**Interview #1  (Total time = 1:00:05)**

R: We will begin by reading the first question. You can read along as I read aloud. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an example of this with an arrangement with sum 16. Prove that the smallest possible value for the sum is 14. Take a minute to review that if you'd like.

Lisa: [pause, reading the question (25 sec)]

R: Do you have any questions about the problem or need any clarification?

Lisa: No.

R: Okay. Then you can begin working. You can work on either that paper or the notebook paper you have there. Just try to describe to me aloud what you're thinking and what you're doing along the way.

Lisa: Okay. [Draws a pentagon on separate sheet of paper, labels with the labels from the example pentagon, pause (45 sec)]

R: Can you tell me what you're thinking about?

Lisa: I was looking at numbers that add up to be 14. [draws box around 5 1 8 on the pentagon]

R: Oh, okay.

Lisa: That are in the same area of each other. [pause, writes the numbers 1-10 vertically below the pentagon, 35 sec]) So, I am looking at different combinations that make up 14 with three numbers.

R: Okay.

Lisa: [pause, writes 9 3 1, crosses off 1 puts 2, writes 7 5 2, crosses out, (35 sec)]

R: Can I ask why you discarded some choices?

Lisa: Um, poor arithmetic.

R: Oh, okay.

Lisa: [pause] Um. [pause (5 sec)]Then they, all sides have to equal 14?

R: That's right.
Lisa: Okay. [pause, writes 10 3 1, draws new pentagon putting 10 3 and 1 on it. Adds in 2 and 9, crosses them out and switches their order (40 sec)]
R: Okay, let me stop and ask you about your reasoning. You just switched around your 9 and the 2 and I just want to know what you’re thinking.
Lisa: By arranging them differently the 9 is less useful in reaching 14 than it would be for the 2. Then I can use it with other numbers as well.
R: Okay.
Lisa: [pause, writes in 5 and 8, crosses out and switches their order (25 sec)]
R: Do you want a Kleenex?
Lisa: No, I'm fine. I always- sorry.
R: No, not a problem.
Lisa: I always sniffle.
R: I figured coming in from the rain.
Lisa: [laughter, pause, writes 8 4 2 and 7 5 1 on list below second pentagon, adds 6 5 4 and 7 6 1 to this list (50 sec)]
R: So you are just experimenting and trying to find-
Lisa: Trying to figure out what-
R: An example of 14.
Lisa: An example, yeah.
R: Okay.
Lisa: [pause, crosses out 2 from 8 4 2 and changes 1 to a 2 in 7 5 1, changes 8 to 7 on the second pentagon, pause, writes 2 9 3 10 and 1 on the first pentagon above previous numbers (1 min, 15 sec)] That’s a 2. [pause, looks over her work, makes list in center left of page, 3 10 1, 1 9 4, etc. (65 sec)]
R: I think you should keep pursuing that. I think you might almost be there.
Lisa: [looks over her table/list, finishes list (10 sec)] Yeah.
R: So you just kind of built the pentagon but in a table form there right.
Lisa: Yeah.

R: And connecting the edges.

Lisa: Exactly.

R: So what have you found? You have found an example of 14, right?

Lisa: Right. 3, 10, and 1 and then 1, 9, and 4 and then 4, 8, and 2. 2, 7, and 5. 5, 6, and 3 [puts her new numbers on the example pentagon on the original question sheet].

R: Okay. And so, have you answered the question? Prove that the smallest possible value of the sum is 14.

Lisa: I have not.

I: So you have shown what so far? You have shown that it’s possible.

Lisa: I have shown that it can be 14.

R: Okay, so how would you go about showing that you couldn't get anything less than 14?

Lisa: [pause] Well, I’d just try it with 13 because the smallest possible sum with three numbers would be 10, 1, and 2. But then also-

R: So clearly it couldn’t be anything less than 13.

Lisa: Right.

R: Okay.

Lisa: [pause, writes 2 10 1 crosses out and writes 1 10 2, and 1 9 3 on right side of page, pause (1 min, 20sec)]

R: What are you thinking?

Lisa: I am going through largest number to smallest number [circles 10, 9, and 1]. And I am just trying to figure out getting 13 with adding 8 and [pause] to get 13 with 8 you need 2, 8 and 3 [writes 2 8 3 on her list]. And that would take up both 2s and 3s [circles both 2s and 3s]. So it would make a triangle instead of -

R: But is that the only possibility? [pause (10 sec)] So clearly 10, 1, and 2 is the only way to get 13 with 10. Right? And 9 and 3 and 1, is that the only way to get 13 with 9?
Lisa: [pause] I could use 2 and 4. [writes 9,2,4, crosses out] Right? No.

R: No, I think you're right. I think you have gotten the only way to get 9 with some combination because we need two different numbers. So then you move down to 8 and you had 2 and 3 and that doesn't work because we've made a triangle.

Lisa: And you can do 1 and 4 with 8 [writes 1 8 4] but still the 1s are both used already. So, it would be stuck at 8.

R: Okay. Alright, so if you were to present this formally for a class, for an assignment let’s say, how might you write this up formally?

Lisa: The solution, or-?

R: Yeah, the solution to why it's not possible to get 13.

Lisa: Um, [pause (45 sec)] I'm not quite sure. Um. [pause (15 sec)]

R: But you're convinced that it's not possible, right? Just by our trying, your trying the triples -

Lisa: Yeah.

R: -and you've exhausted all possibilities and it can't work just with those first three, right?

Lisa: Right.

R: Okay. So, somehow we could rewrite that just that you have tried all of the possibilities and each time you come up with using the numbers too many times.

Lisa: Yeah. Well, I would come up with the different ways to make 13 with 10 and 9 and then, um, show them that that's only those two possibilities [points to her list on the right] for those with three digits and then I would show that you would have to have a different 5 to go with 8 and the only way to get 5 with 8 would be two and 3 or 1 and 4 and I would show that it would create either a triangle or else use the 1 a third time.

R: Okay.

Lisa: I couldn't really think of any other way to explain it.

R: Okay that sounds good to me. Alright. Are you satisfied with your solution and ready to move on to the next problem?

Lisa: Yeah.
R: Okay. Let's do that. If you want to just tear off that top sheet, maybe, and we'll keep it with that. [researcher collects papers, gives student next question] Okay. Now, this is a different problem with a new definition of something coming up. We call a positive integer $N$ a 4-flip if 4 times $N$ has the same digits as $N$ but in reverse order. [pause] Do you understand what's meant by a 4-flip?

Lisa: [pause] Unh, uh.

R: So we'll read it again maybe in a little bit of a different way. So it's a positive integer, it's just a number, that when you multiply it by 4 you reverse all the digits.

Lisa: Okay.

R: So, you get a new number but it's, the digits are exactly reversed. Does that make sense, when you multiply by 4?

Lisa: Mm, hmm.

R: Okay. So, can you try to go about proving that there are no two-digit 4-flips? [pause] So, no number that's two-digits long that when you multiply it by 4 it will flip the number around.

Lisa: [pause, writes $136 \times 4 = 544$ (50 sec)]

R: So, you've looked at 136, but that's a three-digit. And we'll get to three-digits but what about just two-digits.

Lisa: Okay, well I was just trying to see how it flips it. I don't quite understand how. [pause, writes $36 \times 4 = 144$ (5 sec)]

R: So, these are only special numbers, right? What would you expect to see if it was a 4-flip? So if 36 was a 4-flip when you multiply it by 4-

Lisa: Then I would expect to see 63 [writes 63].

R: Exactly. [pause] Now you can convince yourself, but prove that there aren't any two-digit ones and see how you might go about proving that there aren't any of two-digits.

Lisa: [pause, writes $4*n = yx, N = xy$ (1 min)]

R: Okay, so you represented it as a two-digit number and kind of getting a visual of what it means to flip it?

Lisa: Right.

R: Okay. [pause (1 min)] Can you tell me what you're thinking?
Lisa: Well, I was just kind of thinking of, um, ways that for- I don’t know what I was thinking.

R: [pause (25 sec)] What would you like to get right now?

Lisa: Well, in order for it to be a two-digit 4-flip the solution would have to be two-digits long right? [writes \( yx \) again]

R: Right.

Lisa: And- um, so in order for that to happen if you are multiplying something by 4 it would have to be smaller than- if we are just thinking of numbers it would have to be smaller than 24, right? [writes \( \leq 24 \)]

R: Right.

Lisa: And so- and by multiplying it by 4, wouldn't necessarily flip it because if you multiply 24 by 4 you, the solution you would want to be 42. But when you multiply 24 by 4 it's not 42.

R: And you know that right away without doing it right?

Lisa: Mm, hmm.

R: And what's your insight into thinking that it definitely won't be 42.

Lisa: Because 4 times 25 is 100 and minus 4 is 96.

R: Okay. [pause (5 sec)] So, you limited down the possibilities, right? You really only have the digits 10 through 24 –

Lisa: Right.

R: –that we're looking at, okay.

Lisa: And it wouldn’t be 10 because 01 is not 4 times 10. Then you can go through and show that all of them are like that.

R: Okay. Would you like a calculator?

Lisa: No, that's okay.

R: Okay.
Lisa: 23 you'd like 32, [writes 23 → 32] but in actuality you get-[continues list of numbers down the page] 22 you want 22, which doesn't work - 21 isn't 12, you can’t go smaller. [pause] 20 you would want 02, which wouldn’t work either. So, now we are back up here and 23 times 4 is 92 and 19 isn’t going to be 91. And then 18 you'd want 81 and 21 was 84, so that’s not gonna work. And 17 isn't 71. [pause] You get 68. 16,64-[adds to the list, continuing until 10 (35 sec)] So, it wouldn't work for any of them.

R: Okay. So, you just limited your cases by considering what couldn’t be when you multiply by 4 you would get more than two digits and that limited it to just a few cases and then you just tried the cases.

Lisa: Right.

R: You had some good arguments inside. You didn’t even necessarily multiply each one before you decided that it wasn’t true, right?

Lisa: Right.

R: Okay. So you're convinced and I would say that's a good proof not in a final to turn-in form, but a good start to a proof that there are no two-digit 4-flips.

Lisa: Right.

R: Okay. So, the next question is a lot like this one, just going on to the three-digit case. [researcher collects papers, gives student next question] So, thinking of the 4-flips just as we just defined them, prove or disprove the following statement. There are no three-digit 4-flips.

Lisa: [pause, reads the question, writes ≤ 599 (15 sec)]

R: So, can you tell me what your initial thought is on this?

Lisa: Well just looking for the most that any three-digit 4-flip could be without being 1000 which would be four digits.

R: Okay, and if you would like to look back at any of this work to kind of-

Lisa: That's okay.

R: You certainly can do that at any time. [pause] Okay, so you have limited it to 599, why 599?

Lisa: Because 4 times 600 would be 1000.

R: Think about that one again.
Lisa: 4 times 6 is 10.
R: 4 plus 6 is 10. 4 times 6 is 24.
Lisa: Oh, that's right, huh? That's right, so that's not right. [crosses out the 599]
R: But, you can do a limiting case, right? You just have to pick the right one.
Lisa: That's right.
R: No big deal. [pause, researcher gets out calculator (20 sec)] If you have any desire to use a calculator, cause three digits gets to be a little long.
Lisa: [writes ≤ 249] Yeah. So, it has to be less than 249.
R: Okay, because 250 gets you a 1000, right?
Lisa: Mm, hmm.
R: Okay. [pause] But clearly, case-by-case this time is not going to work.
Lisa: Yeah- I won't-
R: Cause we have 100 through 249 and we don’t want to try that many. [pause] So, what’s your thought? Do you think that this is a true statement or not? Do you think that there are three-digit 4-flips or not? What’s your idea?
Lisa: I'm not sure if there would be or not because when you do a three-digit 4-flip, the center digit would be the same. [writes 4*xyz \rightarrow zyx, points arrows to the 2 y's]
R: Mm, hmm. [pause (20 sec)] So what are you thinking about? [pause (20 sec)]
Lisa: Trying to devise another plan?
R: Okay. [pause (5 sec)] Numbers with zero in the middle [writes 0] would satisfy the \( y \) staying in the same place because no matter what you multiply that by it's always going to be zero in the center.
Lisa: Well, it's probably not. [pause] It could be.
R: Cause we could always have carry over from the other number, too.
Lisa: That's true. [pause (30 sec)]
R: So, what else might you try?
Lisa: You could limit the last number to less than or equal to 2. [writes \( \leq 2 \)] That way you wouldn't have carry over and change the center number.
R: Okay.
Lisa: [pause] So, it has to be less than 249 and the last digit cannot be greater than or equal to 2. [writes \( \leq 249 \)] again and draws an arrow to the last digit and writes \( \leq 2 \), pause (40 sec)] Then, I was thinking about what you can say about the middle number. [circles middle number in 249] As far as like eliminating it or [pause] fixing it so that the solution no matter what you multiplied it by- when you multiply it by 4, that you'd wind up with the same number. [pause]
R: So, you're thinking of still your case that there's no carry over.
Lisa: Mm, hmm. Right.
R: So, if there is none then what can the middle number be?
Lisa: Right.
R: So, we said that zero would work. Are you just running through the other possibilities in your head?
Lisa: Right. [pause (20 sec)]
R: So, did you find any other digit that would work in the middle if there wasn’t any carry over?
Lisa: [pause (15 sec)] Yeah, there's no number that works other than zero. [pause, writes 0 over the 4 in 249 (5 sec)] So, this is going to be less than or equal to 2, then this also – well its already got to be less than or equal to 2 [draws arrow to first digit of 249 and writes \( \leq 2 \), pause (20 sec)].
R: So, can you keep going with that idea?
Lisa: Well it wouldn't work because if it was 202 times 4 you would get 808 [writes this calculation down]. Which doesn't – the center stays the same but the other two aren’t flipped and 101 would be the other option and times 4 it would be 404 [writes this calculation down]. So it wouldn’t work.
R: Okay, and so – um, [pause] you've mentioned that the first digit has to be less than or equal to 2, right?
Lisa: Mm, hmm.
R: And we clearly knew that because we already explained that there was supposed to be three digits not four so when you multiply by 4 that's the most you can get and so that actually gives us our limiting condition on the last number as well, right?

Lisa: Right.

R: Cause it's supposed to be the same. And so you said so if it's less than or equal to 2 there's no carry over so that middle number can only be zero -

Lisa: - Right -

R: - in order to stay the same and so the only 2 possibilities are 202 and 101. And you've tried both of them and they don't work.

Lisa: Right.

R: Okay, so do you feel convinced that it wouldn't, with some thought, it wouldn't be difficult to write this up formally and that it would be a proof that there are no three-digit 4-flips.

Lisa: Um. [pause] You could write in a proof. I'm not sure how it would be, other than looking at your limiting-

R: [pause (15 sec)] Would you say your ideas for formulating this one came from your work from the first part?

Lisa: Part of it did.

R: Okay, what other ideas came up?

Lisa: The limiting came from it [circles the $\leq 249$ at top of page].

R: The limiting it to less than 249.

Lisa: Mm, hmm. And then from there, I kind of went with the second limit [circles the $\leq 2$ condition on the last digit]. And by eliminating that I knew that the center had to be zero. So-

R: Okay, good. Let's move on to the last portion of this problem. [researcher collects paper, gives student next part of question] Which is finally an example for you of the fact that a four-digit 4-flip does exist. The number 2178 is a 4-flip. And just as we have defined them so far, I'd like you to prove that it's the only one for four digits.

Lisa: [pause, writes $xyzm=mzyx$, then does the calculation of 2178 *4, uses calculator (25 sec)]
R: So you are just checking the numbers just to make sure.
Lisa: Yeah. [pause]
R: So, it does indeed work.
Lisa: Right. [pause (15 sec)] So that's going to have to be less than or equal to 249, 2499 [writes xyzm ≤ 2499].
R: Okay.
Lisa: And then- [pause (45 sec)]
R: So you're trying to think of other limiting conditions?
Lisa: Yeah, I was thinking of the lowest number 1000, that one doesn't work because it would be 4000 when you multiply it by 4. [pause (10 sec)]
R: So there's a lot of numbers there, right?
Lisa: There is. [pause] 2499 doesn't work [circles 2499, pause (25 sec)].
R: So, where would you go next? [pause (35 sec)] Tell me what you're thinking about.
Lisa: I was just thinking about how this all gets flipped when you multiply it by 4 [points to her calculation of 2178*4]. And the carry overs and everything else. [pause (35 sec)]
R: So, what else would you like to try?
Lisa: Um- Let's see. [pause] The last number [circles the 9 of 2499], if we're looking at 2000 and something has to be when it multiplied by 4 has to be a multiple of 2. Or has to have a 2 at the end of it, so that would include 8.
R: Oh, I see what you're saying. So you're saying that if I multiply 8 by 4 I get 32 to get my 2.
Lisa: Right.
R: Okay.
Lisa: And- [pause (15 sec)] it could be 8, 3, [pause] and that's it. The last number can be 8 or 3 [draws arrow to last number and writes 8,3], if it's 2 [pause] and it can't be- It has to be 2000 or more [pause, adds 2000 ≤ before xyzm ≤ 2499 (10 sec)]. Because you can't multiply any number times 4 and get 1.
R: Okay.

Lisa: And, um- [pause, writes 2 y z 8 and 2 y z 3 (40 sec)]

R: So you're thinking about the different possibilities, if it starts in 2 and ends in either 8 or 3?

Lisa: Right.

R: Cause you limited it down that far?

Lisa: Mm, hmm.

R: Okay.

Lisa: So, that could be anywhere from 21 1 through 9, for this. Or it could be 22 1 through 9, 23 1 through 9 – [writes list 21 8, 22 8, etc.]

R: - So, it's still a lot -

Lisa: - and 24 1 through 9 and then you also have that for 3s as well [puts 3 after each of the above (5 sec)].

R: Mm, hmm. So there's still a lot of possibilities.

Lisa: Right.

R: Okay. [pause] Anything else you could try? To further this.

Lisa: I was thinking for the next number you would have to have a 1, 2, or 3. 1, 2, 3, or 4 at the end of it to multiply by 4. [pause (10 sec)] 9 wouldn’t work. [pause (15 sec)]

R: Then, are you remembering that there could be some carry over?

Lisa: Mm, hmm.

R: So, you're considering 9 with both the possibilities of 8 and 3. And checking the carry over and seeing if you can get a 1, 2, 3, or 4? [pause (35 sec)]

Lisa: So, 9 won't work because 8 times 4 would be 32 and 9 and 4 is 36 and 3 is 39 [writes this calculation]. Or else 3 and 4 is 12 and 9 and 4 is 36 plus 1 is 37 [writes this calculation as well].

R: Okay.

Lisa: [writes 32 + 3 = 35, pause (30 sec)] So, 8 would work with 3. [pause (15 sec)]
R: [audio tape side 1 ends, researchers flips over tape] There you go.
Lisa: And - [pause, writes \(28 + 3 = 31\), makes lists if 3 and if 8 (20 sec)]
R: So, you're considering the possibilities of the middle term?
Lisa: Right.
R: Okay. And you have eliminated 9 from both of those lists?
Lisa: Mm, hmm.
R: Okay.
Lisa: [pause (5 sec)] So, 7 would work with 3, no with 8. [pause] And it would work with 3 as well. 6 wouldn't work. [pause (30 sec)] 5 would work [pause (10 sec)] 4 won't work. [pause (10 sec)] 3 would work with 3, [pause] but it wouldn't work with 8. [pause (10 sec)] Hmmm, and 1 wouldn't work. [pause, has now put in 8 7 5 3 in if 3 list and 7 5 2 in if 8 list, she did the calculations in her head (15 sec)]
I: So, you have narrowed down the options.
Lisa: Not by much though.
R: There is still a lot there, right?
Lisa: [laughter] Yeah. [pause, writes list on side 2 \(y\) 8 3, 2 \(y\) 7 3, etc. and 2 \(y\) 7 8 etc.(40 sec)]
R: Okay, so you're just trying to fill in that last digit, right?
Lisa: Right.
R: And considering what those possibilities might be?
Lisa: Mm, hmm.
R: Okay.
Lisa: And \(y\) has to be greater than or equal to 1 and less than or equal to 4 [writes \(1 \leq y \leq 4\)].
R: Okay.
Lisa: And we know it works for this one with 1 [points to 2178, draws arrow to 2 y 7 8 and writes 1, pause, writes 1 over 2 y 8 3 and 3 over 2 y 7 8 (25 sec)]

R: So, what do the 2 and the 1 represent?

Lisa: Where are you at? Here?

R: Yeah.

Lisa: Oh, this is just to remind me of the carryover that you’d have multiplying 8 times 4 and 3 times 4.

R: Okay. [pause] Oh, so that's a 3. Okay.

Lisa: Sorry.

R: Not a problem, I just couldn’t see it.

Lisa: [pause, writes 2y58 *4= 8 _32, (1 min 10 sec)] This combination wouldn’t work because you can't multiply a whole number by 4 and add 2 to get 5.

R: Okay.

Lisa: So, this isn't a possibility [crosses out 2y58 on list, does calculation for 2y28*4, (15 sec)]. This number would have to be 1 and times 4 is 4 plus 1 is 5.

R: So, you limited it to 1 because you already knew that 1 was the thing you wrote down there, right?

Lisa: Right because when you multiply 2 times 4 plus 3 would be 11. So, you know that since the digits are all flipping that the 1 is supposed to be the y.

R: Okay.

Lisa: Then it doesn’t work for [crosses out 2y28 from list]- And you do the same for all of them. [does calculations for 2y83 (5 sec)]

R: So, in fact, we don’t have that many possibilities to check, right?

Lisa: Right. And 3 times 4 is 12, 13, so that won’t work either. [crosses out 2y83 from list]

R: Okay.

Lisa: [pause, does calculation for 2y73, (20 sec)] That doesn’t work because it would be 38. [crosses out 2y73 from list, does calculation for 2y53, (10 sec)] Cause again, it'd be 6.
[crosses out 2y53 from list, pause, does calculation for 2y33, (50 sec)] That wouldn't work. [pause, redoes calculation for 2y33 off to the side, (20 sec)] This should be 3. [draws arrow to answer of 2y33*4, the first digit is 9, she circles it and writes 3]

R: So, you've checked all of your possibilities now.

Lisa: Right. [crosses out 2y33] And the only one that works is the one that's given.

R: Okay. So, can you describe the process that you went through to complete this question? Kind of where you went and steps you took.

Lisa: Well I was just thinking, first I figured out what the number would look like if it was 4-flipped [points to where she wrote *xyzm=yzmx].

R: Okay.

Lisa: And then I went through and I limited it so that it would remain a four-digit number [points to the ≤2499]. And then, um, I went through and I checked the first number's got to be either 2 or 1 and I realized that it wouldn't work with 1. So, I knew that the number had to be greater than or equal to 2000. And then, um, by using that I figured out the first number [circles the first digit 2 in the list 21_8, 22_8, etc.] and then I looked at, um, the last number in the one that they gave us and realized that the 8 times 4 had to have the 2 in the last digit and then I went through and figured out what other number when multiplied by 4 would have a 2 in the last digit. And so it was 8 and 3 and then I went, realized that since it is 2000 and 2400 the number the second digit had to be 1, 2, 3, or 4.

R: Okay.

Lisa: And then I went through and used the same process with 8 and 3 times 4 and to find out what numbers you could get that would end in a 1, 2, 3, or 4. And then once I did that I went and looked through, I wrote out the options for 3 and 8 as the last digit. And then I just solved them from there.

R: Okay, so just kind of a continual process of eliminating answers and then finally just checking the last possibilities.

Lisa: Right.

R: Okay, good. Alright, I think that's all the problems we're going to have time for today. But I do want to ask you a couple of questions to follow up. [pause] Can you identify any strategies you used in constructing these proofs? Things specifically you might say you generically you used?

Lisa: Limits. I used a lot of limits.
R: Okay, so limiting things down -
Lisa: - Right. -
R: - and specifically basically on all of question 2 that's what happened, right?
Lisa: Right.
R: Okay. How about in question 1? Was it kind of the same thing? That was with the pentagon.
Lisa: Um, [pause] not really. I just used the idea of- um, you can't build up you have to work down with the numbers. You have to start with the larger numbers because they are less useful than the smaller numbers are.
R: Okay.
Lisa: And so I just kind of worked backwards on those.
R: Okay. And looking for examples and finding one for 14 and then finding that 13 didn’t exist -
Lisa: - Right. -
R: - from there. Okay. Um-do you ever recall being taught or using these kind of strategies before?
Lisa: Limits I have, but not the working backwards, I don't really-haven’t done too much with that.
R: Okay. Any other strategies you can think of that you may have used here?
Lisa: Examples. A lot of examples.
R: Okay.
Lisa: I think those help a lot. Kind of it helps get an idea of what you are looking for.
R: Okay. Great. *End of interview
Interview #2  (Total time = 1:01:05)

R: We'll begin by reading the first problem. There's your copy you can write on that as much as you like. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16. So, this is one example. Prove that the smallest possible value for the sum is 14. I will let you take a second to read over that for yourself.

Ellen: Prove that the smallest possible value for the sum is 14. Right now it's 16. So, 1 through 10. So, it's the sum of three numbers [pause] and the numbers can't be repeated, I assume.

R: Correct.

Ellen: So, I would imagine then the smallest possible sum would be sort of some optimization of the largest with the smallest numbers. Does that make sense?

R: Mm, hmm.

Ellen: [laughter] Really?

R: Totally, yeah.

Ellen: Is that right? No, don't tell me. [pause] Is it legal to have a rock star? [referring to her drink]

R: Sure. [laughter]

Ellen: So, to do this proof I would try and first of all find that arrangement, for which the sum is 14.

R: Okay.

Ellen: Um. So 1, 2, 3, 4, 5 [circles the numbers 1 - 5 on the example pentagon] – so the sum of 16. Well I'm gonna – if I put 1, 2, 3, 4, 5, on vertices [draws new pentagon and puts 1 - 5 on the vertices] and then so, 1 and 5 that'd be 6 so for that to be 14 – [laughter, writes in 8 between 1 and 5]

R: Not a problem.

Ellen: So, 9, that is not going to work because that would need to be 5. Oops, I erased. Um. So [crosses out 2 changes to 3 and writes 10 with it (5 sec)] 10 is the big one and to make that 14 is 1 and 3. And then 9 would be [crosses out the 3 first written and changes it to a 2] 3 and 2 and 8 – [pause (10 sec)] and lets see. Hmmm. [pause, crosses out the first 8 written (10 sec)] So, I'm looking for a minimum. I don't know, I'm not – I've
never done geometric proofs like this. Is this one of the things, like are these supposed to be completely new to us?

R: Yes. That's the hope.

Ellen: It is. This one is completely new to me. This arrange – or this sort of. Okay, so – so for 10 what are my options to sum to 14. I can do 2, no I can’t do 2. I can do 1 and 3 [starts list on side of page, 10 \{1,3\}]. I can do. That’s it. There's not really any other option for 10. As for 9, I can do sums of 5, I can do 2 and 3. I can do 1 and 4 [adds 9 \{2,3\} and \{1,4\} to her list, continues to add throughout the following paragraph]. That's about it. Then, if I do 8 – it'd be 5 and 1, 4 and a 2. 7 we can do 3 and 4, 5 and 2, 6 and 1. [pause (15 sec)] Sum to 8. 3 and 5. 2 and 6, 1 and 7. [pause (5 sec)] 5 need to sum to 9, I already have that. [circles 10,9,8,7,6 in list, pause, draws new pentagon and beings to fill in, puts in 1 10 3, (25 sec)] So, I can do a 9 here and that would be 2. [puts in 9 2 next to 3] Sometimes I admit when I am doing proofs I go kind of sloppy. I get used to doing MatLab where you don’t have to have it exact. You can kind of think you have the right idea and plug and test it and see if it works.

R: Mm, hmm.

Ellen: So, that's what I mean by kind of sloppy.

R: What's really interesting is your paper looks a lot like mine when I started this problem.

Ellen: Oh really?

R: Everybody does it really differently and so it's really interesting that yours looks so much like mine did when I started. [laughter]

Ellen: Cool. And you are a successful mathematician, I am really encouraged. That's cool. Alright, so 7, yeah that's not going to work. [erases 9 and 2] Yeah, I don’t think it’s that. I think, oops I erased.

R: Oh, that's okay. So you just retried your 9 and 2 and 3 and –

Ellen: – Yeah, look at that –

R: – you're rethinking your choice of that. But your 10 and 1 and 3 was forced in terms of at least that needs to be on some side, right?

Ellen: Right. And 10 is not going to be on a vertex because that will be more toward the maximum end of things –

R: – Okay –
Ellen: – because it was shared. I would imagine, to minimize, you would want the low numbers to be the shared ones.

R: That makes sense.

Ellen: Um. So, I’m kind of going along with the thought that 6 through 10 have to be on the sides. So it’s like, it’s kind of interesting. 9 and 8 have two sets, 7 and 6 have three sets. [pause] So, if I do 1 and 4 with the 9, then I might you know, this shares with 6 or 7. If I do 6, that would be a 2 and that would be a 7. [Finishes filling in the pentagon]

R: Hey.

Ellen: Ta – da.

R: You got one.

Ellen: Yeah, that's fun. Oh, but that is not proving it.

R: So you have found an example of 14.

Ellen: Yes.

R: Right?

Ellen: Yes.

R: So clearly 14 works.

Ellen: Yes.

R: So, now how do you go about proving that that's the smallest one that works.

Ellen: It has something to do with 1 3, 5, 2, and 4, being on the vertices [circles the numbers on the vertices]. Because– oh. Um.

R: You can use that paper right there if you want.

Ellen: So, the sum is equal to 2 vertices plus a side value [writes “sum = 2 vertices + side”]. So, that would be s and that would be v [labels vertices as v and side as s]. And so we have 5 times 2v plus s is equals some minimum [writes 5(2v+s) = min]. How do we find the minimum? We take the derivative. [laughter] But somehow that doesn’t seem relevant here. [pause (15 sec)] Okay, so– That's not quite right [crosses out 5(2v+s) = min], I don't think, because they're not all the same value. So, to minimize the value of the sides that's the same thing as minimizing the sum of our convention.

R: Okay.
Ellen: So, so the total sum [writes Total sum = ] around the pentagon, pentagram, pentagon?

R: Pentagon.

Ellen: I could put a star in there. [laughter] So, it's, um, so there's v 1 through 5, vertices sides 1 through 5. So we have 2 times v1 plus v2 plus v3 plus v4 plus v5 plus side 1 plus side 2 plus side 3 plus side 4 plus side 5 [adds these variables into the Total sum =]. So that's got to be the minimum because – excuse me – [pause] For values 1 through 10, if the smallest values are doubled then that's the smallest value.

R: Okay.

Ellen: How am I trying –? How do I write that up? I mean, I think I'm barking up the right tree, but I just don't have eloquence to make it–

R: Okay, so you are saying that all the vertices need to be 1 through 5 because they are the things that are doubled. So if we want to be minimum –

Ellen: – Right. –

R: – they're the things that need to be doubled. [ pause (5 sec)] Okay.

Ellen: [writes “\(v_1, ..., v_5 = 1, ..., 5\) That's not a very tight proof. So, there's going to be 6 through 10 because – [pause, writes \(s_1, ..., s_5 = 6, ..., 10\) because the \(v\)'s are doubled whereas the sides are just added once.”] (30 sec)

R: Okay.

Ellen: So, 5 factorial plus – I don't know, am I beating a dead horse? [writes 5!]

R: That's not 5 factorial, though, because it's plus.

Ellen: Thank you. [erases 5!]

R: Because that would lead in the wrong direction. That would get something too big right there.

Ellen: So 9 plus 12, 13, 14, 15, [writes \(2*15 + 6+7+8+9+10\) 6 through10.

R: And you're welcome to use the calculator. No need for the arithmetic to get in the way.

Ellen: Yeah, but there should be. So let me think about this a little more. I'm being sloppy. So I want to show the smallest possible value for the sum is 14. [pause] If I
didn’t know it was 14, how would I generate that. [pause, uses calculator, writes that 2*15 = 30, (15 sec)] Incidentally, I didn’t need the calculator to do that. [laughter] I was checking that 1 + 2 + 3 –

R: Not a problem.

Ellen: Just in case. So, my total is 55 [writes 55 under the 6+7+...]. So, okay. So, that's a minimum sum, total sum. [writes min total sum]

R: So, that’s the total of all of them?

Ellen: Yeah.

R: Including the 30?

Ellen: Mm, hmm.

R: Okay.

Ellen: No. [laughter] Wait a minute. It’s 70 [crosses out the 55, writes 70] that sounds better.

R: That sounds better.

Ellen: Okay, and so. [pause] So, one side is – [pause, writes $v_i + v_{i+j} + s$, then looks back and crosses it out, (25 sec)] I bet these tapes are fun to listen to. Lots of dead air. Wow. [laughter]

R: So, what are you trying to find? You are trying to relate 70 to what? To the 14?

Ellen: Okay, so it’s going to be. Regardless we are going to have 1 plus 2 plus 3 plus 4 plus blah, blah, blah, blah plus 10 [writes 1+2+...+10] and then plus some subset– 5 times. No not five times. Some subset of 1 through 10. You are going to have five elements from that. [writes +( 1+2+..)]

R: Okay.

Ellen: So I mean, the proofs that are obvious are always the hardest for me. [laughter] Because it's obvious that you would want to add the lowest ones.

R: Okay, that’s fine for the purposes of what we are doing today.

Ellen: Okay, cool.

R: That's good. So why is it that 14 gets to be that lowest sum then? [pause] So I'm equally convinced that the 1 through 5 have to go on the vertices.
Ellen: Okay.

R: So why is it that it couldn't be 13 if I rearranged those somehow?

Ellen: Because, um, [pause] because the total minimum sum is 70 and if you divide that by 5 you get 14.

R: Okay, that makes total sense.

Ellen: Yeah.

R: So there's no way to get less than that?

Ellen: Right. Right. [writes 70/5=14 with a check mark]

R: Okay. Good, good. So, ironically, what’s really cool is that this is exactly the way I did the problem.

Ellen: Oh, sweet.

R: And I’ve interviewed like five other people in my pilot study and one other person this morning and nobody has ever done it the same as me. So, I was kind of like, oh was I beating up a dead tree or something? But no – that was really interesting.

Ellen: Did other people get it in different ways?

R: They did, they got it in different ways, just in really interesting ways that I didn’t expect.

Ellen: Sweet. Like how? Like, what’s one other way that you saw?

R: Um, trying different combinations. Like the last girl that I just interviewed, she said okay, so if I had 13 that’s obviously the smallest because that’s 10, 1, and 2, that’s as small as I can go.

Ellen: Oh yeah.

R: And then that one’s forced, and then 9 would have to go with um, 1 and 3 –

Ellen: Right.

R: – and that forces 8 to be either 2 with 3 or 1 with 4, and either creates a triangle or it uses 1 three times. And so, it can’t work with less than that. But I don’t know that I would have felt satisfied with that kind of–you know what I mean? I was like yeah, that makes sense.
Ellen: So, that proves – okay yeah, so the proof was just asking for –
R: Proving it can’t be less than 14.
Ellen: Oh, okay.
R: So, she found 14 in a similar way. I mean, she kind of – it took her much longer to
find 14, but then she just tried a few more possibilities. So, what I do want to ask you
though is, have you ever seen anything like this before. You said nothing geometric like
this.
Ellen: No, I haven’t really done any sort of geometric thing since sophomore in high
school, which was a very long time ago.
R: [laughter] But do you think that any other classes may have had an influence on the
methods you used here? Like if you were to recap what you tried.
Ellen: Well, no, just sort of logically, whatever logic you'd call it, what I used. Um.
Minimization, optimization, I don’t know. My background's engineering so there's a lot
of, you know spatial things that we, maybe not explicitly but implicitly worked a lot with.
R: Okay.
Ellen: Piping optimizing, piping routing, reactions stuff like that. It was chemical
engineering.
R: Okay, I think perhaps your background highly influences what you try here because
my background is in optimization –
Ellen: – Oh, yeah, cool. –
R: – so I immediately tried to minimize and kind of come up with a formula. So I have a
feeling that could be where yours came from too.
Ellen: One of my favorite engineering classes was an optimization class.
R: Oh, okay.
Ellen: So I have never taken any formal mathematics like optimization, but I'd like to.
Cause I think it'd be fun.
R: Okay, very cool. Alright. So that's that one. What I need to do is decide how much
of this one I want you to do. Um – Okay, I'm going to skip to this question next.
[researcher gives student next question.] Just so I get a variety of responses. I get
different questions and stuff like that. So the traditional chessboard consists of 64
squares an 8-by-8 chessboard. Suppose dominoes are constructed so that each domino
covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard
with dominoes covers every square of the chessboard without overlapping any of the
dominoes. Okay? [pause] So, consider a generic chessboard of size $m$-by-$n$. I would
like you to prove that the generic chessboard of this size has a perfect cover if and only if
at least one of $m$ or $n$ is even. So, I will let you read and think about that.

Ellen: Oh, okay, so if and only if we have to prove both directions so, um – So, I'll start
with suppose we [writes this out on the paper, continues writing what she is stating aloud]
–suppose a generic chessboard of size $m$ by $n$ is perfectly covered. Then –then every
square of the chessboard is covered by non-overlapping dominoes. Since each domino
covers exactly two adjacent squares. [pause, stops writing, draws picture of a domino
then erases it, (30 sec)] How do I want to say it?

R: Okay, tell me what you want to say first so I can get an idea of what you’re thinking.

Ellen: Since each domino covers exactly two adjacent squares every square is covered
either $m$ or $n$ has to be even. But, I mean that's true but I don’t feel like I am proving it
by stating that.

R: Okay, so what's your feeling as to why it has to be true?

Ellen: Um, because each domino is a 1 by 2. Ahhh!

R: Okay.

Ellen: Each domino covers a 1 by 2 generic chessboard. [writes the previous sentence (5
sec)] Kind of a mini-chessboard. [pause (5 sec)] So, since— [pause (5 sec)] some sort of
like an integer divisible thing.

R: Okay, so tell me what you’re thinking then.

Ellen: So [continues writing] since the board is completely covered there exists some $k$
that's an integer such that $k$ times 1 times 2 is equal to $m$ by $n$. Hmm.

R: Okay.

Ellen: [still writing statements down] And that implies $k$ times $2k$ equals $m$ by $n$ um, and
since we have equality. Okay, without loss of generality— [laughter] Let $m$ equal $k$ and let
$n$ equal $2k$. Thus, $n$ is even.

R: But that wouldn’t have to be true would it? That $m$ would have to be $k$ and $n$ would
have to be $2k$. You can get a product with two different numbers can't you? But I think
you’re there, I think you know what you mean. You mean that you've got what on the
left hand side? [pause] So, according to what I've seen you write and what you've said,
you said basically you've got an even number on the left hand side, right? Because you
said you've got $2k$ over there, right?

Ellen: Right.

R: And, so why does that tell you that one of $m$ or $n$ has to be even because it doesn’t
have to be exactly that number.

Ellen: Um, I don’t know.

R: Do you see what I am saying?

Ellen: No.

R: Okay, for example. Like, um, I don’t know, two numbers multiplied together so like
um 4 times 5 is 20, but so is 10 times 2.

Ellen: Right, oh, oh.

R: So, just because two things multiply to be the same doesn’t mean they are the same.

Ellen: Right, yeah, okay. So, but one of them has to be even. [crosses out statement that
$k \times 2k = m \times n$ and $m = k$ and $n = 2k$]

R: Resisting your erasing, thank you.

Ellen: I worked in the lab for a long time, so lab book etiquette's like that too. You are
not supposed to erase anything.

R: Oh, okay. Want to know what you have tried I suppose.

Ellen: Yeah, or to know where the flaws in your thinking generally crop up. It's a good
habit. [pause (5 sec)] So why is that? Well. Oh, okay well maybe we could do it in a
different way. So we could divide both sides by $k$. [laughter]

R: What's your idea there?

Ellen: Well, [pause] if there's no uncovered squares and each domino is 1 by 2 either $m$
or $n$ has to be even.

R: Because what would happen if it wasn’t?

Ellen: Oh, proof by contradiction.

R: [laughter] I didn’t mean to imply that but I was just asking.
Ellen: But if it wasn’t then if there was an odd number of squares then oh, yeah, okay I can just say if there was an odd number of squares. Yeah, proof by contradiction.

R: So, why would an odd number of squares because odd-by-odd–?

Ellen: Is odd.

R: Gives odd. Okay.

Ellen: And you can't cover completely an odd-by-odd, or an odd number of squares with non-overlapping dominoes.

R: Okay that makes total sense. Okay, so are you finished with the proof?

Ellen: No, no. I’ve got to prove the other direction and I have to tidy up this one. So – [pause] Do you want me to finish writing this side up?

R: If you would like to you, you certainly may, but you don’t have to. So you've expressed basically – so, one of them has to be even because if they were both odd we couldn’t get an even number. Right? That's basically what you said?

Ellen: [writes “one has to be even b/c if they were both odd then the board would have an odd # of spaces which couldn’t be completely covered” (15 sec)] Yeah. If they were both odd – odd spaces which couldn’t be completely covered, couldn't be perfectly covered. [crosses out completely covered, continues with “be perfectly covered by 1 x 2 sized dominoes (proof by contradiction)” (10 sec)]

R: [videotape ends, researcher begins to change tape] Well, that's rude. The tape ended. It's supposed to be 2 hours. [pause] Go ahead. Keep going with what you are doing. [pause (10 sec)] It's just a pain to change the tape.

Ellen: So if I was, you know, doing this for a class or something I would rewrite this out as a proof by contradiction and neaten it up. But, um, now I will just prove the other direction. Is that okay?

R: Okay, yeah.

Ellen: Okay, so, next suppose – on a generic chessboard [begins writing what she is stating aloud] suppose a generic chessboard is of size $m$ by $n$ and let $n$ equal $2k$ where $k$ is an integer, positive. [stops writing] Excuse me. [videotape resumes here] It might be easy to do another proof by contradiction. Um – [pause, begins writing again, (20 sec)] So, since one of the dimensions is even, there are an even number of squares.

R: Certainly true, mm hmm.
Ellen: [pause, still writing while talking aloud] Since there are an even number of
squares, 1 by 2 square dominoes can be arranged to perfectly cover the board and show
that by – um – [pause, (10 sec)] So, \(m\) by \(n\) is equal to some \(2k\). [stops writing, crosses off
the initial assumption that \(n = 2k\), (10 sec)] I'm going to let one of the dimensions be
even.

R: Okay.

Ellen: Just sort of edit it there. [resumes writing] So, \(m\) by \(n\) is equal to \(2k\), where \(k\) is a
positive integer and – [pause, stops writing, (10 sec)] So, what I'm thinking and what I'd
like to do is show that if we have \(2k\) squares and we successively subtract two squares we
would get to zero eventually.

R: Okay.

Ellen: Thus showing–

R: So, you're trying to show somehow that you have a perfect cover, right?

Ellen: Right.

R: Okay. So, you're thinking that to do if you just successively take off the places where
the dominoes go that it will show that.

Ellen: Yeah, maybe I don’t know that.

R: So you have to demonstrate a perfect cover, right? Is there any other way you can
think of to demonstrate a perfect cover? So, this is really what [your instructor]’s talked
about in class is an existence proof, right? That's what you are going for is the existence
of a perfect cover.

Ellen: Okay.

R: How else might you demonstrate a perfect cover? [pause (5 sec)]

Ellen: Um, – I don’t know.

R: My concern is that by just taking away dominoes you can't say for sure that the places
that you took them away from that you won't leave a spot, you know, that can't be
covered.

Ellen: Right, like they have to be adjacent.

R: [pause] So, what might you try to go about doing this?
Ellen: Then another way to make a perfect cover could be, would be constructing an \( m \) by \( n \) matrix using 1 by 2 dominoes.

R: Okay. Can you show me what you mean?

Ellen: So, since there are an even number of squares, dominoes perfectly cover the board. Um – So I mean I could say a line of even dimension and certain number of dominoes wide thing.

R: Okay, keep going with what you are thinking there and show me what you mean.

Ellen: Show by writing or drawing?

R: Either one. Drawing is totally fine. Show me what you mean.

Ellen: If it's \( m \) by \( n \), there's \( m \) rows and \( n \) columns. [Draws picture of dominos lined up] Let’s say \( n \) is even sort of, I would just line them up in the even dimension and then fill in the even or odd dimension with the \( 1D \) side. That way that would be constructing an \( m \) by \( n \) cover that can be superimposed over–

R: Any graph right? Any chessboard.

Ellen: Any chessboard of \( m \) by \( n \).

R: Okay, yeah that totally makes sense so you've basically constructed what it would look like in general, you just have to lay the dominoes out on the even side.

Ellen: Right, okay, that's how you say it.

R: And then you for sure can put them on, right? [pause] Long way on the even side or however you would like to say that, but I um, yeah. You've shown me what you mean there.

Ellen: [starts writing again] Lay the dominoes on the even side, along the even dimension. And do that for each row.

R: And then it won't matter, right?

Ellen: Right.

R: Yeah, that totally makes sense.

Ellen: Okay. Yeah, so that's the better way to do it. So, it's basically showing that one exists.

R: Right. So, now are we finished with the proof?
Ellen: Um. I think so. So, prove that the generic chessboard of size $m$ by $n$ has a perfect cover if and only if at least – [re-reading the question] Does it – I don’t know. It's almost like well if $m$ or $n$ isn't even, like if they’re both odd. I almost feel like I need to show that it wouldn't work if they're both odd, but if I did the first part as proof by contradiction that would do that too.

