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# Rational Approximations of Sine and Cosine

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**ABSTRACT:** In this paper, we use elementary methods to derive a rational function over the integers to approximate the trigonometric sine function on the interval $\left[0, \frac{\pi}{2}\right]$. This formula can then be used to derive a quartic polynomial with a root close to $\frac{\pi}{3}$, providing an interesting algebraic approximation to this value. A more accurate rational function over the reals is then computed using numerical optimization. This new formula, while more accurate, provides a worse approximation of $\frac{\pi}{3}$ in the corresponding quartic equation, showing the trade-offs in local vs. global approximation. This paper is accessible to undergraduates and illustrates a combination of mathematical constructions used in Algebra, Calculus and Numerical Optimization.
**Keywords:** rational approximation, quartic polynomial roots, trigonometric functions

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# Introduction

Beginning students of mathematics often hope for a simple way to compute the sine or cosine of an arbitrary angle. In fact, the trigonometric functions are (usually) among the first functions students encounter that do not have simple algebraic (i.e., rational) representations. It is not until experiencing a sufficient amount of calculus that students are taught that these functions can be represented by their MacLaurin (Taylor) series and even later that they come to understand them as (complex) analytic functions [1].

Before the advent of modern computing, simple approximations of the elementary functions may have been useful as a computational efficiency. In the modern age, they can still serve as a recreational pass-time and instructive tool. Interestingly, it is the very advent of modern digital computing that encouraged Kovach and Comley [2] to provide their simple approximation,

$$\sin(x) \approx C_1 x + C_2 x^2 \mathrm{sgn}(x) + C_3 x^3 + C_4 x^4 \mathrm{sgn}(x),$$

where sgn is the standard sign function and $C_i$ are constants provided in context. Kovach and Larson [3] follow up on this original work by studying further approximations to the trigonometric functions using polynomials with rational powers and coefficients and limited to the region $[-1, 1]$. The results are then compared to Lanczos' economization method [4] for finding a modification to a truncated Maclaurin series with the same number of terms. Their method results in an expression

$$\sin(x) \approx 1.0053607|x|^{1.0022}\mathrm{sgn}(x) - 0.1638897|x||^{2.8628}\mathrm{sgn}(x),$$

which has relative error bounded below 0.00075 for $x \in [-1, 1]$. Likewise, they compare their method to that of Hastings [5], who approximate $\sin\left(\frac{\pi x}{2}\right)$ on $[-1, 1]$ with the polynomial,

$$1.5706268x - 0.6432292x^3 + 0.0727102x^5,$$

which has maximum relative error 0.00011 on $[-1, 1]$. In contrast, Kovach and Larson find the approximation,

$$1.5708268x - 0.6478298x^3 + 0.0770030|x|^{4.85}\mathrm{sgn}(x),$$

which has relative error 0.000022.

Despite the fact that these approximations are no longer necessary thanks to the efficiency of modern computing architectures, there is a certain delight in finding simple approximations for the trigonometric functions using only a small number of algebraic expressions. An example of this is Bhaskara's approximation to the sine function, which is discussed and justified by Stroethoff [6].

In this paper, we use simple geometry and a few additional facts to derive a rational expression over the integers for the sin function on the interval $[0, \frac{\pi}{2}]$ that has relative error bounded above by $\approx 0.0003681$. We then show that this resulting construction allows us to derive a quartic polynomial with a root close to $\frac{\pi}{3}$ and analyze the relative error if we allow the rational function to have real coefficients. Thus, the paper provides a recreational and instructive blueprint that could be used in a class for exploring geometric methods, approximation, solvable quartic polynomials and numerical optimization.

# 1 Derivation of the Approximation

Consider the diagram shown in Fig. 1 showing a circle of radius $r$ with an isosceles triangle formed from two radii. It is immediately clear that,

$$k = 2r\sin\left(\frac{\theta}{2}\right).$$

Azim et al. [7] present an interesting formula that approximates the ratio of the length of the base of the triangle to the arc $a$ on the interval $\left[0, \frac{\pi}{2}\right]$. We discuss a modified proof of the approximation from [7] and then use it to construct an approximation of $\sin(\theta)$ using a rational expression.

