Tall & Vinner, 1981) remains the most influential and most cited theoretical construct in the field. His edited volume *Advanced mathematical thinking* (Tall, 1991a) consolidated the work of leading scholars who were conducting educational research in advanced mathematics, calling attention to the field and providing it with direction. David Tall has been a prolific scholar who continued to make seminal contributions in the area of advanced mathematical thinking until his recent passing.

There are many aspects of David's conception of advanced mathematical thinking that continue to shape how research in this area is conducted. First and foremost, David believed that advanced mathematical thinking was something unique, special, and worthy of serious mathematics education research. Advanced mathematical thinking was not simply a natural extension of elementary and secondary students' mathematical thinking. Advanced mathematical thinking was qualitatively different, and constituted a breach from students' prior ways of reasoning. Understanding how students could use their prior ways of doing mathematics to inform their engagement in the new endeavor of advanced mathematical thinking was a major focus of David's later work (e.g., Gray et al., 1999; Tall, 2008). A further legacy in David's framing of advanced mathematical thinking is pluralism: Because there are multiple ways that mathematicians engage in mathematics, there must be multiple trajectories for students to learn to engage in advanced mathematical thinking. Also, because advanced mathematical thinking is a human activity with psychological, social, and mathematical dimensions, advanced mathematical thinking can be researched productively using multiple theoretical perspectives (Tall, 1991b). With the

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benefit of hindsight, I believe David's early insistence on pluralism played a substantial role in the growth of the field, since it invited numerous researchers to participate in the enterprise, regardless of their theoretical commitments.

Four things that I learned from David Tall about advanced mathematical thinking

David Tall had an enormous influence on how I conducted my research on advanced mathematical thinking. From 2001 to 2003, I was an assistant professor of mathematics at Murray State University. As a young scholar, I was committed to learning more about undergraduate mathematics education research, so I made sure to read at least two new mathematics education articles every week. Unfortunately for me, Murray State did not have a very extensive library in mathematics education. If memory serves me correctly, *Journal for Research in Mathematics Education* was the only significant mathematics education journal they housed, and *JRME* at the time rarely published work in undergraduate mathematics education. The best resource that I had available to me was access to David Tall's academic homepage, where every article that David had written was available for download. David was a wonderful writer who expounded on a wide range of topics, so I never became tired of visiting his webpage and reading his articles.

In 2004, I attended the ICME conference at Copenhagen, eager to meet David. I recall being in the computer laboratory checking my e-mail when I heard a large British man named David brashly shouting to anyone who would listen that he could not get his computer to work and demanding that his difficulties be resolved. "This has to be David Tall", I thought, so I introduced myself to him. When David heard my name, he immediately said, "Oh yes, I've been meaning to talk to you" as if we were old friends. He had read an article that I recently published that built on his work (Weber, 2004), and for the next half hour, he proceeded to discuss the integration of my ideas and his. It will come as no surprise to those who knew David that he was not shy about saying where he thought I missed the mark. These anecdotes illustrate several reasons that David was so influential: He was what us Americans would call a salesman, someone who unabashedly and effectively pitched his product to anyone who would listen. David was egalitarian in this respect. He disseminated his work so everyone could have access to it, and he enjoyed talking about it. I've found ICME to be a conference where scholars often go to see and be seen, and spend time finding the right people to talk to. Do you know who David liked to talk to at a conference? Someone who wanted to discuss his ideas. This was the case even if, *especially if*, that was a young scholar who might utilize his ideas and push them further.

I'll discuss four ideas from David that had a profound influence on my subsequent research. My list is obviously idiosyncratic and provincial and centered around my research interests. I am not mentioning some of David's most important ideas on technology, procepts, and met-befores. Doubtless, other scholars drew other lessons from David. But I hope the spirit of some of the lessons I learned will show how David was a unique thinker who offered vast insights.

First, in Tall and Vinner's (1981) classic concept image paper, the authors remarked, "For each individual, a concept definition generates its *own* concept definition (which might, in a flight of fancy be called 'concept definition image')" (p. 153). For the field, the big takeaway from Tall and Vinner's paper is that understanding formal concepts in advanced mathematics goes well beyond being able to recite and use concept definitions. For me, I became obsessed with the link between concept image and concept definition: how can concept image influence proof writing when proofs (supposedly) can only appeal to definitions and cannot appeal to concept images? The notion of "concept definition image" was central, as this observation showed to me that it is productive to think about how students and mathematicians integrate their concept images with concept definitions.