R: Yeah.

Ellen: So then I'd be happy with that.

R: Okay, so rewritten for class purposes or something –

Ellen: – Right. –

R: – you’d probably rewrite it a little bit, but then you’d feel satisfied with what you've got.

Ellen: Right, right. So sort of showing, so when it's at least one it might be easier to come at it from the angle none.

R: Okay, and that would be the direction kind of that you showed right there [pointing to the first part of the work that was done].

Ellen: Yeah, yeah.

R: So, if there was a perfect cover, so you have shown if there's a perfect cover then it has to be that at least one is even and you might do that by contradiction by saying what if they're both odd.

Ellen: Right.

R: And keep going. And then you've shown the other direction that if at least one of them is even you can construct a perfect cover.

Ellen: So, in this one though, I mean I guess I would still feel like, well I don’t know. Like I would have to write out the proof by contradiction to see if it felt satisfying. Like– So if they're both odd it doesn’t work but then what's, let's see – [pause] And I guess I did this on one of the exams too. Like I proved like two cases by contradiction and then went ahead and showed that the actual case worked.

R: Oh, okay.

Ellen: Just because it felt better to me and I don’t know for rigorous mathematicians reading a proof it's probably not as important to them but –
R: Okay. So basically, that's what you think you might want to try here is that you have three cases, you have the one where they're both odd or –

Ellen: Well I would only go two cases, they are both odd or at least one is even. Because I feel like at least one is even would satisfy.

R: Okay, and so you've demonstrated over here the construction of if at least one is even, then it can be done. [pointing to the second portion of the proof] And then over here you've got the idea of why if they're both odd it can't be done [pointing to the first portion of the proof].

Ellen: Right.

R: Okay. So, you might rephrase it a little bit.

Ellen: And if they're both odd it can't be done. But then I'd like to show with one even, it could be done. So I guess if that's what you mean by the three cases, then it would be three cases. But, not the –.

R: I was just thinking of even - even, even - odd, and odd - odd. Yeah.

Ellen: Even sort of swallows things doesn’t it?

R: Yeah.

Ellen: Once one thing's even it's all even from there on out with that little 2.

R: Yeah, and that works. Okay, alright perfect. Can, um, you identify anywhere where you may have used strategies similar to this? So, you've talked about the test in 305 that you've seen some cases before. Anywhere else you may be able to identify anything that you've used?

Ellen: Yeah, 305. You know, the if-and-only-if sort of converse type proof. You know with both directions. Um, um, yeah, then I don’t know playing dominoes and checkers.

R: Okay.

Ellen: So, I got a good visualization there.

R: Okay. Any other, any strategies you can identify? I'm not, I don't remember when we started if we should go for another one or not. Maybe we could start a third one. But any strategies you can identify here? Anything you could say you tried and used?


R: Okay.
Ellen: Yeah, that helps. And, you know, this is one of those where at first I didn’t see the contradiction thing, but once I see that then it becomes a little bit easier. Like I find with these proofs it’s, oh I can just be completely stuck but then one little thing changes and boom it's a flood.

R: Oh, okay.

Ellen: Then, it’s possible.

R: Anything else you've tried that you can identify?

Ellen: Well, I see a lot of pictures in my head.

R: Okay.

Ellen: I do a lot of visualizations like with the dominoes with little white dots and the chessboard.

R: Okay, great, that's awesome. Okay. Thanks. Let's do a different one.

Ellen: Okay, so there’s three?

R: There's three problems total. So we just did 1 and 3 and now we're going back to 2 and we'll see how far we get on there. This is a multiple part problem. [researcher gives student next question] So, here's the first part. It says we call a positive integer $N$ a 4-flip, if 4 times $N$ has the same digits of $N$ but in reverse order.

Ellen: Cool.

R: Do you understand what we mean then by a 4-flip?

Ellen: Yep, I think so. So, 4 times 12, just for instance, if you had 4 times 12 it would be 48, but it would have to be 21.

R: Exactly. Yeah, to be considered a 4-flip. So, the first part of this problem is proving that there are no two-digit 4-flips.

Ellen: Um. Okay, how would I do that? So a 2, what is a two-digit 4-flip? [writes “two-digit 4-flip” (5 sec), audiotape side one ends, researcher flips over tape] It would be, uh – Okay so I'll let $N$ be represented by a $d_1, d_2$, and a $d$ is [writes $N = d_1d_2$, erases] – what do I want to do, I want $d_1$ for the 1s and $d10$ for the 10s and $d$ is a digit from 0 to 9 [writes $N=d_10d_1, d=0-9$ digit]. So [writes while stating aloud, $4xN = 4(d_10d_1) = d_1d_10$] 4 times $N$ is equal to 4 $d_10$ $d_1$ and we’d want that equal to $d_1$ $d10$. Um. [pause, sigh, (20 sec)] So, I see this as being pretty tricky and being like how to manipulate a number expressed as its component digits.
R: Okay.

Ellen: To me that's kind of like, uh-oh. I'm not sure how to do that. Because they're not multiplied. They're grouped.

R: Oh, I see. Okay.

Ellen: So, it's an order thing. Well, I mean here it's multiplied. But between the d10 and the d1.

R: Right, I understand. [pause] So, you're trying to think of some other way to represent them so that you can multiply by 4 and say what that looks like?

Ellen: Yeah, maybe there's a better way to represent them. Um – [pause (10 sec)] Okay, so what do I know about how to – so say there's 12, 21, 44, 44, 37, 73, [writes these numbers down] so there's not really – [pause (20 sec)] Um, so I want to show that it's not possible 4 then does not – oh I guess we need a question mark. d1, d10 over d10 d1. [writes question mark over equals in 4V equation written previously, then writes 4 = (with a question mark over the equals) d1 d10/d10 d1] So, this gets to proving that there's no sort of thing like this that equals 4. [writes = 4 after the equation]

R: Okay.

Ellen: Oh, and 4 is 2 times 2. [pause, changes 4 to 2*2 (with question mark over the equals), (10 sec)] So– [pause, looking back over her work, (30 sec)] Hmm.

R: Can you tell me what you're thinking?

Ellen: Well, I was starting to think okay, maybe I could show it exists for 2, so, like, if it does – [pause (5 sec)]

R: So, like a 2-flip or something, is that what you're thinking?

Ellen: Yeah, yeah just I don’t know. Just somewhere to start. 24, 42, that's not, 26, 62 [writes these numbers down] So, in order for this to be 2 times 2, [pause] this number has to be divisible by 4 [circles d1 d10 with an arrow and writes “has to be divisible by 4”]. And if we look at all the two-digit numbers divisible by 4 we have 12, 16, [laughter] 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60… [writes list d1 d10, 12, 16, etc. until 64]

R: It's getting to be a long list.

Ellen: Yeah, that's probably not the way to go.

R: Well, what would you like to try? So, you're looking at– and these are divisible by 4 so they represent after you multiply by 4 is that what you were saying?
Ellen: Yeah, so $d_1$ $d_{10}$, okay, so the ones that are even already have to be – Do they have to be even? [pause] No, they don’t.

R: What were you saying, do they have to be what?

Ellen: Oh, if the $d_{10} d_1$ has to be even.

R: Oh, I see.

Ellen: But, it doesn’t.

R: So the original number you're thinking about.

Ellen: Yeah, it doesn't. Yeah, the original number doesn’t have to be even.

R: Okay.

Ellen: I don’t think. But the flipped one has to be divisible by 4. Um –

R: So, it looks like you're restricting some. So, you're just trying to take your possible two-digit numbers and chunk away and restrict them a little bit.

Ellen: Right. Okay, so, so, alright now I'm starting to see something. So $d$ – what do I want? Uh. [pause, (10 sec)] Well I want– $d_1 d_{10}$ obviously will be bigger than $d_{10} d_1$

[writes $d_1 d_{10} > d_{10} d_1$.] Well that doesn’t get me very far. Oh, so what does that mean?

That means I need the – [pause, (20 sec)] one of these digits has to be bigger than the other one. Right?

R: Okay.

Ellen: So, the $d_1$ has to be bigger than $d_{10}$ [writes $d_1 > d_{10}$].

R: And that was with the rationale that when you multiply by 4 the other guy needs to be bigger than the original, right?

Ellen: Right.

R: Okay.

Ellen: Right, so, so I can get rid of a bunch that way. Um – [pause (5 sec)] So, if $d_1$ is bigger than $d_{10}$. I don’t know this one's really hard [crosses out some of the numbers on her list (16, 24, 28,...)].

R: Okay, what else might you want to try?
Ellen: It might just be more like I'm out of time.

R: That's fine.

Ellen: I don't know I'd probably keep going along. I would probably write out all of them and cross off the ones, just as –

R: – Okay. –

Ellen: – and like go through the reasons why and then maybe I would stumble upon a big one as to why it's not going to work.

R: Okay.

Ellen: Because like I can get a bunch gone because well one I know the second one has to be divisible by 4. Then I know the first, the 1s digit has to be bigger than the 10s digit. And so, you know, slowly I'm coming up with –

R: Of the originals, right?

Ellen: Right, so slowly I'm coming up with ways to compare them.

R: Okay, alright, so you're just looking at. So, first you started with, um, writing them down, representing them as \(d_1, d_{10}, d_{11}, d_1\), and then flipping them, considering what happened when you divided them. Just considering how to eliminate some possibilities, right?

Ellen: Yeah, or – [pause (5 sec)] Okay, in order for this to be even it's also these two numbers divided by each other has to be even. So, what does that mean? So, they both have to be even. Maybe they do both have to be even. Do they? Okay, so if this is odd divided by even. Okay. So, even times even is even and even times odd is even and odd odd is odd [writes even*even=even, even*odd = even, and odd*odd=odd]. So the odd odd doesn't really matter. So even divided by an odd is still even [adds to previous writing by dividing right hand side of first by even and crossing out even on the left hand side, same with odd in the second equation]. So, so \(d_{10} d_1\), okay so the original one can still be odd.

R: Okay. [pause (10 sec)]

Ellen: Yeah, okay, that makes sense. Yeah, I don't know. Boy, what do you do? Hints?

R: So, just because basically we're running out of time, right? So we'll stop there, but you'd probably proceed by like you said writing down the rest of the numbers and probably chunking off the ones that –
Ellen: Maybe just for, I mean until I had an insight. I would just sort of plug away at the structure of it.

R: Okay.

Ellen: You know, just looking at the form of the numbers.

R: Okay. Alright, sounds great. I think basically I asked you my follow-up questions already to identify the strategies as you went along. Was there anything new that you tried here in terms of a strategy?

Ellen: Yeah, this kind of representing the digits.

R: Okay, have you ever seen anything like this before? Has it ever come up for you before?

Ellen: {No}

R: Okay, so never had to represent digits or anything like that. So that idea is one that just came from the form of the question maybe or something?

Ellen: Yeah, as a way to understand, (cough) excuse me, as a way to understand the question, the problem, a way to represent it, play with it.

R: Okay. Alright, that sounds good. **End of interview.**
Interview #3  (Total time = 1:06:10)

R: And then we'll start with this problem first. [researcher gives student first question]
I’ll read it aloud and then I’ll give you a chance to think about it. The numbers 1 through
10 can be arranged along the vertices and sides of the pentagon, so that the sum of the
three numbers along each side is the same. The diagram below shows an arrangement
with sum 16, just for an example. Prove that the smallest possible value for the sum is
14. I'll give you a second to think about that.

Shelly: [pause, reading the question, (20 sec)] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; 1 through 10;
the highest and the lowest are at vertices. [referring to example pentagon] Can I start?

R: You may start.

Shelly: Okay, so the highest and the lowest are at vertices and I’m looking to see what’s
at the other vertices. 3, 4, 7, 10, 1, 2, 3, just to see if there’s a pattern.

R: Okay.

Shelly: Um, 1, 3, 4, 7, 10; 7 to 10 is 3, 1 to 3; no, it’s not 3. Um, [pause] 2, 5, 8. 2, 5, 6,
8; 2, 5, 6, 8, 9. So, at vertices I’ve got 1, 3, 4, 7, 10, and on the sides I’ve got 2, 5, 6, 8, 9,
10; 6, 8, 9 [makes list of numbers on vertices and sides]. So, it takes three numbers to
make a side. 16, 16, 16, 9 and 7 is 16. [whispering while counting all sides]. So, we
wanna prove that the smallest possible value for the sum is 14. So, combinations of three
numbers and we’ve gotta use them all up and the vertices get used twice. So, 10 figures
into this one, this one. Um, [pause (15 sec)] so some of the sides are the highest of the
three numbers but some of them aren't. (throat clearing) [pause] So, um, let’s see. [pause
(5 sec)] Smallest possible value is 14. Smallest possible value – if you used 1, 2 and 3,
you’d get 6, but then, you couldn’t get that from a – and, [pause (10 sec)], so let’s make a
14; 10 and 1 would use 3. Okay, which is okay cause, just because it’s all the vertices are
these numbers on this one doesn’t mean it would be with 14. Um, [pause (10 sec)] that's
the smallest possible, 6 is the smallest possible with the three smallest numbers. 10, 1
and 5. 10, 1 and 5 took the extremes and the middle. Um, so the only criteria is that the
sums are the same, the same. [pause (20 sec)] Um, show that the smallest possible value
is 14, so 10 is the highest number so let me work with 10. Um – so I could write here
[referring to blank paper].

R: Certainly.

Shelly: [draws new pentagon, puts 10 on a vertex, (5 sec)] Okay, 10 and we’ve got 16 so
if I'm gonna make it smaller – [pause] well right now I’m trying to make it 14, see how,
what, but I’m thinking that will take a lot of time, [laughter] and that might not be the
clue to proving it. Um, although in the process of doing that, I would probably figure out
what it takes to prove it, but –. Um, [pause, (15 sec)] smallest. Okay, the vertices get
used twice so if I put the highest number on a side, it would only get used once. So, that
would help to bring the number down, if it only got used once. [crosses out 10, rewrites it
Um, so what if I put all of the highest numbers on the sides? 10, 9, 8, 7, 6 and [writes, 10, 9, 8, 7, 6 on sides] then um – what will it take to make 14? [pause, (5 sec), begins new pentagon] Um there’s 1 here, and 2, uh 3. That’s 5 here, that’s 12 and 2, 2 and 8 is 10 and 4 and I get 14 [finishes filling in pentagon]. Okay, so I figured that out.

R: Hey, it didn’t take too long.

Shelly: [laughter] Yeah, right, I thought it would. Um okay – so, so I got it to what they say – um well there the hypothesis is that the smallest possible value for the sum is 14. So the hypothesis is um – so given all these conditions, then the smallest possible value for the sum is 14, trying to use proof language. Um, so I need to prove that. I could disprove it by a counter-example. So, if I can get something smaller than 14, then my job would be done. [laughter]

R: Okay.

R: Okay.

Shelly: So um – but I’m really thinking that I went minimum here, I mean well – let’s see, 10 and 1 and 2. [draws new pentagon, puts 1 10 2 along side] So, let me see if I can do it with putting the highest with – Okay, that gives me 13. [works on new pentagon to get 13] So, again 10, 9, 8, 7, 6 – um, let’s see that’s 3, 5, 2, 4, 2, 3, 4, 5 – 1, 2, 3, 4, 5 – and [whispering] this is 13. So, in my mind with those numbers, the minimum even possible would be 13.

R: Okay.

Shelly: Okay, 9, 10, 11, 12; that would make 10 with two 1s and you can’t use em twice, can you use them twice? I guess, it doesn’t say you can’t but–

R: You can’t.

Shelly: Okay, um, so okay so with that reasoning 13 would be the lowest, now can I make 13 work? Um, so let’s see. 9, 10, 11, 12, 13. You can’t use the 2 again so let’s try 9 with the 1 [changes entries in pentagon, changes 6 to 9]. 9, 10, that would make 3 here. We’re going for 13 [writes “∑ = 13” on the side]. 10, 11, 12, 13, okay. Uh, 9, 12, 13, okay, so… 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 [lists numbers 1 – 10 along the side of the paper, crosses out those she has used]. I’ve used 1, I’ve used 10, I’ve used 2, I’ve used 9, I’ve used – this is 13 10 3 so that can’t be anything but 3. What have I got left in the middle
range? Um, vertices get used twice, so I want the small numbers at the vertices 1 2 3 and I only have 4 and 5 of the small numbers left. Put 4 here and 5 here [writes 5 and 4 along bottom of pentagon]. That's 8 I'm going to 13. That's a 5, that's a 5 again, can't do that. 3 and 4, 4 here is 7, going for 13, that's a 6, that could work. So, 4 here 6 here, 10, 13, 12, 13 [changes around 4 and 5, places 6 between 3 and 4] – Okay, so that leaves um – unless I'm wrong on putting the small ones – One side is 9 and 4. That's what I used, can't do that. Um, 5 and 2 is 7 and 6, 6 is already used, can't do that. Um, prove that the smallest possible value is – so I'm believing now, [laughter] based on my tests that 14 is the smallest sum.

R: Okay.

Shelly: Um –

R: So you observed that 13 would be the absolute lowest just by the basis of having to combine three numbers.

Shelly: Right.

R: And now you're convinced that 13 is impossible.

Shelly: Right.

R: So, how do we prove it?

Shelly: Wait a minute, I just had a little niggling doubt. Um, cause 13 was the smallest cause it had to be 10 and two numbers but what if I took the smaller one. No, 10 has to be with two other numbers.

R: Right.

Shelly: Okay, okay, ya, yep, so not 13, not [writes “NOT” below the 13 case she was trying] – okay this was 14 [writes “∑ = 14” beside the 14 case] – 13 14. Okay, that was 14. So, prove that the smallest possible value for the sum is 14, this is where I always fall down is proving what I know. Because once I know it, that's just as far as my mind wants to go. I could care less about proving it to somebody else. Other than sharing my process with them. So, let's see, let's say that I care about proving that it's the smallest, I mean, ya, I would just go through my reasoning of what I did to prove to myself that it was. Um –

R: So, have you shown beyond a shadow of a doubt, I mean obviously it's true, 13 won't work, by the statement of the problem we're convinced it won't work. But have you actually shown for sure, for sure, that 13 won't work? Or how may we do that? How might you go about proving, say, [your instructor] was asking to do this proof for a homework assignment.
Shelly: Not her!

R: Okay, a teacher –

Shelly: A math monster with {my instructor's} voice coming out of it.

R: Um, so a teacher asked you to submit this for a homework problem so you have to convince them that 13 is not possible. How might you go about doing that?

Shelly: See my dad, my dad says “oh ya, I think it is, prove it”. Then I get motivated.

R: Okay, there you go.

Shelly: [laughter] Um – [pause] by definition of the problem, it doesn’t say it verbally but the way it’s drawn – um, it takes three numbers to make this sum. Um, [pause] so sets, if I take a set with – sets isn't going to work, um, if I take 1 through 10 in a set and I gotta take three of 'em. [pause (15 sec)] It takes three numbers to make the set; a, b, and c. [on new piece of paper, writes $A + B + C =$] A plus B plus C equals – Okay, so we’re, we’re gonna challenge, we’re gonna say 13 [adds = 13]; we’re gonna suppose that it equals 13 and come up with a contradiction.

R: Okay.

Shelly: [laughter] Um, because 13 would be less that 14 but it wouldn’t, proving that this can’t happen doesn’t prove that 12 couldn’t be it or 11 couldn’t be it or 10 couldn’t be it.

R: But you did make the statement that 13 was the smallest because of the 10 and the 1 and the 2.

Shelly: Right.

R: Right? So, I think we can fairly say at this point what is left to be shown is that 13 needs to be shown not to be possible.

Shelly: Um – so let’s suppose, I’m working on getting the language [writes in “Let's suppose”]. Let’s suppose that $A$ plus $B$ plus $C$ does equal 13. Um. [pause (10 sec)] and $A$ plus $B$ plus $C$ [laughter] I don’t even know whether to go there or not doesn't seem useful to me. Okay, um $A$ plus $B$ plus $C$ equals 13. Um, [pause, sigh, (10 sec)] again I would just plug in. I would say okay, we want it to be the smallest, so I would take $A$ and $B$ being the smallest of the possible, the two possible, of the 10 possible numbers. And then what does it have to equal, what does it have to have to equal 13 [writes $1 + 2 + x =$ 13]. And then, $x$ equals 10 and that possible [writes $x = 10$ – I mean, I would just do algebraically what I did pictorially.

R: Okay, so we know one of the sides has to be 10, 2, 1. Right? Can you keep going from there?
Shelly: Well, I mean by process of elimination, I would just do what I did here. Cause now I can’t use 1, 2, or 10, so I have a smaller set to work with. [pause, (5 sec)] I mean, in my mind that’s proof.

R: Okay.

Shelly: Um, [pause, (15 sec)] cases? I mean –

R: So, you tried what you could over there placing numbers until you exhausted the possibilities. And you were convinced that the numbers 1 through 5, that came up, that they needed to be on the vertices, right?

Shelly: Well I had a rationale for it but it might not have been prac – I mean it might not have been true.

R: And what was your rationale?

Shelly: Because the vertices got used twice in sums, and so to keep the number minimum, to it’s minimum, then I should use the smallest numbers twice.

R: Okay, and then once you did that, the 10, 1, 2 was forced. I see that you tried, your first try was the 9 with the 2 and that didn’t work. That couldn’t work, so the 9 had to go with the 1 and that forced the 3, forced the 4 and the 5 and the 4 and the 5 can’t be together, so that must mean this one doesn’t work, if all those decisions are forced?

Shelly: Right.

R: Okay. Anything else you can identify in terms of what you did along the way, steps that you took, any strategies you used along the way?

Shelly: Well I was looking for patterns was the first thing that would help me to just short circuit all the figuring. If I could figure a pattern, if I could see a pattern –

R: Okay.

Shelly: Then I um, although the pattern that I saw here would only be for 16 for the sum equals 16 [writes sum = 16 next to original example pentagon]. Um, so that same pattern might not work for every single possible sum. Um, but I, so first I looked for a pattern in the numbers, like all the odds were vertices or all the evens –

R: Okay.

Shelly: Um, some span between em – 5 to 8 is 3, 8 to 6 is only 2, oh well that breaks that pattern. Maybe it’s an ascending interval or a descending interval or you know, so I checked out a bunch of different patterns that I’m aware of.
R: Okay.

Shelly: And um, none of ‘em seemed to work. And then I tried to see what was in the
vertices and what were on the sides and that actually was confounding because then I
almost got visually stuck into, oh these had to be at the vertices. So, I had to abandon,
that’s why I went to the second page, because I had to abandon that cause that was
actually, I thought, was limiting me.

R: Okay.

Shelly: So I broke out of that box.

R: Okay. You came up with 14 and that led you to kind of thinking about what 10 had to
go with to get 13.

Shelly: Mm, hmm.

R: And that led you to determining that 13 wasn’t possible and so, then you –

Shelly: Like, trying to get this [pointing to her pentagon with sum 14] helped me to
realize that that the highest of the 5, the top 5 of the 10 numbers needed to be on the
sides.

R: Okay.

Shelly: Rather than vertices.

R: Okay, anywhere that you’ve seen anything like this before, or any of the tools that
you’ve used, have you used before?

Shelly: Yeah, um, I taught math for elementary school teachers and I forget them now,
but there were triangles and there were all kinds of pattern games um, Pascal’s triangle?

R: Okay.

Shelly: Um, that we used as exercises.

R: Okay, so that may have been why you tried to look for patterns.

Shelly: The patterns, definitely. And, the specific patterns I looked for came from ones
like intervals. You know–

R: Oh, okay.
Shelly: Or at ascending or descending intervals or um those were all, I pulled out of past experiences.

R: Okay. Anything else you can identify.

Shelly: Math games. Um. [pause (5 sec)] I’m glad that it wasn’t all words. Words confound me.

R: Okay.

Shelly: Um and if I had just been given that [points to statement of question], I would have drawn it myself. So, if that hadn’t been provided, I would have gone to that. And you know, perhaps I wouldn’t have known pentagon, I, I do, but they might have used, you know, the problem might have used some geometric figure that I didn’t know.

R: Okay.

Shelly: So that might have caused me problems. So, it was very helpful to have the drawing.

R: Okay. Great, good insights. Alright, let's gather up this stuff. [researcher gathers papers from student] And if you want to tear that page off, I’ll keep that with this one.

Shelly: Oh, you want it?

R: I do, I going to keep all of these. – [student writes YUK! on page with $A+B+C=13$ work, laughter] Thank you very much.

Shelly: Editorial comment.

R: [laughter] An editorial comment. Um, I need to make a decision on what we should do next and I think my decision will be to go to this Question 3. [researcher gives student next question] Question 2 is almost a whole time in and of itself. So, we'll go to this one. Okay. So, a traditional chessboard consists of 64 squares, it’s 8 by 8. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes. So, do you understand what’s going on here?

Shelly: Yeah, we had dominoes too, but I didn’t assign that one cause I didn’t, it didn’t come easily to me. So, I don’t have a lot of experience with dominoes. But I, I know what it means to cover ‘em.

R: Okay.
Shelly: Um, dominoes are two – 1-by-2’s, okay. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any other dominoes. Yeah, I know the visual on that.

R: Okay, you understand the idea. Okay, so the question below is about a generic chessboard.

Shelly: Mm, hmm.

R: It says if we just have one that’s size m-by-n –

Shelly: Mm, hmm.

R: Prove that it has a perfect cover if, and only if, at least one of m or n is even.

Shelly: Uh. [pause (5 sec)] Hmm.

R: Any clarifying questions? To start with, anything you need–?

Shelly: Okay, so a traditional chessboard, but that’s just extraneous information cause that’s not what we’re working with here.

R: Just to give you, you know, an idea to set yourself up with it.

Shelly: Okay, so we have a chessboard, m-by-n. At least one is even so, um let's say we have a 1-by-2 [on separate sheet, draws 1x2 chessboard, (5 sec)]. Um, and one of those is even and that gets covered. Prove that the generic chessboard has a perfect cover [underlines “has a perfect cover” in the statement of the question] and perfect cover is every square is covered without any overlaps, and no omissions, no gaps. Okay, so 1-by-2, um, has a perfect cover and – [writes “has perfect cover”, pause, (5 sec)] and um – and m is even [writes “and m is even” below the 1x2 case and “mxn” above it]. I’m doing cases here.

R: Okay. [laughter] Looking at some examples, that's good. Okay. Getting some ideas.

Shelly: Um, so at least one, so let me try one where they’re both odd, not, which is not at least one is even. So the case where um how do – I do not like that, we do not like this. Not, um, at least one even, the symbol for not – [writes “¬(at least one even)”]

R: Got it.

Shelly: When I first did logic in math – Okay, not at least one even means um not at least one even, both are odd [writes “means: both are odd”, (10 sec)]. Um, so my m by n let's try 3-by-5.

R: Okay.
Shelly: [writes mxn and 3x5, draws the 3x5 chessboard, (10 sec)] Okay, 3-by-5, so, this is the case. If and only if at least one is even, so this is not – I’m starting with not this [pointing to the statement of the question where it says at least one is even] and so I want to prove that, if and only if – I want to show that um, this can’t be covered.

R: Okay.

Shelly: Perfectly covered. Um, which means to show that I would have to test every– [pause] well, to start trying to cover it, I guess, until that I show that it, somehow until something comes up in my mind.

R: Okay.

Shelly: [begins to cover the board with 1x2 shading] Um, so that’s one, that’s one. [pause] Okay, the area of each domino is two; 3 times 5 is 15 show, um it would come out with an odd one. Okay. So, area of the domino equals 2. [writes “$Area_{dom} = 2$”] Um, area of the 3 by 5 equals 15 [writes “$Area_{3x5} = 15$”]. 15 divided by 2 equals $k + 1$, which equals non-perfect cover [writes “$15/2 = k+1 = \text{non-perfect cover}$”].

R: So, you’re saying because of the fact that it’s 15, it’s odd, it’s got one extra, is what you pointed to on the graph?

Shelly: Oh, I got this wrong. I have to think about this. Um, sorry, um because this is how I think of odd and even and in notation it doesn’t work. [pause] For $k$ being even, if $k$ is even, we want $k$ to be even – number of times, but if there’s a remainder of one, for $k$ equals $2L$ sorry [writes “for $k = 2L$”].

R: Okay, so you’re thinking about it, that you’ve got, you pointed to this square here, this last square and said that that one would be extra –

Shelly: Uncoverable.

R: And uncoverable, so because of the fact that the dominoes have two, this is not coverable.

Shelly: Right.

R: So, in general, how could we say that?

Shelly: Um –

R: So, you believe this won’t work no matter what your choice?

Shelly: Yeah.
R: You know, 3 and 5, 5 and 7, it doesn’t matter what you choose as long as they’re both odd it won’t work.

Shelly: The only case I haven’t tested is that they’re both even but that still um but let me go through that. So both even, the product of two even numbers is even so um and even is defined by divisible by 2 and 2 is the area of a domino, so $2m$ and $n$, if they were both even, it would be perfectly covered.

R: Okay, so how would we go about proving this to somebody who is unconvinced?

Shelly: Um, [pause, (10 sec)] I would just give em, show em a bunch of numbers. [laughter] Um, let’s see, um – [pause]

R: So you split it into cases, right?

Shelly: Right.

R: You split it into the case where –

Shelly: All the possible cases of combinations of $m$ and $n$.

R: Okay.

Shelly: The product of $m$ and $n$.

R: And so you’ve shown clearly, and I think without question, that if they’re both odd, their product is odd so it can’t be covered.

Shelly: Right.

R: Right? You’ve shown that if one of them is even, you’ve demonstrated one example of how to cover.

Shelly: Right.

R: Right? And you’ve also said that if both are even then the product is even and so in theory we could cover it. How would you prove these two cases that we’ve got hiding over here on the left that both even and the one being even? [student draws line between sides of the page] How would you prove that those have perfect covers?

Shelly: Um, well this [pointing to the 1x2 case] could be a special case because it’s got two – it’s got the exact area of one domino.

R: Okay.
Shelly: So, that might have been a special case that misled me to make a general rule about it so um – This was a case of one of them being even. [pause, writes in only 1 even above 1x2 case, (10 sec)] Okay, so an even times an odd, an even number of odds. Well, the definition of even means divisible by 2. So, if you have an even number of odds, or an even number of evens, it doesn’t matter. They’re always going to be divisible by 2 and 2 is the area of a domino.

R: Okay. So, you’ve proved that they have at least an even number in their area.

Shelly: Right.

R: Okay. Does that demonstrate, um, for sure, and for you and for anybody else, I’m just asking a question, does that demonstrate that you do for sure have a perfect cover?

Shelly: Well, it occurs to me that you might not be able to arrange them.

R: Okay.

Shelly: In such a way that they cover it perfectly. Cause you might need one isolated, one here and one here and that wouldn’t make a domino.

R: Okay. So, how could you show that no matter what m and n are that you can find this perfect cover? [pause, (10 sec)]

Shelly: Cause m times n would determine the arrangement. I could always rearrange ‘em into, to, you know, one domino wide by whatever.

R: Okay, can you show me a picture of what you mean?

Shelly: Um, like if, let's see, any product, um, an odd times an even gives you an even [writes 3x4=12 and 4x4=16]. An even times an even gives you an even. And any even can be arranged 2 by 1, 2 3, 4, 5, 6 [draw 2x6 chessboard]. Any even number could be arranged this way but this is not the arrangement of the 3-by-4.

R: Right.

Shelly: So, their product could be the same and it would be a different arrangement and this would automatically be coverable by, perfectly coverable by dominoes.

R: Certainly.

Shelly: But, if it’s in a 3-by-4 – [pause, draws 3x4 chessboard, (10 sec)] yeah because one side is even so that side would always be um, broken down into some number of 2, 2 by’s [traces over where dominoes would go].

R: And then what would you do with the rest of them?
Shelly: The rest of?

R: The dominoes that you need to cover that?

Shelly: Well, they’d just follow suit.

R: Okay.

Shelly: [pause] And a 4 by, yeah, so if, if at least one side is even, then that side can be – that side to one domino depth can be covered with dominoes. And then that can be replicated, let’s see if that side was even – [labels side of chessboard with 4 and 4] so if this side is even, than that side can just be replicated an even number of times. But if this side were odd, which it is in this case, um, it doesn’t matter, it’s still replicated that number of times. Yeah.

R: Okay. Good. And so we could write this up formally if we needed to but I think you’ve got the ideas down.

Shelly: We could, but [laughter] with a little help from –

R: But of course the process is the important part here and we’ve gotten to that part, right?

Shelly: Okay.

R: Okay, so let’s start on the second question and see how far we can get. [researcher collects student papers] Because we’ve still got about –

Shelly: This is fun. I like this. If I don't have to prove it – [laughter]

R: If you can stop there – [laughter, researcher gives student next question] Okay, so this is a slightly different thought here. We call a positive integer, N, a 4-flip. If 4 times N has the same digits as N but in reverse order. So, do you understand what I mean by a 4-flip?

Shelly: Positive integer, let’s say 2, is a 4-flip if 4 times 2 has the same digits as N, but in reverse order. So, reverse order, N must be a two-digit or more integer.

R: Mm, hmm.

Shelly: So um, 12 is a two-digit, let’s say N equals 12 [writes “N=12”]. Okay, if 4 times N, that’s 48, has the same digits as N but in reverse order [writes “12x4 = 48”]. Times 4 equals 48. Therefore, 12 is not a 4-flip [adds “... 12 is not a 4-flip”].

R: Because what would you want if it was a 4-flip, you would have wanted that to equal?
Shelly: 21.

R: Okay. I just want to make sure we're on the same page.

Shelly: So, um –

R: So the question asks, prove that there are no two-digit 4-flips.

Shelly: Oh, right, I forgot about the question. [laughter] I got all involved in that. Prove that there are no two-digit 4-flips. So that's uh, 10 to 99. So, prove that between 10 and 99 there are no 4-flips. So I could go 4 times all of those um and prove it by exhausting the cases. But, and I may start doing that because I'm sure a pattern will emerge if I start doing that. 4 times 10, um, so 10 to 99 [writes 10 - 99] are four-digit, uh two-digit 4-flips, potential, but potential, for um so 4 times 10. [pause, (15 sec)] If 4 times the number is the reverse of the number, 4 times [writes 4 x on the top of the page, pause, (10 sec)] I'm trying to work on both ends at the same time. I don't want to have to go very far before I see this. Ah, 98 divided by 4 is 2 and 2 is already not in 9 8. So um, –

[pause, (10 sec)] Okay, 10 through um, no that's not true.

R: What were you going to say?

Shelly: I was going to say all of the ones with a leading one, like 10 through 19, but it breaks out. [pause, (5 sec)] And I was gonna say 10 through 19 [writes 10-19], any of those times 4 would have a leading digit of 4. So I was gonna try 14.

R: Oh, okay.

Shelly: And see if 41 is but nothing times 4 is, ends in a 1. [videotape ends] So, that won't work. Um.

R: Hmm, so restate what you just said. Nothing times 4 ends in a 1. Can you pursue that a little bit, in what that might mean for this? Are you kidding me? [referring to videotape, starts to change tape] It's supposed to be lasting for four hours. [pause] You can keep thinking.

Shelly: I am. Um, [pause (10 sec)] so these all have a leading digit of 1 and that would have to be the end digit equals something 1. And nothing times 4 equals something 1. There is no – there's no number times 4 that ends in 1.

R: Okay.

Shelly: So, that takes care of those. So if we can follow the same practice 20 to 29. Um, times 4 has to end in 2. And there's lots of products with 4 that end in 2. Um, but maybe not lots. There's [pause] 3 times 4 that ends in 2 and um 8. [videotape resumes] 3 times 4 ends in 2 and 8 times 4 ends in 2 [writes 3x4=_2 and 8x4=_2].
R: Okay.

Shelly: So um, [pause, (10 sec)] but it's still – um [pause, (10 sec)] that doesn't seem to be helping me in my mind. [pause, (10 sec)] 4 times 25 is 100, 4 times 5 – does that – ? Well let me try that, 4 times 23 [writes 23x4 = 92 (written vertically)] 92, no. Um. So, 4 times 28 [does this calculation, written out, then crosses it out, (5 sec)] No. So, I guess – I'm not convinced that that process of elimination eliminates all the 20s. But, the fact that these are larger, um, this one's larger 92 is the largest that this range would give me [pointing to the 20 – 29 range, pause, (10 sec)].

R: So, you've ruled out the 92. Why?

Shelly: The what?

R: So, you said 92 was the largest that the range from – [audiotape side one ends, researcher flips tape over] This was supposed to happen. There we go. So, you said 92 was the largest that the range from 20 to 29 would give you.

Shelly: No. It went, 28 went higher and 29 would go even higher.

R: Okay.

Shelly: But, flipping, to be a 4-flip, 92 is the highest.

R: Okay.

Shelly: Yeah.

R: So, how come?

Shelly: Because in this range, um, 29’s the highest and its flip would be 92.

R: Oh, I see, okay.

Shelly: So, the highest 4-flip, if it were truly a 4-flip, the highest one I could get from 20 to 29 is 92.

R: Got it.

Shelly: Okay, um and I quickly exceeded 92 at 23 even and I wasn’t even you know halfway through this range. [pause] And, I’m not perfectly convinced that I’ve exhausted all possibilities but for the sake of time I would just say okay and move on.

R: And where would you move on to? To the 30s, then?
Shelly: Yeah.
R: Okay.
Shelly: Until I saw more of a pattern emerging, I don’t know. I feel the need to write
down the rule for which I ruled out each one of these. Why, why I ruled out each one of
these [pointing to the intervals of numbers]. So, since I feel the need, I’m gonna do it.
R: Okay.
Shelly: Um rule okay, it’s a contradiction actually. Contradiction for 10 to 19 um, a 4-
flip would end in, [writes “Contradiction for 10–19: 4-flip would end in 1”] oh I hadn’t
considered the possibility that there were three digits here and end in 2 – see it’s useful to
write this down.
R: But would that be a two-digit 4-flip then?
Shelly: Oh right, no. So, that’s already ruled out cause that’s, a two-digit 4-flip has to be
two-digits when it's flipped. 4-flip would end in 1 and there exists no x –[adds “and no
x”, laughter]
R: That’s fine.
Shelly: No, because the no x part um, – would end in 1 and no x, and no 4 x exists
[changes x to 4x and adds “ends in 1”]. Okay, then the contradiction for 20 to 29 [writes
20 – 29] was because um the only product of 4 from 1 to 10 is um– 8, 3, 3 times 4 ends in
2 cause we need it to end in 2 and 8 times 4 ends in 2. Um, something’s bugging me
about um, I think I’ve left out a piece here. The flip has to end in 2 for this range [points
to the range 20 – 29], so it's, they’re all gonna be 80 plus something, plus 4 times
whatever the ones place is, so these are all, all of these products, the product of all of
these and 4 is going to be from 80 – cause that’s 4 times 20, to 80 plus 36 um, um,
[pause, writes 80 – 80+36, (25 sec)] to 116, oh, okay and it can only go up to 99 to be a
flip so it could only go up to 92. [laughter]
R: Well, and even probably a little bit higher than that but that’s where, I thought when
you originally said the 92 [pointing to the calculation with 23x4=92] you had pulled it
from here. I did not see that you had pulled it from there, that’s why I asked why the
other one was ruled out. What was the “oh” you just had, was it’s a 116, “oh”? 
Shelly: Well, we established that a flip, a two-digit flip could only be two digits.
R: Okay.
Shelly: So, 99 is you know, 4 times each of these has this range but the fact that it can
only be two digits stops this at 99 [writes 99 above 80 – 80+36].
R: Okay.

Shelly: And then I went back from 99 and said oh but it has to end in 2 so 92 is the highest.

R: Oh, I see, okay.

Shelly: [laughter] So, um and 92 was, oh, this is cool, this is working. So, and 92 was 23 times 4 so I only could consider 20 to 23, anyway.

R: Okay.

Shelly: So I mean, so then that shorted this down to 20 to 23 [writes 20 – 23 under the 20 – 29 range] because 23 gave me the highest possible range in terms of set ranges.

R: Okay.

Shelly: In terms of function ranges. [laughter] So um, so nothing above 23 would work. Oh, so pretty soon we’re gonna run out of two-digit products. [laughter]

R: Cause those weren’t just the 20’s you’re saying, right? You’re saying even more than that; nothing above –

Shelly: No, I was just thinking in terms of 20s, but now I’ve moved down and realized that that stops everything above 23. Well, not above 23, um, because 23 was, see that’s why I stopped for this range because um 23 was only good to get me 92 which had to end in 2 but the next one could end in 3 so that would take me to 93. So, I might have to consider it doesn’t stop at 23.

R: Okay.

Shelly: What – I mean what I have to consider cause 30 to 39 [writes 30 – 39 on list of ranges], let’s say, it only stops me at 23 in this range um because um because 30 times 4, oh that was 120 okay, so yeah so it does stop at 23. I had to go beyond it to make sure it stopped there.

R: Okay.

Shelly: Um –

R: So, so far, you’ve ruled out all the 10s.

Shelly: Mm, hmm.

R: 10 through 19. You’ve ruled out everything above 23.
Shelly: Well, let's see 24 might be, it won't be a 92, it won't be this. It would fit for one thing but it won't fit for another.

R: Okay, so you’re saying it would be okay cause it would still be two digits.

Shelly: Right.

R: 24, but it wouldn’t fit cause wouldn’t end in the 2.

Shelly: Right.

R: Is what you’re saying?

Shelly: Yes.

R: Okay.

Shelly: Yep. And then no, nothing above um, 24 times 4 is 16, [writes 24*4=96] 96, 25 times 4 is 100. Uh, so 24 nothing above 24 will work because it exceeds two digits.

R: Okay.

Shelly: The flip would exceed two digits. So, um, so we've ruled out everything.

R: Okay. And, and then you ruled out the 20 to 23 for your other reason, cause it can’t end in 2, right? That was why you said you ruled out everything because you'd ruled out those already.

Shelly: Right, because the first thing times 4 that ends in 2 is a 3, so –

R: Okay. And you tried 23 and it didn’t work.

Shelly: Right. Well, wait. [pause] Oh, it was a two-digit and it ended in 2 but it was not the flip of this.

R: Yeah, so it satisfied two of the rules but not the very last one which was the definition of the 4-flip.

Shelly: Right.

R: Okay, well if you don’t mind and have time, I would love for you to try one more portion of this problem. Would you be willing to stay?

Shelly: Of this problem?

R: Yeah.
Shelly: Prove that there are no two-digit 4-flips.
R: So there’s another part.
Shelly: Oh, a b.
R: Yeah.
Shelly: Um, I have class at 2.
R: Okay, so just like for five more minutes or so.
Shelly: Sure.
R: 5, 10 more minutes. Okay. Cause we can't exactly go back to it once we’ve stopped.
Shelly: Right, cause then I might be thinking about it.
R: That’s right. I just want to know where you go next. So, the next portion is prove or disprove the following statement. There are no three-digit 4-flips.
Shelly: Hmmm, [pause, (10 sec)] so three-digit 4-flips are 100 to 999? [writes 100–999] Um, [pause, (10 sec)] 4 times 100 is 400. So there gets to be a place in here where it exceeds three digits. Um, ah, what’s 4 into 999? [does the division of 4 into 999 using long division] Um, I may loose track of why I’m doing this.
R: Well there’s a calculator there, if you like.
Shelly: No, actually, this helps. Um, [pause] 4 into 9 goes twice (inaudible) [calculating]. Um, (inaudible) [calculating (20 sec)]. Okay, now why did I do that? Because I wanted to know what number in this range was gonna exceed three digits. And it’s 249. So, I now only am considering 100 to 249. [writes 100 – 249]
R: Okay.
Shelly: Because 249 times 4 would give me 999. So, that’s the last possible one. This interval is closed.
R: Okay.
Shelly: So um, that cuts down what I have to, okay, I guess I could go through the same process. I mean I’m using stuff I learned from doing the two-digit one. Um, but I’m not wanting to do that.
R: It’s why I asked if you could stay for the second one, because I think it’s enlightening to know like how you proceed past –

Shelly: Right.

R: – the two-digit. Because the two-digit, once you narrow down your window, it’s easy to try just what’s left. But now that you’ve narrowed down this window, you still have 150 numbers to try –

Shelly: Right.

R: – that’s too many to just guess and check.

Shelly: Yeah.

R: So, where might you go next?

Shelly: Um, well all the 100s flipped have to end in 1.

R: Okay.

Shelly: And um, nothing times 4 equals 1, that still applies.

R: Okay.

Shelly: I’m thinking if there's a remainder of whatever 4 times something, 4 times nothing is going to end in 1. So that cuts out the 100s, so that’s 100 to 199 [writes 100–199]. Um, so now I’m 200 to 249. [writes 200–249] Um, I’m okay this range is okay to get three places [writes “OKAY to get _ _ _ 3 digits” next to 100–249].

R: Okay.

Shelly: [pause (5 sec)] Okay, these um, not because um, nothing times 4 equals 1 [writes “NOT : nothing x 4 = _ _ 1” next to 100–199]. Um, so I’m down to 49 of em.

R: Okay.

Shelly: Um, these have to end in 2 and 4, I don’t want to go through all those.

R: Okay, so your thinking again of the fact that in order to end in 2, our last digit needs to be 3 or 8, just like last time?

Shelly: Well see that’s why I didn’t want to, I wanted to diverge from that test that we did with two digits, because I’m not clear what happens with this middle digit.