Apply the law of cosines to the triangle in Fig. 1, to conclude that,

$$k = \sqrt{r^2 + r^2 - 2r^2\cos\theta} = r\sqrt{2(1 - \cos\theta)}$$

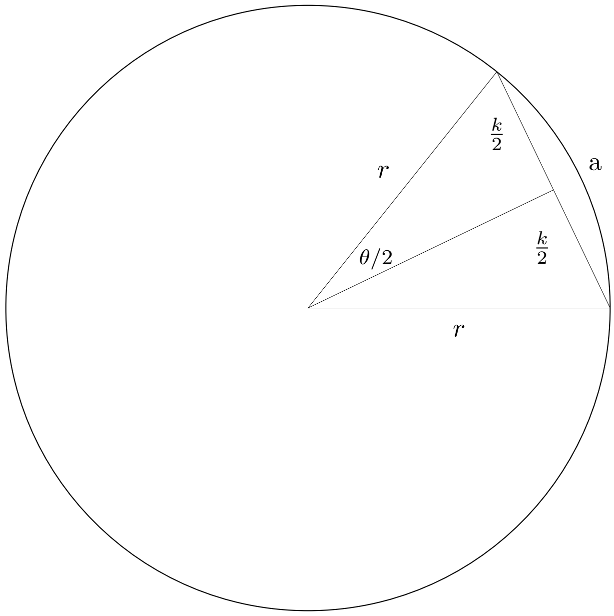


Figure 1: Geometrical structure used in the construction of the approximation.

The quantity,

$$f(\theta)=\frac{k}{r\theta}=\frac{\sqrt{2\left(1-\cos\theta\right)}}{\theta},$$

has second order Taylor approximation,

$$f(\theta)=\frac{k}{r\theta}=1-\frac{\theta^2}{24}+O\left(|\theta|^4\right)\approx\frac{24-\theta^2}{24}.$$

Note that we have a second (shared) Taylor approximation,

$$\frac{24}{\theta^2+24}=1-\frac{\theta^2}{24}+O\left(|\theta|^4\right)\approx\frac{24-\theta^2}{24}.$$

Therefore,

$$f(\theta)\approx\frac{24}{24+\theta^2}.$$

The insight in [7] is to observe that a minor adjustment, offers a better approximation on $\left[0,\frac{\pi}{2}\right]$ to $f(\theta)$, namely,

$$\frac{\sqrt{2\left(1-\cos\theta\right)}}{\theta}=f(\theta)\approx\frac{23}{\theta^2+23}. \tag{1.1}$$

To see why, notice that,

$$\int_0^{\frac{\pi}{2}}\left|\frac{23}{\theta^2+23}-f(\theta)\right|d\theta\sim 0.00083<\int_0^{\frac{\pi}{2}}\left|\frac{24-\theta^2}{24}-f(\theta)\right|d\theta\sim 0.00099<$$
$$\int_0^{\frac{\pi}{2}}\left|\frac{24}{\theta^2+24}-f(\theta)\right|d\theta\sim 0.0021.$$

Thus, the heuristic approximation in Eq. (1.1) provides a measurable improvement over the Taylor approximation.