Second, David's analysis of mathematical proof (Tall, 1989) indicated a problem with the popular notion that a proof was a very convincing argument. David argued that not all convincing arguments would qualify as proofs, as proofs require deductive reasoning from clearly formulated definitions and statements[1]. We should not alter what proof is to fit it into our preferred characterizations, but appreciate the richness of proof as it exists in mathematical practice. Further, David noted proof might not be appropriate for all students or in all situations. For instance, if mathematical concepts have only been defined intuitively or via example with students (which is often appropriate with young children), then they will not be able to prove statements about those concepts. There also may be very good intuitive, technology-based, or empirical arguments that are convincing to students and are perfectly good to use in the classroom, even if they are not proofs (c.f., Tall, 2001). What this taught me is that proof is a special *type* of argument that is valuable for some purposes and some people. However, while proof is important in mathematical

[1] I agree with David that not all convincing mathematical arguments are proofs. I have also said that not all proofs are completely psychologically convincing, since there may be doubt that steps within a proof are carried out correctly. In Weber et al. (2022), I illustrate that this is how mathematicians view proof.

practice, mathematics educators should not worship proof. Proof is not the only argument that can provide conviction to students and it is sometimes not the best means of doing so.

Third, Pinto and Tall's (1999) have conducted great research on how students can learn advanced mathematics either by "giving meaning" to concept definitions (i.e., using their pre-existing concept images to make sense of the new definitions) or by "extracting meaning" from concept definitions (i.e., logically parsing concept definitions and drawing consequences from definitions to build concept images at a later time). The point that I drew from this is that there is no optimal learning trajectory to learning advanced mathematics. As a teacher, I learned that it behooved me to look at the resources that students brought to the table and build from there, even if this was decidedly not the way that I learned or thought about the mathematics in question.

Finally, David offered a critique of APOS theory that stuck with me. I wish I could quote this critique verbatim from a manuscript that David had written. Alas, I could not find David's critique in print. I'm afraid I will have to paraphrase based on my recollection. David said: "In APOS, there is a P for process. But where is the P for *picture*? Where is the P for *problem*? Where is the P for *proof*?" I knew that David had enormous respect for APOS theory, as it simultaneously offered a perspective of what advanced mathematical thinking is, how students learn it, and how they can be taught it. My interpretation of his commentary is that we should resist the temptation to reduce advanced mathematical thinking into our favorite theoretical framework. If a theory cannot account for all of the wonder of advanced mathematical thinking, including pictures, problem solving, and proof, then that theory is too small for advanced mathematical thinking, and we must complement that theory with other perspectives.

Two disagreements with David Tall

There were two issues where David and I never saw eye-to-eye on, but in recent years, I am coming around to appreciate David's perspective. The first is that I sometimes critiqued David for being exceptionally broad and vague with his theoretical constructs. For example, a student's image of a concept was anything that came into the student's mind about a concept aside from its definition. Couldn't David be more precise about the structure of concept image and how it played into students' cognition when performing mathematical tasks? David rejected this critique. Instead, he took my critique as a point of pride. David boasted to me that his constructs that were simple catchy phrases that were vaguely defined but easily understood. This vagueness was a virtue to David, since it meant that everyone would use his construct. At the time, I thought this was cheeky—David was again being the salesman who wanted to get his citation count up. Now I'm not so sure. What David was doing was inviting other mathematics educators to play with his ideas, and develop them using their own perspectives and expertise. If David had, say, defined concept image in terms of semiotic representation systems, then those who used David's idea would mostly be semioticians. This would reduce David's citation counts, of course, but more importantly, it would limit the impact of concept image and stifle the development of the construct. The field would be worse off for this. As an author and a reviewer, I sometimes push myself and others to be more precise in the theories that they introduce. I am coming to see that initial vagueness may be a virtue.

The second source of disagreement is the role of data in theory development. I would often use data to constrain my theory generation. I would deliberately design studies where I could collect data that could potentially falsify my hypotheses. I did so because I was worried that my theorizing might not align with how students or mathematicians thought and behaved, so I would collect data to see the extent that my theories aligned with the real world. David was not interested in this, and he was always puzzled why I was. David did value data, but only as the grist for his mind to analyze and further his theory generation, not to constrain the ideas he had already produced. The moral that I draw now is that, much as there are different ways to do mathematics, there are also different ways to conduct mathematics education research, and us mathematics education researchers will inevitably gravitate toward approaches fitting our personalities and capabilities. In my research career, I've often learned the painful lesson that the hypotheses that I was sure of turned out to be false. I used data to constrain my theories since my past errors had made me humble. In contrast, David did not have as much use for this approach. For David was rarely wrong and not demure.

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