R: Okay.
Shelly: I don’t know whether the 3 has to be here or here or the 8 [indicating the middle and ending digits of N]. So, I could test that. [pause] Something in this range [referring to the range 200–249] times 4 has to equal something, something, 2 [after 200–249 writes x4 = __2]. Okay, um, [pause, (25 sec)] I’m drawing a blank because I really want to leave and I want to eat and so I’m distracted.

R: Sure, totally, yeah, but I, I’m excited that you got to apply these rules to the next stage and I got to see how you could take those rules and apply it to the next stage. So, let me just ask you about any techniques, anything that you can identify here that you’ve used, that you may have seen before or not or –

Shelly: Um, [pause, (10 sec)] the only thing that comes up in my mind is my wanting to do it longhand um, and your offer of the calculator I, distracted me, so then I had to consider and I had made comments about it would be quicker or you know, wanting, wanting to go for speed –

R: Mm, hmm.

Shelly: – so I’m thinking my instinct was no, I want to do this by hand. And I, and I was distracted by considering well, use the calculator and it, and it derailed my thinking of doing this by hand –

R: Okay.

Shelly: – so I, so I’m wondering why, something about my process has to do with, it’s facilitated by doing stuff by hand.

R: Okay.

Shelly: Just the age of the student I am, I guess. I learned this stuff without a calculator, so –

R: Um, so you’d like to do it without a calculator. And you – and you made the comment –

Shelly: But it somehow it helps my process.

R: That’s what you said, yeah.

Shelly: Yeah. Even though, getting bogged down in a procedure, I could sometimes lose track of why I headed there. Which is why I make these notes and stuff, so that I could go back and say, oh, where was I and why did I think about doing this?

R: Okay.
Shelly: But, I was able to recapture, even in the middle of doing this calculation, why I was doing that.

R: Okay.

Shelly: And it was because of multiplication and division being you know, reciprocal operations.

R: Okay. **End of Interview
Interview #4 (Total time = 58:46)

R: Okay, we’re gonna start with this problem right here.

Jon: Okay.

R: Problem 1 is a good place to start, generally. Alright, I’ll read this problem aloud, you can read it to yourself. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement, as an example, with sum 16. Prove that the smallest possible value for the sum is 14.

Jon: Hmm.

R: Do you have any questions or need any clarification?

Jon: The smallest sum of these three numbers along the [pointing to the sides of the pentagon] – would be 14?

R: That’s right, around the entire thing.

Jon: Okay. Hmm. [pause, draws new pentagon, (25 sec)] 1 to 10. [pause, writes “5 sets of 3”, tries 10 2, crosses out 2 and writes 10 3 1, (65 sec)] Hmmm, any hints?

R: [laughter] Not really, but, so you’ve formulated what 10 goes with, right, 3 and 1?

Jon: Mm, hmm, to make it the smallest one. It's just because 10 has to be the, is the largest number in there. So then the 9 would go – hmm. [writes 9 4] Can only use the number once. [pause, (15 sec)] No, I guess I can use the numbers more than once. [crosses out 4, writes 9 2 3, begins to place numbers on pentagon, puts in 3 9 2 and then 1 and 10 with the 3, pause, (20 sec)] Hmmm, but I don’t know if I’m gonna be able to prove it, rather than just show it.

R: So what you’re going for right now is a picture of 14? Is that what you’re kind of experimenting with?

Jon: Well I, yeah, I’m, what I’m trying to do is write it out, see what it looks like first, I guess.

R: Okay.

Jon: Cause I don’t, like I said, I’m not so sure if I’m going to be able to prove it, necessarily, so I want to just prove it to myself visually, more than anything.

R: Okay.
Jon: 8, 1 4. [writes 8 1 4 off to the side, adds the 8 and 4 to the side with 1 on the pentagon, pause, (25 sec)] There’s probably tons of combinations, though.

R: So, you’re thinking of all of the different sets of three, and where they go, right?

Jon: Right. I just, I guess I’m treating it more like a puzzle right now than as a, as a proof.

R: Okay.

Jon: So, [pause] 5, 6, 7. [writes 5 6 7 in list below other numbers] Okay.

R: Just note, 8 and 1 and 4.

Jon: Mm, hmm.

R: Add ‘em again.

Jon: Hmmm – [laughter, changes the 5 to a 4 on the pentagon and in the list below]

R: I just didn’t want it to mess you up so bad that it’s unrecoverable or something.

Jon: Thank you.

R: You’re welcome. [pause (5 sec)]

Jon: Hmmm. [crosses out the 9 and 2 on the pentagon and switches their positions, (5 sec)]

R: Okay, can I just ask you to give me some insight as to why you just changed the 9 and the 2?

Jon: Well, cause I know that I’m trying to get to the lowest number possible, or the lowest number possible. So, the only way I could get the lowest number possible is if these numbers on the corners, which have, which go into two different formulas, so and another thing is this 9 just would – couldn’t go with a 6, 7, or 4, that’s obvious. But also just, I was just looking at my other corners and saying that it probably gotta be my smallest numbers. So, I guess I could just probably throw in the 4 there and see, just see if it works out you know. [puts a 4 on the remaining corner of the pentagon]

R: Okay.

Jon: Which it doesn’t look like it’s going to but I, but like I said, I was just thinking the smallest numbers to go into all of em. [pause, (15 sec)] Hmmm. [pause (5 sec)]

R: So, why are you stuck here?
Jon: Well, cause I can see, I can see that I’m not going to be able to, like I said, there’s probably way too many combinations of this for me to treat it like a puzzle. So, now I’m, basically I’m trying to see how I could try to take combinations of 3 out of the 10 numbers, but then I’m trying to think of how I can have five sets of three.

R: Okay.

Jon: Not really – I’m contemplating how I can throw that into an equation right now.

R: Oh, okay.

Jon: I don’t know if it’s necessarily possible. [pause, (10 sec)]

R: How about your thought process as to why an equation?

Jon: Well um, just, just cause it has to equal 14. So, if I know I have to have, you know, three numbers that equal 14 [writes $14 = x + y + z$], but see, but I’m choosing out of 10 numbers and I don’t – I mean, I didn’t study before I came into this, [laughter] so, I didn’t know what to expect, you know.

R: Of course.

Jon: So I’m, I’m just thinking of some sort, it seems like the equation’s gonna be really big but I’m just trying to put it into the simplest terms possible for my brain to comprehend, I guess.

R: So, you’re thinking really big in terms of trying to deal with all 10 numbers.

Jon: ‘Cause 10, at first it doesn’t seem that much, but I mean, 10 numbers isn’t that much, but then the fact that I’m having to use the numbers two times [points to the corners], some numbers only once [points to the edges], and then I’m, my point is to prove that the smallest value combination is 14, so I’m [pause] hmm, [pause, (10 sec)] well [writes a list to the side: “19 8 27 34 40 45 49 52 54 55”, circles the 55, (25 sec)]

So.

R: Can I ask what just happened there?

Jon: I just added all my numbers together to see what the total is. But – [pause] but like I said, I gotta use different numbers multiple times, so I just use all the – [writes in a list 3, 2, 1, 5, 4, 15, (15 sec)] So, these would be the smallest numbers that could be at my, these intersection points.

R: Okay.
Jon: And they add up to 15, but I don’t know if that necessarily has any bearing on the situation.

R: So, the 55 that you circled, that’s the addition of the numbers –

Jon: Of all of them.

R: – 1 through 10?

Jon: Mm, hmm.

R: Okay.

Jon: I think.

R: I think that’s right.

Jon: So maybe go – 8 plus 9 plus 10, of course [writes 13, 21, 30, 40 in a list to the left of the 3, 2, 1–list, pause, (30 sec)]. Hmmm.

R: Can you tell me what else you’re thinking about?

Jon: Um, I was just thinking of, I knew that of my small numbers that I’d have to use two of ‘em, but only, but then only one of my big numbers. So I was thinking I could maybe – hmmm – so then I got 1, 2, 3, 4, 5, so the average would be 8 [writes 8 below other work], 5, which would be 3 so that would be 11 [writes 3 by the 8], but I’m not using, it'd be 6 [crosses out the 3 and writes 6], cause I’m using two, two out of the five. So 6, that equals 14, but it doesn’t necessarily prove anything. Hmmm.

R: Interesting, so what, explain that to me what you just did.

Jon: So, I added up the small numbers and they equal 15. Well, when I’m trying to get any, when I’m trying to get a combination of three, I’m gonna have to use two of, two of the five numbers, I have to use ‘em.

R: Okay.

Jon: So, then I divided it by 5, which was 3, multiplied by 2 to get my 2/5, which was 6.

R: Okay.

Jon: So, then on the other hand, these other ones add up to 40, my big numbers and I can only use one of ‘em. So, I divide that by 5 and I get 8.

R: Okay.
Jon: And my average, so at least the average of ‘em would be 14.

R: Okay.

Jon: But that’s not necessarily a proof, per se. But it kind of shows that that’s what the average, that does show what the average is in a way. Hmm. [pause, (30 sec)]

R: So, where would you like to go from there?

Jon: I’m trying to think of how I can, how I can say in words, to kind of like, just like in the proof class where you have to say well this is A or whatever and I have to be able to choose one of those, you know, some choose some element of A in here [draws lines next to the list of 13, 21, 30, 40, to the side writes \( \subseteq \)] but only one of ‘em, but then choose two of ‘em [draws lines next to the 3, 2, 1, 5, 4 list]. And I don’t know how to – [pause, (5 sec)]

R: You trying to reformulate what we were just talking about?

Jon: Yeah, I’m trying to formulate it into words.

R: Okay.

Jon: Like, I’m trying to see, see how I could develop it into a proof.

R: Okay.

Jon: ‘Cause I don’t know – [pause] ‘cause otherwise I’m, I’m kind of stuck right here. But I’m just trying to see how I could – [pause] trying to, I guess I’m trying to think in my head how I can develop a proof on this. ‘Cause at least I’ve gotten it down to – at least I, at least I’ve kind of proven it to myself that it’s at least possible for all the smallest sums to be 14.

R: Mm, hmm.

Jon: Then I go to, hmmm. [pause (30 sec)] Okay, let’s try this. [writes list to the left of 6, 7, 8, 9, 10 and writes in the possible combinations to make 14 using two numbers from 1–5, draws new pentagon, fills in 5 6 3 and 10 1, then 9 4 with the 1, and finishes it correctly, (1 min 35 sec)] Hmm. [pause (10 sec)] Well, there’s my thing.

R: Okay, so you’ve found one with 14.

Jon: Mm, hmm.

R: Cool. So, we at least know it’s possible to get 14.
Jon: I do. So, so maybe the route I should take is that, okay, so if, I see. So, the reason why we can’t get say a 13, is cause we can’t use a 0. And if we built it higher then other things would have to be lower? Or no? ‘Cause this is all 16. [pause] So I guess I could get a 13 with a 10. [writes combination of 10 with 2 and 1] But the smallest possible value for all sums is 14.

R: So, you’re saying the smallest we could possibly get incorporating that 10 would be the 13?

Jon: Would be 13, but then everything else would be a little bit higher, probably.

R: So, is your thought, then, that 13 must not be possible?

Jon: [pause] Let’s see. [writes list of combinations with 13, (45 sec)]

R: So, you’ve written down just the different possibilities to add to 13?

Jon: Mm, hmm.

R: Okay.

Jon: [draws new pentagon, pause, (35 sec)] Oh, that’s gotta be 10 and 1 and 2. [pause] Let's just see, throw in – [fills in numbers around pentagon from the list, (15 sec)] Hmmmm, Okay. But then they all equal 12, except for this one. [pause] 13. [pause, (25 sec)]

R: So, what are you thinking? Just going through some possibilities in your head?

Jon: Well again, I keep going back to this [points to his list above of 13, 21, 30, 40, and 3, 2, 1, 5, 4], where I want to, where I just want to pick one of the big numbers and two of the small numbers, but I don’t – [pause] I’m trying to think of how that – [pause] again, like I said, I’m just trying to put it into words, ‘cause an example isn’t a proof. [pause, (10 sec)]

R: Okay, would you like to try a different problem and leave this one for a while?

Jon: Sure.

R: Okay.

Jon: Want me to tear this out?

R: Yep, please. Put those over here. [researcher collects papers, gives student next question] Okay. Let’s go to Question 2. It’s a brand new idea. We call a positive integer $N$ a 4-flip if 4 times $N$ has the same digits as $N$ but in reverse order. Do you understand what that’s saying?
Jon: So like [writes 1221], and if that was, hmm. [pause (5 sec)] Same digits as N but in reverse order. So it’d be like [writes 124], and then if, this times 4 was 421 [writes 421].

R: Exactly.

Jon: Okay.

R: Yeah, that’s exactly what it would be.

Jon: Okay.

R: Okay, so part a is prove that there are no two-digit 4-flips.

Jon: No two-digit 4-flips, okay.

R: Do you know what the question’s asking you for?

Jon: Mm, hmm.

R: Okay.

Jon: [pause, (20 sec)] So again, I’m thinking of examples, so like, so, so at least, right off the bat any digits that are like 22 can’t be in it. Or 33, 44. And otherwise I’m looking for something like 14 to 41 or something like that but, but that would work. [pause, (20 sec)]

Hmm. [pause, (15 sec)]

R: Tell me what you’re thinking about.

Jon: Well, I’m thinking that if I can try to prove that, hmmm, I guess I don’t know if I’m trying, maybe I should try to prove that there is, that there is one. And then since I can’t find one, that means there isn’t. But, I guess it kind of seems like the same, the same idea even if I took the converse or whatever, the – [pause] like if I try to prove there’s not, like if I, just like when we do normal equations, er like when we do normal proofs, when I try to prove that there is, when I want to prove that there’s not something, I prove that there is. And then if I can’t prove that there is, then you know, the other option is that it’s not, or –

R: Okay.

Jon: – visa versa. So, I have to go from a two-digit to another two-digit. [pause] So, it’d be – [writes \(xy = 4^*N\), (10 sec)] so it’d be 4 times, it’s not \(x, y\) [writes \(4^*xy = yx\), (5 sec)]. I don’t know how to say that in digits. It’d be – [pause, writes \(4(x+y) = yx\), (30 sec)]

Hmm, it still isn’t quite what I want. [pause, (25 sec)] Hmm.

R: Tell me what you’d like to find here.
Jon: Well what I’d like to find is, I’d like to try to find the, the 4-flip that does exist. But I don’t – [pause (5 sec)] but I, I’m assuming it doesn’t exist.

R: Okay. [pause (15 sec)] And how about, can you go any further with what you were trying to do with the digits?

Jon: What do you mean?

R: So, you were trying to express it some way, it looks like –

Jon: Mm, hmm.

R: – you’re searching for a formula or something you can solve.

Jon: Mm, hmm.

R: Okay, do you think you can go any further with it?

Jon: Um –

R: Anything else you can think of to do?

Jon: Well I’m trying to think of – [pause (15 sec)] cause I know what I’m thinking but I just can’t express it in mathematical terms. Um, cause I would want to say that x is greater than or equal to 10 [writes x ≥10] but – hmmmm, but then it’d have to be less than 25? [writes 25 > next to the x]

R: Hmm, where’s the 25 coming from?

Jon: Well, 25 times 4 is a 100, which then makes it a three-digit number.

R: Okay.

Jon: [pause (10 sec)] And then I’d want y to be between 1 and, – 0 and 10, I guess [writes 0 < y < 10, pause, (20 sec)] So, I guess the only way I can really think of proving this is by, by actually just going through and showing by example.

R: Okay.

Jon: But that’s really–?

R: – Which ones do we need to check? –

Jon: What’s that?
R: So, which ones are we gonna need to check here?

Jon: We’ll just 10 through, 10 through 25.

R: Okay.

Jon: Cause those are the only positive integers that would still be a – that are both two, that would be two-digit but then we multiply it by 4, it’s not, not a three-digit. [begins list of 10 – 40, 11 – 44, etc.]

R: Okay. So, why don’t you start trying it, maybe something else will come out for you. [pause, student finishes list (35 sec)] So, clearly, none of them work.

Jon: Clearly.

R: Right? Okay, so tell me what you did when working through this problem. What strategies did you use, because you found a solution, right?

Jon: Yeah.

R: Whether brute or not, it’s, it’s a solution.

Jon: Sure.

R: True?

Jon: Well, this is what actually what I immediately thought of doing.

R: Okay.

Jon: But see, I wanted to try, but I wanted to think, try to think of it in a non, non-brute manner. [laughter] But, but apparently my mathematical knowledge is not up to snuff when it comes to this problem cause I didn’t, don’t know how to put it. Cause like, when I, like here I was wanting, it’s not \( x \) times \( y \), I was wanting it to be 4 times some number, \( xy \), would equal \( yx \).

R: Okay.

Jon: But it’s not, but you know what I mean–

R: –Yeah, I definitely know what you mean –

Jon: – where \( x \) is the 10s digit, \( y \)’s the 1s but–

R: So, what else, let’s say that you’re not in the situation where you don’t have resources and stuff. What may you have tried to do at that point, when you were searching for that?
Jon: Definitely would’ve looked to see, um, you know, how, how to represent that mathematically. How to represent um, a two-digit number, but you know, like 18 or whatever, where \( x \) is 1, \( y \) is 8, but \( x \) is in the, in the 10, in the 10s category. I don’t know how to –

R: Okay, so let me be your resource for that.

Jon: Sure.

R: Okay, so we’ll pretend we’re out and you’re just doing it normally and somewhere on the internet or something or you would ask somebody and they’d be able to say, “Oh, I know what you mean to do here, you need to represent it as 10\( x \) plus \( y \)”. [interviewer writes 10\( x \)+\( y \) on the paper]

Jon: Mm, hmm.

R: And \( x \) and \( y \) are single digits.

Jon: Mm, hmm.

R: Is that kind of what you were going for? Like does that represent what you wanted it to represent?

Jon: But then the thing is, is that I want it to equal 10\( y \) plus \( x \) [writes = 10\( y \)+\( x \)].

R: Right. You want what to equal that, 4 times that, right?

Jon: Mm, hmm. [writes in 4 with parentheses around the 10\( x \)+\( y \)]

R: Okay, so see if you can proceed with that.

Jon: [writes 40\( x \)+4\( y \)=10\( y \)+\( x \), (15 sec)] Hmmm. Then I guess I just, just to see what happens– [writes 39\( x \)=6\( y \), \( \frac{39}{6} \) \( x \)= \( y \), pause, (10 sec)] But then 39 over 6 is not a, is not an integer, or vice versa. [writes \( x = \frac{6}{39} \) \( y \), (5 sec)] 6 over 39 is not, either.

R: Is there anything you can multiply there to get an integer?

Jon: Multiply this by?

R: So, for instance, like is there a number, \( x \), could go in there?

Jon: [pause] Um, say that again?
R: Well, can you use any of your other things that you’ve learned along the way in this
to prove that there’s no way that 39 over 6 times x is ever gonna be an integer? Can you
use any of the other things that you know about those two digits, x and y, to prove that
that can never happen?

Jon: To prove that some number x times 39 over 6 won’t be an integer?

R: Right.

Jon: Well – [pause] the only way, the only numbers I can multiply that to get it to be an
integer would be ano–, would be another, another fraction. Or a, or a um, or a common
multiple. [pause, (10 sec)]

R: So, let’s see if any of these strategies play out in the next question. Okay?

Jon: Okay.

R: Why don’t you tear off that sheet; you can set it to the side in case it’s useful.

Jon: Okay. [researcher gives student next part of question]

R: So, the next portion of the question is this one. Prove or disprove the following
statement. That there are no three-digit 4-flips.

Jon: Hmmm. Okay. [writes 4(100x+10y+z)=100z+10y+x, then 400x+40y+4z=, then
399x+30y-96z=0, pause, (1 min 25 sec)]

R: So, what now?

Jon: Hmmm. [pause, (30 sec)]

R: What are you thinking?

Jon: Um, well I think I want to do the same thing and get, you know, [writes
399x+30y=96z] just get one on, you know, and get em on each side of each other, I guess.
And then, I can divide both sides by 3. [writes 133x+10y=32z, pause, writes 10y=32z-
133x, then y = 3.2z-13.3x, pause(1 min 15 sec)]

R: Where would you like to go from there?

Jon: Hmmm, well I guess I was just quickly thinking of uh, some examples of where this
minus that would be, would still be an integer.

R: Okay. [pause] So, what do you need to try there, if you’re going to try to find that?
Jon: Um, well I don’t know. I was just, basically I was gonna guess and check but I was kinda – I was, immediately I contemplated putting in a 0 for x cause I, try to keep this um, positive.

R: Mm, hmm.

Jon: But if x is 0, then it’s not gonna be a three-digit number. So, it’s gotta be something.

R: Okay. [pause, (10 sec)] So, you wanted x to be 0 so that you made sure that it stayed positive?

Jon: Just to make, just to make sure that my, my 3.2 z would be positive.

R: Okay.

Jon: And just for simplicity sake, you know, multiplying one of em by 0. So, I mean, since I’m just guess and checking anyway. But like I said, if I can’t have it as 0 or it’s not gonna be a three-digit number anymore. So, it’s gotta be at least 1. But now I’m thinking I’m gonna, I want it as a 2 [writes x = 2 to the side] so it could have the 0.6, which this can also have.

R: Hmm, okay.

Jon: But see – [pause, (10 sec)] hmmm – [pause (10 sec)] but then this number can’t be over 9, or it would put that over a three-digit number.

R: [pause (5 sec)] So, tell me what you’re thinking in reference to that.

Jon: Well it’s um, so the only number I can, I have to multiply this by 9 otherwise it’s gonna be smaller than 26.6 [writes in 26.6 under 13.3x].

R: Okay.

Jon: Which would be 27, 28.8 - 8? 8. [writes 28.8, solves for y, (20 sec)] 2.2. [writes y = 2.2]

R: Okay, so recap what just happened there. You put in x being 2.

Jon: Put in x being 2, with the only directive of making this a 0.6.

R: Okay.

Jon: Because I can make, in the hopes that I could make this multiply some number by 3.2 to make it a 0.6.
R: Okay.

Jon: But then I didn’t look, but I did, I didn’t look deep enough cause if I would have just thought about it, I would’ve seen that I would’ve had to multiply this by 3 to get it to be a 6 there.

R: And that’s not big enough.

Jon: And that’s not big enough, not by any means.

R: So, you’ve seen that x being 2 doesn’t work.

Jon: Mm, hmm.

R: Because it forced z to be 9 and it didn’t work.

Jon: Mm, hmm.

R: You also, to recap what you did, you saw x being 0 wouldn’t work, because then it wouldn’t be a three-digit number, right?

Jon: Exactly.

R: So, what else could x be? Can we keep going in that line?

Jon: Well, yeah, that’s what I was just, that would be my next thing, would be to put it to 1 or to 1’s the only other option, really.

R: Okay. And why is that, that it’s the only other option?

Jon: Well if it was any, if it was any higher than a 2, then I couldn’t pick a z that would be – my z couldn’t be big enough, otherwise it would break this rule. Cause if, cause if my z was over 10, then it would be a four-digit –

R: – Okay. –

Jon: – flip. So, my only options is 1 and 2 for x.

R: Okay.

Jon: See now this 3’s pesky cause I know I’m not gonna, no matter what I multiply 3.2 by, not gonna get a 3. Unless if my z’s, ah, not an integer. And see now I’m starting to wonder if, well, my z can’t be – [pause] I guess I was just, immediately just thought if – but x can’t be there, I see. Hmmm, so with that said, is it necessarily wrong for my y to have a 0.2. Cause then it would just become –
R: Well, let me ask you to go back and think about what x, y, and z represented. [pause, audiotape side one ends, researcher flips over tape, (10 sec)] So, what do x, y, and z represent?

Jon: They represent the 100s, 10s, and 1s placement. Oh, my left hand side of my equation [labels LHS].

R: Okay.

Jon: But then their places swap-swapped, at least x’s and z’s, while my y stays the same. But so my, just my thought is, is that because my x’s and z’s, they most definitely have to be an integer, because, because they’ll become the one spot on the other side of the, of our equation, and if it’s 2.2 for x or z, my number’s gonna be a 102.2 or whatever. But I’m just thinking that, is it necessarily incorrect if my y digit is 2.2, since it’s not–?

R: Why don’t you go ahead and pursue that thought because you’ve found, you know what x and z and y have to be, the only possible scenario, right?

Jon: Mm, hmm. [writes z = 9, y = 2.2]

R: So, go ahead and pursue what that would mean those numbers to be.

Jon: [writes 924, (5 sec)] See then, because my x is still 2, but I’m adding on the extra 0.2 of my y. And there’s no way I can – I [laughter] and I definitely can’t multiply 924 times 4 and get 429. [laughter]

R: Or vice versa.

Jon: Right.

R: So, what’s your conclusion?

Jon: So now, well now I know I have a limit of my – I get – just like my last one. So I was able to catch it on the first move, not this one. That the largest number 250 ‘cause 2, well, 249 [writes 250 and 249].

R: Mm, hmm.

Jon: Cause that’s the only way I can multiply a number by 4 and still be a three-digit number. So, I forgot to keep that in, in mind. Okay, so with that in mind, my y, where it’s y equals – [rewrites y = 3.2x-13.3x, pause, (10 sec)] if my x is between 1 and 2– [writes 1 ≤ x ≤ 2] My y doesn’t matter, my z doesn’t matter. [writes 0 < y < 9 and 0 < z < 9, (5 sec)] That there’s no way I can multiply – hmm. Well no. – [pause, (10 sec)]

Okay, so my–bad. Okay, so this is my right hand side of the equation [circles the 924]. So it’d really have been 290. [pause] So, it can’t be 290 or 2– [pause (5 sec)] Okay, and that doesn’t qualify as the flip, I see. Cause that’s another thing is when you have the 0.2
in there, can’t have it because it, it’s gonna mess up the flip on the other side. Cause this
would be my, my left hand side and when I multiply that by 4 – it’s not gonna be 232 or
924.

R: Okay, so what’s your conclusion?
Jon: Um, my conclusion? [pause]
R: So, have we answered the original question, prove or disprove the following
statement. There are no three-digit 4-flips.
Jon: [pause] Okay. Well, it’s too weak, I don’t think I’ve proved anything but in my
mind, I don’t think that there exists a 4-flip in the three-digits.
R: Okay.
Jon: [pause] Only because I can’t, cause if I’m dealing with fractions, it’s never gonna be
a positive integer and that’s always gonna mess up, always gonna mess up. When I
multiply it by 4 and switch it around.
R: But you’re not convinced?
Jon: Well, I’m not convinced cause I’m not–
R: You don’t like it?
Jon: – yeah, I don’t like it at all. Cause I haven’t, it’s not – unlike this one where I could
at least see it, you know. Where I can at least see every number and I’m just not, not
adventurous enough to do –
R: I wouldn’t be either. –
Jon: – 100 through 249.
R: So can you describe at all any strategies you used here? Anyways that you went
about solving this problem?
Jon: Well, I guess the first and foremost was getting on the internet [laughter] and
finding out what I needed to do, you know, to better, you know, look for reference
materials. That’s even what I do in my math classes; that’s what I’m always doing is
looking back. Cause I’m not, I’m not a very good math receptacle, so I have to keep, I
have to look in my books to see what I, what I’m forgetting.
R: Okay.
Jon: Um, I think the number one strategy for both of these is to realize that – to know the limits, the limits of your um, domain, as it may be. To know that your x can’t go above 2, or ah, 2.5, I guess, as if our, our y and our z are what, are both 0. If 2.5 isn’t an integer but, or 2.49 or whatever. [pause] Cause that just, that just really slowed me down once I got to this part here [points to the equation y = 3.2x-13.3x]. Cause I was thinking that I could still, I could swing the – I guess I kind of found out anyway that my x couldn’t be any greater than a 2.

R: Mm, hmm.

Jon: Cause it would have to be on that side, too. Cause my 9–

R: Yeah, you found that kind of in two different ways, didn't ya?

Jon: Mm, hmm, yeah, both on the different sides of the equation.

R: Mm, hmm.

Jon: On part a versus part b.

R: Anything else you can note as a significant strategy or something that you could name as I did this or I used this?

Jon: Mm, hmm. [pause] I don’t know. I think I’m, I think what I’m doing here is pretty simple math. You know, I don’t think that the strategy I’m using is, you know, using the simplest, using the simplest strategy possible. But mostly cause I can’t think of the, any difficult, not necessarily difficult, but better equations or tricks to use for the time being for this equation.

R: Okay.

Jon: So, I’m just putting it into the simplest terms that I can think of and then try to work it out from both sides of the equation. Which I can’t, which I can’t ah, rectify it. So, I guess that would be the, I guess just for both sides is that since I wasn’t able to, since I wasn’t able to, like I said, to rectify both sides of the equation with each other, I just have to assume, even though I’m not, even though my assumption is based on some rather sketchy proof work, that these 4-flips don’t exist.

R: Okay. So, I noticed that in all three problems, you wanted to form an equation.

Jon: Yes, I definitely wanted to but the problem was in all three problems, I couldn’t develop an equation. –

R: Not necessarily. –
Jon: So my next, my next step was always to go through and look to see what, to look at examples.

R: Okay.

Jon: Cause once I’m able to look at the examples then at least I’m able, like especially for this one [points to Question 2 part a], I’m able to actually see that they don’t exist.

R: Mm, hmm.

Jon: And um, and just because the equation, even if I can’t think of the equation, but at least the idea process for all of ‘em are still pretty simple.

R: Okay.

Jon: You know, so I’m able to use at least the um, the outside of a superficially simple idea and use and show myself the examples. Cause I guess what I’m just saying is that if I can you know, just like with any math problem, if I went and did all these problems and then just found out that that one does work, then I just keep trying to trudge away until I can find the one that works. But, that, I guess that’s where my problem is on both these, is that since I see that it doesn’t work, I can’t, I don’t know how to develop the equation to show that it doesn’t work.

R: Okay. And so, what triggered the desire for an equation? Is that something from just past experience or is it because you’re working with numbers?

Jon: It, I guess to be honest, it’s because I’m taking that proof class right now so when I think of prove something, I’m immediately thinking like I need to put em into equations and try to prove or disprove the equation, rather than – so maybe it’s my thought on what a definition of a proof is.

R: Okay.

Jon: Cause if I, so maybe if my, if what I thought, if I thought I could prove, by just showing you examples, like, cause like I said, I would have done this right off the bat – [points to the previous work on question 2 part a, trying all 10–24]

R: Mm, hmm.

Jon: Cause I, I mean that was the first thing that struck me, cause I wanted to see the examples, but I wanted to tinker around with the idea of the equation first off because that’s what I, that’s what I think of, when I think of proofs is, is some equation, not English words in a sentence, I guess.

R: Okay, interesting. Alright, great! Well thank you very much for your insights.

***End of interview
Interview #5  (Total time = 55:05)

R: We will begin by reading through this first problem. The numbers 1 through 10 can be arranged along the vertices of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with the sum 16, just for an example. Prove that the smallest possible value for the sum is 14. So, is there anything that you need clarification on in the question, or do you have any questions at all? I’ll give you a second to read through it.

Beth: I can’t think of anything. [pause] Are they listed once?

R: Yeah, everything just once.

Beth: Okay, I understand what they’re asking.

R: Okay. Proceed in any way you would like to.

Beth: Okay. Prove that the smallest possible value of the sum is 14. Okay, I am just going to look at combinations of the numbers 1 through 10.

R: Okay.

Beth: Because, um, we're dealing with, all the different combinations of three of them since we’re talking about, um, we're summing up three at the most.

R: Okay.

Beth: Yeah, three numbers, so, [pause] huh, but I sorta have to think about all of the sides because, like if I just summed 1,2 and 3, that would be 6, [writes 1,2,3 – 6] but I mean that wouldn’t ever be – Okay, what they're saying is 6,7,8,9,10,11,12,13 could never be a combination so you would start with 14, so I guess that maybe I’ll try to find how they arranged that.

R: Okay.

Beth: I mean, this is wasting time, but I don’t know. [draws new pentagon]

R: Time we’ve got, its no big deal.

Beth: Okay, so right now though I’m just trying to understand the problem, if we're going how like {my instructor} does things. So, I understand what they're saying by the sum is 14, so um I’ll just start with 10, [writes 10 on a vertex of the pentagon, pause, (5 sec)] so then the small numbers have to be around it, there it's 16 [points to pentagon given], so 10, I’m thinking 1,2,3 and 4 [points to the places around 10] have to be here or something. And then, I don’t know, [puts 1 and 3 along a side with 10, crosses out, pause, (15 sec)]
R: So, you wrote 1 and 3 because that added to 14, right?

Beth: Mm, hmm.

R: But you scribbled it out, and how come?

Beth: Because when I was thinking 2 and 4 if I put here it would be 16, so that wouldn’t work, and we can only use numbers once [pause (10 sec)]. Okay, we need ways to make 14 maybe? [pause (5 sec)] Okay, let’s say that I can do this. [points to the pentagon she drew]

R: Okay.

Beth: The main thing is though realizing that 13 and under can’t work. [pause] And let me see how they did their's. 9, 10 these are the big numbers, [circles 10, 9, 8, 7 on the given pentagon] And [pause (5 sec)] So, let's see, we’re doing 14, hum, okay well, this is a tricky one. [laughter, pause, (20 sec)] I guess these have to go here [puts 3 and 1 on her pentagon with 10, pause, (10 sec)]. Can we use 0? No [pause (15 sec)]. Okay, if that doesn’t work, maybe that means that 10 doesn’t go right there. [draws new pentagon with 10 on a side] Maybe we can share that big of a number. [puts 1 and 3 on vertices next to 10, pause, puts 2 on another vertex, crosses out, then puts in 4 and 9, pause, (45 sec)]

Hmm.

R: So, what are you thinking about?

Beth: Well, I’m noticing here. [points to the 8 and 7 on the given pentagon] They had 2 consecutive numbers and then they added 1. So, I was thinking about 6 and 7 and 1 that equals 14 but my 1 is here so I already used it so, then I was wondering. Okay, I can’t put a 6 here [points to next vertex after 4] ’cause that would be 4 and 6 that is 10 and I need another 4 but I used the 4.

R: Okay.

Beth: Can I put a 7 here? [still looking at vertex after 4] That means that I would have to put a 3 here and I can’t. So, if I put a 5 here, that is 9 and 9 to 14 is 6, I guess that I could do that, [puts 5 on vertex and 6 on side of pentagon] so I'm left with, I have to use a 2, 7, 8 [writes 2 7 8 off to the side] and – this is like a, this reminds me of Sudoku.

R: Yeah [laughter].

Beth: I’m a lover of those, so – So, I’m thinking a combination of two things that can go here [points to side of pentagon with 3 at the vertex] that have to add up with the 3. Hmm, 11, 12, 13, it's not looking good. [pause] Hmm [pause, writes in 7 and 2 with the 5, (10 sec)] That works and then 8 and 5 is 13, so this wasn't successful. [pause] I’msort of um well if I can’t get this to show 14, then as far as I know that could not work.
R: Okay.

Beth: So –

R: So, what would you try if you came to a stumbling block like this? So, I'll advise you to go back and just check over your work a second.

Beth: Okay.

R: Check over those numbers that you've written down.

Beth: Are you saying that I added wrong somewhere? Could be possible. So, this is 14, this adds to 14, oh, this adds to 15 [adding 4 and 6 and 5]. So, yes that's a good thing to do if this fails, to see if I did all of the algebra right, which oftentimes I don't.

R: [laughter]

Beth: So, what was I thinking here? Oh, why did I do that? 9, okay so I can't have these consecutive things here [pointing to the 6 and 5, crosses them out]. So – let's think of something else we can use. [pause] 8 and 2 [writes in 8 on the side and 2 on the vertex], that's 12 and 14. So, now the numbers that I have left to go with are 5,6,7. [writes 5 6 7 below, pause, tries different combinations, first 6 and 5 with the 3, (15 sec)] Okay, 2 and 9 plus 7, that doesn't work. [pause, (10 sec)] Okay, I wish I had a pencil.

R: [laughter]

Beth: Let's draw a new picture. [draws new pentagon] I think I’m a very visual learner, so it helps if things are neat for me. [writes 1 10 3 across the bottom]

R: Okay.

Beth: [adds in 9 and 4 with the 1 and 8 and 2 with the 4, (5 sec)] So thus far, [pause (10 sec)] I can’t put 7 here [points to remaining vertex] because then all I am left with is a 5 or a 6 and that would equal 15 or 16. But, 7 goes here and a 5 can go here. [writes in 7 on vertex and 5 between 2 and 7] ‘Cause 7 and 5 is 12 and 14, but 6, that doesn't work. [pause, draws new pentagon, (15 sec)] No, sometimes that frustrates me ‘cause you see I'm gonna have to do it all again. I’m thinking that you have to do it all again since – I mean it’s going to be moderately different from this is.

R: Why don’t you keep going with where you’re at? I think you're almost there, I think you may have left out one possibility or something, so keep trying.

Beth: Okay, so everything worked except the 6 right here, so maybe if I try switching some of these vertices, the ones that are on here. [pause] Oh wait, let’s see, oh then I would have to switch a lot of things, if there was a 1 here that would be 7 and 8, and the 6
would work, if there was a 1 here [points to vertex where 3 is at]. So, that means that
there would be 3 here [lightly writes in 3 where 1 was], and that doesn’t work. So let’s
see, 3 if we switched the 2 here and the 4 here [switches 2 and 4 around] we would have
4, 9 and that wouldn’t work. Okay, let me see if there's a pattern up here. [referring to
example of sum 16, pause, (30 sec)] Um, I don’t know. I think that at this point I would
usually do homework with someone and I would ask them what they have.

R: Okay.

Beth: Because sometimes maybe they got this step out of luck or something, but I think
that I would go and like try a new one.

R: Okay.

Beth: Try a new –

R: A new problem, or –?

Beth: Pentagon. Like assigning new numbers. So, I’m going to do that. [pause] I don’t
know if there should be a strategy of keeping small or big numbers on the vertices.
Sometimes I was stuck with lots of big numbers just to put back in the middle.

R: Okay.

Beth: So, maybe I'll think about that when I am going through this time. [pause (5 sec)]
Okay first, the only way for a 10 to be with someone is to be with a 1 and 3, I think that's
a definite. Because 2 wouldn’t work. Oh! Well, maybe you can say, since they are
talking about the smallest sum for a side is 14 [circles the number 14 in the question], um
since you know that 10 is your biggest number – Okay, here is another insight I might try
to do, I’d say, okay I’ll take, – I’m trying to find – um. [pause] Okay, what I was first –
This is the first thing that I wrote down [pointing to where she wrote 1,2,3 – 6], um I
know isn't possible cause, the numbers left 4,5,and 6 [writes 4, 5,6] – there’s no way to
make 6 with that since 6 is going to be on one of the sides, there's no way to get less.

R: Okay.

Beth: So I’m thinking, the smallest combination maybe can be this. [writes 10-1,2]

R: Okay.

Beth: I don’t know why because the smallest combination obviously would be this
[points to the 6], but um the smallest thing a 10 could go with is this, that's what I’m
trying to say.

R: Okay.
Beth: And since 10 has to be included in one of the sides, we could try to go from there.
So –

R: So, your theory is that the absolute smallest is most likely 13.

Beth: Mm, hmm.

R: Okay.

Beth: So, maybe I can try to think of why that doesn’t work.

R: Okay.

Beth: So, I’m just going to take a whole new approach. [switches to new sheet of paper, writes 1,2,10] So 1, 2 and 10, we’re left with 3,4,5, 6,7,8, and 9. [writes list of the remaining numbers] And so now I’ll – oh but, I can use 1,2, 3 – There’s going to be 5 of these [puts dashes for places for each side of pentagon in a list, (5 sec)], and each one – like say, um 1, 2, 10 was my first side [writes 1, 2, 10, on side of blank pentagon already drawn on previous page], the next one can use one number from the previous. Oh ,this is gonna be, I’ll do a picture, just so I can keep track, too. [draws new pentagon, writes in 1, 2 ,10 along one side, pause, (10 sec)] So, I guess my thinking is, so if I can prove that there's no way, I mean, I should still try to get this right first [pointing to her attempts at getting 14] you know an induction thing –

R: Okay

Beth: – so you have to prove that it is true for the $N = 1$.

R: Okay.

Beth: So, my sort of thinking is prove that this is true for 14 before you try to disprove that it's, it can’t work for 13.

R: Okay.

Beth: But, I mean, this is something that I would definitely go back to [points to pentagon for 14 she started] if I were to write a formal proof. So, I’m just going to try to see if this can jog anything. [goes back to case of 13]

R: Okay.

Beth: Um [pause (10 sec)] yes, it depends on which order you are in too, because right here 10 pairing with the next, say like the smallest things, it comes nowhere close to our number here we are trying is 13. [writes 13 and put a box around it at the top of the page]

R: Okay.
Beth: So, maybe 10 has to be on the middle. [switches 10 and 2 on pentagon and in list]
R: Okay.
Beth: [pause (10 sec)] Okay, so – It seems like a lot of guessing and checking to me, I
don’t know if that’s –
R: Okay.
Beth: – because I am not seeing a smarter way to do it? So, I sort of feel foolish.
R: Don’t feel foolish. Remember these problems are meant to be challenging.
Beth: Mm, hmm.
R: They are meant to get you to think and have to work, so don’t feel like you're missing
something. Okay? [laughter]
Beth: Well, I don’t want {my instructor} to be sad.
R: Well, she won't see any of this, so you don’t have to worry about it.
Beth: I know, I know. So, I’m just going to try a random number. We're going to try to
do 13. [puts 8 next to the 2 on the pentagon] 8 and 10, so this has to be 4. [writes 4 next
to the 8]
R: 13.
Beth: You’re right. It has to be a 3. [changes 4 to a 3, puts 2, 8, 3 in the list] Okay, 3 and
8. [crosses off 3 and 8 from list of what numbers are left] 1, 2, 3 [pointing to the vertices
of the pentagon]. Okay. [pause] So, right now I know that this can’t work because – well
when a problem says like prove that the smallest possible value is 14, do we assume that
they are right?
R: I guess that's a personal choice, to assume or not. So, what's your inclination, given the
way the problem is worded?
Beth: To prove that it is true.
R: Okay.
Beth: So, right now I’m knowing that this is not going to work. [points to her attempts at
13] But, I’m wondering how to prove that –
R: Okay.
Beth: – for all combinations and all orders –
R: Okay.
Beth: – which could take a long time, so that’s what made me think that guessing and checking is sort of um not an efficient way.
R: Okay. So, is there any way that you can proceed with this with using some smart choices somehow, like is anything forced or anything like that?
Beth: Mm, hmm. Like saying –
R: ‘Cause you said for sure, 10 has to go 1 and 2.
Beth: Mm, hmm.
R: Can you pursue that any further? Any other choices?
Beth: That have to go together? Um, let’s look at the next biggest number, 9. 9 can be with – 11. [below other work, writes 9, 2, crosses off 2] Okay, it can’t be with a 2 because if it is with a 2 it would be 11 and it would have to be with another 2 and you can’t repeat.
R: Okay.
Beth: So um, 9 can’t be with a 2, [writes 9-2 at top of page] let’s see if it can be with anything else, if it was with a 1, it could be with a 1 and 3. [writes 1 3 next to 9 in list at bottom of page] So, let's think of – and that’s it because if it was with a 4 you would need a 0 to be 13 and we don’t have a 0. [writes 9 4 0 in list]
R: Okay.
Beth: So 9, 1, and 3 have to go together. Um, this is my in no particular order – [writes 1, 10, 2 to the right]
R: Okay.
Beth: – column, just if you want to know. So, 9, 1, and 3 have to go together. [adds 9, 1, 3 in list] Okay, so let’s go from there. What we have left, so we used – okay, I can, I reuse, so let’s just see what we have left, we have 4, 5, 6, 7, and 8. [writes list of numbers that are left] Let's see what 8 has to go with. [writes 8 in list on bottom right under 9 4 0]
Um, let’s see it can’t go with 5 because that would be the same thing with 0, [writes 8 5 0] um so it can go with a 4 and a 1. [writes 8 4 1] That makes it 12 and 13, and then to go with 2, it would have to be a 3. [writes 8 2 3, pause, (5 sec)] And those are the only combinations.
R: Okay.

Beth: So 1 – okay 10 has to look like that and 9 has to look like that. [On list on right hand side of page, towards the top, writes 10 = next to 1,10,2 and 9 = next to 9,1,3] 8 could look like 8,4,1 and 8,2,3, [writes in 8 = 8 4 1, and 8 2 3, pause, (10 sec)] maybe I'll keep going, because maybe I'll see a pattern –

R: Okay

Beth: – that will show why it can’t add up to 13 or something. [pause (5 sec)] You can reuse. Let's see. [crosses 8 off of list of numbers that are left] I don’t know why I’m crossing these out, because we reuse them sometimes. Oh, but um it can’t be this one because I can only be used at the most twice because it is on a vertex (sic). So, we know that that has to be like that. [crosses 8 4 1 off list to the right] So, let’s look at 7. [writes 7 = ] 7, the combinations are – could be a 7, can’t be a 6 because it would be like that. [On list at bottom of page, writes 7 = 7, 6, 0] So, 7, 5, and 1 [writes 7, 5, 1, pause] and we can cross that out because that 1 is already used. [crosses out 7,5,1, continues list] 7, 4,2 – 7,3, it can't be 7,3, and 3, and I think that that's the only combinations.

R: Okay.

Beth: So, 7 has to be with 7,4,2. [In list to the right, next to 7 = writes 7,4,2] Oh, but look. We already used 2 twice, so maybe by saying, – so I would sorta say okay that proves that 13 can’t be done like that.

R: Yeah, because all of your decisions were forced right?

Beth: Right.