Now, applying this approximation from [7] and using some trigonometry, we know that the ratio of the length of the base of the triangle to the length of the arc $a$ is approximated by,

$$\frac{k}{r\theta}=\frac{2r\sin\left(\frac{\theta}{2}\right)}{r\theta}=\frac{2\sin\left(\frac{\theta}{2}\right)}{\theta}\approx\frac{23}{\theta^2+23}.$$

This implies that,

$$\sin\left(\frac{\theta}{2}\right) \approx \frac{23\theta}{2\left(\theta^2+23\right)}.$$

Suppose $\theta = 2y$. We arrive at an intermediate approximation,

$$\sin(y) \approx p(y) = \frac{23y}{4y^2+23}.$$

We can improve this approximation by using an interesting result from Hesselgreaves [8] that,

$$\sin(3y) = 3\sin(y) - 4\sin^3(y).$$

Let $x = 3y$ and substitute $y = \frac{x}{3}$ into $p(y)$ to obtain,

$$\sin(x) \approx 3\left(\frac{69x}{4x^2+207}\right) - 4\left(\frac{69x}{4x^2+207}\right)^3 \triangleq f(x), \tag{1.2}$$

after simplification. This is a rational approximation of $\sin(x)$ on the interval $\left[0, \frac{\pi}{2}\right]$, by construction (see Fig. 1). The surprisingly small absolute and relative errors are shown in Fig. 2. Numerical analysis shows

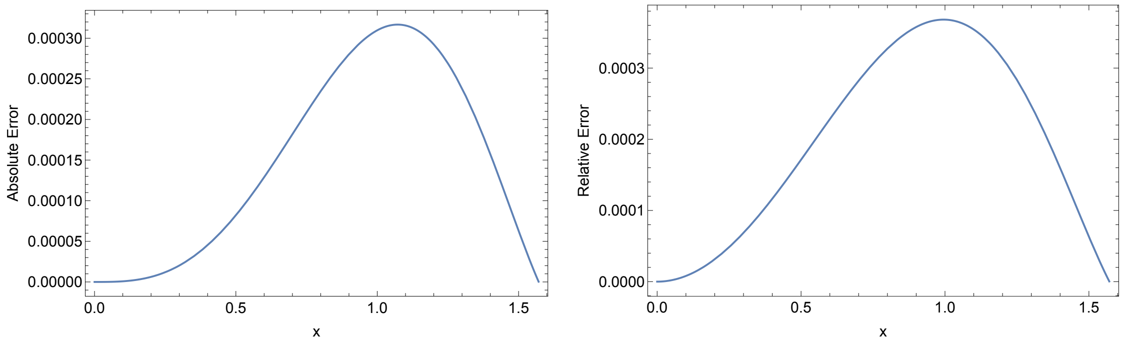


Figure 2: (Left) The absolute error and (right) the relative error show that Eq. (1.2) provides a good approximation for $\sin(x)$ on $\left[0, \frac{\pi}{2}\right]$.

that relative error is bounded above by 0.0003681, which is better than Lanczos' method [4] but not as accurate as Hasting's polynomial [5], which uses real coefficients (as opposed to integer coefficients). We will address this in the sequel.

## 2 Polynomial over $\mathbb{Q}$ with Root Near $\frac{\pi}{3}$

It is well known that $\pi$ is not algebraic, see, for example, Hilbert [9]. However, Eq. (1.2) gives the potential for a novel approximation of $\pi$ using algebraic functions. Compute $f'(x)$ to obtain,

$$\cos(x) \approx f'(x) = -\frac{207\left(8x^2\left(8x^4-9108x^2+471339\right)-8869743\right)}{\left(4x^2+207\right)^4}. \tag{2.1}$$

This approximation for $\cos(x)$, while not as good as the approximation for $\sin(x)$, has reasonable relative error less than 0.000731 on the interval $\left[0, \frac{\pi}{3}\right]$ as shown in Fig. 3. The fact that $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ yields the approximation,

$$\frac{1}{2} \approx -\left.\frac{207\left(8x^2\left(8x^4-9108x^2+471339\right)-8869743\right)}{\left(4x^2+207\right)^4}\right|_{x=\frac{\pi}{3}}.$$