R: Nothing was ambiguous at all.

Beth: Right.

R: Okay.

Beth: So, then maybe I can try doing that for 14 –

R: Okay.

Beth: – instead of trying to do a picture. [starts new page, writes 14 in a box at the top] It’s cool, it's sort of just like actually Sudokus.

R: It is, kind of.
Beth: Because, if you know that you can only have 1 through 9 in one row and, hmm. Okay, so let’s start with 10 again, so like we said over here, the only way it would be a 1, 10 and a 3, [writes 10 = 1,10,3] because if it was 2 it would be a repeat. Okay. So that's for sure, so 9 can be written 9, 1, and 3, 9 can’t be 9, 2, and 2. [writes 9,1,3, and 9,2, off to the side]

R: You need 14.

Beth: Yeah that’s right, 11 okay. 9, 4, and then 1. [writes 9,2,3 and 9,4,1 in side list, pause, (5 sec)] And it can’t be 9, 5, so those are the only combinations. [writes combinations in main list] Okay, for 8. We can have 8,1,5; 8,2,4, and 8,3, no that can't work. [writes 8,1,5, and 8,2,4 in side list, writes 8,3, crosses off] No, I already have 8, 4, 2. And then, 8,5,1. Okay, that's, that’s it for 8, [writes combinations in main list, (5 sec)] so I guess what I'm looking for here is – once I have, um, once I go 10 through 1, to find out all the combinations that, unlike here [points to work on 13], where since 4 – I was forced to stop because I said since this combination involves [referring to the last combination with 7 in the list for 13] – The only thing this involves was a 2 and since we already used 2 twice, we can’t have that. So, I’m looking for, since right here [referring to the current main list and the possibilities with 9], maybe like these two will be the only ways and then each of my final things I chose will have, um, like 3 and 3 will be – these two will be the only ones with 3 in it so they share a vertex (sic). That’s what my goal is for um –

R: Okay, check the math on that first one.

Beth: Okay, um 4 so we already have that, it is supposed to be a 4. [changes 3 to 4 in main list with 9, then crosses off 91 4]

R: Yeah.

Beth: Okay, yeah little things like that could mess you up on here.

R: Yeah.

Beth: So, am I done with 8? Yes. This must be boring for you.

R: No, just difficult not to input more, that’s the only thing.

Beth: It’s a very cool thought for a thesis or um, it’s a cool idea, especially if you're going to teach, you know?

R: Yeah, it is really interesting to watch how different everybody approaches everything.

Beth: Huh. 8,9 – [writes 7,1,6 in side list] Okay, so, I can’t talk and do like math in my head.
R: Totally fine. [laughter]

Beth: 9 and 5, [writes 7,2,5 and 7,3,4 in side list, (10 sec)]. So, those are the only combinations. [writes combinations for 7 in the main list, (10 sec)] Okay. 6,1,7, 6, 2, 6 so that wouldn’t work, right? [whispering, inaudible] [writes 6,1,7 in side list, then 6,2,6 which she crosses out, continues, (10 sec)] I’m just going 1,2,3,4,5 just to be systematic, even though it’s obvious that you can’t repeat just to let you know. [works through combinations with 6 on side list, crossing off those which are repeats or not possible] So, (whispering, inaudible as student works through combinations) and 6, 6 we can’t do and 6, 7 we’re back at that and you can’t do 6 and 8 because, like we said, can’t use 0. So, we have 6,1 and 7; 6,3 and 5, oh yeah that was a repeat. [writes 6 = 6,1,7, and 6,3,5 in main list] Okay. [pause (10 sec)] I won’t, I can’t obviously strike things out until I can see everything. Um, so 5, 1, 6 (mumble, working through numbers aloud but inaudibly) [writing down combinations with 5 on side list, (15 sec)] now I can’t do it, here 6 okay [writes 5 = 5,1,8, 5,2,7, and 5,3,6 on main list, (10 sec)] So, then a 4 (Mumble, inaudible) [working through combinations with 4 on side list, writes 4 = 4,1,9, 4,2,8, and 4,3,7 on main list, (50 sec)] What is there left? Okay 3, – 3, 1,10. Oh my gosh, have I been forgetting 10 the whole time?

R: {No}

Beth: No just –

R: Why not? Because that’s really the only option with 10 in it, right?

Beth: Right. Okay with 1s and 3s and I just hadn’t gotten there. Okay. Right I knew that. 3,2,9 (mumble, working through numbers) [works through combinations with 3 on side list, (30 sec)] Oh, I already have that though. Okay, so do I have any repeats? [moves to new sheet of paper] Do you want me to keep this here?

R: Yep, that’s fine.

Beth: So, you have 3 equaling, 3, 1, 10, 3, 2, 9, and 3, 4, 7 [writes 3, 1, 10; 3, 2, 9; 3, 4, 7; and 3, 5, 6 on main list]. 2, I can’t do it, because it would have to be an 11 and we can only use 1 through 10.

R: Okay.

Beth: [works through combinations with 2 on side list] 2, 3, 5, 9, 2, 4, 5, (mumble) 2, 7, 5 we have 2, 2, 9. Okay, so these are the options. 2, 3, 4, 5, (mumble) 2, 3, 9, 2, 4, 8, and 2, 5, 7 on main list] And then 1, we know is 1, 10, 3. Hold on, we’ll just see. 1 and 2 can’t be done. [works through combinations with 1 on side list] 1, 3 I have, 1, 4 would be 5. 9, 1, 5, 1 and 6, 7, (mumble) 8, 9 yeah so I think that these might be my only ones. And I already have that one. (mumble) [writes 1 = 1, 10, 3; 1, 4, 9; 1, 5, 8; and 1, 6, 7 on main list, (10 sec)] Okay. So, now comes the part I guess that’s kind of fun. You know that this is the only way [puts box around 1,10,3 on previous page] and
now maybe what I can do is I see that a lot of them have 1s in them. And I wonder if let's take 8, if we look at 8 and 6, these have the least amount of options, they can either have the one with a 1 in it or the other one. So I guess, what should I be thinking here? I guess it's sort of guess and check, like if it is, let's say we picked this one [points to one of the options with 8], that's kind of hard, because then, [pause (5 sec)] Okay, let's see how many numbers are going to be shared. There's 1, 2, 3, 4, 5. [counting vertices on original given pentagon] So, I'm going to have 10. [counting] Okay, I'm going to have 10 groups, 10 ways to equal 14, that's what I'm looking for. And two of each are going to share one number.

R: Okay.

Beth: [pause (10 sec)] Alright, [pause (5 sec)] so maybe I don’t have to find out – Okay, I'm asking myself this not you, because I know that you can’t help me. But, do I actually need to find out which group it is, or do I just need to somehow prove that it can be done? There's nothing that forces, there's nothing that, um, like in this case we couldn’t –

R: So, how could you go about proving that it can be done?

Beth: Proving that what can be done?

R: That you can make the choices that you want to make, that you can, you never hit the roadblock you hit with 13. How might you go about proving that?

Beth: Okay. [pause] Well, [pause] I would just have to at least find one way to make 5 pairs.

R: Okay.

Beth: That only, that each group of two shares, um, that each group of two shares one number. [pause (15 sec)] So, I’m just going to try to make a random grouping to see if I am even thinking right, if that's possible. So, I'm just going to – I'll just start and go down and we will pair this 1, 10, 3 with 1,9,4. [writes new list at bottom of page, lining up numbers that are the same starting with 1,10,3, then1,9,4] And maybe I’ll draw a picture for this one. [draws new pentagon] I's our shared person. [writes in 4 9 1 10 3 around pentagon] Well, I guess the way I ordered these – You see another thing that I am thinking about is, I wrote this order really doesn’t matter written down here [pointing to her newest list] but it could matter on the picture.

R: Okay.

Beth: So, let’s see how this works out. So, I'm gonna find one that has – okay, I have to make a list of what I, the numbers I used. [writes 10 9 to the side of the new list] So I used one from the 10s, and one from the 9s, so I need something, let’s start with 4 in it. Okay, let’s try this one. [writes 2 and 8 next to the 4 on the pentagon (8 on the vertex)]...
don’t know if this is going to work, so 4, 2, 8. [adds 4,2,8 to new list] Let’s try something with 8s now. [pause (10 sec)] Wait.

R: You just hit a little roadblock, right? So, tell me about the roadblock that you just hit. It's “Wait”, hold on –

Beth: Alright, well I'm, I have 10 of these. Um, often times it's gonna happen you get, so um um robotic that you forget what something might mean. You know what I mean?

R: Oh okay.

Beth: So I was just saying to myself, okay, 10 through 1, I only have to pick two more.

R: Oh, mm hmm.

Beth: So, I guess as long as I use all of the numbers once and if I just pick two more random ones that would just prove that there is one way.

R: Okay, if you can make them work in the picture, right?

Beth: Yeah, right.

R: Okay.

Beth: So, I’m going to see what I should look for. I have 1,2,3,4,5,6,7. [writes 5,6,7 below pentagon] Okay, so I have to look with, including these vertices, 3 and 8's in them. [writes 3,8 before the 5,6,7] I have to look for, to see if there’s any two ways to include these 5 numbers so I can put them here. [pointing to the open spots on the pentagon]

R: Okay.

Beth: So, let’s just think. 3, 5,6,7,8. 3 – 3,5,6 – There’s one that could potentially work. So, if I have 3,5,6, we would be left with 8.

R: So, where can I look to see what 8 could go with? [pause (5 sec)] What can 8 go with?

Beth: Well, I mean I wrote here. Also, should I – automatically include – oh, since I wrote these down that 8 can only go with these, but then why did I write down 8 can go with these?

R: Well look at what this says, this says 8,4,2, right? [pointing to the listing for 2]

Beth: Mm, hmm.

R: And this says 4,2,8. [pointing to the listing for 8]
Beth: Oh, okay so this is 1, 5, 8.

R: I don’t want you to get too lost like you said the kind of getting to be a robot.

Beth: Okay, alright, so since the only thing that 8 can go with is a 1 and a 5, or a 2 and a 4 – 1 and 5 – and since we both used them – I used them already then this is a no go.

R: At least the way you've written it, right?

Beth: Mm, hmm.

R: Okay.

Beth: Okay, so let me think. I’m trying to think of what I could, um, think about going on this next round so I don’t have to go through a million of these.

R: Yeah.

Beth: [pause (5 sec)] This is difficult. I want to try to simplify it so that – I want to try to find like maybe the one easiest way that I know works.

R: So, maybe try looking back at the pictures and see if you can go back to what you were doing over here on this first page and see if you can incorporate it into what you are doing here, just because you're seeming to hit a little bit of a roadblock, right? You want some more ideas.

Beth: Okay, well what I was doing over here, I was just trying to – this is my random plug in to try to find a way to prove that 14, the addition of 3 on each side would be 14. So, the one thing they have in common here is that 1, 10, 3 is the only way and that's a for sure side [pause] and 10 has to be in the middle [pause (5 sec)] just because it doesn’t go with anything else besides 1 and 3.

R: Okay.

Beth: Um, incorporate this stuff? [pause (10 sec)]

R: So, you just said to me that 10 has to go in the middle because it can’t go with anything but 1 and 3 right, so you are having problems with it on the vertex, right? Can you use that thought that there could be problems with certain numbers being on vertices to correct what you've got here [points to last pentagon drawn] so that you can continue going? So, you hit a problem because you couldn’t find something else to go with 8, right?

Beth: Mm, hmm.
R: Can you use that thought that you just told me about 10 having to be in the middle because it can’t go with anything else?

Beth: [pause] Um, you mean like maybe I could take out this and try to find something, a different combination with a 1 in it that maybe 8 was in the middle? Since I’m having problems with it on the vertex?

R: Yeah, what about that?

Beth: Something like that.

R: Can 8 go in the middle right there?

Beth: Oh [laughter] yeah, I forgot about – okay so let’s try that. Oh well. 8 and 2. [switches 8 and 2 on the pentagon] So, now we can look at combinations – So, what went wrong is that when 8 was here the only things that it could go with were already used.

R: Mm, hmm.

Beth: So, now 2, we'll look at what 2 can go with and see if there's something not used, so 2 and 3 is already used. 5 and 7. Okay. So, we could use 2, 5 and 7, since 5 and 7 aren’t used. So, we could go with that, so let’s try to put this here. [writes 7 in middle and 5 on vertex on side with 2] And the only way – Okay, my number that's left is what, 8? Nope, it is 1, 2, 3, 4, 5, 6. And if this were true, it could be just 2, 5, 7? And this doesn't work, or we could try, 7, 5, 2. Huh. I got one.

R: It works? [audiotape side one ends, researcher flips over tape] The tape recorder even had an ah, ha moment, right?

Beth: Okay, so I just found a way to prove that – let me just write this over before I do a little conclusion.

R: Okay.

Beth: [redraws pentagon with numbers correct for sum 14, reads off numbers] 1, 10, 3, 6, 5, 7, 2, 8, 4, 9. Alright, so I just found a way after writing down all possible ways of using a number to combine it with 2 others to make 14, I found a way which proves – in which all of the sides add up to 14.

R: Okay.

Beth: So, since I proved for 13, but since we hit a roadblock when I tried to find all combinations of 7, since there was only – there were none that were left since you could only share – okay, I'm not being – Alright, basically since I showed one way to add up to 14, and if I continue then maybe I have to show that I hit roadblocks for addition of 12, 11, 10, 9 and all the way down to 6 since that's the lowest number.
R: So, where would you start if you were to look at 12, what would be the first number combination that you would try, you would work with? If you were to try to get 12 and hit roadblocks?

Beth: In trying to —?

R: So, where’d you start here and where’d you start here? You started with what number?

Beth: 10.

R: 10.

Beth: So, I can’t use 10 to go with 12, because 10,1, and 2, the lowest combination is 13 so I’d have to start with 9 so —

R: But 10 has to be placed somewhere, right? So, what conclusion can you come up with?

Beth: Oh, oh yeah, okay so, since for 12 and under you cannot, um, add the numbers 1 through 10 up to make 12, since you can’t use 10 and you have to use all numbers, and so I would say that for the numbers 12 and under —

R: Okay.

Beth: - there is no way to use all of the numbers; because you can’t use 10 right off at 12 and consequently you can't use them below.

R: Okay.

Beth: So, I prove that 12 and under is impossible, so then I would say okay, I would re-summarize what I did for 13; which was we started with 10, I started with 10, and I found ways to use 10, and I found there was only one way. And when I hit that roadblock, which forced 9 to only have one combination and it forced consequently since we used — we already used our 1, since these have to be that way, we already used, we already have that 1 as a vertex (sic), so that crossed out this possibility since there were only 2. And then, since both 2s were used we said, okay all of these possibilities can’t happen. [pointing to the options with 7]

R: Okay.

Beth: So, basically I proved that 13 cannot be an addition on all sides.

R: Okay.
Beth: And since I did show at least one way for 14, then I – you know what? Yeah! I'd say it's the smallest. I was thinking about doing something with 15, but you wouldn’t need to do that.

R: No, you don’t really need to do that. You certainly, I think it is possible to find it just as a note, I think 15 is possible, you can find it. Um, but it's certainly not asked for in the proof, right?

Beth: Right.

R: So, how about any strategies you used, anything that you could pinpoint and give a name to as to how you proceeded through this problem?

Beth: Strategies, okay well definitely in the beginning I was guessing and checking.

R: Okay.

Beth: And then by guessing and checking seeing a picture, I saw okay, the only way – 10 can only be in one combination. And so that sort of led me to this, okay well if you found out the only one possibility for a combination involving 10, you could do that with all of the other numbers.

R: Okay.

Beth: So, that was a strategy. Another strategy maybe is to disprove what they're not talking about. For, you know, 13, 12, 11, 10, to prove that it isn’t possible for them.

R: Okay. Would you say that you had to convince yourself that it was true at some point? Did that occur at some point?

Beth: That what was true? That this is true? [pointing to the statement of the question]

R: Mm, hmm.

Beth: Um, yes, because usually um if this might not be true it would say prove or disprove.

R: Okay.

Beth: So, I just sort of took their word that it was true, because we have to prove it.

R: Okay. And at some point you looked for patterns, right? –

Beth: – Um, hmm. Oh yeah. –
R: – you kind of talked about differences in numbers, you looked at patterns in there and that's right, right?

Beth: Yep, I didn’t find any patterns.

R: And you looked for examples with guessing and checking, you used a strategy here of thinking about how small it could be and then reached the determination that 13 was as small as you could go and we kind of reiterated that later in talking about the proof that it couldn’t be 12 or less.

Beth: Mm, hmm.

R: And then you went to something, I think that you went over here next to the 13 and did a bit of work there, to prove that that wasn’t true, and then I think that led you here [pointing to work on lists for considering 14], wouldn’t you say? To thinking about each possibility and then being able to put them in the –

Beth: Right, let me see, I forgot what I did. This is the 2nd page, do you want me to label the pages? Are you going to keep them?

R: Yeah, I’m going to keep them. I know what order they came in.

Beth: Okay.

R: That is what the video is for –

Beth: Okay, yeah –

R: – is to tell me what order they came in.

Beth: So, I sort of came here because this was a rough idea of “okay, I need 10”, okay no. This is my thought that I know it can’t work for 13 so maybe if I go through I'll see that it can’t work.

Beth: So, when I starting writing random numbers down, I think that that led me to say, “Okay wait, 10 can only do this and what can 9 be?”

R: And let’s say that you had somebody else in here to work with, what things would you maybe have done differently had you been working with somebody else?

Beth: I don’t know, I think maybe if they randomly, cause you could – like what if I randomly got – did this [pointing to the completed pentagon for 14], if I guessed that at first, I don’t think that it would have been as structured or as clear of a proof, sometimes.
R: Okay.

Beth: I love working with other people, but realizing right here, once you worked through it yourself maybe it is a little bit easier and more explicit to explain it as a proof?

R: Okay.

Beth: Cause if I jumped, if I figured out, okay, 14 works, I might not have known where to go.

R: Okay. That makes sense.

Beth: Cause maybe they would have gotten it, but if I asked them, they could have also have already done something like this. [pointing to her lists of combinations]

R: Okay.

Beth: Maybe it was a different way.

R: Alright, sounds good. Okay. ***End of Interview***
Interview #6  (Total time = 56:20)

R: Alright, let's start with this first question. I'll read the question aloud as you read it as well. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with the sum 16 – just for an example. Prove that the smallest possible value for the sum is 14. I’ll give you a second to read through that.

Lily: [pause, reading through question (15 sec)]

R: Okay, you may proceed however you want.

Lily: [pause, writes “pentagon 5 sides”, pause, writes “x1 + x2 + x3 = 14” (65 sec)]

R: Can you want to tell me a little bit about what you're thinking?

Lily: Basically I, I know that a pentagon has five sides. So you have to go from there and you know that on each side there’s three numbers, so that’s why I have x1 + x2 + x3 = 14, just because they said that the smallest sum is 14.

R: Okay. [pause (25 sec)] Where would you like to go from there?

Lily: Well, basically I know too that the numbers along the sides are going to be whole numbers 1 through 10.

R: Okay. [pause (30 sec)]

Lily: I guess the problem too, it's just like a puzzle almost, where you have to find where each of the numbers go in to equal 14 –

R: Okay. So, are you thinking of ways to make it –

Lily: Mm, hmm.

R: – to get 14.

Lily: Yeah.

R: Okay.

Lily: And I mean that you know also too that each number can only be used once.

R: Mm, hmm. [pause] So what are you trying in your head?

Lily: Um, I’m just kind of going around on almost like they did on the pentagon and just seeing where numbers can fit –
R: Okay.

Lily: ‘Cause it helps me to look at stuff like visually like that.

R: Okay.

Lily: [pause] I guess the other tricky part too that is kind of stumping me is on the ones on the actual angles versus the numbers on the sides, like those are used in the addition of both sides.

R: Okay. [pause (20 sec)] Can you tell me or write down what you're trying?

Lily: Um, yeah. [draws pentagon] Basically I’m thinking like the larger numbers are going to have to go actually in the middle – [points to the middle of each side]

R: Okay.

Lily: – just because like it doesn’t make sense since the smallest sum they say is 14 like on these ones [points to the example pentagon] it's 10+2+4 which equals 16 but if you have 10 on a corner then you're going to have 4, well you're going to have to have like a 3 and a 1 or which leaves only a 2 to be left here and I mean it's not possible because they say the numbers are just 1 through 10 so you can’t have a 0 on that corner –

R: Okay. [pause (10 sec)]

Lily: I mean is this true, too? Is the sum 14? Is the smallest –

R: Yes. Yeah, that is.

Lily: Okay.

R: And how might you go about convincing yourself of that?

Lily: So, I mean, you are going to know for sure that if that's true, you're going to have to have 10, 3 and 1 on one side –

R: Okay.

Lily: I mean because 10’s the biggest number, and if you can only use the numbers, well I’m assuming that you can only use the numbers once. Is that right?

R: That's true. Yeah.

Lily: So, then you’re going to know, I don’t know what order these are going to be in, but you are going to have 10, 3 and 1 on one side. And I mean I guess, too, [writes 3 10 1 on
one side of her pentagon] that’s why I said that um, 10’s gonna have to be in the middle
because to equal 14 you're going to have to, if you have 10 on the corner, you're going to
have to have 2 and 2 which you can’t use and I’ve already used 3 and 1. [pause] So,
basically on this side left, you have to, this is going to have to equal 13 if you have the 1
on this corner and then 11 on this side [writes 13 and 11 outside of pentagon on
appropriate sides, pause]. So I guess the choices that we have that add up to 11, you’ve
already used the 10 and 1, we could use 2 and 9, we already used the 3, or 7 and 4, or 5
and 6. [writes these combinations down in a list] I mean cause that’s all your possible
combinations I'm thinking.

R: Okay.

Lily: – on that side. [pause] And I guess it's kind of just taking awhile because it's almost
like a guess and check problem –

R: Okay.

Lily: – well that’s the easiest way for me to figure it out I’m thinking. [pause (10 sec)]
I’ll just try it and see if they work, I’ll just put the 2 right here which (inaudible) [puts 9
and 2 on pentagon] (more inaudible figuring) So, for that side your only possibilities
would be a 4 and a 5. This is actually – [writes that the bottom side needs 12 more, tries
more combinations, writes 5,7 and 3,8, pause (25 sec)] And we can only make the 3 and
the 8 with those two [points to where the 3 is already written on the pentagon, writes in 5
and 7 on bottom of pentagon, pause, writes 4,6,8 off to the side (25 sec)] I'm just writing
these numbers over here because if I chose these 9 and the 2 for this first one those are
the only ones that we have left to use.

R: Okay.

Lily: [pause] Alright, I guess I’ve come to the conclusion too over here that this was the
wrong choice because none of these [points to the 4,6,8] with the 7 will add up to 14.

R: Okay.

R: Okay, so where would you go from there?

Lily: So, then from there I guess I can eliminate this choice right here [points to 2,9 in
list on left hand side] and work on these two [circles 7,4 and 5,6 in the list, draws new
pentagon on another page, writes in 3 10 1 on one side (20 sec)]. I guess the other
problem is too, I mean, you could flip these [points to the 3 and 1 on the pentagon] which
may change just where they're placed on the pentagon [pause (15 sec)]. So – I guess I
could try that, too. [writes 2 9 on the pentagon in different order than previously done]
And that side to total 5. [refers to what needs to go with 9 to get 14, writes 5 below
pentagon and lists 1,4 and 2,3] I guess we can come to the conclusion that that's going to
be 1 and 4 or 2 and 3 and since we already used a 3 here and a 1 here it’s not possible
either.
R:  Okay.
Lily:  – so you can eliminate the 2 and the 9 from that side. [draws new pentagon, writes in 3 10 1] So, I'll work with the 7 and the 4 now. [writes in 7 and 4 on the pentagon with the 3] So, on this side you'll have 10. [writes 10 below pentagon] I mean you can take out, you obviously can’t use 1 and 9 since you used the one right there [points to the 1 on top of the pentagon], and the 3 and 7 have already been used. [writes 2,8 below the pentagon, then writes in 2 and 8 on pentagon, pause, writes 6 to the side with list of 1,5 and 2,4 (25 sec)] I already used the 1 here and the 2 here so [pause] I can’t use the 3 twice, so maybe I can just switch these and just see what that comes out to – So that means that side is going to have to equal 12 now. [writes 12 to the side] The 10 we already used so the 10 and the 2 – [writes 10,2 and crosses out] and 9 and 3 has already been used [writes 9,3 and crosses out] And the 8 has already been used, the 8's there. [writes 8, 4 and crosses out] So, basically we can eliminate the 4 and the 7, I'm thinking too now.
R:  Hmm.
Lily:  [pause, draws new pentagon, writes in 3 10 1 (10 sec)]
R:  So, it’s kind of a guess and check, but you’re eliminating it systematically.
Lily:  Mm, hmm – yeah.
R:  Okay.
Lily:  So I guess that I have to find – [writes in 5 and 6 on pentagon, writes 6 and lists 1,5 and 2,4 below pentagon (15 sec)] I guess my only choice would just be the 2 and the 4 since 1 and 5 have already been used. [writes in 2 and 4 on pentagon] So, that side adds up to 10. [writes 10 and lists 1 9, 2 8, and 3 7 to the side, proceeds to check each of these, pause, switches the 2 and 4 around, crosses out the 10 list, writes 12 (45 sec)] Oops. [scratches out the 2 and 4 and the list with 6 below as she sees it didn't add to 14 (10 sec)] So, that should add to 8. [writes 8 and lists 1 7, 2 6, 5 3, proceeds to mentally check these options (45 sec)] So, we see, so none of these are checking out, so I guess, unless I made another arithmetic error, I'm gonna have to switch those two around. [refers to the 5 and 6, draws new pentagon, writes in 3 10 1 and 6 and 5 with the 3. Writes 9 with list 1 8, 2 7, 3 6, 4 5, below pentagon (35 sec)] 1 and 3 and 5 [crosses out the options with 1, 3, and 5 in them, writes in 2 7 across bottom edge of pentagon. Writes 7 to the side with list 1 6, 2 5, and 3 4, considers these options, then switches around the 7 and 2. (50 sec)] This side would add to 12. [writes 12 to the side, pause, writes in 4 then 8, considers, then changes the order to 8 then 4 on the corner, pause, writes in the last number, 9 (30 sec)] So, I guess here's the right thing. [circles correct pentagon]
R:  Alright, you’ve got it.
Lily: So, I guess that basically I just used elimination, I mean cause I just found it easiest to start with the greatest number –

R: Okay, yeah.

Lily: – and kind of worked from there. And I guess that it made it easier that you could only use the one number, each number once.

R: Mm, hmm. And went through it fairly systematically.

Lily: Yeah.

R: Let’s go back to the original question, it said to prove that the smallest possible value for the sum is 14. You have found one with sum 14.

Lily: Mm, hmm.

R: Have you sufficiently proved the problem?

Lily: No, but basically you can say that it is gonna, well [pause] I guess, you know that you are going to have 10 on one side for sure, and the next smallest number you could have 1, plus I guess you could have 2, 13 and so I guess that you would need to prove that you wouldn't be able to find 13 –

R: Okay.

Lily: – on all of the sides.

R: Okay, so how would you go about doing that?

Lily: Um, I guess that I would just take this. [draws new pentagon, writes in 1 10 2 (15 sec)] I guess I would see if there was any combinations, that make it 13. So, that's gonna have to be 12. [writes 12 to the side] and the other problem that you run into like last time was if you want this on the corner, but if you have this on the corner here [pointing to the 10 moving to the corner position], then your only, your next two smallest numbers are going to be 3 and 4 because you have already used 1 and 2 there [pause] and 17 does not equal 13 obviously, so you know that you are going to have to keep the 10 in the middle from that.

R: Okay.

Lily: And then if we do keep it like this or like we have this side’s gonna have to equal 12. [lists 2 10, crosses out] So, if we have. We've already used those. So, 9 and 3 [writes 9 3 in the list] So, I guess we could just plug those in to here [pointing to the side with 1 on the corner, writes 9 and 3] That one would equal 13 [referring to the next side] if we have a 9 and a 3. [pause] And over here, we would have to have that equal to 11
So, we can't use the 10 and 1. [writes 11 to the side and lists 2 9] 2 and 9. [pause] We used the 3 already. [writes 3 8, and 4 7, writes the 4 and 7 on the pentagon, writes 5, 6, 8 off to the side (20 sec)] And those are just some numbers that we haven't used yet. [pause] I guess [pause] basically wherever you stick these numbers [circles 5 and 6] they're not going to sum up to 13.

R: So you circled 5 and 6; they make 11.
Lily: Mm, hmm, yeah, 11 plus 3 is 14.
R: Okay.
Lily: But then you're gonna have to go also and maybe check, maybe if we switched these two around just because we want to keep the smallest on this corner right here since 5 and 6 are, add up to 11 and we want to have the smallest number possible to see if that side will equal.
R: Okay.
R: [pause] And then if we put the 4 in the corner we run into the same problem as right there, but then I guess just to be on the safe side we should probably just check [draws new pentagon] since we have the choices of a 9 and 3, we did that up here, we should probably just check 8 and 4 just to be on the safe side [writes in 1 10 2 and 8 4 on pentagon on side with 1 (10 sec)]. And since this side needs to add up to 11, [writes 11 to the side and lists 2 9, 3 8, and 4 7] We have 2 already, so 2 won't work. And a 3 – and a 4 and 7 which we already used so I guess it's not possible.
R: Okay.
Lily: So, I mean, this is the only way that we would be able to get that to equal 13 and we proved that with both, with the remaining numbers we have for on this side and this corner and that side it's not going to equal, the smallest amount we can get that to equal would be 14.
R: Okay. So you feel confident that you could, with a little bit of work, write this up for an assignment or something.
Lily: Mm, hmm, yeah – probably.
R: ‘Cause you proven then that you can find 14, 13 is the absolute smallest –
Lily: Mm, hmm.
R: – and it’s not possible.
Lily: Yeah.
R: – So that means that 14 is the smallest.

Lily: Yeah.

R: Okay, um, how about anything you used during this process, can you identify anything, any strategies you used?

Lily: Um, mainly I used, I first kind of started out with a guess and check, I mean just to figure out what possible combinations you could have there and then from there I kind of used the elimination whether we could keep that possibility or not.

R: Okay.

Lily: But I mean with each possibility you have two numbers and you have to check both combinations of 'em.

R: Okay.

Lily: Just to be sure.

R: Have you ever seen anything like this before?

Lily: {No}

R: Okay. Is there any other place that you have pulled any of the things that you have used in this problem?

Lily: Um, it kind of almost reminded me, I don’t know what those one puzzles are called, but like with all of the squares and they're divided –

R: Oh, the Sudoku?

Lily: Yeah.

R: Mm, hmm.

Lily: That’s cause I just like actually learned about those like a week ago and that’s almost what it reminds me of.

R: Okay. Just eliminating some possibilities and needing to fill in the numbers and stuff?

Lily: Yeah.
R: Okay, cool. Let's move on to another problem. I'll take those sheets and this.
[rsearcher collects papers, gives student Question 3] Okay, we're actually gonna go to
another problem, skip problem 2 for right now, go to problem 3. A traditional chessboard
consists of 64 squares 8 by 8, suppose dominoes are constructed so that each domino
covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard
with dominoes covers every square of the chessboard without overlapping any of the
dominoes. [pause] Does that all make sense? Do you want to read through that a second
and wait for the question for a second?
Lily: Yeah. Sure. [reads question (10 sec)] Okay.
R: Okay. Makes sense?
Lily: Yeah.
R: Consider a generic chessboard of size m-by-n. Prove that the generic chessboard of
size m-by-n has a perfect cover if and only if at least one of m or n is even.
Lily: [pause (20 sec)] Okay.
R: Do you understand the question?
Lily: Yeah.
R: Okay. What would you like, where would you like to go first?
Lily: Um, I think I just want to prove to myself that um, one does have to be even so I'd
just take, I mean you obviously can't have a 1-by-1 with the dominoes but –
R: Okay. Mm, hmm.
Lily: [draws picture of 3x3 chessboard, proceeds to shade in 1x2 areas (20 sec)] Alright,
so that just proves to me that one does have to be even, cause with a 3-by-3, both of those
are odd and no matter what way you lay this first domino, they're gonna cover the border
but then you're gonna have one square left in the middle.
R: Okay.
Lily: And you need two of 'em. [labels the 3x3 case, pause (15 sec)] I guess the smallest
one where this would actually work would be 1-by-2 or if you just lay the domino across,
it's gonna cover it. [draws 1x2 shape and shades in, pause (50 sec)] Or I guess what's
kind of stumping me is that I know it works, it does not work for a 3-by-3 and I know it
works for a 1-by-2. But I guess the hard part for me is kind of like proving it for the
generic, just in m-by-n matrices.
R: Okay.
Lily: Because it’s impossible to prove like all the [laughter] combinations. [pause (40 sec)]

R: What are you thinking?

Lily: Um, I was just kinda in my head thinking like of possi-, the possibilities just like of the different matrices.

R: Okay.

Lily: With, with um, just like with $m$-by-$n$, with both of them being odd like in this example [points to the 3x3 case] or of one being odd or one being even [points to the 1x2 case] or both of ‘em being even. And I’m kind of thinking like of the smaller ones.

R: Okay.

Lily: Just because it’s easier to visualize.

R: Kind of drawing a picture in your head?

Lily: Yeah.

R: Okay. [pause (10 sec)] And you’re checking that this statement is true –

Lily: Mm, hmm, yeah.

R: – for those things that you can, finding covers of those –

Lily: Yeah.

R: – that are supposed to work.

Lily: [pause (35 sec)] And I guess another thing that I’m just kind of wondering about is the statement if it’s easier to just prove it directly or maybe try something like the converse or try contradiction –

R: Okay.

Lily: – or some other way.

R: [pause] So, can you show me how you wanna proceed from there?

Lily: [laughing] I guess I don’t know how I want to proceed.

R: Mm, hmm. So, you’re convinced that it’s true?
Lily: Yeah.
R: Right?
Lily: Yes.
R: Okay. Can you say maybe why it’s not true for the odd-by-odd case?
Lily: Um, [pause] I guess because a 3-by-3 you’re gonna have nine squares [writes =9
next to the 3x3 label]. But, when you have the domino [draws one domino to the side],
since they’re divided into two squares, you’re always gonna have an even number.
R: Okay.
Lily: So I guess that’s why it, I guess that’s why you have to have one of those be an
even number.
R: Okay.
Lily: Either the row or the columns. Because with the dominoes, you’re always gonna
have an even number.
R: So you’re saying you want the product of the two things to be even –
Lily: Mm, hmm.
R: – cause that’s the only way that you can get it to cover the dominoes, right?
Lily: Yeah. Right.
R: And so that means for sure that one of ‘em has to be even.
Lily: Yeah.
R: Okay.
Lily: [pause, labels the 1x2 case (15 sec)]
R: What else needs to be said to prove that there is a perfect cover if and only if at least
one of $m$ or $n$ is even?
Lily: [pause (15 sec)] Like, what are you asking then?
R: So you've, you've said and we haven’t written down formally or anything, but I think
you could probably do it, to say that um, if they’re both odd it can’t work.
Lily: Mm, hmm.

R: Because their product is odd and so they can’t be covered.

Lily: Yeah.

R: Right? So does that prove the entire statement? [pause] So, you’ve shown that if they’re both odd, it cannot have a perfect cover.

Lily: [pause (10 sec)] You mean would I have to show too that if, if one of ‘em’s even then it can have a perfect cover?

R: Okay. How would you go about doing that?

Lily: I mean cause, since this is an if and only if statement, we have to prove it both ways.

R: Yeah. So, how would you show that part?

Lily: Um, I guess because we know, since the domino’s divided into two, it’s always gonna have to cover an even number of squares.

R: Okay.

Lily: A product of two, um, two even numbers is always gonna be an even number. And the product of an odd and an even number is always gonna be an even number.

R: Okay.

Lily: But I don’t know if that’s the best way to prove it.

R: It certainly shows that it’s possible.

Lily: Mm, hmm.

R: Right?

Lily: [writes \( m \times n \)] So I guess you would almost need to divide this into cases.

R: Okay.

Lily: [writes case1: one odd and one even, and case2: both even, proceeds to work on case 2. Writes \( 2k(2l) = 4(kl) = 2(2kl) \) (40 sec)] And then by definition, this one is also gonna be even if they’re both even. [Goes back to case 1, writes \( 2k+1, 2l \), then below]
writes \((2l)(2k+1) = 4lk + 2l = 2(2lk+l)\) (35 sec)] Then by definition that is even, so it would be even.

R: Okay. Do you have an idea of a picture of how these would be laid out in those circumstances?

Lily: Um – [pause]

R: So do you have a picture in your head of what an even by an even would look like and how you could lay out the dominoes?

Lily: [pause] Well, if it’s an even-by-even, you should always be able to lay the dominoes going like this way [draws a domino horizontally] because you’re always gonna have an even number.

R: Okay.

Lily: And so then – I mean you can put as many here as you want for it to be even as well. [draws more dominoes below the last one in a column]

R: Okay.

Lily: And I guess and even by an odd, you can lay ‘em out the same way [starts drawing dominoes horizontally], but then this isn’t gonna matter, like however many you build up. [draws two more dominoes below, making 3 rows]

R: Okay.

Lily: So I guess, basically, on this the, the column number isn't gonna matter because you would have it be either even or odd.

R: Okay. And if we had, say, 4 instead of 2 across, how would you –

Lily: Then you could just add a whole ‘nother domino here [adds domino beside the stack of three].

R: Okay. Okay, so you think you could write that up formally and –

Lily: Yeah, I guess I have –

R: And have you proven what you intended to prove?

Lily: I think we’ve proven it, my problem is just like writing it up in the exact wording.

R: Okay.
Lily: I mean because we’ve already proved that it can't be odd because there is gonna be a leftover square, since the dominoes have an even number of squares. And I guess, basically, what I’m trying to say here is that depending on which way you lay ‘em, cause I mean you can also lay ‘em vertically, and that’s not gonna matter either, just depends. [draws another two dominoes to the side, stacked vertically] But, either the row or the column, one of those two has to be even.

R: And so you can make your pattern work?

Lily: Mm, hmm, yeah.

R: Okay, alright. You feel good about it at that point?

Lily: Mm, hmm.

R: Okay. Can you, um, identify any strategies that you think you may have used along the way?

Lily: Well, I think the first part I like had to prove the problem to myself.

R: Okay.

Lily: And I think drawing the picture and just like seeing it visually why it would work and why it wouldn’t work helped. [pause] And then I guess you kind of have to think about how many total squares there are in like an odd-by-odd matrices or even-by-odd or an even-by-even one.

R: Okay.

Lily: Whether the product is even or odd.

R: Have you seen this anywhere before?

Lily: {No}

R: Okay. And has anything you’ve ever seen before, um, directed your path in what you did here?

Lily: Um, I think just the kind of things that we’ve been doing in abstract math with um even and odd integers, just setting em up like 2k, for an even versus like 2k +1 for an odd.

R: Okay.

Lily: I think that helped a lot, otherwise I wouldn’t have known where to go with this one.
R: Okay. And trying cases and stuff like that?

Lily: Mm, hmm, yeah. Yeah, and that’s something else we’ve worked on.

R: Okay, alright, that sounds good. Let's try another one. [researcher collects papers, gives student next question] So, let’s see, hang on a second. We'll just rip off this one.

Let me have you try this one. So, let $x$ and $y$ be two integers. We say that $x$ divides $y$ if there is an integer $k$ such that $y$ is equal to $kx$. Consider the integers $a$, $b$, and $c$. Prove the following. If $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$. [pause (10 sec)] Do you have any questions about the problem?

R: No.

R: Okay.

Lily: [writes “there exists $m$ such that $b = ma$, $b/a = m$” (30 sec)] I guess I'm just showing that, oh wait, I did that backwards. [starts to erase writing, but stops and re-reads her work, doesn't erase]

R: So you're starting at looking at $a$ divides $b$?

Lily: Yeah. [writes “there exists $n$ such that $c = nb$, $c/b = n$” (30 sec)] And $b$ divides $c$.

[pause (30 sec)]

R: What are you thinking?

Lily: Um, I’m just kinda trying to figure the possibilities, like with actual integers but –

R: Okay. [pause] So can you write out what you were trying?

Lily: Yeah. [pause (5 sec)] 3. [On question statement, writes in 3 for $a$, 12 for $b$, and 24 for $c$, pause, writes “$12/3 = 4$, $24/12 = 2$, then $24/3 = 8$” (35 sec)] I guess I’m just trying to think of examples where it works out.

R: Okay.

Lily: [pause (30 sec)] But I guess I’m try – right now I’m just kind of thinking how to prove this in like a general form.

R: Okay.

Lily: Because once again you can't go through all – [laughing] every possible number. [pause (25 sec)] And I guess, since this is a 12 and it can divide 24. I mean, you could also do something like with prime factorization, that’s kinda what – [pause (15 sec)] I guess, basically, what I’m thinking is that since 3 can divide 12 and you know that 12 can
divide 24 and this [points to the 12] can be divided by 3, if this comes out [points to the
24/12], then obviously 24 can be divided by 3 since it can be divided by 12 –
R: Okay.
Lily: – which was divided by 3. [pause, looks over what is written on separate paper (55
sec)]
R: So, what are you thinking now? You are looking at your generic forms –
Lily: Um, yeah. I was just kind of thinking how the generic forms relate back to the
actual integers that I plugged in.
R: Okay.

Lily: [pause, writes in “12 = 4(3)” and “12/3 = 4” next to generic forms on separate
sheet, proceeds to write in “24 =2(12)”, “24/12 = 2”, audiotape side one ends during this
time, researcher flips tape over (45 sec)] So, if this – [writes “if” before there exists m
statement] and – that [writes “and” before there exists n – and writes “then” below (15
sec)] I guess – [writes there exists y such that c/a = y (40 sec)] I guess I’m just kinda
having problems like relating the general form back to the actual integers.
R: Okay. [pause (20)] What are you thinking about them?
Lily: Um, I’m kind of just thinking like if it would help to like substitute anything in for
these to get further. [points to the c/a=y]
R: Okay. [pause (5 sec)]
Lily: I mean just to prove like that they have a common factor.
R: Mm, hmm. [pause (15 sec)] What might you try?
Lily: Um – I was trying to do something like with like this 24 on top, this c, to maybe
substituting this in – to try to pull out like the b, ‘cause I mean both of ‘em, both of them
have that in common or –
R: Okay.
Lily: [pause] So, maybe I’ll just write that down just to see where it gets me [writes y =
c/a = nb/ b/m = (nb)*m/b = mnb / b (35 sec)]. Well, I guess by that, we can see that they
have the b’s in common.
R: Okay.
Lily: Which means they’re both divisible by $b$, which is just an integer. [pause, writes $mn$ (10 sec)]

R: What does that tell you?

Lily: [pause (10 sec)] By that you know that it can be simplified and I mean both the, the um numerator and the denominator have a common factor.

R: Okay. [pause] And what do you have left? $mn$ is what you’ve written, right?

Lily: Mm, hmm.

R: What does that $mn$ tell you?

Lily: [pause (5 sec)] That’s the um, [pause] that’s just those two products, it’s a product of [pause (5 sec)] the integers of what they were divided. [pause (10 sec)]

R: Can you conclude anything from that or were you, would you try something else?

Lily: I don’t know, I’m kind of stumped right now. I don’t think this, like this didn’t get me where I wanted to go. But I don’t know if this part’s helpful too.

R: Okay. [pause (10 sec)] What are you thinking about what you see there?

Lily: Um, I guess I’m kind of just trying to figure out like exactly what this means [points to the $mn$].

R: Okay. [pause (15 sec)]

Lily: Oh, I guess then, if I go back and substitute these in $b$ over $a$ – times – $c$ over $b$. [writes in $= b/a \times c/b = bc/ab$] And the $b$ cancels out. [crosses out the $b$, writes $= c/a$] I guess that’s what we were trying to prove. [pause (15 sec)] I guess that proves that $c$ can be divided by $a$. Well, [pause (25 sec)]

R: Any other thoughts about it?

Lily: No.

R: Okay. [pause] Are you convinced that it’s true?

Lily: Mm, hmm.

R: And that you could do something with it.

Lily: Yeah, it would just take a little while, I think.
R: And maybe where you’re going is –
Lily: Yeah.
R: – is possibly a direction to head?
Lily: Yeah.
R: Alright, how about any strategies you’ve used in trying to solve this problem?
Lily: I mean, kind of just like proofs, like showing that there, there exists something. I mean kind of like the just substituting in –
R: Okay.
Lily: – what we have, what we know, for sure. I mean cause we know these two are true [points to the statements about existence of m and n]. Then we have to prove this [points to statement about existence of y].
R: And in the process you looked at an example –
Lily: Mm, hmm.
R: – right? To try to relate these –
Lily: Yeah.
R: – kind of convince yourself that it was true –
Lily: Yeah.
R: – maybe see where to head?
Lily: Yeah. And that’s my first step I usually have to take is just proving something is true.
R: Okay, alright. Um, have you seen anything like this before?
Lily: {No}
R: Okay. And did you use anything um from abstract or anywhere else that you may have seen before?
Lily: Um, I mean, just kind of like the basic concepts of proofs and like substitution.
R: Okay. [pause] Okay, anything else that you can add to that?
778
779  Lily: No. Not really.
780
781  R: Okay. ***End of Interview.
Interview #7  (Total time = 58:23)

R: Let’s start, actually, at Question number 2.

Sam: Alright.

R: Okay, we call a positive integer, \(N\), a 4-flip if 4 times \(N\) has the same digits as \(N\), but in reverse order.

Sam: Okay.

R: Okay. Does that make sense; do you have questions with that?

Sam: No.

R: Okay. Prove that there are no two-digit 4-flips.

Sam: Okay. [pause, reads the question, (10 sec)] So, what I would so is, I probably, I don’t know if I’d wa – tell me to write stuff down when I’m doing it. But um [pause], two-digit 4-flips would mean that one, there’s only so many two, ah, two-digit numbers that are divisible by 4. Um, and of those, [pause (10 sec)] let’s see, so we have a finite set, obviously of, of two-digit numbers that we’re working with. Um – actually there’s 25 of them.

R: Okay.

Sam: So, ah, first I would work through, or try and figure it out, knowing there’s only 25.

R: Okay.

Sam: I might just, if I got frustrated, –

R: Okay.

Sam: – just start chugging away –

R: Okay.

Sam: – and seeing what comes up. Umm, actually there’s only 23 of them because the first – let’s see here, there’s probably even fewer than that I’m thinking. There’s 25 numbers that are divisible by 4, but \(N\) needs to be a two-digit number as well. So, 10 would be the first one.

R: Okay.
Sam: So, 40 would be the first number that would be, that would have a chance of being a 4-flip. Um –

R: And we actually call N the 4-flip, so like –

Sam: Okay, in that case 10 would be the 4-flip.

R: – 10 would be called – Yeah.

Sam: So, at that point, given that there is even fewer than I thought before, I would probably just start chugging away.

R: Okay, go ahead. I can –

Sam: So. Okay, so.

R: You can certainly do the smaller ones by hand. [researcher gets out calculator]

Sam: Yeah.

R: But if you needed it, it's there, a calculator.

Sam: [pause, writes 10(4) = 40, 11(4) = 44, (5 sec)]

R: And so you know –

Sam: So, it would, so actually our numbers will go 10 through 99.

R: Okay.

Sam: Is that what the question is asking, I assume then?

R: Yeah.

Sam: ‘Cause we were talking about N being the 4-flip.

R: Right.

Sam: As you said

R: Mm, hmm.

Sam: So there’s actually more than, than I thought. [pause (5 sec)] So, it would probably be easier to try and figure it out then to do 8, 9 different, different ones. [pause, writes 10, 99, (15 sec)] So between 10 and 99, those are the numbers [pause] possible for a 4-flip; those are the N’s, right?
R: Mm, hmm.

Sam: And – [pause, writes 10*4 = 40, 99*4 = 396, uses calculator, (20 sec)] So, we can’t use these, so the highest one we can use would be 25. [writes 25*4] No, 24. [crosses out 25, writes 24*4 = 96, crosses out 99*4 = 396, pause, (10 sec)] Knowing that, I probably would just start cruising’ through ’em.

R: Okay, go ahead.

Sam: Okay. [pause (5 sec)] Obviously – [writes 10*4 = 40, 11*4 = 40, 12*4 = 48, lists 13, 14, 15, (10 sec)] So, none of the, none of 10 through 19 [writes 10, 19] can work, ‘cause 4 times anything would be even. None of 30 through 39 would work, [writes 30 39] well that, we decided just to 25.[crosses out 30, 39]

R: Okay, so you're reasoning was because you have to flip em and the flips would be -

Sam: – ‘Cause you have to flip em and the flips would be odd –

R: Okay.

Sam: We can’t use them.

R: Okay.

Sam: Well, no, I think that's true – yes. So, we have 20, 21, 22, 23 and 24 to check. [pause, lists 20, 21, 22, 23, 24, checks each times 4, (5 sec)] And they’re all gonna be 80 and above. Or 80, 84, 88, – 96, 92. So, none of those, none of those are flips. And those are the only possibilities of flips.

R: Okay.

Sam: So, there are no two-digit 4-flips.

R: Okay, so to recap what you did, you first um, you tried a few different things, but eventually you got to the point where you said, well it has to be less than 24 because otherwise it won’t be two digits anymore –

Sam: Mm, hmm.

R: – so then you have 10 through 24, you eliminated anything that started with a 1 ‘cause when you flip it, it ends with a 1 –

Sam: It ends with a 1 –

R: – it ends with a 1 and that can’t happen when you multiply by 4 –
Sam: Mm, hmm.
R: And then you just tried 20 through 24 and none of them –
Sam: – And none of them worked, so, numbers 1 through 9 don’t work because they
won’t be two-digits. Numbers above 24 won’t work because their answer will be in
three-digits –
R: Okay.
Sam: So, we’re left with those and numbers 10 through 19 would be odd when we
flipped.
R: Okay. Alright, sounds good. Um, can you identify any strategies you may have used
along the way?
Sam: Um, initially I wanted to decrease the number of possible choices; I misunderstood
the question, originally, thinking that the flip was the answer and not N.
R: Okay.
Sam: Um, so I went astray a bit there. Um, then I thought the choices were 10 through
99, not real, so my strategy was to minimize the number of choices that there were –
R: Okay.
Sam: – so that whether or not I used any kind of [pause] um, number theory or anything
like that, I would maybe make it a reasonable number of um, answers that I could do –
R: – Okay –
Sam: – just brute force at the end. [points to where he computed 20, 21, etc.] Um, I
realize when multiplying 99 times 4 that the answer’s three digits, which tipped me off to
the fact that 24 would be the highest N that you could use to get a two-digit 4-flip.
R: Okay.
Sam: Um, and then, as I started, with, and then, at that point I was figuring 4, 15 was not
too bad to go through.
R: Mm, hmm.
Sam: And as I started doing them, it didn’t make sense that ‘cause we were in the 40s, I
was looking at the flips here [points to where he computed 10, 11, and 12*4] and
realizing that, should have known before, but that anything multiplied, the answer would
have to be, the $N$ would have to be even if we’re flipping because our answer, no matter what we multiply by 4, is gonna be even.

R: Okay.

Sam: So, whenever we flip the teens, um, we would get an odd.

R: Okay.

Sam: Which got me down to five answers.

R: Have you, have you seen anything like this before?

Sam: Um, I couldn’t, I don’t think so. Probably something similar but I couldn’t recall it off the top of my head.

R: Okay. Um, is there anything that you’ve done before that may have given you hints as to what to do in this problem?

Sam: Well, I would say, if anything um, understanding the problem first. Um, once I understood that and then making it, for me, I’m not very strong on the theory part of stuff, so when that breaks down, if it’s within reason to try by brute force, I’ll do it.

R: Okay.

Sam: Um, so initially I wanted to see you know just my thoughts that would make this an easier problem. Like I said, where minimize the number of chances that – of the number of possibilities for $N$ that would even be possible –

R: Okay.

Sam: – to make a two-digit flip. Um, but yeah.

R: Okay.

Sam: Nothing, I mean just the process of trying to simplify a problem by understanding it first so –

R: Okay.

Sam: – as I walked through, I understood it more.

R: And where did learn, or pick that up along the way? Do you remember?

Sam: I would say mostly in my college math classes from 305 and on.
R: Okay.

Sam: ‘Cause I was not exposed to proof-based math in high school –

R: Okay.

Sam: or in my undergraduate college, which is not in math. Um, so, I’ve learned a lot since I’ve taken 305 about taking problems like this where you might not know how to solve, but you can try some things to –

R: Okay.

Sam: – at least give it a shot.

R: Great. Alright, let’s work on one that’s sort of related to this, well, that’s very related to this. [papers from first question collected, next question given to student]

Sam: [reads the question] So, same thing as before, prove or disprove the following, there are no three-digit 4-flips. Okay. Well, knowing what we’ve done before, I know that our initial number, our $N$, can only, will only, the first thing I would do is say, it can only be 100 through 999. [writes $N$ 100 → 999] And then I would say um, 4 times $N$ equals 999 to see what the highest value of $N$ we could get. [writes $4*N = 999$]

R: Okay.

Sam: [pause, uses calculator to compute what $N$ is] so 249, which makes sense, so it’s actually to 249 [crosses out 999, writes 249].

R: Okay.

Sam: Now, once again, we can take out all the 100s [writes 100 and strikes through it] so we’re left with $N$ from 200 to 249 [writes 200 → 249]. Um [pause (5 sec)] and in fact, we need to have at least two, we can take out 200, all the 201s, no, we can take out 200 out of that. 210, 220, 230, and 240 [writes 200, 210, etc and strikes them out] because they don’t make a three-digit flip ‘cause of the 0’s.

R: Okay, mm, hmm.

Sam: [pause (10 sec)] And then, um, I’d probably, [pause (5 sec)] well probably the easiest thing to do would be [pause, figuring, in head not on paper, (20 sec)] Obviously, one way to do it would be to go through it by brute force. I’m trying to figure out if I can think of a way to not have to do that.

R: Okay.
Sam: [pause (15 sec)] Well, from 200 through 225 or 224 [pause, writes 200, 224 = 8], we’re gonna end the beginning number would be 8. [pause, figuring on calculator, (40 sec)] So, for 200 through 224 [pause, writes 200 = 800, 224 = 896, (5 sec)] the beginning number would be 8, so when we flip it, the last number would be 8 –

R: Okay.

Sam: – for all of those numbers. So, we would only have to look at um, let's think here [pause (20 sec)] the numbers that finished in 8 here [writes 208 and 218 between 200 and 224 previously written, pause, (5 sec)] ‘cause if they flipped they’d have to be 8.

R: Okay.

Sam: So, there’s only two, and 208 times 4. [pause, uses calculator, (10 sec)] does not equal 802 [writes *4 ≠ 802, beside 208] and 218 times 4 ah, does not equal 812 [writes *4 = 812, beside 218]. So, those two are out.

R: Okay.

Sam: Above that, from 225 to 250 [writes 225 and 250], they’ll end in 9 or it’ll begin in 9 [writes 225(4) = 900]. Well, we only go to 249 so [pause, changes 250 to 249, uses calculator, adds (4) = 996, (15 sec)] So, when they flip, they’ll be odd, so none of those numbers can be, can uh, [pause (10 sec)] same as before.

R: Okay.

Sam: [pause, looking over work, (20 sec)] Well, when they flip, they’ll end in 9 and none of the answers between there will end in 9. So, none of the numbers from 225, none of N from 225 to 249 will, can equal an odd number when they’re multiplied by 4.

R: Okay.

Sam: So, the numbers 225 through 249 [writes 225 – 249 →9] when multiplied by 4 create a number that begins with 9 and then that’s flipped. [draws arrow for flipping] Okay, so that doesn’t matter. [pause] It creates a number that ends with 9, so we just need to check the numbers that are betw – that have a 9 on the end. 229, 239, 249 [pause, writes 229, 239, 249] 229 times 4 [uses calculator, writes (4) = 916], 916, and we're gonna get the same [using calculator, writes (4) = 956 and (4) = 996, beside where he has written 239 and 249, respectively, pause, (20 sec)].

R: Okay.

Sam: So, that’s how I would do it.

R: So, can you recap your answer?
Sam: Yeah, so like part a), I wanted to minimize the numbers that I was dealing with –
R: Okay.
Sam: So, I realized that to get a three-digit answer, well \( N \) had to be three digits for it to be a flip, and the answer of 4 times \( N \) had to be three digits, as well, so that limited us to 100 through 249, ah ‘cause 250 times 4 is a thousand, which is not a three-digit number anymore.
R: Okay.
Sam: Um, let me think about this here – 100 through 199, when flipped, if \( N \) is 100 through 199, when flipped will end in um a, oh, will end in a 1 and none of those numbers when multiplied by 4 will can end in a 1 ‘cause it will always be even –
R: Okay.
Sam: – so 100 through 199 were canceled. That left us with 200 through 249. Um, all the numbers between there that ended in 0 are canceled, although you don’t necessarily need to do that. Ah, the numbers 200 through 224, when multiplied by 4, create a 800 number.
R: Okay.
Sam: 800 somethin’ somethin’ – um so we only have to check the numbers in that range that end in 8. Um, because when those are flipped, they’ll begin in 8 –
R: Okay.
Sam: – creating that requirement. And the only two numbers in that range are 208 and 218 and their answers did not create a three-digit flip.
R: Okay.
Sam: Um, similarly with 225 through 249, multiplying any of those numbers by 4 we get a 900 answer. So, when we flip our 4 times \( N \), we’ll get 9 as our last digit so we only have to check the numbers between 225 and 249 that end in 9, which are 229, 239 and 249 and none of those numbers um, – fit the requirement on it.
R: Okay. So, pretty much building off the first problem – right?
Sam: Mm, hmm.
R: Okay, alright one more similar to that. [papers for this question set aside, new question given to student] Okay. Prove that the number \( N \) equals 2178 is the only four-digit 3-flip, or 4-flip, sorry.
Sam: Okay. First thing I would do is, that I’ll do is prove that 2178 is a four-digit flip, so 2178 times 4 equals 8712 [writes 2178(4) = 8712, using calculator to check], which is a flip, so check [writes check mark by previous calculation]. Um [pause] once again, with four digits, we’re gonna, because it’s worked before, I’ll probably just keep using it.

R: Okay.

Sam: Um, you have to have at least a four-digit number to start off with, which would start with 1000 and end with 999, or 9999. [writes 1000 → 9999] Um, however, when multiplied by 4, it also has to be a four-digit number. Um, so we can, ah, limit it to 2 4, um, what would it be? [pause, uses calculator, (5 sec)] 2499 [writes 2499, crosses out 9999] are the beginning possibilities that I would take a look at.

R: Okay.

Sam: So, we’ve narrowed it to a certain number. Um, once again 1000 through um 1999 [writes 1000 → 1999] won’t work because it began in 1 and when flipped [pause, adds to above “= when flipped end in 1(odd)”] end in 1.

R: Okay.

Sam: Which is odd, so it won’t work. So, we’re left with 2000 through 2499 [writes 2000 → 2499]. Um, 2000 through [pause, writes 2000, uses calculator, writes “-2499”, (5 sec)] 2249 [uses calculator, adds to -2249 “when x 4 8000”], um, when multiplied by 4 create a 8000 number.

R: Okay.

Sam: Obviously in this case there is a lot more choices to choose from. But it limits us at least to those numbers that between here and here [highlighting where he wrote 2000 – 2499, he adds “end in 8”], that end in 8 and then we’re left with [pause, uses calculator, (5 sec)] 2250 to 2499 [writes 2250 – 2499], which would be um [pause], which will give us numbers that end in 9 [pause, writes next to 2499, “x 4 end 9”, (5 sec)] so we only have to look at those numbers that end in 9 [under 2250-2499 adds “end in 9”].

R: Okay.

Sam: Now, in this particular case, there’s a lot more of them then there were with the three-digits so we’re gonna have to try something’ else. Um, but we’ve limited it at least to a more do-able number than 1500. [pause (25 sec)] So, with the 2000s when we flip them, they’re gonna end in 2. [writes “2000 → end in 2”] Or 2000 to, well with any of them, they'll end in 2. Or wait, when you flip them. So, [pause, writes “x 4 = 2”, (5 sec)] when multiplied by 4 the only ending digits [pause, writes “ending digits”, (5 sec)] that would end in 2 [pause] would be [pause] 3 times 4 is 12, and 8 times 4 is 32 [writes 3 x 4 = 12 and 8 x 4 = 32, pause]. So, since we’re in the 2000s and when we flip we end in 2
and when we multiply by 4 we want our nu-, our last digit to be 2. Um, 3, the last digits of 3 and 8 are the only ones that will work. Since, 1 times 4 is 4, 2 times 4 is 8, the only 2 between 0 and 9 that will end in 2 will be ah 8 and 3. So, none of these would work [referring to 2250-2499 ending in 9, crosses this out]. Because they, we were just gonna look at the ones that ended in 9.

R: Okay.

Sam: So, we’ve narrowed it down to the numbers the 250 numbers there between 2000 and 2499 that end in 8.

R: Okay.

Sam: Ah, that will allow us to do that, and 2178's one of them, so one less. [laughter]

R: [laughter]

Sam: Um. [pause (15 sec)] Okay. [pause] At this point you could, if you wanted to spend a lot of time doing it, use brute force; I’m not gonna do that. So, I would start thinking about what the numbers looked like when you multiplied them by 4.

R: Okay.

Sam: So obvious, um, 2 oh, 2008 times 4 is 8032 [pause, writes 2008(4) = 8032, (5 sec)] um, which isn’t correct, but it’s fairly close, as far as this being 2308, so we’re 300 off and I would see if it got any closer by increasing by 10 [writes 2018(4)] . Which it probably shouldn't [pause, uses calculator, writes = 8072, (10 sec)] 8 0 7 2 [pause]. So, now we’re a lot farther off so I would say that [pause, writes 2098(4) = 8392, (15 sec)] after the next one you won’t have a 0 anymore so none of those will work from 2019 times 4 [writes 2019(4) = 8076, pause, (10 sec)] well, once you got up to that certain point [pause (5 sec)] What did I do there? Sorry, 2028 [crosses out 2019 and 8076, uses calculator, puts in 2028(4) = 8112, (5 sec)]. So, none of 2000, none of the 2000s will work.

R: Okay.

Sam: Step two, check and see what happens with the 100s when we do it.

R: Okay.

Sam: So, 2108 is once again [pause, writes 2108(4) = 8432] it’s gonna be larger than 8012. So, that won’t work [pause, writes 2108 = 8012, (15 sec)] So, already by the time we get to 2098 [circles where he wrote 2098 above] and multiply it by 4, 8000-, we’ve come to the point where 8392 is our total and the highest we’re going is 2498 [writes 2498] for the numbers we’re looking at. Um, which the answer would be 8924 [writes 8924]. So we can disclude anything below that or – [pause, uses calculator, writes
divided by 4 under the 8924, (10 sec) 8924 [pause (5 sec)] 42 [replaces 24 with 42 in 8924, pause, uses calculator, writes that 8942/4 = 2235, (5 sec)], 2235. So, that would be the largest, um, number that we could have.

R: Okay.

Sam: Because multiplying that by 4 um, is the last number that will give us a number smaller than or equal to 8942, which is the largest number we can have because 2498 is the largest N that we could have.

R: Okay.

Sam: Okay, so now that leaves us, let’s see here again [pause (15 sec)]. So it’s, if there was one, it would be ending in 8 between 2098, or 2108 [writes 2098, scratches out, puts 2108 and 2228, (5 sec)] and 2228.

R: Okay.

Sam: Um – [pause (15 sec)] And any of the [pause (15 sec)] and actually, it would have, in this particular case [pointing to 2098(4) = 8392], 2938 – How do I put this in words? 2098 times 4 equals 8392 and every time we increase a number, this number is gonna increase. So, this number is the lowest number and when we flip it, that we know right now and when we flip it becomes 2938?

R: Okay.

Sam: And 2938 is outside of our range of possible numbers that we’ve already figured out. So, none of the numbers in between there should work.

R: Okay.

Sam: So, I think that should do it, if I was to write it down, neatly.

R: Okay, so can you recap what you’ve done?

Sam: Yeah. First, I checked to make sure that it truly was a four-digit flip [pointing to where he checked 2178(4) = 8712], which it is. Um, similarly to the previous two problems.

R: [video tape ends, researcher changes tape] This thing is really going to tick me off, I can tell you that much right now. [laughter, referring to video camera] Alright, then stop. [referring to video camera still] Go ahead and explain it to me.

Sam: Okay, I limited to the numbers that we knew at least having a chance of being a flip. Meaning N is four digits and 4 times N is four digits.
R: Okay.

Sam: Which limited us to 1000, from \( N \) equals 1000 up to 2499. Ah, we took out all the numbers that are in the 1000 range, to 1999 because they’ll end in 1 when they’re flipped and any \( N \) times 4 will be even. So, um, we got rid of 1000 numbers there which limited us to 2000 times or up to 2499. Um, the next thing I did was realize that when multiplying the numbers 2000 through 2249 by 4, we get a number that’s in the 8000 range.

R: Okay.

Sam: So, any number has to end in 8 for that range to, um, for it to be possible to have a flip.

R: Okay.

Sam: Similarly, the numbers 2250 through 2499 will be in the 9, their answer times 4 will be in the 9000 range. So, we only had to look at the numbers that ended in 9, in that range. Um, [pause (5 sec), videotape resumes] um, because our number begins with 2, is in the 2000 to 2499 range, we know that it would have to end in 2, that the result of multiplying by 4 would have to end in 2. And the only possible numbers to multiply, last digit to multiply by 4, that would give us a number that ended in 2, were 8 and 3. So, we could discount all the numbers that ended in 9. We’ve already discounted all the numbers that ended in 3. Um, so we ah chose to, initially I chose to look at the numbers that ended in 8, ah, in the range 2000 to 2249. Um, so I began by just taking a look and seeing what the patterns were of the numbers as we began with 2008 and moved up. Um, and noticed that um, after the 2008 and 2018, ah the answer of \( N \) times 4 was greater than ah, 8100. So, that 100s digit, which would become [pause (10 sec)] So, that was a little erroneous, my assumption there, but, I assumed before that because of the 100 there was only two, well, because there was only two numbers that had the 0 in the 100s digit that we could discount all the numbers from 2000 to 2100 [pointing to his work on 2008 up to 2098] because of the 0, but that’s necessarily the case. Because that would be the 10s digit, not the 100s digit. Um [pause] however, all we would have to figure out is we would have to – going back and looking at it now, I might begin at 2008 [writes 2008] and find an answer until I, to see if there was one –

R: Mm, hmm.

Sam: – until I got to an answer that was, since our highest number was 2 4, or actually was 2249, or 2248, [writes 2248] actually ‘cause it ends in 8. The largest result that we could have, when multiply \( N \) times 4 would be 8422 [writes \( N(4) = 8422 \), pause, (5 sec)] and you could, and I would notice, I’ve noticed that [pause (15 sec)] What did I do there? [pause] 2248 would be 8422. So, um I noticed that 2098 multiplied by 4 is 8392 [pointing to where he calculated 2098*4] 2108 times 4 equals – [writes down 2108(4), uses calculator, writes = 8432] 8432, which is higher than the number we had. So, I did
not compute a few here [pointing to where he calculated 2008 to 2098]. I would have to
compute them to, to see.

R: Okay. We don’t have that many possibilities to compute.

Sam: No, but now we’ve come down to all I would have to do is divide 2248 by 4 [going
back to where he wrote 2248 in the middle of the page, he writes divided by 4, pause] to
get the [uses calculator, pause]. No, 8422, sorry. [pause (5 sec)] To get the highest
number that could be used –

R: Okay.

Sam: – in order to find that which is 2105, meaning obviously 2208’s gonna be above it,
so 2098 would be the largest and 2008 would be the smallest [pointing to his calculations
of 2008 and 2098]. So, we’ve narrowed it down to 10 or to 10 different options of which
I’m comfortable plugging and chugging to see if it works.

R: Okay. Okay, anything new or different from the other ones?

Sam: Um, yeah, we have bigger numbers now, so and so in this particular case, I started
using what the beginning number was to limit what we had to look at [pointing to his
work to eliminate the 1000s at the beginning].

R: Okay.

Sam: Um, ‘cause when we multiply, if these are our only choices [points to his
beginning statements about 2000-2499] for the numbers that we can use, from 2000 to
2499 after I’ve discounted the 1000s, the number has to end in 2, which gives us only
three choices.

R: Okay.

Sam: Or two choices; 3 and 8, so –

R: – Okay, that was new compared to the rest of them. –

Sam: So, that was new, um, also um, realizing that it ended in an 8, we could discount
these and knowing that 2248 was the maximum that we could have, if I were to flip it
[pause (5 sec)] and divide by 4, I could figure out what the largest N is that I could use, so
we’ve, so –

R: Okay.

Sam: – limited it from 2000 to whatever that N is, which is another smaller number, in
this particular case.
R: Okay.
Sam: So, I didn’t do that before.
R: Okay.
Sam: I’m sure that there’s a way to have it done in a couple seconds, but I don’t have it [pushes paper aside].
R: Well, what would you be searching for otherwise? I mean if you’re not –
Sam: I mean that’s the, I would be searching for some sort of um, [pause] check. Some sort of theorem, some sort of postulate that I could come up with that I, that I could just um, prove on its own, separate from the numbers that would tell me that that was true.
R: Okay. Alright.
Sam: Which I’m not very comfortable with, which is why I do things the way I do.
R: What you do is fine, totally fine. Let’s see, I need to decide which other one to do. I think I’d like to do this first problem now. [researcher gives student next question] This will be the last one we tackle for the day.
Sam: Okay.
R: The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same.
Sam: Okay.
R: The diagram below shows an arrangement with sum 16, for an example. Prove that the smallest possible value for the sum is 14.
Sam: Okay. Well, I think I should know this ‘cause of the classes that I’m in. Um. [pause, reads the question, (10 sec)] Well, there’s three numbers. There’s 10 to, there’s 10 total numbers, I through 10 [writes “1-10”]. And three numbers have to be added together [writes 3, crosses it out], so a plus b plus c to equal a number [writes $a + b + c = 14$]. They say prove that the smallest value for this is 14, Okay. Um, [pause, reads question again, (10 sec)] so it’s just along the side, not side angle side or side vertice side that matters. So, I was gonna try something along those lines. Um, my guess would be we need to prove this by contradiction. By assuming that we could make it 13 or 14, 13 or less.
R: Okay.
Sam: And finding a contradiction –
R: Okay.

Sam: – involved there. So that’s probably how I would go about doing it. Um, [pause (5 sec)] so [pause, writes “Prove by contradiction”, (10 sec)]

R: Go ahead. [audiotape side one ends, researcher changes sides of cassette tape]

Sam: Okay, so, um the key would be to put together the right statement for what we’re doing here [points to the $a + b + c = 14$] so that we can take the contradiction, the contrapositive of that. Not contrapositive, but um [pause (15 sec)] Okay, so [pause (5 sec)] if we have – how I would think about it is we’ve got five different sums that we have to do and if, [writes 5] if we’re trying to prove the contradiction that the sum could be 13 or less, we would say that 5 times 13 equals [writes 5(13), uses calculator] 65 [writes = 65 next to 5(13)].

R: Okay.

Sam: So, what we need to do is show that no matter how we put it, the sum, [points to the pentagon given] the total sum of each of the sides [pause] probably whether or not they add up to the same or not is greater than 65.

R: Okay.

Sam: ‘Cause if the total sum is greater than 65, then no matter how we, how we arrange the numbers –

R: Okay, how might you go about proving that?

Sam: Well, the numbers at the vertices get counted twice. And the numbers on the edges get counted once. [pause (5 sec)] So [pause], just to check what I would do is take the, since they get counted twice, we’d want the smallest numbers counted twice and the largest numbers only counted once.

R: Okay.

Sam: So, the smallest numbers being 1 through 5 (throat clear) um, totaling, so I’d do 2 times 1 plus 2 plus 3 plus 4 plus 5 [writes $2(1+2+3+4+5)$], and then 1 would add 6 plus 7 plus 8 plus 9 plus 10 to that [additionally writes $+ 6+7+8+9+10$]. So, 12, 15 times 2 is 30 [writes 30 under the $2(1+2...)$] and 10 plus 9 plus 8 plus 7 plus 6 is 40 [uses calculator, writes 40 under the $6+7+...$], which equals 70 [adds = 70 under the above writing]. So, the minimum number, the minimum sum of adding up all the sides would be 70 and if we had five sides that sum 13, we would have a number smaller than that, so it’s not possible to do 13 or any number smaller.

R: Okay.
Sam: Now I can’t prove that, so it is possible [stresses the word possible] so it says prove that the smallest possible value for the sum is 14. So, the smallest possible value for the sum, ‘cause 14 times 5 is 70.

R: Okay.

Sam: But it is possible it’s not necessarily going to happen.

R: Okay, so how might you prove that it could happen?

Sam: Um, well I’d have to come up with [pause (5 sec)] in that case, [pause], we’d have to come up with an arrangement [pause (5 sec)] of numbers such that 2 times the sum of five of them [points to his calculations of 2(1+2...), pause, (5 sec)]

R: You think you could try to do that?

Sam: Um, plus the rest [pause (5 sec)].

R: So, can you use these ideas that you’ve had so far and see if you could come up with one?

Sam: Ah, yeah we could, I mean for me it probably would be beginning just by attempting to try some different combinations that would work.

R: Okay.

Sam: Um –

R: Go ahead, let’s see how far you can get.

Sam: Okay. I’m trying to think whether or not it would be valid to try and do it just by finding, putting numbers into this set [points again to his calculations of 2(1+2...)] and this set [points to 6+7+...] that equal 70.

R: Okay.

Sam: And if that’s the case, then trying to arrange them on the pentagon so that it worked.

R: And so, you have the numbers 1 through 5 with the 2 and the numbers 6 through 10 otherwise –

Sam: – Which equal 70. So, actually that’s the only way that it would work.

R: Okay.
Sam: So, we need to come up with – [pause, draws pentagon, (5 sec)] So, 1 2 3 4 5 are going to be on the-

R: Okay.

Sam: – on the vertices. The question is where they go. And we need each side to add up to 14. So, the smallest [pause (5 sec)] other one would be 5 or would be 2. We couldn’t use 5 or we could use 5 – [pause (10 sec)] Okay. [pause] So, if we did [pause] I don’t know why, but I think that I would probably try and match 10, the longest edge to 1.

R: Okay. – And what would that have to go with? –

Sam: – ‘Cause it is the smallest amount. – So, if we put 10 along one edge [writes in 1, 10, and 3 on one edge of the pentagon], it would have to go with 3. Um, [pause (15 sec)] So, probably I would just try one. A lot of times trying stuff for me leads me to at least some kind of generalization [writes 9 next to the 1] but if I try the next largest one, 9, we’d be left with 4 [writes 4 next to 9] and we’d be left with 2 and 3 to place here [points to the remaining two vertices]. If we place 2 here [points to the vertex closer to the 3 already placed], it’d have to be 9, so it can’t be 2 or a 5. Sorry. So, if we put 2 here [points to the other open vertex], it would have to be 8, which would work [writes in 2 and 8 along the side with 4]. So, for that part. And that leaves us with 5 [writes 5 on the last vertex], which means this has to be 7 [points to the spot between 5 and 3], wait, 14, this would be 6 but this [points to the spot between 2 and 5] would have to be 7 [adds up all sides to see they add to 14, (10 sec)].

R: Good.

Sam: So, there it is.

R: So, you found 14 being a possible, you also proved that 13 or less is not possible. So, you feel satisfied that you’ve proved the statement?

Sam: Yeah.

R: Okay.

Sam: The statement was prove that the smallest possible value for the sum is 14. To me, that statement, to me when I read that statement [pause (5 sec)] since it says possible, doesn’t necessarily mean that the sum of 14 is going to work.

R: Okay.

Sam: But that it could work given the, what we showed before.

R: Okay.
Sam: That the smallest, the smallest possible value for vertices and edges is 70. And 14 times 5 which is the sum of each of the edges, um, is 70 is well. Meaning it could be possible.

R: And that also led you to –

Sam: That led me to try this. I feel like I got lucky on a first attempt and the number of possible combinations of things is, is large enough and maybe it would work, you know. If I didn't put 10 and 9 there, if I used –

R: You’re 10 and 1 and 3 was forced. Um, but I think your placement of the 9 next was key in finding it so quickly. ‘Cause everything past that was forced for you but that was the choice to be made.

Sam: Yeah, yeah. But that’s what I mean like, I chose, at some point down the line, I chose something which worked –

R: Mm, hmm.

Sam: – if I hadn’t have chosen that, I might have been, oh I'd have to try – assuming 10 was correct, I would have to try – if I put 6 there and it didn’t work, I could try 7 there; if it didn’t work I could try 8 there and it didn’t work, I could try 9. But if 10 wasn't correct, then I would have to see that all four of those choices didn’t work.

R: Okay.

Sam: And then move on.

R: Have you seen anything like this before?

Sam: Yeah. Similarly, we just had a problem – how I got this was from a problem we did in another class.

R: Okay.

Sam: In discrete optimization doing the pigeon-hole principle. So, that’s how I knew, kind of, what to do there.

R: Excellent. Okay. ‘Cause that was a great insight into the problem. That’s great.

Sam: But beyond that beyond that I don’t, like I said, I probably would have quit given my – I probably won’t have tried this even –

R: Okay.
827 Sam: Given my understanding of –
828
829 R: Of the problem itself?
830
831 Sam: – of the statement. That was the case.
832
833 R: Okay that sounds good. *End of interview
Interview #8  (Total time = 51:10)

R: We’ll start with Question 2. Here’s your copy. We call the positive integer, $N$, a 4-flip if 4 times $N$ has the same digits of $N$ but in reverse order. Prove that there are no two-digit 4-flips. So, I’ll give you a second to read that.

Julie: [reads question, (10 sec)] Okay.

R: Do you think you understand what that means?

Julie: Yeah. Um. Positive integer. [writes “positive integer” and 4N, (15 sec)] Okay. So, that would be if – [pause (5 sec)] So, it would be like if you had 4 times 123 is equal to 321, right? [writes 4*123=321]

R: Right.

Julie: Okay.

R: Okay. So now the question is to prove that there are no two-digit 4-flips.

Julie: Okay. [pause (10 sec)] Well, there’s only 99 of them. [laughter] Go through ‘em. [pause, looks back at question, (15 sec)]

R: What might you want to try first?

Julie: Well, I guess I’ll start trying here. [writes 4*11=44, 4*15=60, 4*20=80, 4*25=100, draws line and writes “stops here”, (35 sec)] So, then we have 10 through 24 to work with. [writes 10-24]

R: Okay. So, you noticed that 25 got you three digits?

Julie: Yeah.

R: And so that was too big? Okay.

Julie: Yep. And [pause (5 sec)] let’s see. [pause, looking at her work, (10 sec)] So, it’d have to be something and it couldn’t be, couldn’t be 10 through [writes 10-19] 19 because they’re going to have to – oh wait. Oops, don’t know what I was thinking. [begins to erase 10-19] No erasing, just kidding. [laughter]

R: You can just cross it off.

Julie: Okay. [crosses off 10-19] 10 gonna give us 1, which isn’t going to work. [writes 10-01] So, it’s going to have to, have to – [pause] Can I just go through them?

R: Sure, you can do whatever you like.
Julie: Alright. [begins list, writing 11-11, 12-21 (48), 13-31 (52), grabs calculator and uses to do multiplications, then writes 14-41 (56), 15-51 (60), 16-61 (64), 17-71 (68), 18-81 (72), 19-91 (76), 20-02 (80), 21-12, 22-22, 23-32, 24-42, (1 min 40 sec)] These are increasing. [draws arrow to the right of the list, and writes incr.] So, that’s gonna be 84, 88, 92, 96. [writes in 84, 88, 92, 96 in their places on the list] So, that’s not going to work. 

R: None of them work.

Julie: Is that alright?

R: Mm, hmm, certainly.

Julie: Alright.

R: Okay. Um, can you tell me any strategies you used along the way?

Julie: Um, well, in the beginning I did a few examples. [draws line on the side of her examples 4*11=44, etc.] Just to organize my work, to see where I was gonna start, and after I did that, I eliminated a bunch that I wasn’t even thinking about in the beginning.

R: Okay.

Julie: And that’s when I narrowed it down to I guess 10 through, 10 through 24 and then [looks over work, (5 sec)] hmm, and then I just did it the long way.

R: Okay. Have you ever see anything like this before?

Julie: Um, probably, I can’t think of anything specifically though.

R: [laughter] And can you think of any, um, problems or a time that you may have used similar strategies that may have guided you in what you did?

Julie: [pause] Hmmm, I probably do it all the time without even thinking about it. Just guessing and checking a few to eliminate a bunch, and then narrowing it down from there.

R: Okay.

Julie: But I can’t really think of anything specifically.

R: Alright, let’s work on the next one. It has to do with this as well. I’ll just grab that piece of paper. [researcher collects work from previous question, gives student next question] So, the next statement is, thinking of 4-flips just as we have, prove or disprove the following statement. There are no three-digit 4-flips.
Julie: Okay. [reads the question, begins to write 3 dig. (10 sec)] So, it’s – [pause, writes 100-999, (10 sec)] Okay, it’s gonna be 100 through 333 [writes (100-333) with a star beside it], cause after that it gets too big. [begins list of examples, 4*100 = 400 (001), 4*101 = 404 (101), skips a line, then writes 4*123 = 492 (321), 4*125 = 500, (55 sec)] I can eliminate everything that ends in 5. [to the side, writes “elim. – everything ending in 5”, (15 sec)] Multiples of 5, I guess.

R: Okay. And why is that?

Julie: Hmm, cause when they are multiplied by 4 it’s gonna end in a 0, which would make it only two digits when we flip it.

R: Okay.

Julie: [writes “end in mult. of 10” under last statement, (10 sec)] And the same thing with multiples of 10. [writes “mult. of 10” under last statement, then continues list 4*126 = 504 (621), 4*127 = 508 (721), 4*128 = 512 (821), 4*129 = 516 (921), 4*221 = 884 (122), uses calculator for some multiplications during this time and pauses after each line before continuing, (1 min 50 sec)]

R: So, what are you looking for?

Julie: Um, a pattern.

R: Okay.

Julie: [pause, looking at her work, then draws line across page, pause, (50 sec)]

R: What are you thinking about?

Julie: Um, I was thinking of the last digit and what that has to do – how the number’s gonna start or how the numbers gonna end. [circles 8 of 128 in list and makes arrow to 821]

R: Okay.

Julie: To see if that’s going to go at the beginning of the 4-flip, or what could be the 4-flip. [pause, writes 100-333 to the side, (25 sec)] Multiples of 5, multiples of 10. [pause, begins list at bottom of page, writes 4* _ _ = 1, 4* _ _ = 2, 4* _ _ = 3, ..., 4* _ _ = 9, then goes back to fill in right hand side of these equations, first line = _ _ 4, (55 sec)] So, the 4-flip is going to have to be a number that starts with – 4 [draws arrow and 4], or that’s not possible, since it’s going to be 100 through 333.

R: Okay.
Julie: Same thing with 2. [marks X next to lines with 4* _ _ 1 and 4* _ _ 2 (for reference purposes, call these lines 1 and 2)] 3’s going to end in a 2. So, that’s a possibility. [puts circle next to line 3] Ends in 6. [writes = _ _ 6 on line 4, marks X next to line 4, writes = _ _ 4 on line 6, (5 sec)] Nope. [marks X next to line 6, writes = _ _ 8 on line 7, marks X next to line 7 (5 sec)] 8. [writes = _ _ 2 on line 8, marks circle next to line 8, writes = _ _ 6 on line 9, marks X next to line 9, pause and looks over her work, to the right, begins new list with numbers ending in 3, writes 4*103 = 412 (301), 4*113 = 452 (311), 4*123=492 (321), uses calculator for some multiplications, (1 min 15 sec)]

R: So, you’re just looking at some examples to get some ideas.

Julie: Yeah, to see if there’s a pattern. I’m getting closer.

R: Okay.

Julie: [writes 4*133 = 532 (331), 4*143, pause and looks at list, (15 sec)] That’s just gonna get bigger. [pause (5 sec)] They’re not – starting here they’re just getting bigger. [pause (5 sec)] So, none of the 100s will work.

R: And why did you deduce that?

Julie: Um. [writes “100s won’t work” next to this list] Because I already figured out that a 4-flip is going to start out with 300 and something.

R: Okay.

Julie: And I started with the lowest 100, 103. And we multiply that 4, by 4, started with 412 and it’s just increasing after that.

R: Okay.

Julie: Which means it’s not going to start in the 4-flip, which is 300. It’s going to be in the 100s more. [starts new list with 4*203 = 812 (5 sec)] And the same thing’s going to happen with the 200s. They’re just going to start really big too, and get bigger. [writes “incr.” below last equation, then writes “200s won’t work” next to equation, (15 sec)]

So, nothing that ends in 3 works. [starts new list with 4*108 = 432 (801), 4*128 = 512, 4*168 = 672, (35 sec)] However, things that end in 8, can have a bigger 4-flip. When we’re looking at 801, and when you are multiplying things by 4, these numbers are getting bigger. So, it looks like it could possibly work. [writes 4*188 = 752, 4*198 = 792, rewrites 4*199 = 792 to the left when she ran out of room, then writes 4*208 = 832 (802), (35 sec)] Oops, I just passed it. [pause, looks at work, (10 sec)] These are just going to increase. [draws arrow below 832 and writes “increase”, (5 sec)] And, that’s it.

I’ve already surpassed it, so none of these will work either. So, I don’t think any of the three-digit 4-flips would work.
R: Okay. So, you think that you would prove the statement that none of them work? Right?

Julie: Yes.

R: And you have narrowed it down to having to end in 3 or 8.

Julie: To – Yes.

R: And then you tried all the 3s and noticed that you already at 103, got things that were too big –

Julie: Right.

R: – when you multiplied by 4. And so, you eliminated all the 3s, and then you looked at 8s and a similar kind of strategy, looking at what you get?

Julie: Mm, hmm. And I figured out the point where one passed the other. Between 198 and 208.

R: Okay.

Julie: Do I have to do more?

R: Um – Well, other than writing it up, do you feel confident that you have found your solution?

Julie: [pause] Yes.

R: Okay.

Julie: Am I wrong?

R: So, you’re at the point where all you would have left to do is write this up formally, the ideas that you’ve expressed?

Julie: Mm, hmm. Yes.

R: Alright. Okay. Um. Anything new or different than you used from the first part of the question? Any new strategies?

Julie: Um, no. I just had to be a little more general. Cause it was the larger spectrum.

R: Okay.

Julie: And eliminate things small, like the lower.
R: Okay.

Julie: The lower increments. [laughter] But, it’s pretty much the same.

R: Okay, one more question in the same light. [researcher collects papers from this question, gives student the next question] And that is this one. Prove that $N$ being 2178 is the only four-digit 4-flip.

Julie: Okay. [writes $4*N=$ and “N-4 dig”, pause, looks at question, then writes 1000-9999, pause, writes 1001-, pause, looks at work, checks some calculations in calculator, (1 min 20 sec)] Oh. [pause (10 sec)]

R: So, what are you looking at right now?

Julie: I’m trying to figure out what the, um, numbers go through. Cause with the other ones, we can cut off a bunch of them because they wouldn’t be 4-flips.

R: Okay.

Julie: Cause when you multiplied by something it would be way, the um, the answer would have more digits.

R: Okay.

Julie: And I can’t think of what. [laughter]

R: Well, that’s no good.

Julie: Um. That looks like it’s the whole thing. [writes 9999 next to 1001-, pause, (10 sec)]

R: So, how else can you go about investigating that? Because you seem troubled by that answer.

Julie: Yeah. I am. Um. [pause (10 sec)] Investigating what?

R: So, you’re wanting to know what numbers get so big that when you multiply by 4, there’s more than four digits. Right?

Julie: Yeah. Oh. [sigh, erases the 9999 last written, writes 2500 in it’s place] It goes up to 2500.

R: Okay.

Julie: That’s, it goes up to 2499. [changes 2500 to 2499]
R: And you already eliminated the 1000.

Julie: Yes.

R: So, what was your reasoning?

Julie: Um. Numbers that end in 0 –

R: Okay.

Julie: - multiples of 10 are going to only have three digits. Same thing for fives again. Eliminate multiples of 5 – [writes elim in new column and underneath writes “mult of 5”, and “of 10”, (5 sec)] and multiples of 10. So, why don’t I just try the same thing. [makes list $4*1_1=..., \ldots 4*1_9$, begins to fill in right hand side with $\ldots _4$, $\ldots _6$, (60 sec)] Okay. And so, since it only goes up to 2s, you can only use anything that would end in 1 or 2. So, we can eliminate 1s, 2s, 4s, 6s, 7s, and 9s. [marks X next to lines 1, 2, 4, 6, 7, and 9 in this list] We’re left with 3s and 8s again.

R: Okay.

Julie: [starts new list, writes $4*1003=4012$ (3001), $4*1013=4052$ (3101), uses calculator for multiplication again, (40 sec)] Okay. So, once again, anything that ends in 3 is going to be, too, when you multiply it by 4, it’s going to be too big to make a 4-flip.

R: Okay. Same reasoning as before?

Julie: Yeah.

R: Cause you’re already bigger than 3000 and they will all be 3000 numbers?

Julie: Yes.

R: Okay.

Julie: [marks X by line 3 of previous list, pause, looking at work, (20 sec)] There’s only going to be 1 because these pretty much meet in the middle [pointing to her list of things that end in 3]. I mean, as these get bigger [draws arrow down from 1013], then these get bigger really fast. And these get bigger slightly [draws arrow down from 4052] It’s when this one surpasses this line [referencing the two arrows just drawn], there’s going to be a 4-flip. And so 2178 is going to be the only one [writes 2178 to the side of these arrows].

R: Okay, why don’t you look into that a little bit further? I’m just wondering what happens.
Julie: [pause (10 sec)] Oh, let’s try 2168, 2178 which is our 4-flip, and 2188. [writes 4*2168, 4*2178, and 4*2188, makes calculation, writes = 8672 in first line, (10 sec)] And we would have wanted 8612. [writes (8612) to the side] And this one will be [does calculation, (5 sec)] 8712, which is what we wanted. [does last calculation, writes = 8752, (10 sec)] and what we would have wanted is 8812. [writes (8812) to the side] So, this will be the bigger and this is the smaller. [points to 8672 and 8612, respectively, labels with b and s] This is the bigger and this will be the smaller [points to 8812 and 8752, respectively, labels with b and s]. So, this is where, where it meets in the middle. [circles 8712] These are going to continue – [pause] getting bigger um – gradually. [pause] Is this making sense?