This, in turns, yields a reducible polynomial over the rationals,

$$128x^8 + 39744x^6 - 13026096x^4 + 851495328x^2 - \frac{1836036801}{2} = 0,$$

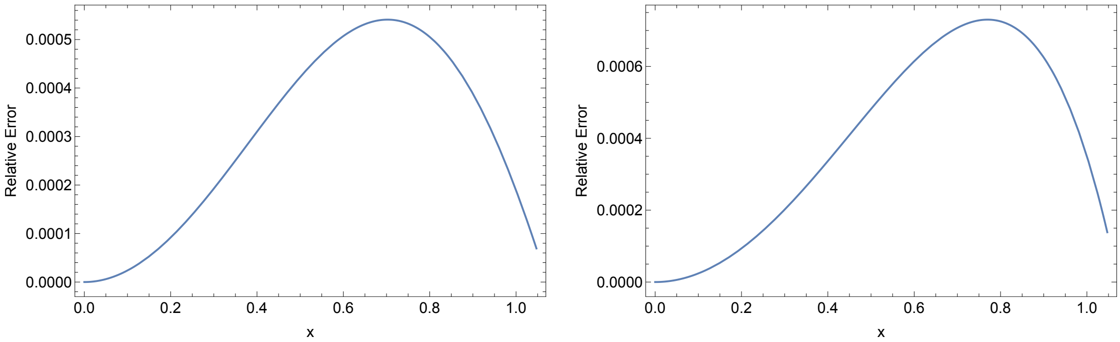


Figure 3: (Left) The absolute error and (right) the relative error show that Eq. (2.1) provides a good approximation for $\cos(x)$ on $\left[0, \frac{\pi}{3}\right]$.

with root near $\frac{\pi}{3}$. Let $u = x^2$. The equation reduces to a quartic,

$$128u^4 + 39744u^3 - 13026096u^2 + 851495328u - \frac{1836036801}{2} = 0,$$

which can be solved in radicals. This yields the unusual (though untidy) approximation in radicals,

$$\frac{\pi}{3} \approx \sqrt{-\frac{69}{8}\left(9 + A - \sqrt{\frac{B}{C}}\right)},$$

where,

$$A = \sqrt{3\left(103 + \frac{284\ 2^{2/3}}{\sqrt[3]{11813 + 3\sqrt{10414929}}} + \sqrt[3]{23626 + 6\sqrt{10414929}}\right)},$$

$$\begin{aligned} B = 3\Bigg(& 20082 - \frac{161312\sqrt[3]{2}}{\left(11813 + 3\sqrt{10414929}\right)^{2/3}} + \frac{29252\ 2^{2/3}}{\sqrt[3]{11813 + 3\sqrt{10414929}}} + \\ & 103\sqrt[3]{23626 + 6\sqrt{10414929}} - \left(23626 + 6\sqrt{10414929}\right)^{2/3} + \\ & 1134\sqrt{309 + \frac{852\ 2^{2/3}}{\sqrt[3]{11813 + 3\sqrt{10414929}}} + 3\sqrt[3]{23626 + 6\sqrt{10414929}}}\Bigg), \end{aligned}$$

and

$$C = 103 + \frac{284\ 2^{2/3}}{\sqrt[3]{11813 + 3\sqrt{10414929}}} + \sqrt[3]{23626 + 6\sqrt{10414929}}.$$

This approximation has relative error less than 0.000077 and an absolute error less than 0.000081.

## 3 Improving the Approximation with Real Numbers

Since Lanczos [4], Kovach et al. [2, 3] and Hastings [5] all use rational numbers in their approximation, it is worth concluding by determining whether the rational function in integers in Eq. (1.2) could be improved if we used rational numbers instead. To that end, we minimize the function norm,

$$J(a, b, c) = \int_0^{\frac{\pi}{2}} \left[\left(\frac{ax}{a + cx^2} - \frac{bx^3}{(a + cx^2)^3}\right) - \sin(x)\right]^2 dx. \tag{3.1}$$

with constraints $205 \leq a \leq 209$, $1314035 \leq b \leq 1314037$ and $3 \leq c \leq 5$, to simplify the search space. To accomplish this in Mathematica™, we use the following two steps. First, define a numerical objective function.

```
ComputeOBJ[a_, b_, c_] := NIntegrate[
(-((b x^3)/(a + c x^2)^3) + (a x)/(a + c x^2) - Sin[x])^2,
{x, 0, Pi/2}];
```