R: Yeah, but why don’t you try one more example? Try like 2218 or something like that.

Julie: [writes 4*2218=8872 (8122), looks at work, (20 sec)]

R: So, does your theory still hold?

Julie: Um. [pause, looks at work, (5 sec)] No, it doesn’t because –

R: What happened?

Julie: Um. This is ending in a smaller number [circles the 18 of 2218] than these are ending in [circles the 88 of 2188], and so we can start with a smaller number. [pause (5 sec)] So, I probably have to prove it for each of the – [pause, looks at work, (5 sec)] 10 through, 10 through 24. And figure out if each one rules out of. Or the four digits that start with 10 through 24 –

R: Okay.

Julie: – is what I mean. So, I know 21 works. [pause]

R: So, your idea is finding that midpoint for each set of the 100s –

Julie: Yes.

R: So, like 2100s, and 2200s –

Julie: Mm, hmm.

R: That each one of them will have that meeting point.

Julie: Yeah.

R: And 2100 will be the only one that actually works.
Julie: Yeah.

R: Okay.

Julie: [starts new list, writes 4*1018=4072 (8101), 4*1098=4392 (8901), using calculator, (50 sec)] The 10s aren’t big enough.

R: So, describe what happened there. You were looking at the 10s –

Julie: They need to start with, they need to start with an 8000 [underlines the 8 of 1098] and they’re only getting up to like 4000. [writes 4*12]

R: Okay. [pause] So, you jumped to 12 and skipped 11, right?

Julie: Yeah.

R: With the insight of?

Julie: We need to be getting to – actually, we need to go all the way up to – 20. [erases 12, writes 2008, (5 sec)]

R: And your thought there is?

Julie: Because those are going to reach 8000.

R: Okay.

Julie: [does calculation, writes = 8032 (8002), (10 sec)]

R: So, you’ve in fact eliminated a lot.

Julie: Yes.

R: Now you just have 20, 21, 22, 23, and 24.

Julie: No, I’ve narrowed it down to – 2000-2499. [writes 2000-2499 underneath the 1001-2499 she wrote earlier]

R: Okay.

Julie: [writes 4*2018=8072 (8102), labels 8032 and 8002 with b and s, respectively, and 8072 and 8102 with s and b, respectively, (30 sec)] So, there our 20s just passed that point. [draws line between 8032 and 8072]

R: Okay.
Julie: This one was a bigger number, this one’s a smaller number [pointing to 8032 and 8002], and then it swapped. Which means we passed that equilibrium where it was gonna, a number would have worked. We already figured out where it was for 21. So, we can go 22. [writes 4*2208=8832 (8022), 4*2218=8872 (8122), using calculator, (35 sec)] And these ones are already too big for them to work. [pointing to 8832 and 8872] Um, when we multiply these numbers [puts box around 2208 and 2218], what we get is too big for our 4-flip to be able to work [puts box around 8832 and 8872].

R: Okay.

Julie: And so, the same thing will happen in 23s and 24s. [writes 23s and 24s, (5 sec)]

R: And specifically, you’re going to get, because you’re already at 8800s, right?

Julie: Mm, hmm.

R: And you know you want to start with 8, but these guys are going to start with bigger than 8s.

Julie: Even with 2308, we want to start with. [writes 4*2308] Yeah, those will start with 9s so they’re not going to work. [writes 9000s to the side]

R: Okay.

Julie: So, we can eliminate 23s, 24s, [puts X next to 23s and 24s] so that just leaves our 2100s to give 2178.

R: Okay. So, just to recap what you did, you limited it down to just 2499. You went again through the choices of the end digits, motivated probably by the last problem, right?

Julie: Mm, hmm.

R: And saw, with a little bit of experimentation that 8 was the only ending. And so, you knew what you wanted, so you could eliminate – at some point you eliminated all the 1000s because they weren’t high enough.

Julie: Mm, hmm.

R: And then, you could eliminate everything above 22 and a little bit because they were too big.

Julie: Mm, hmm.
R: So, you were able to narrow it down and focus in until you got to the 2178 as the only possibility. And so, with a little bit of work, you feel confident you could write that up and you would have a full solution?

Julie: I think so.

R: Okay. Anything new or different from the other parts of the problem?

Julie: No. I don’t think so. Once again, it just got bigger.

R: Okay. Is there any, um, outside related stuff that you’ve seen that influenced what you did here? Do you think?

Julie: Hmm, I don’t think so.

R: Okay. Let’s spend a few minutes on one more problem, okay? [researcher collects papers] I’ll just grab that sheet from you. Thank you very much. [researcher gives student new question] So, this is a different problem altogether. A traditional chessboard consists of 64 squares, 8-by-8. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes. [pause] So, do you understand what is meant by that, by a perfect cover?

Julie: Um, yeah, I think so.

R: Okay. Consider now a generic chessboard, it’s just \( m \)-by-\( n \). Prove that this generic chessboard has a perfect cover if and only if at least one of \( m \) or \( n \) is even.

Julie: Okay.

R: Alright, any questions about it?

Julie: [pause] Nope.

R: Okay, and proceed however you’d like.

Julie: Okay. [reads the question, (10 sec)] Okay. [draws dominoes, (25 sec)] Okay, so these are dominoes covering the chessboard.

R: Okay.

Julie: And this side [referring to the length or top of the drawing] could be – let’s say that \( m \) has to be – Prove that a generic chessboard \( m \) times \( n \) [rereading the question] – so at least one of them has to be even. So, our choices are that \( m \) is even [writes \( m \) even], hmm case 1 [labels this as case 1]. \( m \) is even, \( n \) is odd [writes \( n \) odd], or case 2, \( m \) and \( n \) are both even [writes case 2, \( m,n \) even]. Okay. [pause (5 sec)] And [draws more
dominoes) it has to be even because these cover two spaces. So, 2 [writes 2 above one
domino, pause, rereads the question, (15 sec)] This is our $m$ and it has to be 2 times $s$.
Hmm. [writes $m = 2s$ above the top of the dominoes] This doesn’t make any sense.
[erases and redraws some of the dominoes] I didn’t mean to erase. [crosses out the $m =
2s$, pause, (10 sec)]

R: So, describe to me what you’re thinking.

Julie: Well, a domino has to cover two spaces for each domino. So, you know that if the
domino’s laying it’s long way, then it’s going to cover two spaces. And if you want the
dominoes to fit perfectly then there has to be an even number of spaces. So, that will be
the side where it always has to be even. Then, it doesn’t matter because the other side is
just the short side and it can cover, I mean that’s only one space.

R: Okay.

Julie: So, that, it can be even or odd, that side doesn’t matter because Case 1 and Case 2
both work.

R: Okay.

Julie: I just don’t know how to explain why, I mean in terms that it has to be even. It’s
cause it covers two spaces, it doesn’t matter how many dominoes there are – [pause]

R: So, can you think of another way to prove it? So, you’ve demonstrated that one exists
if at least one of them is even. Right?

Julie: Yes.

R: And now you’re trying to say, why is it that it has to be that one of them is even,
right?

Julie: Yes.

R: How else could you go about that?

Julie: Um. [pause (5 sec)]

R: Any alternative ways that you can think of?

Julie: To go about?

R: Proving that it has to be even. So, you’ve shown even works.

Julie: It has to be even for our domino to fit, because it takes up two spaces, because it
takes up an even number of spaces.
R: Okay.

Julie: And an even number times – two spaces times however many dominoes is always going to be even [writes 2 spaces*dominoes = even].

R: Okay.

Julie: [pause] But I don’t know how to put it in generic terms.

R: So, describe to me why the fact that the dominoes take up an even number of spaces total tells you that one of the sides has to be even.

Julie: [writes even # spaces total → even, (10 sec)]

R: I’m being intentionally obtuse, so – [laughter] You’re not barking up the wrong tree, I’m just trying to get you, trying to get a further explanation of it.

Julie: Mm, hmm. [pause, looks at question, (5 sec)]

R: So, why is it that a total number being even tells you that at least one of the sides has to be even.

Julie: [pause] You have a square that’s odd times even it is always going to be even. [draws square and labels top even and side odd and writes “= always even”]

R: What do you mean by that, it’s always even, what’s even?

Julie: The total. [writes “total” over always even]

R: Okay.

Julie: [pause] Even, even, even-by-even is always even [writes even x even = total even, (5 sec)]. But odd-by-odd, obviously gives you an odd number [writes odd x odd= total odd, (5 sec)] Which wouldn’t be a square, or not a rectangle. [writes → not a rectangle, (10 sec)]

R: [researcher flips over audio tape] Okay, so you just said an odd times an odd gives a total number being odd. And that’s bad because we have – [pause (5 sec)] that gives us the total being odd, why is that bad?

Julie: Because the dominoes are 2s, basically two spaces, a space is s. [writes “space = s then dom = 2s”, (2 sec)] And, domino is 2s, and if our total is odd, 2s won’t go into it. [writes “total odd/2s ≠ chessboard”]

R: Okay. So, no matter what you try, you’re always going to –
Julie: [finishes writing last statement, (5 sec)] Does not equal chessboard.

R: So, your – your dominoes can’t cover it?

Julie: Right.

R: Okay.

Julie: But if they’re even. So, let’s, well, let’s say our odd is $2s+1$, times $2s$, that doesn’t work [writes “total odd: $2s+1/2s \neq$”, then changes s to k]. But if our total is even, well I guess we call it something $2k$. But, if our total is even, then it’s going to be $2k$ divided by $2s$ [writes even: $2k/2s = k/s’”], which will always work. [pause] Does that make sense?

R: Mm, hmm.

Julie: Okay.

R: So, let’s go back to the problem. The problem said: prove that the generic chessboard has a perfect cover if and only if at least one of them is even. So, you’ve demonstrated a cover if at least one of them is even. You’ve described verbally how it works, that you can cover it. You’ve described that both being odd doesn’t work, and shown why that’s true. Okay, so, with a little write-up, do you feel confident that you solved the problem, that you have a full proof with write-up?

Julie: [pause] Yes.

R: Okay.

Julie: I did all the cases.

R: And the third case being that they’re both odd?

Julie: Yes.

R: Okay. Alright. Have you ever seen anything like this before?

Julie: Um. Maybe, well –

R: Does it remind you of anything?

Julie: This year is really the first time that I’ve worked with proofs with even and odd, in 305.

R: Okay.
Julie: And um, this is the closest I come to working with how things fit in the – [pause] depending on whether it’s even or odd.

R: Okay.

Julie: And different cases and things. So, this year’s probably the first that I’ve gotten to do proofs that are close to what this is.

R: Okay. And you think your ideas from 305 carried over a bit into this proof?

Julie: Definitely.

R: Like what?

Julie: Um. Like using $2k + 1$ and $2k$ [circles $2k+1$ and $2k$] to explain an even and an odd number.

R: Okay. And maybe doing it by cases?

Julie: Yep.

R: Cause you’ve seen that a lot in 305, right?

Julie: Yeah.

R: Okay. So, again, you drew an example. You started with a small example and you built up from it. Kind of looked at all – and you immediately went into, um, breaking it into cases and solving from there.

Julie: Mm, hmm.

R: Okay. Anything else you want to note about strategies that you used?

Julie: I don’t think so.

R: Okay. Well, why don’t we stop there. *End of interview.*
Interview #9  (Total time = 52:23)

R: …can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16, for an example. Prove that the smallest possible value for the sum is 14.

Maggie: Ahhhh, okay.

R: Any questions about the problem?

Maggie: Huhhh, ummm [pause] you’re proving that smallest possible value is 14. And you’re using all of the numbers?

R: That’s right. 1 through 10, once.

Maggie: Okay. Um, not yet –

R: Okay.

Maggie: – but I’ll start thinking and then I’ll – okay. There’s five sides [pause (10 sec)]. So, we have 10 numbers and there’s three numbers on each side. Sooooo, [pause] um how to go about a proof? [draws new pentagon (5 sec)] 10 numbers and there are three numbers on each side, right? [writes “10 #'s, 3#'s on each side = 14”]

R: Mm, hmm.

Maggie: And each side equals 14, right?

R: Mm, hmm.

Maggie: Or well we're proving that that's the smallest you can get. [pause (25 sec)] I don't know how to do this.

R: Can you tell me what you’re thinking about?

Maggie: Well I’m thinking about just that fact that you have this number line [writes the numbers 1 through 10 in a line] and the smallest sum that you have to, that you can have, would have to involve at least, well one of the sides would have to be with 1 and 10 [circles the numbers 1 and 10] and so on and so forth going in [motions taking pairs going towards the center of the list of numbers, pause (10 sec)] 3 – So this – We're trying to prove that it should be 14 so your smallest side would have to be [pause] with 10 and 1 and 3. [writes 10,1,3] But, why would you use 3? [pause] How is that possible?

R: So what’s your question? So you, why would you use 3 so versus –
Maggie: You can only have three numbers and the only way to use the number 10 is to have it with 1 and 3. ‘Cause you’re not duplicating the numbers, right?

R: Right.

Maggie: Okay. [pause] But then, oh, um, okay.

R: What was the oh?

Maggie: The oh was the fact that the 1 can be in a vertex (sic) so it can be used with 9. Or the other numbers, like such.

R: Okay.

Maggie: Okay, so the 10 would have to be in the middle because it can only have one arrangement.

R: Okay.

Maggie: (deep breath) [draws new pentagon] Um, in order to prove this, can I just show?

R: So, you’re working on finding 14 and showing that it is possible.

Maggie: Right.

R: Okay.

Maggie: Do you want an actual formal proof, I mean to say that the 14 is the best?

R: Well, work on that and we’ll see where we go from there.

Maggie: Okay.

R: That’s fine.

Maggie: ‘Cause I don't know how I would – [writes in 3 10 1 on the pentagon] Okay. So, then [pause (5 sec)] Well, I guess the next one, just 1. [writes 9 4 next to the 1 on the pentagon] It seems to me that you want your larger ones in the middle.

R: Okay.

Maggie: [pause, looking at pentagon (10 sec)] That doesn't work out I don't think. But if you did – [pause (10 sec)] 5 and 6 and the 3 would work there. [writes 6 and 5 next to 3 in the pentagon (10 sec)] And then, 5 and 7 and 2, 7 and 2 would go in there. [writes 7 and 2 on the side with 5] And then what do we have left? An 8. Oh, that works. Okay. [writes 8 in the last spot]
R: Nice. Okay, so we have one with 14.

Maggie: Right. So why is that the smallest possible value that we could possibly get?
And for that I would, could I just argue that because all the larger numbers are not being used twice [circles the numbers 6 through 10], those are your five largest numbers not being used – hmm. Do you kind of follow what I'm trying to say?

R: Yeah.

Maggie: ‘Cause in this [points to the given pentagon with sum 16] this is a larger sum because some of these top five larger numbers are being used twice. So in order to get the smallest sum, the larger numbers have to be in the middle. ‘Cause they're not being used multiple times.

R: Okay.

Maggie: [pause] Then you want me to write that down?

R: Yeah, well I, the verbal argument is, is good, is sufficient. Um, why is it that a rearrangement of the vertices wouldn’t give you a smaller one?

Maggie: Because there aren’t [pause] – oh, oh you mean like switched around and such?

R: Yeah. [pause] So, I get the argument that you want the biggest numbers on the sides.

Maggie: Right. Because there are only certain numbers that go with those big numbers, you only have so many choices, you only have –

R: Okay, why don’t you proceed in trying to formulate something else and see what happens.

Maggie: Formulate something else?

R: Or can you get less than 14? I guess is what I'm saying.

Maggie: Oh.

R: And you can argue that in any way you want.

Maggie: Can you get less than 14? Okay. Well. [pause (5 sec)] But our largest smaller number [pointing to the 5] is already with a number, what if we switch those around?

Okay. [pause] So you want to put – hmm. [pause (10 sec)]

R: What are you thinking?
Maggie: Well I’m thinking that, in order to get a smaller number, you could possibly have 10 with 1 and 2. [writes 10,1,2]

R: Okay.

Maggie: But your next smallest number – 9 – we’ve already put it with the 1 [pointing to the pentagon with sum 14]. If we’d perhaps had put it with the 2, then it would have been like 9, 2 and the next smallest number it could possibly be is with the 3 which is 14. [writes 9,2,3 →14]

R: Okay, so the 10 the 1 and the 2 gave you 13, right?

Maggie: Right. [writes = 13 next to the 10,1,2] But there’s no way for us to get 13 with the 9. [pause] Do you see that? I could have had. [draws new pentagon] If we had the 10 and the 1 and the 2 there. [puts 1 10 2 along one side of pentagon] Um – well if we, no wait – What was I trying to say? [pause] Well then if we put 9 with the next smallest one is 3. [writes 9 3 along side with 1 on pentagon]

R: Okay.

Maggie: So, okay that's 13. Okay. [writes 13 next to two sides of pentagon] Um, so our next largest one is 8. We would want to put that with that. [writes 8 next to the 2 on the pentagon] But in order to get 13, you have to put 3 again.

R: Okay, is there anywhere else 8 could go?

Maggie: Well it could go here [writes 8 next to the 3 on the other side of the pentagon] But then you'd have to do 2 again.

R: So, can you argue that you found that there isn’t a possible way to do 13, for sure?

Maggie: Yes. ‘Cause either way, you put the 9 on there. The only way to get 13 with then is to do it with 1 and 2.

R: Okay.

Maggie: Since we can’t do multiples of numbers.

R: Okay, and then the only way to do it with 9 –

Maggie: Mm, hmm.

R: – had to go with?

Maggie: The 1. Because you can’t do 9 with 2 and 2.
R: Okay.

Maggie: So, it has to go with the 3 but then once you put the 8 on either side, you get a multiple of the other ones.

R: Okay.

Maggie: Which we are not supposed to do.

R: Okay. [pause] And you for sure need 8 on those other two sides, right?

Maggie: Yeah.

R: I mean it couldn’t go on a on a third one?

Maggie: Because the 8 can’t be with 4 or anything else because then it’s already gonna be over –

R: – Okay. –

Maggie: – 13, once you add everything else. You want your smallest numbers with your largest numbers in order to get the smallest combination.

R: Okay. I think you could probably right that up formally, just what you’ve said verbally, right?

Maggie: Right.

R: I mean, if necessary.

Maggie: Yes.

R: So, do you feel satisfied you’ve solved the problem? You’ve found one with sum 14 and you’ve shown that, clearly, 13 would be the only other option –

Maggie: Right.

R: And 13 is not possible.

Maggie: Yes.

R: Okay. Have you ever seen a problem like this before?

Maggie: Um – yeah like they have those Sudoku things, or whatever. I don’t do them but I’ve seen them. [laughter] Finding sums of the lines and stuff like that.
R: Mm, hmm. And this reminded you of that?

Maggie: Yeah, just rearranging numbers to get different sums.

R: Okay. Um, what strategies would you say you employed to do this?

Maggie: What strategies? Um [pause] I don’t know. What kind of strategies? [pause]. Um, just thinking about how the numbers can combine together. Since we were trying to get the smallest sum –

R: Okay.

Maggie: – then you need the largest numbers to go with smaller numbers to get the smaller sum.

R: Okay. Anything else you can think of? [pause] If not, that’s fine.

Maggie: No, that’s it.

R: Okay. Any class where you’ve done anything that gave you any insight into this problem? [pause] Or any time you’ve done anything that gave you insight into what you did here?

Maggie: Um. [pause] I don’t think so. I mean, when I was younger I used to always do little puzzles.

R: Okay.

Maggie: Like this, but –

R: Alright, so nothing that jumps out at you right now?

Maggie: {No}

R: Okay, let’s move on to another question then. If you want to tear that page out, I'll take it. [pause, researcher collects papers, gives student next question] I think we’re gonna skip this one for right now and go to this one next. Okay. Question number fi – 3 – page five. A traditional chessboard consists of 64 squares, it's 8-by-8. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes. Consider a generic chessboard of size $m$-by-$n$. Prove that the generic chessboard has a perfect cover if and only if at least one of $m$ or $n$ is even.

Maggie: [pause] Okay.
R: Do you understand the problem?
Maggie: Yes.
R: Okay.
Maggie: So since you have one domino covers two squares [writes “one domino covers two squares”] and we want to prove that an $m$-by-$n$ at least is even. [draws rectangle, labels $m$ on side and $n$ on top and writes “@ least one is even, pause]. Okay. Well, like just picturing it in my head, I can picture dominoes going across like this. [draws picture of dominoes lying on their short side]
R: Okay.
Maggie: And you can, each of them is basically covering up two squares there. So – um, you can, you could potentially add them on this way [draws arrow in a vertical direction] in sets of 2. So, that direction would have to be an even side. [indicates left side and writes even] But then, this way, they’re one length across? [indicates the bottom]
R: Oh, okay.
Maggie: You could potentially have an odd side there. [writes odd side across the bottom]
R: Okay.
Maggie: Okay. Now – [pause] I don't know exactly, hmmm, I’m not sure.
R: Not sure?
Maggie: What to do from here. [laughter]
R: Okay.
Maggie: How would you go about saying that, I guess.
R: Okay. So you’ve shown that there’s a way to cover it. You’ve drawn a picture of how you would cover it so long as one of ‘em’s even, right?
Maggie: Right. Yeah.
R: Okay.
Maggie: And this could be even or odd depending on how far you make it go out.
R: Okay.
Maggie: So –

R: So what, if anything, is left to prove to show that it has a perfect cover if and only if at least one of $m$ or $n$ is even?

Maggie: I don’t think anything is. I mean –

R: Okay, so you’ve covered um, you’ve covered any board that’s even by either even or odd, right?

Maggie: Yeah.

R: Okay.

Maggie: And then it could be even and even, you just go one more –

R: Okay.

Maggie: – if you want it to be even. [referring to the bottom of her picture]

R: Okay. So why is odd-by-odd a problem?

Maggie: Because the fact that the domino is shaped with two. [draws new domino]

Well, let’s see, because – oh, okay, I got ya. Um, one side you could, let’s say if we had a row, going this way and these guys are each dominoes. [draws row of dominoes laid lengthwise] 1,2,3,4.

R: Mm, hmm.

Maggie: But then you put them sideways, right, so that’s one and one and this side is odd [labels side as odd] but in order for it to be an actual rectangle, then this side needs to be even. [labels bottom as even]

R: Well, rectangles can be odd-by-odd. I mean you could have, in fact, a square that’s 3-by-3.

Maggie: Right but then you would have to um. It would – [pause] um. Oh, odd. Like you're saying put one here. No, that's not right. [pause] That is two, so this was three and that was four. [referring to the bottom row of dominoes, pause (5 sec)]

R: So you're kind of seeing visually you're gonna have a problem, right?

Maggie: Right, because if you put one more so that you have an odd number, you’re left with a little space here, so it’s not a totally filled in rectangle and you can’t put another
domino there, because dominoes take up two squares where you as you only have one square left in there.

R: Okay. How can you prove that in general that that’s gonna happen any time you have odd-by-odd?

Maggie: Um [pause] because an odd number times an odd number is an odd number [writes odd x odd = odd] and dominoes are in sets of two which are even numbers.

R: Okay.

Maggie: So, they're gonna be in an area that is an even number.

R: Okay. So, they can’t possibly cover an odd area?

Maggie: Right.

R: Okay. So, now, what if anything, is left to prove?

Maggie: Well, we’ve proven that it can’t be odd and odd [pause (5 sec)]. So, okay, if we do if and only if – [pause, reads the question and her work softly to herself (10 sec)] And then you can have even and even. So, there’s basically four cases and we’ve shown even and odd, even and even, and the reverse of even and odd and then odd and odd.

R: Okay.

Maggie: So, I think we’ve covered all of our bases.

R: Okay.

Maggie: I think so.

R: Okay. And then just left to, write it up?

Maggie: Yeah.

R: So, have you ever seen anything like this before?

Maggie: Um, I feel like I have but I don’t remember where. And not necessarily like [pause] um, not necessarily as a visual example but I know a lot of like the number theory problems that we did in that class had to deal with whether a number was odd and even and how they interact with each other.

R: Okay. Considering the cases and stuff?

Maggie: Yeah.
R: Okay. Um, anything, any strategy wise that you’ve used here? That you can think of.

Maggie: Lots of visuals.

R: Okay. Drawing some pictures, looking at examples –

Maggie: Mm, hmm, yeah.

R: So you’ve pulled in a little bit of your ideas from number theory and trying even and odd cases and how they interact.

Maggie: Yeah.

R: Okay. Anything else you can think of?

Maggie: I don't think so.

R: Okay. Let’s move on to another question. [pause, researcher collects papers, gives student next question] Question 2 – we call a positive integer a 4-flip if 4 times N has the same digits as N but in reverse order. [pause] Do you understand what that means? I'll let you read it and think about it for a second.

Maggie: Oh. [pause (5 sec)] Okay, so you’re saying you have some number and if you multiply it by 4 then it’s like [pause] Yeah the reverse order. So –

R: So, part a) prove that there are no two-digit 4-flips.

Maggie: Okay. So no two-digit 4-flips. [writes “no 2-digit 4-flips”] So, because a four or two-digit 4-flip. [pause (10 sec)] Hmmm. [pause (10 sec)] Looks like this, you have these two digits. [writes “”] Then you have 4 times these two digits [writes 4*”=] supposedly equals, like if – Err, maybe digit 1 and digit 2, 1 and 2 [labels the spots “_1 ___”] It's gonna be 2 and 1 digit 2 and digit 1. [writes “= _2 _1”] So, why is that not possible with the two-digit? Hmmm, hmm, hmm. [pause (10 sec)]

R: Can I ask what your thinking about?

Maggie: I thinking that I'm not sure where to even start. Um. [pause (10 sec)] I guess, um just how would you represent a digit of a number to be able to show it in the reverse?

[pause (5 sec)] Hmmm. Because when you multiply by 4 [writes “_1 _2” and the number 4 below them], what basically happens is it kind of depends [pause] Let's see. Like digit 1, 0 times 4, plus digit 2 times 4. [writes “_1 0 x 4 + _2 x 4”, pause (10 sec)]

R: So how else can you represent that? You're wanting to come up with a representation.
Maggie: Mm, hmm.

R: And so far you have that you want digit 1 to represent the 10s place.

Maggie: Right.

R: And digit 2 is the ones so you, you said you can think about that like digit 1 with like a 0 attached to it.

Maggie: Mm, hmm.

R: Times 4 plus the other guy times 4.

Maggie: Right.

R: Okay. Is there any way to say that?

Maggie: Oh, I don’t know.

R: To attach a 0 to something.

Maggie: Oh, times 10.

R: Okay, so how about you try that.

Maggie: Okay. [writes “_1 x 10 x 4”] And then that's also times 4. [pause] And digit 2 times 4. [writes “+ _2 x4”]

R: And now that you have that representation, what did you want to do with it?

Maggie: Um. [pause (10 sec)] I’m not sure. Um, well you have digit 1 basically times 10, [pause (5 sec)] and you want that to equal your digit 2 times 10 plus your digit 1. [writes “= _2 x 10 + _1”]

R: Okay.

Maggie: Okay. [pause (20 sec)] I don't see how this is helping me.

R: Okay, what would you like to do next?

Maggie: Um –

R: Can you go any further with this or do you need to start somewhere new, or what would you like to do?
Maggie: Yeah I feel like this is getting me nowhere. [pause (5 sec)] ‘Cause I don’t see how there’s a way [pause (5 sec)] ‘cause ideally we would be able to arrange this and get to this point. [referring to starting with the left hand side and working towards the right hand side of the equation, pause (10 sec)] I mean, right? Is that what to do? You can't tell me? [laughter]

R: I won't give you too much input. But, if you wanted to go further with this, what would you do with it?

Maggie: [laughter] I don’t know. [laughter]

R: Or if you'd like to give it up then that's totally okay, too, and you can start somewhere new, if you'd like.

Maggie: Yeah, but then I don't know where I would start. So, if I wanted – [pause (5 sec)] Well, I feel like I’m having issues with this because supposedly it's not possible. [pause] So – but how do I prove that it's not is the issue. [pause (20 sec)] Hmm.

R: What else are you thinking?

Maggie: I’m thinking that you could, I mean you just go back to the, if you write like in general format, you can say that for n digits [writes “n-digits”] and you have your digit 1 times 10 to the n minus 1, plus digit two times 10 to the n minus 2, and so on and so forth [writes “\(-1 \times 10^{n-1} + -2 \times 10^{n-2} \ldots\)”] until you get to n – n. But I don't think that helps me out either. But potentially you could do that.

R: Okay. That'd be just a generic for any length.

Maggie: For any, length of a number. [pause] So then why does it work that you can multiple it by 4, well, I guess there are only special numbers that do this, huh? It's not every number. [pause (5 sec)] Well, that potentially equals – opps – digit n times 10 to the 0, ‘cause that's n minus n plus. [writes “\(- n \times 10^0\)” with n-n above the 0] It's really strange that it does that. This number is obviously bigger. [referring to the right hand side after multiplying by 4, laughter]

R: You mean when you multiply it by 4 it gets bigger?

Maggie: So that means that –

R: Sure.

Maggie: Well it means that your last digit here is, has to be larger than your first digit for this.

R: Okay.
Maggie: I don't think that helps me at all though. [laughter]

R: Well, remember we’re only dealing with two digits.

Maggie: I know.

R: Right, so maybe life is simpler in two-digit land, so –

Maggie: [laughter] So, this number has to be bigger than this number. [referring to digit 2 being larger than digit 1]

R: Okay.

Maggie: And is has to be – wait, does it have to be 4 times? Wait, hold on. That number [points to digit 1, pause, (5 sec)] Well, that number times [pause (10 sec)] this number times 4 [pause (10 sec)] well, it has to be either – Are we talking integers?

R: Mm, hmm.

Maggie: Positive integers?

R: Yes.

Maggie: So, this first number has to be either 1 or 2. [writes 1,2 above digit 1 in the middle of page]

R: Okay. And your reasoning?

Maggie: Because 3 times 4 is 12, that's gonna give you something in the 100s place.

R: Okay.

Maggie: Okay. So [pause] and this number [points to digit 2, pause] There, um, hmm. If we had a 1 there [referring to digit 1] and we multiplied it by 4 that would give us 40. [pause] The second digit though, so the second digit has to be a 1 or a 2, also. Is that true? [writes 1,2 above digit 2] I’m not sure that’s true, though. Well, yeah, it has to be a 1 or a 2 otherwise it’s gonna, oh I don’t know [pause] it’s gonna go into the 10s place, therefore adding to the first digit, which therefore is gonna give you a different digit here [points to the ones place of the right hand side] then the first one.

R: So, say that again, illustrate that again for me.

Maggie: Like if you multiply this second digit, um by 4, it has to be a 1 or a 2 because if it’s like a 3, then you’re gonna get 12, it’s gonna add into your 10s place, [pause] which is going to, um, make this number which has been multiplied by 4 – oh wait, that number
– oh, hmm. Well, it’s gonna bump up this number. [referring to the 10s place of the right hand side]

R: Mm, hmm.

Maggie: By a 10s place. So, it’s gonna be different. Oh, I am so confused. Then this one. [points to the ones place of the left hand side] That one, that digit. [pause] I think I confused myself even more. [laughter]

R: So, let’s go back to the point where you said the first digit can only be 1 or 2.

Maggie: Okay.

R: Okay?

Maggie: Okay.

R: And you reason that if you multiply anything more than that by 4, you get over 100 and that's bad.

Maggie: Right, yes.

R: Okay. So, where can we go from there? We know the first digit is only 1 or 2.

Maggie: Can we just do cases?

R: Okay. What would you like to do?

Maggie: So, if you had the 1 there, then the second digit when you multiply by 4 um, should be a 1 [writes “should be 1” under digit 1 on the left hand side], but there’s no way to multiply a number over here [points to right hand side] to get 1 because it’s an even number, 4 is an even number. You’re gonna get an even number out –

R: Okay.

Maggie: – on that side. So, 1 you can't use.

R: Okay, so we’re down to the first digit has to be 2.

Maggie: Okay. So, if it’s 2 [pause (5 sec)] thennnnnn – um – if it’s 2 then this is going to end up being 2. [circles the ones place on the right hand side]

R: Okay. [laughter]

Maggie: And then [pause] you need to multiply this digit [underlines ones place of left hand side] by 4 in order to get a 2 there. [pause (5 sec)] So, um, multiplying by 4 and I
don't know. Well, if you go through all the numbers, you could find out which numbers
work.

R: Okay.

Maggie: 1 and 2 don’t work but 3 does. If you do 3 times, if you, but if you do 23, then
you’re ending with 92 [writes these numbers down, pause] and this number [points to the
92] is way too large.

R: Okay.

Maggie: So, which one, 4 and 8 is 32, so I can say, okay it could be 8. But that's gonna
be a hummungo number too. It’s gonna – it’s gonna be a three-digit number.

R: You’re looking at 28?

Maggie: Mm, hmm. [writes 28]

R: Okay.

Maggie: Yeah. ‘Cause 8 and 4 is 32, 112. [writes 28*4=112] That is not right. So, 9
and 4 is 36 and those are the only integers. Therefore, it does not work, because we have
exhausted our possibilities.

R: Okay. So, can you recap your solution for me?

Maggie: Yes. In conclusion, [laughter] the first, okay, we’re looking at a two-digit 4-
flip, so your first digit must either be a 1 or a 2, since any multiple of an integer
thereafter, will get you a number that is in the 100s place.

R: Okay.

Maggie: Your second digit must, uh, multiply by 4 to give you, oh wait – hold on, back
up. We first need to say that we can’t use the 1 because the last digit cannot be 1 when
you’re multiplying by 4, ‘cause 4 is an even number there's no way to get an odd number.

R: Okay.

Maggie: When multiplying it. So, therefore it has to be 2, so we need a number for our
second digit that when multiplied by 4 to result in a 2 at the end of our number. And, the
only numbers that make that possible are 3 and 8. But they don’t work out, so there’s
gonna be two cases we went through and it does not give you a flip.

R: Okay. Have you ever seen a problem like this before?
Maggie: Umm, not like this but I mean like when I was talking about all this stuff – [points to where she wrote about \( n \)-digits]

R: Mm, hmm.

Maggie: This is like number theory stuff all over again.

R: Okay.

Maggie: Looking at series and all that stuff. Yeah.

R: [laughter] Okay. What strategies would you say you used here?

Maggie: [laughter] Um – what strategies did I use? [pause (5 sec)] Just kind of writing it down to see what exactly was going to happen when you have the digits in place. I think that helped me [pause] to kind of see what was gonna happen to them when they are flipped.

R: Okay. And this time around, you didn’t go to anything specific until much further in the problem, right?

Maggie: Right, yeah.

R: So that was a little bit different. You, you immediately started pulling –

Maggie: general, sort of –

R: – what do you think, maybe from number theory? Do you think is that what you were thinking of the general form of, or where do you think you started there?

Maggie: I don’t know. [pause]

R: What gave you an inclination that you might want to do that?

Maggie: The general form versus a –?

R: Yeah.

Maggie: Because you have 10 digits for each of these and I didn't want to go through all of those.

R: Okay.

Maggie: And, I guess in my mind I needed to go through a general form in order to figure out that even this first one was only limited by 2 and then, I guess my, I wasn’t
thinking [pause] about specific cases until then. Until I could simplify it to something I
could actually work with.

R: Okay.

Maggie: As a simplified – as like a specific case.

R: Okay. And maybe the digits place, maybe from the definition of it, because you need
to flip the digits –

Maggie: Yeah.

R: So, you wanted to be able to work with it, somehow.

Maggie: Right, yeah.

R: Okay. Alright, why don’t we tear that page off. [researcher collects papers] The next
one deals with the same stuff. [pause, researcher gives student next question] There you
go. Prove or disprove the following statement. There are no three-digit 4-flips.

Maggie: Uh – [laughter]

R: So, we've bumped up a level. (Maggie) apparently likes it. [laughter] So,

Maggie: Okay, well, okay, so if we have three-digits [writes __], this one has to be 1
or 2 [writes 1 or 2 below the first digit, laughter] but it can’t be 1, or wait, are we
multiplying by 4?

R: Yes.

Maggie: Okay, so it can’t be 1. So, it could be 2. Oh, this is gonna be icky.

R: So, we’ve narrowed down one digit, it has to be a 2.

Maggie: Yes.

R: That’s good, that’s a good start.

Maggie: One out of three. [laughter] Okay. So, the next digit, um if we multiply it by 4,
it's gonna give us these. [writes = ___] This has to be a 2 [writes 2 in the first blank on
left hand side] That has to be, hmm, yeah [writes 2 in the last blank on the right hand
side, pause] So, when this same type of idea, when that number is multiplied by 4 it has
to end in 2, which is only 3 or 8. [writes 3 or 8 above 3rd digit on left hand side]

R: Okay.
Maggie: Okay? Mm, hmm. The middle part, though, let's see. I mean how is that going
to work? So, this number has to be a 3 or an 8. [writes “3 or 8” above first digit on the
right hand side] Okay. So [pause (5 sec)], if it was a 3, if it was the case that it was a 3
[writes 3 below equation], so 2 something and then 3 [writes 2 _ 3], we want it to equal 3
something 2 [writes = 3 _ 2]. But any time you multiply by 4, by that number, it’s gonna
give you at least 800 and some. [crosses off right hand side of equation]

R: Okay.

Maggie: So, that is not going to work. Okay, if it’s 8, on the other hand…

R: So, you’ve concluded 3’s not possible.

Maggie: Right.

R: Okay.

Maggie: If it’s 8 though. [writes 8 in a circle to the side] You have 2 dot 8 [writes 2 _ 8]
that might be possible because it could be 4 times 2 is 8. [writes = _ _ _, writes a 2 in the
last blank] So, that has to be your 8 there. [writes 8 in the first blank on the right hand
side] This middle number –

R: Hold that thought. [audiotape side one ends, researcher flips over tape] Okay, go.

Maggie: [laughter] The middle number is a tricky one though. We’re multiplying by 4.
4 times 8 is 32 [starts multiplying 2 _ 8 vertically by 4, writes underline then 2 on the last
spot of the answer], so it’s whatever this number plus 3 [writes +3 above blank in 2 _ 8] is
supposedly uh, hold on. I want to erase. Can I cross it out?

R: You can cross it out, yes.

Maggie: Okay, 8 and 4 is 32. So, that means 4 times that digit [points to middle digit]
plus 3 [pause] needs to equal that same number? Is that completely – ? No, yeah, it has
to equal that same number. ‘Cause when you multiply by 4 – yeah.

R: So, reason that out with me again.

Maggie: Okay, so when you’re multiplying through, we know that it’s not 3, but it could
be 8 but 8 and 4 is 32. So when you multiply your next digit out, it’s gonna be 4 times
that digit plus the 3 is going to go down in this place. [points to the middle digit of the
answer]

R: Okay.

Maggie: Um, and you want this 4 times 2 to be 8. ‘Cause that’s what we’re going for
over here.
Maggie: So then, that one multiplied all together, that middle number is supposedly the same number, it’s staying the same number. But you can’t multiply that to get that. ‘Cause you’re increasing that number. There’s no way to, it's not possible.

R: Okay. Can you prove that for sure?

Maggie: Yes. Because, well yeah. Because 4x plus 3 is not x. [writes $4x + 3 \neq x$] I can prove that because 4x equals x minus 3 [writes $4x = x - 3$], x does not equal x minus 3 over 4. It’s not possible [laughter]. Because this number is being increased. It's not staying, staying the same [pause] I mean you could [pause] Oh wait, hold on. [pause]

R: So can you keep going with your thought of solving that equation?

Maggie: Yeah, well you could do 3x equals the -3 [writes $3x = -3$] So, x equals -1. [writes $x = -1$]. And x can't equal -3 because we are dealing with positive numbers. And you can’t have a -1 just as a digit.

R: Okay.

Maggie: Yeah. [pause]

R: Alright. So, have we answered the question? Are you, are you proving or disproving the statement “there are no three-digit 4-flips”? [pause]

Maggie: I am proving that there are no three-digit 4-flips.

R: Okay. And describe to me, recap what you’ve done to do that.

Maggie: Okay. So since like our last one this first place couldn’t have anything larger than 2 otherwise you would get into the next 1000s place. So, it has to be 1 or 2. Cannot be 1 because we’re still multiplying by 4, therefore it’s going to be an even number at this end place.

R: Okay.

Maggie: So, then we knew that that was 2 and that was 2 and then like the two-digit one, this last number needs to be multiplied in order to get a 2 at the end. The only way to multiply a 4 by a digit to get a 2 at the end is either a 3 or an 8. Went through the case with the 3 it does not work because any time you multiply 4 times 2, you’re gonna get at least 800.

R: Okay.
Maggie: So, you don't get a 300. Then, with the 8 case, though, um, this middle digit when multiplying by 4 with the added 3 from our 8 times 4 is 32, should equal that same number but the only way to do that is that $x$ is -1.

R: Okay.

Maggie: Which just is illogical.

R: Okay. [laughter, videotape ends here] Alright, is there any um, strategies that you want to point out you used during this part of the problem?

Maggie: Um, a lot of going back to what I had proven the last time and basing that on the, my initial – whatever you want to call it.

R: Okay. Anything else?

Maggie: Pretty much from there, once you got a, it’s kind of the same format with the 3 or 8. Um, then you basically have just your two cases to work through.

R: Okay. It sounds like you’ve pulled in just a teeny bit of algebra but not too much.

Maggie: Yeah.

R: Just a little bit. [laughter] Just to prove that that case is not possible.

Maggie: Right.

R: Okay. Alright. **End of Interview
Interview #10  (Total time = 59:05)

R: The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16, for example. Prove that the smallest possible value for the sum is 14. Any questions about the problem?

Paul: [reads the question, pause] Okay. I don’t, I don’t think so. You were right, this is going to be challenging. Okay. Alright, let’s see here. [reads question again, (15 sec)] So, the uh, the sum of the numbers between 1 and 10, let’s see, what is that? That would be my first step.

R: Okay.

Paul: Like uh– it’s just a series. Like it’s uh, 10 times 11 divided by 2. [writes 10(11)/2] I think. I guess I’ll check that by doing a smaller–

R: Ah, okay.

Paul: –number. [writes 1 + 2 + 3] would be 6, so– [writes $S_{10}$ in front of 10(11)/2, researcher gets out calculator, (5 sec)]

R: I’ll give you the calculator just in case you want it.

Paul: Oh, okay. So, that’s 55. [writes = 55] And if it’s a pentagon, well yeah– [pause] Let’s see. [checking the numbers on example pentagon, (10 sec)] I guess it means that each side would have to be 11, sum up to 11. [writes $s_i = 11$, (10 sec)] Okay. But, uh– [pause (10 sec)] Hmm. [pause (5 sec)]

R: Can you tell me what you’re thinking?

Paul: Well, yeah, I’m kind of having– I guess uh the best way, I guess to go would be to set up an equation with uh– since the vertices I presume are going to be counted twice. At least, I’m assuming that.

R: Okay.

Paul: The sides, uh– [looks at example pentagon, pause, (10 sec)] so I guess, can I write on this guy? [pointing to example pentagon]

R: Oh, yeah.

Paul: Okay. [on example pentagon labels vertices $v_1$ through $v_5$ around the pentagon, (5 sec)] $v_1$, $v_2$– So, each vertex will get counted twice. [pause (15 sec)] Hmm. [pause (10 sec)] Yeah, I’m not sure. I’m not sure at all– [pause, reads the question, (10 sec)]

Maybe there would be a different way to go about doing this, by the uh brute force
technique. Uh. [pause (5 sec)] I guess you could assume with the uh– I might be a little
lazy ‘cause I’m sick, uh. [laughter] I want to say that the smallest value is not 14, and uh
just go from there.

R: Okay.

Paul: [pause] Let’s see smaller than– If you do that, then, uh, I guess you’d have to break
it into two cases, where the smallest value is less than or greater than.

R: Oh, okay.

Paul: I guess that’s be really tough to do. Am I speaking loud enough?

R: Yeah.

Paul: Oh, okay. Okay, good. Uh. Hmm. [looks at question again, (10 sec)]

R: So, what else are you thinking?

Paul: I don’t– Well, that’s – uh, I’m not uh, not coming up with anything really brilliant
at this point, so uh. [pause (5 sec)]

R: So, describe to me again where you were going with the 2 and labeling the vertices.

Paul: Well, I was just going to– uh, since each vertex is counted twice [points to where
he labeled vertices], I was just going to uh– yeah. Twice of every vertex, plus one of
every side –

R: Okay.

Paul: – would have to equal 55 every, every time. Um.

R: Could you pursue that idea any further?

Paul: Well, I guess let’s try.

R: Okay.

Paul: Um. I guess, uh. [writes $2(v_1 + v_2 + v_3 + v_4 + v_5)$, goes back to example pentagon
and labels sides $s_1$ through $s_5$, then adds to last + $(s_1 + s_2 + s_3 + s_4 + s_5) = 55$, (30 sec)] .
Uh. Let’s see I guess, uh, we could, uh [looks at what he just wrote, pause, (10 sec)] We
could say take this [points to equation he just wrote], maybe, now that we have this
established kind of massage it and find expressions for each, each of the sides. Maybe
draw some sort of conclusion about that. So, I guess, $v_1$ plus $s_1$ plus $v_2$ [writes $(v_1 + s_1 +
v_2)$, pause, (10 sec)] Oh, okay. Hmm.
R: What was the oh for?

Paul: Oh, I just do that spontaneously.

R: [laughter] Sounded like you had an idea.

Paul: I did, well, allegedly, not anymore, sorry. Uh. Plus this plus this– [writes \( + (v_2 + s_2 + \) ]Oops, that’s not right. [crosses out equations with sums that he has written, (10 sec)] It’s just [writes \( (v_1 + v_2 + v_3 + v_4 + v_5) + (s_1 + s_2 + s_3 + s_4 + s_5) = 55 \), (20 sec)] .

Hmm. [pause, looking at the last equation he has written, (15 sec)] This uh– \( v_1 \) plus– [writes \( v_1 + s_1 + \), pause, crosses this out, pause, (20 sec)] I uh, if I add all the vertices to both sides, then on the right hand or left hand side – we get all of the, I guess you could say all the combinations.

R: Okay. Can you describe that a little further, what you mean by that?

Paul: Yeah. Um, if you just add all the \( v_i \) plus da ta da \( v_5 \), that quantity to both sides, then you will, the sum on the left hand side of the equation becomes the sum of all of the edges, you know, in question [points to the edges on the example pentagon].

R: Okay.

Paul: That we are trying to find some sort of lower bound on.

R: Okay. So, you’re thinking when you sum up those edges, the vertices need to be counted twice?

Paul: Um, I uh–

R: That was your idea, right?

Paul: Let me just read it again, I think so. [rereads the question, pause, (15 sec)] Yeah, I think so. I might just have to retake English, but–[laughter] So, that’s \( E_1 \) plus \( E_2 \) plus \( E_3 \) plus \( E_4 \)– [writes \( E_4 = E_1 + E_2 + E_3 + E_4 + E_5 \ )] And this is equal to [writes \( = 55 + \) \( v_1 + v_2 + v_3 + v_4 + v_5 \), pause, (10 sec)] I guess.

R: Can you say anymore about it in relationship to what you’re trying to prove? Can you go any further?

Paul: Um. Aside from– [pause (10 sec)] Let’s see, 14 times 5 is uh, is 70. [pause] So, [pause (15 sec)] we have to multiply– The smallest, I mean you know all the vertices, the smallest it could be is if, you know, 1, 2, 3, 4, 5.

R: Okay.
Paul: Which is uh, is 10. So, the smallest case \[ \sum_{i=1}^{5} E_i = 65 \], (10 sec) Equals 65.

R: And that came from? Can you just describe that again for me?

Paul: That just came from the fact that I’m looking at the right hand side of the equation and I’m thinking, how can that be, you know, it’s least possible value. That just occurs when all the vertices are just, have values 1 through 5 respectively.

R: Okay.

Paul: Um, and that sum of 1 through 5 is— No, the sum of 1 through 5— I guess that’d be 15, that’d be 70 then. \[ \text{writes } 1 + 2 + 3 + 4 + 5 = (5)(6)/2, \text{ then crosses out 65 and writes } 70, \text{ (10 sec)} \] That’d be 70. I think I might be on to something here.

R: Okay.

Paul: ‘Cause 14 times 5 is, uh, is 70. Um. So, if all of these smallest possible value, 14, uh— [pause (5 sec)] I guess. My question is— [pause] you know, you could have – [pause]

R: So, why don’t you sum up what you just said again. So, you were saying that you’ve got the smallest case here.

Paul: Right, the smallest case, I guess, to minimize the left hand side of the equation \[ \text{points to equation of } E_1 + \ldots + E_5 = \ldots \] just occurs when \( v_1 \), you know, plus \( v_2, v_5 \) all those vertices, are labeled 1 through 5 respectively.

R: Okay.

Paul: That sum is 15. So, then the right hand side is 70.

R: Okay.

Paul: And, coincidently enough 14 times 5 is, is 70. We have five terms there. I guess I can’t – [pause]

R: So, what can you conclude then from what you found?

Paul: Um, I’m not sure much can be said. Um, hmm. I guess we can say that the, the average [pause (10 sec)]. We could say that the average possible value, okay that the average value for each edge when you minimize everything would be 14. But, I guess it doesn’t say that that could be the minimum, but that, that could be. That could also, you know, not be the minimum. So, I don’t know.

R: So, could you think about what happens if 14 isn’t the minimum. So, you’re questioning –
Paul: If 14 isn’t the minimum, I can think of an example where it uh, where it wouldn’t be. If you had like the vertices 1 and 2 and you had 6 between them, you know that, that wouldn’t be the minimum value at all, or that, yeah you know, 14 of course is greater than that 8, or 9. I think there’s something –

R: So, can you pursue that example a little further?

Paul: You want me to do that?

R: Yeah. You can just tear that sheet off. And work on the next on, that’s fine.

Paul: Okay. [tears off sheet of paper, starts on new sheet, draws pentagon, (5 sec)] Um. So, if you have 1 and 2, and if this were 6 [labels two vertices with 1 and 2, puts 6 between them], you have 1 through 5 here [labels vertices with 1 through 5] and that would be 9 [puts 9 as the sum of side with 1, 2, and 6]. Um. [pause, looking at pentagon he has drawn, (5 sec)]

R: And where would you place the rest of the numbers in that one?

Paul: Well, um, it would uh, let’s see. 6, 9, 5, um. [adding current sums along each side, (5 sec)] So, you need 7 and 8, 9, 10 [writes 7,8,9,10 off to the side]. Uh. Well, depending on, uh, you know other than 9, the least, the next greatest number is, could be 12 because 2 and 3 are going to go with those two vertex combination. You go with the lowest side in the middle and you get, you get 12 [writes 7 on side between 2 and 3, writes = 12 next to this]. And after you do that, well they’ll all be greater or equal to 14. Yeah. They’ll have to be, so–

R: So, is that the same as the example here? [researcher points to his pentagon and the example pentagon from the question]

Paul: Is it the same?

R: Meaning is it the same set–up, does the same thing kind of happen?

Paul: Uh. [looks at both pentagons, reads the question, pause, (15 sec)] Oops. I just, uh, I think I just completely misread the problem. Yeah, I did.

R: So, you’ve reread it, and what does it say?

Paul: I’ve reread it and, you know, each three-digit sum has to be the same.

R: Okay.
Paul: It has to be, you know 16, or whatever. And we’re trying to show that the– uh, okay. Okay, well, after we, taking that information into account. I think I solved the problem.

R: Okay.

Paul: I think I solved the problem. [picks up previous sheet of paper he was working on]

R: So, now that you know that each of those edges has to be the same amount, what can you say?

Paul: Correct. Well, I can say that when I take the least [points to his equations at the bottom of the page], the lowest case or whatever, minimum case when this is 1, 2, 3, 4, 5, all of these have to be the same, so I could just say, like $5E = 70$ [writes $5E = 70$]

And just say $E$ equals 14 [writes $E = 14$]. So, that has to be the minimum value.

R: Okay.

Paul: I think I solved it.

R: So, there’s no way to get anything less than 14 then?

Paul: No. I don’t think so. Just uh –

R: So, that’s what you – you can say from there then that, right?

Paul: I can say that just because, uh, um, yeah, I think I – yeah.

R: Okay.

Paul: Because, I’m just going to say I can and leave it at that.

R: Okay.

Paul: I think my reasoning might be a little fuzzy.

R: So, let’s go back to the problem and just read that one more time. Prove that the smallest possible value for the sum is 14.

Paul: Mm, hmm.

R: Okay? So, I’ll just ask you, and I’ll ask you this on each question, if you feel that you’ve solved it completely and that you’re satisfied with your solution at this point?

Paul: Okay. Um. I think I am, I’m really ashamed that I misread the problem like I did. But, I think I am. I think I used a little too much algebraic manipulation and not, uh, not
math uh, you know logic. I guess there’s logic in algebra, but, I enjoy like massaging
equations out versus looking at– I don’t like to look at the big picture unless I have to.

R: Okay.

Paul: I kind of like to work with equations, I guess. That might be something you glean
from this [points to his work on the question].

R: Okay. That’s a good insight into it. A very good insight.

Paul: Yeah, yeah.

R: Um, have you ever seen a problem like this before?

Paul: No, I haven’t.

R: Okay.

Paul: I have not.

R: And so, you described that you went straight to equations because that’s what you’re
comfortable with, basically, right?

Paul: Right, right.

R: And so that’s what you are likely to do in problems when you see them like this, is try
to find an equation like that?

Paul: Right.

R: Is it– can you say why that might be? Or is it experience or just what you’re
comfortable with, or what do you think?

Paul: Uh, I’ve always just been uh pretty strong in algebra and kind of a lot weaker in
geometry.

R: Okay.

Paul: You know, where you sort of have to look at the big picture. And uh, I don’t
know.

R: Okay.

Paul: I always seem to start out with equations like this. And if you need some insight, I
guess I try to go with that last, or you know. I don’t know, I guess it’s just the way I,
where my strength lies.
R: Okay.

Paul: I just go do what I’m good at. I don’t know if that’s sufficient.

R: Okay, sounds good. Alright, well let’s put this problem away for now.

Paul: That’s good.

R: And we’re going to move first actually to problem three before we go back to problem two, if we have time. [researcher collects papers and gives student next question] A traditional chessboard consists of 64 squares, 8–by–8. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes.

Paul: [reads the question, (10 sec)] Okay.

R: Okay? Prove that a generic chessboard of size m–by–n has a perfect cover if and only if at least one of m or n is even.

Paul: Mm, hmm. Okay. [reads the question again]

R: Where would you like to start?

Paul: Um.

R: Or, I guess I should say, do you understand the question? Is there any questions you have for me?

Paul: Yeah, uh, um. [reads the question] Adjacent, uh, that just means you can’t have any diagonal coverings or anything like that?

R: Right.

Paul: Okay.

R: So, it’s just a domino, just like a normal domino.

Paul: Right.

R: It’s just that it covers exactly two squares.

Paul: Okay. A perfect covering– [reads question] Okay, okay. [pause (5 sec)] Well, this would be a uh, bi–conditional statement, so I guess we have to do two things, I guess. Prove one direction then prove the other direction.
R: Okay.

Paul: [pause, looking at the question again, (5 sec)] So, um, I guess I’ll, uh, go in this direction [draws forward arrow] and assume that, uh, the chessboard of size those dimensions that have a perfect cover and show that they have to be even, one of them has to be even. So, I’m going to assume one’s true.

R: Okay.

Paul: [writes “$m \times n$ has a perfect cover”, (10 sec)] So, uh. [pause (10 sec)] So, I guess, uh, $m$ times $n$ would be the, I’m going to say area, although that’s probably terrible notation. [writes “$|m \times n|$ = area of chessboard”]

R: Okay.

Paul: Area of the chessboard. I would just I guess, $s$ times. [writes $m$, crosses out, writes $s(2)$] Uh, I’m messing my variables. All I’m saying is that the area of the chessboard, uh, is a certain amount of horizontal pieces and a certain amount of vertical pieces, and each piece counts for two, two units.

R: Oh, okay, you mean dominoes?

Paul: Yeah, dominoes, yeah, I’m sorry. Yeah dominoes.

R: Okay.

Paul: So, I guess I would do something like, [crosses out $s(2)$, whispering while he writes, writes $s(2m) + t(2n)$, pause, crosses out last line, pause, looks back at question, (1 min 20 sec)] Hmm.

R: Can you tell me a little about what you’re thinking?

Paul: Well, if I’m just thinking, I guess if, I don’t know, if both of them were odd, that it uh – [pause (10 sec)] it would be, uh, if we just show that it’s impossible to have both $m$ and $n$ odd –

R: Okay.

Paul: – then we’ve proven one of the uh –

R: Okay, so what’s your feeling as to why it’s impossible for both of them to be odd?

Paul: Um. Well, uh. [pause, looks back at question, (30 sec)] Hmm.

R: So, go ahead and talk out what you’re thinking here.
Paul: Okay. Okay.

R: Whatever’s running through your head I’d like to know.

Paul: Alrighty. Okay, okay. Well if, okay, well if both of them were odd, then your
area’s going to be an odd number.

R: Okay, and what were you thinking about when you reached that conclusion? I mean,
just what was going through your head? Just thinking.

Paul: Well, I was – [pause] I’m not sure, I’m sorry. I’m not sure.

R: Well, that’s fine.

Paul: But, right now I have a clear direction.

R: Okay. Go ahead.

Paul: If both of them are odd, um, then your number is going to be odd. So, you just
can’t, I mean you can’t divide that into 2. So, you cannot take portions of 2, and cover all
of that.

R: Okay.

Paul: And that contradicts the fact that uh, the chessboard has a perfect cover.

R: Okay.

Paul: So, the one time when that, what do you call that, the conditional statement or
whatever, is false. You know, that can’t happen, because the board’s a perfect cover, we
assumed that.

R: Okay, so how do we prove that if it’s odd–by–odd it has an odd number of squares?

Paul: [pause] Oh, well, I mean if you just, you can just say m is like 2k plus 1, where k
is an integer [writes $m = 2k+1$, $k$ in $Z^+$], a positive integer, and you say the same thing– I
mean, is this what you want?

R: Yeah. Yeah.

Paul: Oh, okay. 2j plus 1 where j is a positive integer [writes $n=2j+1$, $j$ in $Z^+$]. So, $m$
times n is, uh. [writes $mn=(2k+1)(2j+1) = 4jk+2k+2j+1$, (10 sec)] And that’s of the form
[writes = $2(2jk+k+j) +1$], you know, that since integers are closed under operations
that’s an odd integer. [writes that it exists in odd] That’s the definition of an odd integer.
R: Okay.

Paul: Um. And it’s, you know, just looking at that form, it’s, uh, it’s impossible to, uh, to divide an integer by 2 ‘cause you obviously can’t factor a 2 out from 1, um, at least with integers. So, because of that we know that, you know you’re going to have, you’re going to have what you need. So, that’s –

R: Okay.

Paul: That takes care of that, well half of that. Um.

R: So, that is proved that, that forward direction for you?

Paul: Uh, yes, yes, I–

R: Okay. So, where would you go from there?

Paul: Well, then you just do the reverse and I assume that at least one is even. [writes backwards arrow, then writes “Assume at least m or n is even”, (5 sec)] Um. Well, and um– [pause, looking at his work, (5 sec)] So, if, if m, you know, without, you know, just choose one case without loss of generality I guess.

R:

Paul: [writes WLOG] You know– uh, I have to write this, I never know how to use it properly. [laughter] m is 2k and n is – [writes m = 2k, and n = 2m+1] you know, so, [writes forward arrow, the mn = 2(2km+k), (10 sec)] So, mn has to be an even integer, [writes exists in even] so, um, so it’s possible to you know to divide by 2. You’re going to have an even number of compartments of 2.

R: Okay.

Paul: So, you can perfectly divide it up.

R: Okay.

Paul: Um. I don’t know, I think that I’m satisfied with that.

R: Okay. Alright. Have you ever seen anything like this before?

Paul: Um. I’ve seen problems like this before, [videotape stopped here, but resumes later] but they kind of required, kind of induction–type of proofs. A few with chess pieces.

R: Oh, okay.
Paul: Like dominoes and trominoes and things like that.

R: Really?

Paul: Yeah, yeah, I’m in discrete math, I like discrete optimization, so –

R: Okay.

Paul: Although, I don’t think you can induct on this.

R: Well, I think you found, you know, you found a proof without induction, so –

Paul: Yeah.

R: You know, so you’re okay with that.

Paul: Yeah, right. I think so.

R: And trominoes are probably going to be a little bit more difficult than dominoes, right?

Paul: Right, yeah, yeah.

R: ‘Cause you don’t know how to they fit them together necessarily, right?

Paul: Right, exactly. Yeah, so this is a little easier.

R: Okay, and so your work in those classes kind of informed your work here?

Paul: Yeah, it did, yeah. Yeah, I was um– I really didn’t understand like how to prove things when I took 305.

R: Oh.

Paul: ‘Cause I had, well I had a terrible teacher. So, I just kind of learned by taking some other 300 and 400 level math classes. Like how to go about doing it and –

R: Okay.

Paul: Um. So –

R: What other strategies, or what strategies did you use in solving this problem, do you think?
Paul: Well, I, for my first direction or implication I used contradiction. I remember just saying well if we assume that that’s false, and that’s what I did. And it contradicted our assumption that, that chessboard is a perfect cover.

R: Okay.

Paul: So, so I think I used that.

R: Can you tell me in your head as your thinking about this, are you thinking about specific numbers, are you drawing pictures, are you just thinking about equations, or –?

Paul: I’m just, uh, more of equations, not really pictures. Uh, yeah. So – I guess this might reinforce the notion that I’m more really of an equation aficionado, so.

R: Okay.

Paul: I don’t know.

R: Can you think of any other strategies that you used in here, or is that really it?

Paul: I guess maybe just a little brainstorming in the beginning.

R: Okay.

Paul: I really wasn’t sure where I was going. And that long pause at the beginning was kind of my, was kind of feeling out the problem.

R: Okay.

Paul: Which I guess it something, I guess that should probably be encouraged, so [laughter].

R: So, when you’re brainstorming what is happening, what are you thinking?

Paul: Um, well um. Well, first I try to reread the problem, because that’s kind of a problem for me.

R: Okay.

Paul: We’ve shown that – Um, I mainly just try to go through, you know, the different proof techniques, you know. We have induction, contradiction, which are more of the, kind of the fancier ones. And then you have brute force, I guess brute force through algebra, uh –

R: So, you’re kind of looking at each of those and thinking about whether they fit the problem or not?
Paul: Yeah, right, yeah. Yeah, how could they fit the problem. And also how hard or easy would it be for me to do it.

R: Okay.

Paul: You know, so. Yeah I guess, just kind of get out my playbook I guess, you know –

R: Okay.

Paul: And see what uh, what I could use. So –

R: Those are great insights, thank you very much.

Paul: Oh, no problem.

R: Okay, let’s try another problem. We’ll try one more. [pause] Alright, now we’ll go back to Problem 2. It’s [inaudible, referring to time] after, so we’ve still got a little bit of time. So, we call a positive integer $N$ a 4–flip if 4 times $N$ has the same digits as $N$ but in reverse order. And it wants you to prove that there are no two–digit 4–flips.

[video tape resumes]

Paul: Okay.

R: Any questions about what that means?

Paul: [reads the question] No, um. Not at all.

R: Okay.

Paul: I don’t. Yeah. It’s fairly easy to understand. Uh. [pause (10 sec)] Well, I guess one of the most obvious things to me, I guess just thinking about the problem would be to assume that there exists a two–digit 4–flip.

R: Okay.

Paul: And try to draw some sort of contradiction. Um. But the digits, that’s kind of tricky. So, uh [sigh]. I guess. 10 to the 1, [writes 10^1] oh, I’m writing on this.

R: Oh, that’s fine on that paper, too.

Paul: Are you sure?

R: Yeah. Either paper is totally fine.
Paul: Okay, alright. Um. We have ab as our two–digit 4–flip. [writes ab] Then– [whispering while he writes \(10^1(a) + 10^0(b) = \), pause, (25 sec)] So, that equals, well– uh. [crosses last line out] Um. [writes \(4[10^1(a) + 10^0(b)] = 10^1(b) + 10^0(a)\), pause looking at his work, then writes \(10^1(4a) + 10^0(4b) = 10^1(b) + 10^0(a)\), (35 sec)] Um. And uh, the leading terms, so it would kind of have to imply \(b\) equals \(4a\). [writes \(b = 4a\) and \(a = 4b\), crosses out] Maybe not, no, that’s not necessarily true.

R: Oh, I see. So, you were kind of looking at like the 10s to the 1s place?

Paul: Yeah, right, although you really can’t say that.

R: Okay.

Paul: Um. [pause (5 sec)] Or, you can use the congruence classes I guess, I don’t really know. [pause (15 sec)]

R: Can you tell me what you’re thinking of doing?

Paul: Um, maybe using congruence classes, um.

R: And what do you mean by that?

Paul: Congruence classes? Um.

R: In what way are you wanting to use them, exactly?

Paul: Uh, that’s more of a brainstorming statement, I really don’t know.

R: Okay, so the congruence classes of what?

Paul: Um. Well of I guess \(ab\), \(a\) and \(b\) being digits in our two–digit number. [pause, looking at work, (30 sec)] I guess this is– Maybe–if this is \(a, ab\) is going to be congruent to \(b \mod 10\). [writes \(ab\) is congruent to \(b \mod 10\)] Oh, it would be \(4ab\) would be \(4b \mod 10\), which would be, I don’t know if you can reduce that [crosses out last line]. Doesn’t seem very fruitful. [pause (35 sec)]

R: What else are you thinking?

Paul: Um. Something about a uh– [pause] Oops. [videotape ends]

R: I knew that was going to happen. [researcher changes videotape, pause, (10 sec)] Talk me through what were you thinking.

Paul: Oh, okay. Something about putting a bound, you know if you do have a two–digit thing, 4–flip. [writes \(10^1(4a–b) = 10^0(a–4b) \leq 9\) Uh.
R: Okay, a bound on it how?

Paul: Uh, I’m not, not too sure. [writes $a, b \leq 10$, pause, (50 sec) videotape resumes]

R: What’d you just write there? So, you combined? [audiotape side 1 ends] I see the first step. You combined the terms?

Paul: Yeah. And moved some things.

R: Okay, and then that second part?

Paul: Well, just by definition [crosses off $a, b \leq 10$, writes $0 \leq a, b \leq 9$, researcher flips over audiotape at this time] $a$ and $b$ are, you know, between 0 and 9. Just ‘cause they’re digits.

R: [audiotape resumes] Oh, I see.

Paul: So, um, you know– um. [pause (10 sec)] This uh, that [points to right hand side of the last equation he wrote] would at most be, uh – [pause (10 sec)] 9. And that, uh. [pause (5 sec)]

R: What at most would be 9, you’re pointing to –?

Paul: You know, something, $a$ minus $4b$, when you have, you know, these constraints on $a$ and $b$.

R: Oh, I see.

Paul: This would, this would at most be 9. [again pointing to his equations]

R: Okay.

Paul: So, we can say that, uh, that’s less than or equal, um, to 9 [writes $\leq 9$ beside his last equation]. But, you know– so we have 10 times $4a$ minus $b$ is less than or equal to 9 [writes $10(4a-b) \leq 9$, for all $a, b$], for all $a$ and $b$. You know, between 1 and 10. Uh, 0 to 9. And it’s uh, and it– I think I’ve solved it, just because $4a$ minus $b$, uh– [pause, writes $4a-b \leq 9/10$, (15 sec)] Yeah. So, so $4a$ minus $b$ has to be negative or 0 [writes $4a-b \leq 0$, pause, (10 sec)]. Yeah. And that, that can’t – [pause]

R: So, how could $4a$ minus $b$ be less than or equal to 0? That’s the question you’re mulling over, right?

Paul: Yeah, yeah, it could.

R: Okay.
Paul: It very well could, but not always. You know, this is—[pause] that’s not for all \(a\) and \(b\), that’s just for one of them. [crosses out the for all \(a,b\), pause, (15 sec)]

R: Can you go any further than that?
Paul: Um.
R: So, you said it’s true just for one of them, right?
Paul: Yeah.
R: So, you need to know—
Paul: Yeah, then \(b\) would be greater than—oh yeah, there it is. [writes \(b \geq 4a\)]
R: Okay.
Paul: Because, yeah. This would imply that \(b\) is, this statement here [points to \(4a-b \leq 0\)] would imply that \(b\) is greater than \(4a\). But, \(b\) by definition is between 0 and 9.
R: Okay.
Paul: That can’t happen. Well—[pause] Shoot, well—
R: What are you thinking?
Paul: Well, I mean that could be like 1 or something.
R: Okay.
Paul: You know, it could be 1. So—[looks at work, sighs] I think that’s, I mean, something. I think this might be the right approach. I think that given enough time I might be able to figure it out.
R: Okay, so you’d just keep going with these equations and keep manipulating them.
Paul: Yeah, I’d—
R: ‘Cause right now you have a bound on \(a\).
Paul: Yeah.
R: ‘Cause you know it could be 1, what else could it be?
Paul: Well, it could be 2. Well, it could be 1, 2, yeah just 1 and 2.
R: Okay.
Paul: $a$ could be 1 or 2. Oh.
R: So, that restricts us quite a bit.
Paul: Yeah, I would say just test those cases.
R: Oh, okay.
Paul: Okay, and just show--
R: Plug them into your equations and show--
Paul: Yeah, just show that, you know, if $a$ is 1, you know, the two equations. [starts on new sheet, writes $b \geq 4a$] So, $a$ is 1 [writes $a = 1$], $b$ is oh, 4, 5,-- [writes $b = 4, 5, 6, 7$] That seems fairly arduous.
R: Okay, anything else we could do with it though?
Paul: Um. Well, you could just, you could just plug it in. You know, plug--
R: Plug $a$ equal to 1?
Paul: Yeah. Well, I’m thinking if we set $b$, uh-- [pause (15 sec)] Well.. [pause (5 sec)]
R: What are you looking at?
Paul: Yeah, I think, you know, I’m kind of thinking if we take this result-- [points to $b \geq 4a$]
R: That $b$ is greater than or equal to $4a$?
Paul: Yeah, and plug it in maybe to a more general equation. Getting something more of, you know, an absurd amount.
R: Okay.
Paul: Something that’s like a full blown contradiction. Maybe, but I have a feeling I’m kind of going in circles here.
R: Well, we’ll probably stop there for time.
Paul: Okay, okay.
R: But, how about anything, anything new or different that you did here versus the other problems? Any new strategies?

Paul: Um. [pause, sigh, (10 sec)] I think I was still cranking equations, I don’t think that changed.

R: Mm ,hmm.

Paul: The strategy was different. I don’t think I did any sort of like contradiction proofs, uh–. [pause] I don’t uh– Oh. [pause (5 sec)] You know. I guess if you just consider the numbers, two–digit numbers between 1 and, well 10 and 25.

R: Okay, so you limited it to 25 because when you multiply by 4 you get too big?

Paul: Yeah, yeah, and I probably should have –

R: And you can do that?

Paul: Yeah. And uh, let’s see, with those numbers. Yeah.

R: And with what you’ve done you can actually limit it further, right?

Paul: Yeah, you can limit it further, yeah, you can go– If ab, if ab is our number, uh, then it uh–

R: You actually only have a few cases to try, right?

Paul: Yeah, well what I’m thinking is, ab is our number we started with. [writes ab] And that’s um [pause] I’m just trying to, could you give me some, can I go with this?

R: Oh, yeah, go ahead.

Paul: You know, and b is greater than or equal to 4a, um. [writes b ≥ 4a, pause, (10 sec)] I guess we’d still have to try, yeah I guess try the cases.

R: ‘Cause you’ve limited it to 10 to 25, right?

Paul: Yeah, 10 to 25 [writes 10, 25 and ab] and then you know you can, like you were saying, because of this constraint, you can limit it to, uh, 14. [writes 14, pause, (10 sec)] Yeah, yeah.

R: 14 to?

Paul: Ok, let me see. Then if a is 0, b is 1, b is 2. [writes a = 0, b = 1, b = 2] a is 1, b is 4. Oh, yeah. b is 5, – [writes a = 1, b = 4, b = 5, b = 6, b = 7, b = 8, b = 9, (5 sec)]
R: Okay.

Paul: And then if $a$ is 2 [writes $a = 2$], $b$ would be, the lowest $b$ could be is 8 [writes $b \geq 8$], which is above 25, so you know, just--

R: You actually just have--

Paul: Yeah, you just have 14 through 19 to check and just check those. And I don’t think--

R: And obviously there’s not one, so if you were to check em--

Paul: I think that’s the way I’d do it.

R: Okay.

Paul: So, um--

R: Um, okay. Did I ask you if you’ve ever seen anything like this before?

Paul: I have not.

R: Okay. Um, any classes or anything that you’ve done before that led you, um, in the direction that you went with this?

Paul: Um.

R: Like you pretty quickly formulated what the digits look like and pulled them apart.

Paul: Yeah, I’ve taken a number theory class, I think, here just a 200 or 300 level number theory class.

R: Okay.

Paul: Where we did proofs I guess, taking things like digits, or you know everyday things, remainders, things you know about and putting them in different ways. You know, like remainders, you translate them into congruence classes, you can do all sorts of things and stuff with that. Yeah, so number theory I guess. [pause] Yeah, probably just that.

R: Okay. *End of interview
Interview #11  (Total time = 51:35)
R: …can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16, just for an example. Prove that the smallest possible value for the sum is 14.

Sandy: Um, okay.
R: So, you can take a second to read through that if you want to.
Sandy: Okay. Do I have to start?
R: Yeah, you can start whenever you like. If you have any questions, you can ask me.
Sandy: Um, okay. [reads question, pause, (20 sec)] So, I would just have to rearrange these to where um, the lines, the sum is 14 instead of 16?
R: That’s right, that’s what that means. Mm, hmm.
Sandy: Okay. [reads question again, pause, writes down the numbers 1-10 under pentagon and writes “Find 5 ways to get total of 14 with 3”, looks back at this, crosses off “with 3”, then draws new pentagon, placing 4, 10, 1, 7, and 3 on the vertices as they are in the example, pause, crosses off entire pentagon she has drawn, (2 min)]
R: Okay, so you were thinking of starting with the same vertices as they have in theirs?
Sandy: Yes.
R: I noticed you looked at 4 and 3 –
Sandy: Yeah.
R: – and said, oh I need 7and that’s on the other one [laughing], so never mind.
Sandy: [laughing] Now, I’ll draw a smaller one. [draws pentagon to the side, pause, (10 sec)]
R: So, you’re just thinking about ways to –
Sandy: Yeah.
R: – draw it?
Sandy: I think so. And then I was thinking about just like putting ‘em all out and doing 14, like finding 5 ways of making 14, but that wouldn’t really work because on the vertices you’re using each one like twice [points to the vertices of the example pentagon].

R: Okay. So, oh you were thinking of 5 different –?

Sandy: Like forming 5 ways, yeah.

R: Okay. Okay.

Sandy: Well I guess you can just use the one number twice. Okay. [writes 10 3 1 in list at bottom of page, then adds 9 4 1, 8 4 2, to the side she writes 1 → vertex [sic], 4 → vertex [sic], pause, then writes 7 5 pause then 2, writes 2 → vertex [sic], then 6 5 3, 5 → vertex [sic], pause. Begins to cross off numbers in list of 1-10 above, starts with 1,2, 4,5, then looks back at the list at the bottom of the page, (2 min)]

R: Can you tell me what you’re thinking about?

Sandy: I’m just um, sorry, um finding sums of 14 using 3 numbers and then when I see that ones are being used twice [points to the 1s in her list at bottom of the page] that’s forming a vertex [sic].

R: Okay.

Sandy: And then, like 4 is used twice in each one [points to the 4s] so that’s a vertex [sic]. 5, 2 – [draws new pentagon and begins to fill in with numbers from her list, fills in 1, 4, 2, 5, looks back at her list, circles 3s, then writes 3 → vertex [sic] and puts 3 on the pentagon, (55 sec)]. And then I just noticed that I’ve, um, I have all the numbers, 1 through 10, but the numbers 1, 4, 2, 1 through 5, I guess. Being used twice –

R: Okay.

Sandy: – to be vertices. [fills in the rest of the numbers on the pentagon, (15 sec)]

R: Okay, so you’ve found one with 14.

Sandy: Yeah.

R: Okay.

Sandy: [checks her pentagon] Mm, hmm.

R: Okay, so let’s go back. Prove that the smallest possible value for the sum is 14. So you’ve found one with 14, right?

Sandy: Mm, hmm.
R: So what, if anything, um, do you need to do to finish up the proof? That 14 is the smallest possible value for the sum.

Sandy: Um – [pause, looking at her work, (15 sec)] So, would I try to like prove that 13 could be a possibility?

R: Alright, so what, what would you like to do? [pause, (5 sec)] So let’s say that you go through and you find that 13’s not a possibility.

Sandy: Mm, hmm.

R: What would you do next?

Sandy: Well it couldn’t be less than 10, the sum. Or, it couldn’t be – [pause] no it would have to be – [pause] ‘cause you have to use 3 numbers so the sum couldn’t be less than 13.

R: Okay.

Sandy: ‘Cause you have to use 3 numbers and –

R: You’re pointing to the 10 and the 1 and the 2, right? So –

Sandy: Yeah.

R: So, those are the –

Sandy: The largest and the two smallest.

R: Okay.

Sandy: So, 13.

R: Okay. And so if you proved that 13 wasn’t possible, you would feel confident that then you’re done?

Sandy: Mm, hmm.

R: Okay, so can you proceed and try to prove that 13’s not possible?

Sandy: 13? Sure.

R: And you can use this scratch paper, if you want.
Sandy: Okay. [writes 10 9 8 7 6 in list on new sheet of paper, adds 1 2 next to 10, then 1
3 next to 9, writes 1\(\rightarrow\) vertice \textit{sic} to the side, writes 5 next to 8, crosses out, writes 3 2
instead, 2\(\rightarrow\) vertice \textit{sic} to the side and 3\(\rightarrow\)vertice \textit{sic}, pause, (45 sec)]

R: So, I noticed first you wrote down a 4, right?

Sandy: I wrote down a 5 –

R: Okay.

Sandy: – and 8 and 5 is 13.

R: Okay. And then what’d you do next?

Sandy: Then I switched it to 3.

R: Okay.

Sandy: ‘Cause I, ‘cause I couldn’t use 4 because the 1.

R: Okay.

Sandy: It’s used twice.

R: So, you’re forced, you’re forced to use 3 and 2.

Sandy: Yeah.

R: Okay.

Sandy: [looks at list again, then writes the numbers 1 through 10 to the side and crosses
off 1, 2, and 3, (25 sec)] There is no way I can –

R: Okay, so you’re hitting a roadblock, right? And that’s good, right? [laughter] This is
good. So, what’s your road block with 7?

Sandy: Um, if I use 4, then I need a 2 and my 2’s are already used up.

R: Okay. You used it twice, right, so you can’t use it again?

Sandy: Yeah.

R: Okay.

Sandy: And if I use 5, then I need 1, my 1s are used up. But I can maybe, I don’t think –
[looks back at list again] I just used different numbers here. [points to row with 9] Well
that I have to use. [again referring to row with 9 1 3, then looks at row with 8, (5 sec)] I can’t use 4. But I can use – [looks at list again] Can’t.

R: So, everything was forced, right?

Sandy: Yeah.

R: Then you hit a roadblock.

Sandy: Mm, hmm.

R: So, you could, I think with a little bit of work, you could formally write that up, right?

What you’ve just discovered?

Sandy: Mm, hmm.

R: Okay, so you’ve shown 14 is possible, you’ve shown 13 isn’t possible and you made the note that it couldn’t be less than 13 because of the 10 plus 1 plus 2, right?

Sandy: Mm, hmm, yeah, you have to have 3 numbers.

R: Okay, so have you proven what you set out to prove? That the smallest possible value of the sum is 14?

Sandy: Yes.

R: Okay.

Sandy: Do I have to write?

R: No, you’ve verbally proved it, so that’s cool.

Sandy: Okay, good. [laughter]

R: Um, so, let me just ask you a few questions about it. So, have you ever seen anything like it before?

Sandy: No.

R: Like this question.

Sandy: No.

R: Okay. What strategies do you, would you say you used when doing this? Like you started by trying –
Sandy: I think I just took the top 5 numbers [points to original spot where she wrote down the numbers 1 through 10] and then went down, like I had 10, 9, 8, 7, 6 [points to list at bottom of first page] and I just formed the sum using the other two numbers and then um, when I saw that one was being, one number was being used twice, then I knew that that formed a vertice [sic].

R: Okay. And that’s pretty much the same thing you were doing over here [researcher points to second page], right?

Sandy: Yeah.

R: It was just the same kind of searching for an example, and – okay.

Sandy: And then I crossed the numbers off, as I used them.

R: Okay. Any coursework or anything that you’ve done before that led you to the way that you did this problem at all? That you can think of?

Sandy: [pause] Um, I don’t think so.

R: Okay.

Sandy: Not that I can think of.

R: Alright, let’s move on to the next one. We’ll just tear that page off so I can keep it separate. [researcher collects papers from student, gives student Question 3] Alright [sic]. Next question is Question 3 actually.

Sandy: Okay.

R: A traditional chessboard consists of 64 squares, 8-by-8. Suppose dominoes are constructed so that each domino covers exactly 2 adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes. [pause] Does that make sense, that paragraph?

Sandy: [pause, reads the question, (20 sec)] Okay, kind of, but not really.

R: Well, we’ll move on to this last part and then I’ll let you ask any questions, okay?

Sandy: Alright.

R: So consider a generic chessboard that’s size m-by-n. Prove that the generic chessboard of size m-by-n has a perfect cover if and only if at least one of m or n is even.

Sandy: [pause, reads the question, writes m → even, 2m, n → odd, 2n+1, m x n, (40 sec)] And so, then the chessboard would be like an 8-b- 7? [writes 8 x 7]
R: Sure.

Sandy: Okay. [reads question again, pause, (15 sec)] So, there’s one domino covering two squares?

R: Mm, hmm. So, it’s just a normal domino, just like normal, it just happens to cover 2 squares exactly.

Sandy: Okay. [writes “1 domino covers two squares”, crosses off $m \rightarrow$ even etc. written to the left, (10 sec)] I don’t know why I did that. I think I’m thinking of abstract algebra or math. [laughter, reads question again, pause, looks at work, then draws an 8-by-7 chessboard, labeling sides with 8 and 7, to the side writes “$8 \times 7 = 56$ if it takes 1 domino to cover two squares, then you will need 23 dominoes to have a full” crosses off full, writes “perfect cover”, pause (2 min 30 sec)] Okay, I don’t know. Um, I just noticed that 56 or 8-by-7 there’s 56 squares.

R: Okay.

Sandy: So, if it takes um one domino to cover 2 squares, then if you take 56 divided by 2, you have 23 dominoes, which would, with, and if the dominoes don’t overlap, you would get a perfect cover.

R: Okay.

Sandy: So, I don’t know if that’s right or not. [pause (5 sec)]

R: So, you have an example of an 8-by-7, so an even by an odd, and you noticed it comes out to an even number, so –

Sandy: With 56 squares.

R: – you should be able to cover it with 23 dominoes, you think, right?

Sandy: Mm, hmm.

R: Alright, so where would you like to go next?

Sandy: [pause, looks at question and her work, (5 sec)] Isn’t that all it’s asking?

R: So, have you proved what you set out to prove? Prove that the generic chessboard of size $m$-by-$n$ has a perfect cover, if and only if, at least one of $m$ or $n$ is even.

Sandy: So wouldn’t – [pause, rereading what she wrote, (5 sec)] So, anytime you use an even or an odd value for $m$ and $n$, you get, if the outcome is an even number then you know that you’d be able to cover the chessboard perfectly.
R: Okay, okay. So, when wouldn’t you get even? So you said, if you get even.
Sandy: So, I have to find, okay.
R: Right, that’s what you said, if you get even -
Sandy: I have to find one. Yeah.
R: - it works? Okay.
Sandy: [writes $2x1 = 2$, (5 sec)] If I did a 2 by 1 – [writes $4x3 = 12$, (10 sec)] you’ll always get an even, won’t you?
R: Okay. So, what are you trying? You’re trying even-by-odds, right?
Sandy: Mm, hmm.
R: And you think you’ll always get even, right?
Sandy: Mm, hmm.
R: So, that’s um at least one of $m$ or $n$ is even, you have one of $m$ and $n$ being even, right? How could you prove that that always happens?
Sandy: You’d do this, like let $m$ be even [writes “let $m$ be even, therefore $m = 2m$”] and therefore $m$ equals $2m$ and then let $n$ be odd [writes “let $n$ be odd, so $n = 2m+1$”] So, $n$ equals $2m$ plus 1 and then just multiply – [writes “if take $m \times n$”, (5 sec)] If you take $m$ times $n$, [writes “you get $2m(2m+1) = 4m^2 + 2m = 2(2m^2+m)$”, (30 sec)] which is always even – [writes “which is always even”]
R: Okay. [pause while student finishes writing, (10 sec)]
Sandy: I don’t know.
R: Okay. So what, if anything, um is left to prove? Or do you – or are you finished?
Sandy: And then you can do it vice versa, letting $m$ equaling odd and $n$ equaling even.
R: Okay, anything else?
Sandy: [pause, looks over work, (10 sec)]
R: Or are you good?
Sandy: I think I’m good. [laughter]
R: Um, so, what strategies did you use when you proved this problem?

Sandy: I just took \( m \) an even and odd number, multiplied them together, and found that it takes one domino to cover 2 squares [points to where she wrote this to the side] and so I just took that total number of the whole chessboard that I made by choosing an even and an odd and dividing that by 2 to figure out if it could cover or the if amount of dominoes could cover a full, like have a perfect cover.

R: Okay. So, you looked at a specific example and then kind of generalized from there a little bit?

Sandy: Mm, hmm.

R: Okay. Um, have you ever seen anything like this before?

Sandy: No.

R: Okay. Anything — um, you kind of mentioned that you wanted to use the, kind of the \( 2m \) and the \( 2m \) plus 1 right away from what, abstract?

Sandy: Abstract math.

R: Okay. So, that was just when you were doing proofs and they involved even and odds —

Sandy: Mm, hmm, yeah.

R: — that you split em up like that?

Sandy: Mm, hmm.

R: Okay. Okay, anything else you can think of?

Sandy: No.

R: Alright. Let’s move on. [researcher collects papers and gives student Question 2 part a] Question 2. We call a positive integer, \( N \), a 4-flip if 4 times \( N \) has the same digits as \( N \) but in reverse order. [pause] And the question asks to prove that there are no two-digit 4-flips. Any clarification you need on the question?

Sandy: [pause, reads the question, pause, (25 sec)] Okay, a 4-flip? What exactly does that mean?

R: Okay. So, you read it over again, right?
Sandy: Mm, hmm.

R: We call a positive integer, \( N \), a 4-flip. If 4 times \( N \), so you take \( N \) and you multiply by 4 –

Sandy: Okay.

R: – if it has the same digits as \( N \) did, but they’re flipped, they’re reversed.

Sandy: [pause (5 sec)] Okay, so \( N \) has to be 4 numbers or?

R: Well, we’re looking specifically in this case of a two digit 4-flip. So, the 4-flip just means because we multiplied by 4 –

Sandy: Oh, okay.

R: – it would make it true. But in the first case we’re just gonna look at two-digit \( N \)s. So if you multiply by 4, it’s gonna reverse those digits.

Sandy: Okay, so I want to prove something like take a number [tries to fix pencil, then gets out a pen instead, (5 sec)] Oh, I’ll have to get a pen. Okay, so say I have an \( N \) that has two digits equaling 12. [writes \( N = 12 \)] So, if I take 4 times 12 equaling 48, [writes \( 4\times12=48 \)] then you also want, or you’re also looking for a number where I take 4 times 21 equaling the same number? [writes \( 4\times21 = \)]

R: No, you would be looking for it to equal 21. So, if 12 was a 4-flip –

Sandy: Oh – okay.

R: – instead of 48 we would want to see 21.

Sandy: Okay.

R: So, 12’s not one but if it was, that’s what we would want to see.

Sandy: Okay.

R: And it turns out that there’s no two digit 4-flips and the question is asking you to prove that there’s no two digit 4-flips.

Sandy: Oh. [laughter, pause, looks at work and question again, pause, then writes \( 4\times10 = 40 \rightarrow 01, \) (35 sec)] So, that’s right? 4 times 10 equals 40 and we want to see 01?

R: Right. Exactly.

Sandy: Okay. [pause, looks at work again, (25 sec)]
R: What are you thinking?

Sandy: Well there’s, I don’t think there’s any way that because of this 4, [pause (5 sec)] that the number could be reversed like equaling when it’s multiplied by 4. I don’t know how to explain it. [pause (30 sec)]

R: What else are you thinking about?

Sandy: I’m not sure. [pause]

R: So, how can we prove that this can never happen?

Sandy: [pause (35 sec)] Um, I don’t know, I’m not sure.

R: Okay, what would you like to try next?

Sandy: [pause (10 sec)] Um, how taking the equation. [writes $4N =$] I don’t know what the reverse order of that, though, would be. [pause (30 sec)]

R: What else?

Sandy: I don’t know.

R: What else would you like to try? You tried 10, you tried 12, you tried thinking about what you could think of, in general, to prove it, right?

Sandy: Mm, hmm.

R: Anything else?

Sandy: [pause (10 sec)] I don’t know.

R: Okay. What else, if you were trying to do it for homework, what else would you try to do? Just to try to get an idea of how to do this proof.

Sandy: [pause (5 sec)] Well, I guess. [pause (5 sec)] if I took 4 times 12, [writes $4 \times 12 =$ 21] which would have to equal 21. [pause] And then if I solved for – maybe if I just took that and put in a number.

R: So, it seems like you’re kind of troubled ‘cause you want a way to represent it.

Sandy: Yeah.

R: Right, in the equation?
Sandy: Mm, hmm.

R: And you just can’t quite think of how to represent it and how to flip it and what that means, kind of thing.

Sandy: [shakes her head in agreement] So, if I did 4 times N equals 48 [writes 48 next to where she has 4*N =], or [crosses out 48] what do I want oh I don’t know [pause] or 4 times N has to equal 21. [writes 21 in place of 48, then writes N = 12] So, then N has to equal 12. If you solve for N, 21 divided by 4 is 5 and 1/4 [writes N = 21/4 = 5 ¼ ≠ 12], which is not equal to 12?

R: Okay. So, it doesn’t work for 12.

Sandy: Mm, hmm.

R: What about all the other two-digit numbers?

Sandy: [pause (20 sec)] I could just, I don’t know. [pause, then writes 4*N = 48, N = 84, (10 sec)] N equals 84. You’re dividing. [writes N = 48/4 = 12] There’s no way. I don’t know how to explain it.