We then use the built-in numerical optimization procedure `NMinimize`. Mathematica™will throw a warning because of the definition of the objective, but this can be suppressed by wrapping the call in `Quiet`.

```
 SOLN = Quiet[NMinimize[{ComputeOBJ[a, b, c], 205 <= a <= 209,
1314035 <= b <= 1314037, 3 <= c <= 5}, {a, b, c}]]
```

The resulting function is,

$$h(x) = \frac{206.984x}{3.87901x^2 + 206.984} - \frac{1.31404 \times 10^6 x^3}{(3.87901x^2 + 206.984)^3}. \tag{3.2}$$

This approximation has maximum relative error less than 0.00012, making it numerically consistent with Kovach and Larson's method [3] without using non-differentiable functions. The resulting relative and absolute errors are shown in Fig. 4. Interestingly, while being a better overall approximation, using $h(x)$

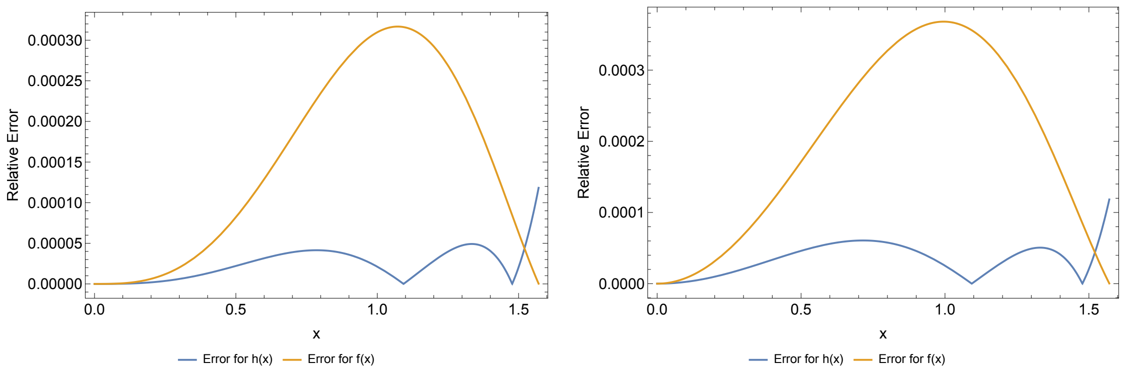


Figure 4: (Left) The absolute error and (right) the relative error show that Eq. (3.2) improves on the approximation for $\sin(x)$ on $\left[0, \frac{\pi}{3}\right]$.

and the steps in Section 2 yields a worse approximation for $\frac{\pi}{3}$ with a relative error greater than 0.00025.

# 4 Conclusion and Remark

In this paper, we derived a novel rational expression over $\mathbb{Z}$ that well-approximates $\sin(x)$ on the interval $[0, \frac{\pi}{2}]$ using classical methods from geometry. Prior work by Azim et al. [7], on which this paper builds, provides an algebraic approximation to the trigonometric functions that includes a radical. Radicals are computationally expensive, and consequently the elimination of these elements represents an improvement over this prior result.

We then used this approximation to derive a solvable octic-polynomial with root near $\frac{\pi}{3}$ leading to a new, but unwieldy, approximation for $\frac{\pi}{3}$ accurate to within 0.000081. We then used numerical techniques to improve the approximation, making it competitive with historically good but non-differentiable approximations. The cost of the improvement in the global approximation is loss of precision in the approximation of $\frac{\pi}{3}$ as the root of a polynomial. This nicely illustrates the tradeoff between local and global approximation accuracies. The steps outlined in this paper could be used as a class project to illustrate interesting connections between various areas of mathematics.

# 5 Data Availability

Mathematica™ code is available from the authors on request to reconstruct the figures and perform the numerical optimization in this paper.

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