R: What are you thinking?

Sandy: [pause] Something that [pause] like dividing by 4, with the number on the other side of the equation, you’re never gonna get the reverse of the N.

R: But why?

Sandy: [pause] ‘Cause it’s always going to be less than –

R: Okay and that –

Sandy: – the value that you need?

R: So, is that always true? I mean, certainly it’s always when you divide by 4, it’s always less –

Sandy: Yeah.

R: – than that original number, right?

Sandy: Mm, hmm.

R: Certainly that’s true. But can you think of an N where flipping it would make that number bigger than the original?
Sandy: [pause (30 sec)] {No}.
R: Okay.
Sandy: ‘Cause like your lowest number that you can use would be 10 to 99 [writes 10→99] of N. And maybe 10 would be. [pause]
R: So, those are all your two-digit numbers, right?
Sandy: Mm, hmm. [pause, writes 4*N = 10] So, N would have to equal 01. [writes N = 01, pause, then writes N = 10/4 = 2 ½, (25 sec)] So, whatever 01 is, so 10 would –
R: Well, it’s just 1.
Sandy: Okay, so that’s just 1. So, this is one case where the number would be greater than the number you needed.
R: Okay.
Sandy: So, I guess I can.
R: Is there anything else you can think of to do?
Sandy: [pause, writes 4*N = 11, N = 11, N=11/4 = 2 ¾, pause, (35 sec)] I can’t think of –
R: No?
Sandy: No. [pause]
R: Okay. Can you think of anywhere else to go or anything else to do?
Sandy: {No}. [pause] I just think that there’s just one case where your numbers gonna be greater than the number that you need in the reverse order.
R: Okay.
Sandy: And that’s, um equals 10 when you set this equation equal to 10 in all other situations it’s gonna be less than.
R: Okay.
Sandy: ‘Cause you’re dividing by 4.
R: Okay. So, um, walk me through what you would want to write down for your proof. So, verbally explain to me how – So, you said that you’re looking for it reversed, so
you’re looking at all these numbers and dividing by 4 and you’re always getting, what
you see is you’re always getting a number less than what you actually wanted when it
was flipped.

Sandy: Mm, hmm.

R: And you just told me that the only time that doesn’t happen is when you’re looking at
10 and you reverse it and it’s 1 and that’s the one case when it’s bigger. Otherwise –

Sandy: Yeah, but you can’t even use 10 because 1’s not a two-digit number.

R: No, it’s not.

Sandy: So you can’t, so your numbers aren’t, you can’t use 10, it would be 11 to 99.
[crosses off 10, writes 11 instead]

R: Okay. And so, you’re thinking of those as the results when you multiply by 4?

Sandy: Mm, hmm.

R: Okay.

Sandy: So then, yeah – So then, every number’s gonna be less than the N you need.

R: Okay. [pause] Alright. Any time, ah, any place you’ve seen anything like this
before?

Sandy: {No}.

R: Okay. Any strategies you used while you were working through the problem?

Sandy: No – from past classes?

R: Yeah, from past classes or anything that you were doing here like describe how you
went through the process. So, you started by trying to get a grasp on what it meant,
right?

Sandy: Mm, hmm.

R: So, you looked at 12 first to see okay, if we multiply 4 by 12, you get 48 and we
would have wanted 21 so just to kind of get an idea of what that meant.

Sandy: [shaking her head in agreement] Mm, hmm, and just seeing different cases using
the numbers like using two-digit numbers so I could see a pattern.
R: Okay. At some point you desired to be able to say $4N$ equals something and reverse it, you just couldn’t quite get that, right?

Sandy: Mm, hmm.

R: Okay and then you went back to trying more to see that pattern. Okay. Alright, let’s put that one away and we’ll try one more. [researcher collects papers and gives student Question 4] Okay. You may or may not have seen this before. Let $x$ and $y$ be two integers. We say that $x$ divides $y$ if there is an integer $k$ such that $y$ equals $k$ times $x$. Consider three integers $a$, $b$, and $c$. Prove the following. If $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$.

Sandy: Okay, so $a$ divides $b$. [writes $a/b$, $b/c$, $a/c$, then pause, writes “$x/y$ if $y = kx$”, pause, (40 sec)]

R: What are you thinking?

Sandy: I’m just, I just don’t understand how $x$ can divide $y$ if this is true. Because – [pause (5 sec)] If I solved for $k$, it would be $y$ divides $x$. [writes $k=y/x$, points to this result] Is that right?

R: Mm, hmm.

Sandy: [pause (20 sec)]

R: So, maybe it’s just the term, so this could be a new term for you – $x$ divides $y$. And what we mean is that when we say that, we mean that that equation is true. And that maybe counter to what you want to write.

Sandy: Okay.

R: But that’s what we mean when we say $x$ divides $y$, we mean that we can write $y$ equals $kx$, we mean that when we take $y$ divide it by $x$, we get a $k$.

Sandy: Oh, okay, so then $a$ divides $b$. It’d be $b$ equals $ka$?

R: Mm, hmm.

Sandy: Okay. [pause, writes $b = ka$ & $c = kb$, $c = ka$, crosses out what she wrote with $x$ and $y$ to the side, (10 sec)] So, this notation’s wrong then or – [pointing to $a/b$, $b/c$, and $a/c$]

R: Yeah, we wouldn’t –

Sandy: Okay.
R: We wouldn’t necessarily – I think it may have led you astray, right?

Sandy: Okay. Yeah. [pause (5 sec)]

R: So, we know we want to prove if $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$. Where would you go next?

Sandy: [pause (10 sec)] I’m going back to this. [points to what she crosses out with $x$ and $y$, pause, looking at her work, (15 sec)] So, $c$ equals $ka$, so this could be rewritten as $b = c$ [writes $b = c$ below $b = ka$, pause, (15 sec)]. And $c$ equals $kb$? Oh. [pause]

R: So, what else might you try to understand the problem and to figure out the proof?

Sandy: [pause] $b$ equals $c$ means that $ka$ equals $kb$ [writes $ka = kb$, pause, (30 sec)].

R: What are you thinking about?

Sandy: [pause (10 sec)] Um. [pause (10 sec)] So, what I’m trying to prove is that these [points to $b = ka$, $c = kb$, and $c = ka$], all these equations hold, or – [pause] Okay, this equation and this equation [points to $b = ka$ and $c = kb$] imply that this equation holds? [points to $c = ka$]

R: Right. Yeah, you’re trying to show that if $a$ divides $b$ and $b$ divides $c$, if you know that those are true then $a$ divides $c$.

Sandy: [pause, writes in if and then between $b = ka$, $c = kb$ and $c = ka$, pause, audiotape stops and is flipped over by researcher, (55 sec)]

R: Can you tell me what you’re thinking?

Sandy: Um. I just said that, I don’t know if I can use this “then” part, use the then to prove um this side [points to the hypothesis of her statement] but I set $b$ equal to $c$ and then since $c$ equals $kb$, then we know that $b$ equals $c$ implies this? [pointing to $c = ka$] I don’t know. [laughter]

R: Okay. Nothing else coming to mind right now?

Sandy: No.

R: Would you like to pursue it further or would you like to leave it?

Sandy: Um. [pause (5 sec)] I just don’t know if there’s anything else, where I could go. [pause (10 sec)] See, and then when I come to here [points to $ka = kb$], I just don’t see how this and this can equal that [points to $c = ka$]. I don’t know where to go from here. [pause (5 sec)]
R: Anything else you might try?

Sandy: [pause (15 sec)] No.

R: Nothing you can think of?

Sandy: No.

R: Okay, well I won’t torture you any further on that one. [laughter] Um, let me just ask you a few questions about it. Have you ever seen anything like it before?

Sandy: No.

R: Okay. And you really want to use those equations, right?

Sandy: Mm, hmm.

R: I mean, so is there any classes that you’ve taken or anything where you’ve done proofs kind of like this where you would have manipulated these equations to come up with it where it would have led you to try those kind of things?

Sandy: No.

R: Not that you can think of?

Sandy: No. Using these equations? [circles \( b = ka \), \( c = kb \), and \( c = ka \)] I don’t think so, no. [pause (5 sec)]

R: And it seems you’re kind of having a hard time maybe visualizing what’s going on here, right?

Sandy: Mm, hmm.

R: And maybe that’s why, because in the first problem you could easily visualize what you wanted.

Sandy: Yeah.

R: And so the proof came pretty easily to you, right?

Sandy: Yeah.

R: But in the second one and this one you’re having a bit of a harder time with it. If you can’t quite get a hold of what it is, right?

Sandy: Like grasping it, yeah. The idea isn’t coming, yeah.
R: So, it’s like even what you’re looking at itself is kind of just outside there, right?

Sandy: Mm, hmm, yeah. Like even though I’ve never seen any of the problems in any of my classes, really, but yeah, these ones I’m kind of struggling with.

R: That first one you could really see it.

Sandy: Yeah.

R: You could see it in your mind’s eye – what you wanted. Okay. Alrighty.

*End of interview*
Interview #12 (Total time = 56:10)

R: The numbers 1 through 10 can be arranged along the vertices and sides of a polygon so that the sum of the three numbers along each side is the same. The diagram below shows the sum 16, for an example. Prove that the smallest possible value for the sum is 14.

Rick: Huh, oh man, a proof. [pause] Okay. [pause, reads question, (10 sec)] So, you want me to talk out loud and everything?

R: Mm, hmm.

Rick: Okay. So, basically I just verified all of this stuff, um that the sum is indeed 16, now I have to figure out – [pause] Prove that the smallest possible value for the sum is 14. Um. [pause] Okay, so what does that mean? Obviously, they’re taking three consecutive numbers, you might say, –

R: – not necessarily consecutive, but three numbers –

Rick: Okay, so I could take 3, 1 and 4.

R: Oh, no, you mean consecutive around the pentagon?

Rick: Yeah.

R: Yes that’s true, consecutive around the pentagon is true.

Rick: Okay, okay, so like this is 15, yeah. [points to 9 4 2, going around a corner]

R: – and they mean just along one side.

Rick: And this is 17, okay.

R: So, just like 4, 2, 10 and 10, 5, 1, and 1, 8, 7 so that we only consider along the side.

Rick: Interesting, so even though like that’s 16 [draws line along a side, repeats with each side], that’s 16. I’m just going to verify this. Ha, yeah that’s 16, that’s 16, 16, 16, and 16, but the smallest possible value for the sum is 14?

R: Right, so you can change those numbers around –

Rick: Ooo.

R: – and make each of those lines you’ve drawn represent 14.

Rick: Okay, I see what you are saying, Okay –
R: So, and that’s the smallest possible, you need to prove that the smallest possible that you could do it is 14.

Rick: [pause] Hmm, so is one of the catches that they all equal each other too?

R: Yes.

Rick: Okay, so like this side is 14, 14, 14, 14, 14?

R: Right.

Rick: Okay, whew, okay. [pause] um interesting [laughter], you’re right this is going to take time. [pause] Um. So, it has to be three numbers adding to 14? [pause] Okay, so 16 times – Oh, I have a calculator. 1,2,3,4,5. [counting sides of the pentagon, then uses calculator] So, they all equal 80. [pause, calculates the total sum of all five sides, writes 80 on paper, (5 sec)] Huh. [pause (5 sec)] Okay, so each one has to equal 14, correct? [points to sides of the pentagon]

R: Right.

Rick: 5 values. [pause (10 sec)] Divide that by 5 would be 16. [referring to dividing 80 by 5, getting 16] Oh man, hmm. [pause] And this is possible?

R: Mm, hmm.

Rick: Okay, and people have solved this?

R: Yeah.

Rick: Within a decent time period? Okay.

R: What are you thinking?

Rick: Wow, okay. So, the sum of all the numbers is 80.

R: Okay.

Rick: And you have to distribute that to 5 sides, is that correct so far? Or is that cheating?

R: [pause] Currently in that picture, that’s true, yes. Your picture has sum 16 on it and that’s one example of what could happen.

Rick: [pause, writes 16, pause, (5 sec)] Say that again.
R: That picture right there. [referring to the example pentagon given]

Rick: Yes.

R: Those things that you said are true, that all of them add up to 80 and that they are distributed 16 per side.

Rick: Mm, hmm.

R: But, we want to know –

Rick: So, it has to be still a pentagon?

R: Mm, hmm.

Rick: It has to be five sides, wow what confuses me? So 80, you have 80 whatever, and you distribute that 5 different ways and yet the sum has to equal 14, so you’re like missing 10, so you are having to take off 10 equals 70 and divide that by 5 and then you get 14, I think.

R: Mm, hmm.

Rick: Oh wow, this is mind breaking. Um, you can’t add numbers?

R: What do you mean?

Rick: I don’t know, I’m trying to – you can’t subtract numbers, let’s put it that way, right?

R: Oh, to get your sums or whatever?

Rick: Yeah, you have to stick with these numbers, you can’t modify them with whatsoever, can you?

R: The numbers 1 through 10 can be arranged along these sides, you can rearrange them in any way you like, but there has to one on each vertex and one on the middle of each side, and they can be rearranged to make all sorts of different sums, but I want you –

Rick: Certainly –

R: – to prove the smallest way that you can do it, gets you a sum of 14.

Rick: And they can be arranged to equal any number of sums, but it has to be 5 sums that equal the same thing, correct?

R: Right.
Rick: [pause] Oh. Fun. I like this. This is good stuff. Am I going to get the answer at the end?

R: I don’t – well I can tell you after we’re done if it is –

Rick: Cool.

R: – if you are not able to find the answers, we can certainly –

Rick: Hmm. [pause]

R: What would you try next?

Rick: Okay, my main block right now is you have the sum equal to 80. You divide it by 5, you’re going to get 16 no matter what. But somehow, you have to divide that by 5 to equal 14. [pause, looks at researcher, laughter, pause, (10 sec)] I mean 7 divided by 70 divided by 5 is 14, right? Yeah. [pause] So, you have to lose 10 somehow, and it has to be a sum, correct?

R: Mm, hmm.

Rick: Can’t do negatives?

R: {No}.

Rick: Okay. Wow, this is cool, I like this. I’m not going to get the answer in the allotted time period, I know that.

R: So, what might you try next, you’re hitting a roadblock with your 80, right?

Rick: Mm, hmm. I’m thinking of some way to ah lessen that to 70. [pause] So all you can do is addition, it’s the only mathematical operation you can do. And yet the sum is 80. And you can’t do negatives. [pause] Okay, so you have to have 9 digits, no 10 digits, consecutive digits. [pause] So, you can take like 10,1 and 3; that’s 14.

R: Okay.

Rick: Uh. [pause]

R: Would you write down what you just said?


R: Don’t erase.
Rick: Oh, duh, duh, see?

R: [laughter]

Rick: Okay let me write that. [rewrites 9+1+4]

R: You can cross it off if you want.

Rick: [crosses out 9+1+4] Okay, let’s start with 1 plus 2, oh can’t [writes 1+2, then erases the 2] 1 plus [pause] 4 plus 9, 2 plus [writes 2+ pause] 3, Eh. [writes 3, erases] Let’s see. 2 plus 5, go with that. [writes +5] So that, so it’s going to equal. [pause (5 sec)] Oh shoot. [writes +7] Yeah, 7. 3 plus [writes 3+] Um, 6. [writes 6] plus, oh yeah. Oh no. [pause (5 sec)] Let’s see. So, I have too much. Let’s see here 2, 3, 4, 5, 6, 7. [checks numbers he has written so far] So, I can use 8 and 10 [writes 8,10 to the side] Shoot yeah that won’t work. Um. [pause (5 sec)]

R: So, right now you’re coming up with things that add to 14, right?

Rick: I’m trying to get, yes, I’m trying to get 5 different ways to add up to 14.

R: Okay.

Rick: With the allotted numbers. Hmm. [pause (5 sec)]

R: What made you stop that process?

Rick: Well okay, so I have 7, 8 digits here, and these are the last two digits 8 and 10, so the current pattern that was going on clearly is not going to work, I think. [pause, sigh, (10 sec)] So, let’s see. [draws pentagon, pause, (5 sec)] I don’t think that’s going to work. [draws another pentagon, wants to erase but stops himself] oops [laughter]

R: You can erase that if you want, and redraw it, that’s fine.

Rick: [erases new pentagon, draws another one, (5 sec)] Okay, this is called the um blood, sweat and tears approach where you just work things out until it works out.

R: Okay.

Rick: [pause] I don’t think that this – Let’s see 1’s, hmm. [pause] Okay, so 1 plus 5, plus 7. [writes 1+5+7 above pentagons] 2 plus 6 plus, um, no, oh yeah, 6 is 8, 7 again, no wait.[erases 7 and 8] Oh, I just totally erased.

R: It’s okay.
Rick: [laughter, writes 8 and 7 again but in right spots] I apologize okay, so 1, 5, 8, 2, 6, 7, so 3 [writes 3+] Can’t use 8. Um, Well, I could technically. See, that’s why I have to start placing things around. [pause] Let’s see where we go with this one. 2. [writes 2, erases it] Sorry. [laughter] I know that this is something that you are not going to mind –

R: You can’t resist.

Rick: I know that this is something that you’re not going to mind, plus you have it on tape, right?

R: Yeah.

Rick: [laughter, puts a few more numbers on the pentagon] Okay, interesting, so [pause, looks at work] Oh, I’m doing it wrong. Can I erase?

R: Mm, hmm.

Rick: [erases all numbers from pentagon] Okay, let’s try this. [writes a set of numbers going around a corner, laughter] I’m like erasing like crazy. [erases number around corner]

R: It’s okay.

Rick: Actually, I’ll just erase that one number right there. [Now has 1 9 4 on one side] There, that’s correct. [pause] So, use 2, 5, 7 here. [writes 2 5 7 along bottom of pentagon] Ooh that’s not going to work ‘cause you need 11 there. I’ll just put the big number on this side. [erases 2 and puts 7 on the corner] I’m totally erasing like crazy, but you know the process I’m going through.

R: It’s okay

Rick: [Laughter] Now, what is this?

R: It’s okay because I’m afraid that if I tell you not to erase it’s going to actually impede your progress even more so.

Rick: You’re very right.

R: Because it’s going to be against your process, so – proceed.

Rick: This is 6, okay. [writes 6 between 1 and 7] This is interesting, oh that nasty 10, oh that nasty 10, I don’t like that 10. [pause] Man, so that 10 has to be between [pause, writes 1 10 3 on other pentagon, (5 sec)] So, it has to be 1 and 3. [pause] It has to be 1 and 3, there’s all there is to it. Okay so now. [pause (5 sec)] So, um, so let’s let that equal that [referring to the 1 10 3 he wrote] Let’s see. I’ll put the 8 over here for now. [writes 8 on corner] I’m going to erase, I’m sorry. This is the blood, sweat, and tears.
method, like I said. [pause] Um. [pause (15 sec)] Okay, so let’s see 9, this has to be 4
[writes 4] no 5 [erases 4 writes 5] Ah I love it, okay, so I have 3 more numbers to work
with. Let’s see here, so I have 4, that’d be 9 and 2. [writes 4, 9, 2 to the side] Okay and I
can’t use 9 there and can’t use 9 there. [pause] Oh, dirty problem, this is a mean
problem. [pause] Um. So, you had people solve this without erasing?
R: Yeah, I mean you might start over, like you could redraw to not erase.
Rick: Yeah.
R: And it might be good not to erase because then you remember what you’ve tried, too.
You can use these, too. [points to extra sheets of paper]
Rick: Oh, yeah that’s right. [grabs new sheet]
R: You’ve got lots of extra paper.
Rick: That’s right I tore a bunch out. I tore out plenty. Okay. [draws pentagon on new
sheet] Uh, it’s always wrong, oh well. [referring to his drawing of the pentagon] Okay.
So, based on this [referring to his previous work], I’m thinking that I have to have the 7
and 8 in the middle somewhere closer. And the 9’s gonna have to be in the middle
somewhere too, I’m guessing, so I’ll –
R: And you make that determination because you were having problems with the 7 and 8
adding to too much was that what I –
Rick: Yeah.
R: – was seeing? Okay.
Rick: 10,9,8,7,6 [writes 10, 9,8,7,6 around pentagon on the middle of the sides, in order]
why not, let’s see what we can do here. Okay, so like I said this is between 1 and 3. I’ll
put 3 here since it’s the smaller one and that equals 1. [writes in 3 and 1 on vertices by
10, putting 3 next to 6] Uh oh. Well, 4 [laugh, writes 4 on the side with 1 and 9] 9 and 5
[writes in 5 with 3 and 6] Uh, oh., I might actually get this. 3 OHHHH! No I didn’t,
shoot.
R: Add again.
Rick: Yeah, I must have screwed up somewhere, 14, 14, 14, [adding up sum of sides he
has put in] 2 [writes in the last number, 2, puts pencil down triumphantly] Yeh!
R: Yeh! You got one with sum of 14, right?
Rick: I did it!
R: Does that prove that the smallest possible value is 14? Are you done with the problem?
Rick: Oh oh, that’s right. Oh man, yeah it says prove that – oh gosh.
R: So you proved that 14 is possible.
Rick: Yeah.
R: Right?
Rick: Okay, now here comes –
R: What, if anything, do you have left to do?
Rick: Here comes the 305 now, the class I’m not doing so well in. Um. [pause (5 sec)]
So, you have three numbers. [pause] Let’s see. [pause (5 sec)] I’m just looking at differences, I don’t know why. [writes 2 and 5 above pentagon, marking the difference between 5 and 7, and 7 and 2, respectively] 5. I’m actually probably making this –
R: Can you describe to me what you goal is here in what you’re doing?
Rick: I’m looking for patterns.
R: Okay, in order to do what?
Rick: Well you’re going to have 3 numbers. That equal, that’s going to equal, well the smallest possible value is 14. So, you’re going to have 3 numbers either way and that value is going to equal x, whatever, so \( a + b + c = x \). [writs \( a + b + c = x \)] Okay. [pause (15 sec)] Oh, let’s see, well when you take 1 through 10, the smallest value that can be is 1,2,3 so it’s 6. That’s the smallest. And 10, 9, 8 that’s 27? So, it definitely has to be between those two values, um. [pause, looks over pentagon with sum of 14, (20 sec)]
R: Can you tell me what you’re thinking?
Rick: What to do next. [pause] What I’ve established is a range of \( x \).
R: Okay.
Rick: I’m trying to think of any other mathematical tools in my very small head, my very small library of mathematical knowledge that is. [laughter] That I could, I could graph something maybe, I don’t know how. [pause, sigh, pause (25 sec)] Is there a time limit?
R: No, we’ll be here for about another half of an hour.
Rick: Okay we are gonna – we’re not going to even finish this problem, you realize this?
Well, most likely.

R: [laughter] – I don’t know, you didn’t think that you would find 14 and you found it.

Rick: You’re correct, you are correct. I’m just [sigh, pause, (5 sec)]

R: Tell me what you are thinking.

Rick: Where to start. So I can graph, and that’s I don’t even know how to graph so I can’t really go there necessarily, I know even know what point that would serve.

R: You found a range for x, right?

Rick: Mm, hmm.

R: You found that 14 is possible.

Rick: I have a range for x and a range for y. 6 to 27. It has to be 10 numbers. [pause (10 sec)]

R: So, you want to prove that the smallest possible value for the sum is 14, so you want to prove that what you found is the smallest possible.

Rick: So, what I’m having to prove, though – [pause (5 sec)] Oh, okay I can handle the proof. I’m having to prove the smallest possible value is 14, but how do I prove that?

Uh. [pause] So, well you could do the long approach and show that 13 doesn’t work, 12 doesn’t work, 11 doesn’t work.

R: Okay.

Rick: Which may not be bad because then you might be able to build a pattern that way.

R: So, why don’t you pursue that idea?

Rick: Okay.

R: So, you said basically you want to show that nothing below it is possible, right?

Rick: Yeah. [pause (5 sec)] Okay, so say the smallest possible value was 13 [writes 13 on page], okay. Why would 14 work? Well, let’s see ‘cause 14 times well yeah, that’s a bad way to think of it I think. [pause] Maybe ‘cause, okay so our first conclusion, well the first thing, um, is we had 80, the sum of the everything equals 80 and there was two numbers, so we subtracted 10 from that and we got 70. So, 80 minus 10 is 70 it’s kind of —
R: Can you relate that to the picture that you’ve drawn?

Rick: [pause, looking at picture of pentagon with 14, (20 sec)] So, the sum is 80 [writes = 80 next to pentagon], we have two numbers, consecutive numbers. [pause (5 sec)]

R: Can you check that for this? [referring to the pentagon]

Rick: That the sum is still 80?

R: Mm, hmm.

Rick: Oh, sure. [pause (5 sec)] Um let me think. [pause] I wonder since, well addition is commutative, right?

R: It is.

Rick: So essentially, it’s going to be 1 plus 2 plus 3 plus 4 plus 5, ‘cause you are going to have 3 plus 10 plus 1 [writes (3+10+1)] for that one, oh never mind, hold on, interesting, okay, so – [writes +(1+9+4)]

R: What was that thought that just occurred?

Rick: Well, you can use each number twice? Well no, wrong way to put it, you can use certain numbers twice.

R: Okay, which ones?

Rick: Um, every other number perhaps, I’m thinking maybe. [writes +(4+8+2) +7] 7, what am I doing? Yeah, okay. [erases 7, writes (2+7+5)] Sorry. [Laughter] I don’t know how people do math without erasing. [writes +(5+6+7)]

R: Maybe I should have forced you to do it with a pen.

Rick: Aw that would have been low. That would have been low. [pause] Okay, so it is every other number that you can use twice. [pause]

R: Okay.

Rick: [pause (10 sec)] Hmm. Okay. Okay, so I’m just trying to prove the smallest possible value is 14, and this is not an easy problem.

R: It’s not meant to be. So, that’s okay.

Rick: [Laughter, pause (10 sec)]

R: What are you observing?
Rick: 5 groups, I’m looking, none of these groups work. Well, essentially 5 groups and there is yeah going to be 14, 14, 14, 14 and 14. [writes 14 under each group of three numbers in his equation] Um, every other number used twice. [pause, bell rings in background, (10 sec)] Okay, it’s 3 o’clock.

R: 2

Rick: 2 o’clock, it could be 3 o’clock before I’m done. [laughter]

R: I’ll stop you before then.

Rick: YEH!!!!

R: But keep going.

Rick: This isn’t bad, it’s just that I’m slow, you know. I don’t mind this at all.

R: It’s totally okay. You’re fine.

Rick: In fact, we could do more after this, I don’t mind.

R: So, you’ve got your groups all laid out, observed that each group adds up to 14–

Rick: Mm, hmm. [pause] But still I have to somehow take into account that I can use every other number twice. [pause] Okay, so we have 15 numbers, is that right? [writes 15 to the right of the equation] Yeah. [pause (5 sec)] 14 [points to the 14 written below the groups of three] We’re only 1 off there. [pause (10 sec)]

R: Can you go any further with adding them all up?

Rick: [pause] Well you certainly can, I mean you could–. Well, it’s commutative, so–

Well, we already established that the total is 70 interesting so. [writes = 70 after equation, pauses, looks up at researcher, (5 sec)] Getting tired?

R: {No}, just watching.

Rick: [pause] So, I’ll draw this over here quick. [draws new pentagon on bottom left of page, (5 sec)] Oh, gosh that’s bad. [referring to his pentagon drawing] That’s why I type up my homework, well that’s one of the reasons. It’s nice and neat and the other reason is you can get just print it, reprint it over and over again without redrawing all of the good stuff. [puts numbers from example pentagon with 16 onto new pentagon, (5 sec)] Okay, so that was 16. So 1,8,7, plus, 7,6,5, [writes (1+8+7)+(7+6+5)] oops, 3 [changes 5 to 3, writes +(3+9+4) ] Plus 3 – plus 4 plus 5 plus 10. [writes +(4+5+10)]

R: That I believe is a 2, just to make sure that we don’t get–
Rick: Thank you [laughter, changes 5 to 2 in equation and on pentagon, (10 sec)]. Plus 10 plus 5 plus 1 [writes +(10+5+1)] Okay and that’s going to equal 80. [writes = 80, pause, (10 sec)] Looking for patterns, looking for patterns. [writes 16 below each set of 3 numbers in this equation] 16, 16, 16, 16, 16. Hmm. So, I wonder if I took – [pause] wow I could take a long time with this problem, ‘cause I wonder if um maybe the highest possible value is 90. [pause]

R: An interesting question, however, we’re not looking for highest, we’re looking for lowest, right?

Rick: This is true, but I’m trying to establish patterns. Okay, so we have 16, 14, try 15. Like I said, patterns.

R: I think it’s probably going to take you awhile.

Rick: Yeah, and not necessarily relevant for this, see that’s how mind works though. I’m–

R: But if you would like to, you may, we’ve got time.

Rick: But I’d like to get to another problem, too. So.

R: Well you can come back for another hour some other time and do some more problems if you want.

Rick: YEH!!!

R: Okay, yeh. That’s a good answer.

Rick: Could it be maybe after the semester’s done?

R: No. [laughter]

Rick: I’ve learned that I need to use my time much more efficiently, though I enjoy this, I still have a lot more math 305 to do. As well as a lot of Calculus 3 to do, as well as a lot of Chemistry as well as Pharmacy, so I have to use my time efficiently. Hmm. Well, hey I get 5 bucks for this right?

R: Mm, hmm.

Rick: Which means nothing, but no, but I appreciate –

R: It’s a little bit, but it is very minor in the scope of things –

Rick: – I do appreciate it –
R: – I get that. [laughter]

Rick: I appreciate helping you out more maybe for the betterment of mathematics as a whole –

R: There you go.

Rick: – I don’t know, yeah. You’ve got to find something else in life. Anyway, I’m good at philosophy.

R: So why is it that we can’t get 13 or 12, or 11?

Rick: Wow, you like totally gave me a huge hint there.

R: No, you asked that question before, you told me that you wanted to prove –

Rick: This is correct, but my mind doesn’t work that well.

R: You told me that you wanted to prove that you couldn’t get 13, and you couldn’t get 12 and you couldn’t get 11 –

Rick: I remember saying that.

R: – all the way down to 6.

Rick: All of the way down to 6, did I say that?

R: Mm, hmm.

Rick: Why did I say 6, oh yeah, because the smallest value that you can add to is 6. Um. [pause (10 sec)] Okay, so let’s just go back, say if it was 13 then I would have to prove that hey you’d be missing some, okay so let’s see. [pause (15 sec)] I’ve reached like a roadblock. I’m trying to think of how to start this because I want to prove, yeah what do I want to prove? Well, I want to prove that 13 wouldn’t work because, say you took 13 –

[laughter] [pause (5 sec)] Okay, so let’s just take 1 plus 3, 2 plus 10. [writes 1+2+10] So, we have – Oh, I shouldn’t have done that. Well. [crosses out 1+2+10] I’m not erasing.


Okay. [writes +6] Okay, this is the way to do it but it’s the long, very long way that’s– [pause] Oh, yeah well yeah, it’s not going to work. This is one of those frustrating things ‘cause it’s not going to work no matter what I try. [pause, sigh, (10 sec)]

R: So, can you think of any way of looking at those that you have just written down, the 1 plus 3 plus 9, 2 plus 4 plus 7, etc, etc. Can you think of any way that that’s going to lead to telling us that it’s not possible or can you pursue that any further?
Rick: I’m not sure yet, I’m going back to my old thing 1 plus 9 [starts new list, writes 1+4+9] just to see what I get. Okay, 1 plus 3 plus 10 [writes 1+3+10], so now 2 plus 4 plus 8 [writes 2+4+8] 2 plus 5 plus 7 [writes 2+5+7, pause]. 3 plus 4 plus 6. [writes 3+4+6] Okay. [pause] So, these all equal 14. [writes equals signs next to each row of this list]. Right? Yeah. So, we used the first number twice, twice, third number twice. Well, clearly, okay – So, you’re obviously going to use the lowest numbers twice. [pause] But, it stops at 5, oh no, yeah it stops at 5. We use 5 twice. We can’t use 6, 7, 8, 9, or 10.

R: Okay, so you made the observation, obviously you’ll want to use those smallest numbers twice. Can you explain that a little bit?

Rick: [pause] Hmm.

R: What you meant by that or how you reached it?

Rick: I – alright so we have the pentagon, or right? Yeah, the pentagon. [clears throat] Excuse me. [pause] So, you have a pentagon. [pointing to the pentagon with sum of 14 at top left of page] Yeah, let’s see so – [pause] You’ve got 5 sides. You have to distribute 3 numbers per side. [pause] Um, so you’re allowed to use 10 numbers in consecutive order. Well, 10 consecutive numbers. Um. So, you have to use – to get the smallest possible value, you’re going to want to use all the smallest numbers twice.

R: Okay.

Rick: [pause] Yeah. Um. Well, that’s the reasoning you get it, but how do I write that down? Well, I won’t write that up necessarily. Um. [pause]

R: What you just said is verbal reasoning is verbal enough to say that because you want the smallest, you want it to be 1 through 5, that’s okay to say it verbally for here.

Rick: Um. Okay. So, the next thing is – [pause (5 sec)] Well, let’s just, well, I can use that reasoning. Okay. [begins erasing list with 1+3+9, 2+4+7, and 3+5+6 above, then stops and crosses it off instead] Sorry.

R: You can start a new paper if you want to, too.

Rick: [pause] Um, so let’s see. [begins new list, top right, 1+3+9, (10 sec)] 1 plus 4 plus [pause] 8. [writes 1+4+8] 2 plus, let’s see, 3 plus 8 [writes 2+3+8] I don’t know, see this isn’t – this isn’t the best way to do this, I know this. Because this is going to take lots of time. [pause] I don’t know how else to do it though. [pause, writes 2+4+7, pause, (20 sec)] I used 2 twice. Um, oh. [pause] I used 3 twice. [pause]

R: Describe to me what you were just thinking.
Rick: Well, just – [pause] hmm let’s see, I was just trying to follow the old patterns, trying to prove that it doesn’t work. [pause] And it’s, I can see that clearly it doesn’t work, ‘cause the old pattern, 1,1,2,2,3,3. I already used 3 twice here, so I can’t put 3 here, I can’t put 4, I can put 5. [writes 5+] And if I use the bigger numbers, I’m going to get way above – I can use 5 twice, but again I have to use 3. [pause] And I’ve already established that I can’t use 6,7,and 8. 6, well I can’t use 6, 7, 8, or 9, or 10. I haven’t used 10 yet. It’s just not going to work. Um.

R: So, you haven’t put 10 in there?

Rick: This is true.

R: Can you try?

Rick: [pause] So, it has to go with 1 and 2.

R: And that’s forced, right?

Rick: What’s that?

R: That has to be. True? 10 has to go with?

Rick: Yes, it has to be, yes it has to, you have to – okay, if you use the number 10, then you have to use two other numbers to add to 13. [pause] So, I’ve already used 1 and 2 twice, and those are the only numbers that work. Yeah, those are the only numbers that work, at least in the numbers 1 through 10. [pause] So, yeah I would just –

R: Can you start this process over again with that thought?

Rick: Uh, which thought was that? [switches to new piece of paper]

R: So, the 10 with the 1 and the 2.

Rick: Sure, why not? [writes 10+2+1] Hmm, I’ll go backwards. Okay, so let’s get 13. So, we’ll go 9. [writes 9+, laughter] I love it. Um. [pause] Wow, what is – I know this isn’t going to work, so I’m trying to figure out a different way to express this. [pauses (10 sec)]

R: I think you might find this way lucrative. I think you’re heading towards a good direction, here. So, I’m going to encourage you to go further, because I think that you’re direction is, might be fruitful, or at least help you make some decisions about some of the other ones, too.

Rick: Hmm, I don’t know where to go first. 10, 2, 1. [writes 10+2+1 again] 10, 2, 1, [rewrites 10+2+1] yep. [pause] No, I’ve already established that I’ve used 10 twice. [crosses off this list of two lines, starts a new list] 10, 9, 2, 1. [writes 10+2+1 in first
line, then 9 in second line, pause, (5 sec) 3, 1. [writes +3+1 next to 9] Okay, so I’ve
already established that I – [writes 8, 7, 6 in lines below the 10 and 9] Okay, there. So,
I’ve already established that I can’t use those more than twice, more than once. Oh.
[pause] So, I’ve already used 1 twice. Oh, let’s just do 2 or 3 plus 2. [writes +3+2 next
to 8]. Okay, so I’ve already used 1, 2, and 3 twice. [pause] And it’s just not going to
work. [pause] ‘Cause if I use, uh, well all I can use now is 4 or 5. But if I use 4 or 5 in
either one of these [referring to the lines with 7 and 6] I have to use 1 or 2. Which breaks
that rule.
R: So, what can you conclude?
Rick: That it doesn’t work.
R: Yeah. Were any of those decisions arbitrary that you just made? The 10 with the 2
and the 1, the 9 with the 3 and the 1?
Rick: What kind of logic?
R: The 8 with the 3 and the 2? Were any of those –?
Rick: Alright, okay –
R: Could any of those have been changed?
Rick: Well, the number but the, the order, I’m sorry.
R: Okay.
Rick: The order, which since it’s all commutative ‘cause it’s addition. So, but no, it
cannot be changed in the sense that I’ve established already that the first, you have 10
numbers, the highest five numbers can’t be used more than once. The lowest five
numbers can be used more than once.
R: At least for low sums, right? That’s what you’re talking about?
Rick: Yes. Yes. Um. [pause] At least for the – okay so the lowest possible value is 13,
which I’ve already showed, well that works, that is. There’s a lot I’ve showed so far.
No, the lowest possible value is 14 that works. So, then I just tried the next lowest value,
which is 13, and I’ve hit a roadblock ‘cause I can clearly see that it does not work, though
because I’m not doing so hot in 305 I can’t really put that in words.
R: That’s okay. How about the next one? So, 13 doesn’t work.
Rick: Yeah.
R: What about 12?
Rick: 12, well. [pause] Okay. What do I have to do?
R: Can you use this process again for 12?
Rick: Well, certainly. You can use this all the way down to 1.
R: Okay.
Rick: [pause] And you hit the same roadblock.
R: So, describe to me what happens with 12.
Rick: Okay. Okay, we’ll take for instance 12. [writes 12 in box to the side] Oh gosh. I’m kind of in a hurry, so I can’t, you know, really – okay.
R: You don’t have to be in a hurry, this is as far as we’re going today, so–
Rick: Oh, man.
R: We only have 10 minutes left, it’s not going to be enough time to–
Rick: I can’t even get one done. Okay.
R: You’re almost there.
Rick: [laughter] Well, 1 plus 1. [audiotape ends side one, researcher flips over tape] No. Well, that’s okay, it’s on videotape anyways, so.
R: That’s right, I pick up what I can from either one. So, 12 we’re looking at, and you’re doing 10. And it has to go –
Rick: Well, 12. Since I’ve already established, since I’ve already established previously that the top five numbers you can use once and once only, but the bottom five numbers you can use twice and twice only. And to equal 12, I use the top number and I can only use the bottom numbers twice.
R: But, can you use, can you use 10 and 1 and 1? Is that okay?
Rick: Well, that equals 12.
R: That’s true.
Rick: But, it breaks all my other rules that I’ve established, kind of.
R: So, looking back at the pentagon.
Rick: Yes.

R: Can you use 10 and 1 and 1 on the side to get 12? Is that legitimate?

Rick: No. Well, [pause] No.

R: Because what would that mean?

Rick: [pause] That means I’ve already used the lowest number twice and the highest number once. Though if I use the next highest number, 9, the only way I can get that to equal 12 is to use 1 and 2. [writes 9+2+] And I can’t use 1, ‘cause I’ve already used it twice for 10. Well, actually it’s the only way I can. [pause, draws new pentagon, (5 sec)] So, 10 clearly has to go in the middle. [writes 10 in the middle of one side of new pentagon] So, we have 1 and 1. So, we have 9, 2. [writes 1 and 1 with 10, and 9 and 2 on another side]

R: So, my question was, is 10 and 1 and 1 legitimate for the statement of the problem?

Rick: 10, 1, and 1 legitimate for the statement of the problem?

R: Yeah, the numbers 1 through 10 can be arranged along the vertices and sides of a pentagon.

Rick: Mm, hmm. I’m not sure I understand your question, though.

R: Okay. Keep going.

Rick: Okay. [laughter] Um. So, we’ll use 8, 2. [writes 8, 2 on bottom] Okay, so we’ll use 7, 9, so we have to use 3. [writes 7 and 3 on right side, pause, (5 sec)] And we have to use 6, 9, 10 [writes 6 in last spot] Hmm, it doesn’t work. [pause] So, it worked for everything but the last one.

R: Did it?

Rick: Well, 7 plus 3 plus 2 is 12. 2 plus 8 plus 2 is 12. 2 plus 9 plus 1 is 12. 1 plus 10 plus 1 is 12. 1 plus 6 plus 3 is 10. [pause] So, no. [pause]

R: Moreover, where’s 4? Where’s 5? [pause]

Rick: [laughter] Oh, my goodness. No, they clearly are not on there. Yeah, I was going off a very, very strange logic there.

R: So, what happened? What happened?
Rick: Well, okay, okay. Since I assumed up here [points to where he wrote 10+1+1] you can use numbers twice, I assumed down here that you can use numbers twice [points to pentagon]. And you can’t, because you use numbers twice because you place, its placement.

R: Because if they’re placed where?

Rick: Yeah.

R: On the vertices, is what you are trying to say?

Rick: Yeah, yeah. [laughter] Golly, yes, I was silly. Well.

R: So, in fact, is 12 possible?

Rick: No.

R: How about 11?

Rick: No.

R: 9?

Rick: No.

R: 7, 6?

Rick: No, no, no, no, no. Nothing below 13–

R: Nothing below 13.

Rick: 14, 14, right? Yeah, 14.

R: Okay, so is there anything left to show, I mean other than writing this up, to prove that the smallest possible value for the sum is 14?

Rick: [pause (5 sec)] I don’t think so.

R: Okay. Have you ever seen a problem like this before?

Rick: I may have, I don’t know.
R: Okay. Um, can you describe any strategies you used while you were working?

Rick: Okay, in the beginning. Oh, any strategies, okay, it doesn’t have to be in order. Well, I’ll go in order.

R: That’s fine.

Rick: Um, well first off I looked for a pattern. Um. [pause] At first, what I did was just the blood, sweat, and tears way, just trying to get numbers to add up to 14, I guess is what we started with.

R: Okay.

Rick: And so then I realized that hey, you can use the numbers twice. That may have been after I realized you could redraw it, too, just start placing numbers in. Um. [pause] So, I realized those two things, I’m not sure which order they were in. It’s on the tape, so it’s all good. [laughter] Um. Okay, from there, I think I went to the other sheet, switched paper, which is always a good thing.

R: I think you went to that one. [researcher points to the second sheet]

Rick: Oh, yes, this is correct. Okay, so, I redrew it, well I drew the initial thing first, just to see how that worked. No, I didn’t, I redrew– Well, I redrew the pentagon and placed my hypothesis, my guess, let’s see, how did I do that?

R: That the 10,9,8,7,6 had to go on the sides?

Rick: Yeah, yeah.

R: So, it happened that your clever placement, that putting them in order–

Rick: Yeah.

R: –happened to work, right?

Rick: Mm, hmm.

R: So, that was a good lucky break.

Rick: Well, it just seems natural. I look for all the breaks I can get, you know?

R: Yeah.

Rick: And then, of course, the smallest number with the biggest number.

R: Mm, hmm.
Rick: Once again, in order. No, that’s wrong, not going in order. [pause]

R: No, but you placed them on the edges–

Rick: Well, in a way I did. Yeah,

R: You put the vertices–

Rick: In a sense I started in order, because I knew the smallest had to go with the biggest number. So, then I realized that hey, I has to go here. Then based off of that, then the next number I guess to 14 and then so on

R: Okay. At some point, you switched to putting them in the equations, right?

Rick: Like so? [pointing to his equations]

R: Yeah.

Rick: Yes.

R: Was there any time, any class, any previous experience that made you want to put them in equations? Was it the idea of doing a proof that you wanted something–

Rick: Well, the idea of figuring out the problem, look for patterns, that’s why I did that.

R: Okay.

Rick: ‘Cause the first thing I said was patterns, you know, so I–

R: Mm, hmm. Okay.

Rick: I kept wanting to establish pattern, you know. So. [pause]

R: And anything else you can think of?

Rick: In regards to?

R: Any other strategies you used?

Rick: Um. Other than blood, sweat, and tears? I like this part, actually. Um. [pause]

So, I tried to see if it works, well I verified that it works the first time. That’s the first thing I did, just verify that this is true. ‘Cause the problem asks to prove that this is, that the smallest possible value is 14. Then, I wrote equations to prove that hey it is 14. And I really didn’t prove that until I drew it in this. [pointing to pentagon] I think. Well, one of these two. [points to two pentagons]
R: Okay.

Rick: Actually, I did such a poor job in those that I went to this, so– [pause]

R: So, we spent some time in the beginning understanding the problem, right?

Rick: Yes.

R: Making sure that we were clear on where it went. And it took a little bit to make sure that there was a clear understanding. And then, you went to some examples, plugged some numbers in, looked for patterns. You went back to the proof and said oh, but I need to find that it’s the smallest and prove it. So, you established some bounds on what it could be.

Rick: Mm, hmm.

R: And you started investigating the further ones, right?

Rick: [Nods to researcher]

R: Okay. *End of interview.*
**Interview #13 (Total time = 54:05)**

Amy: [reading the question aloud] The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. [starts to highlight parts of the statement of the question, highlights “the numbers 1 through 10”, “along the vertices”, “sides of a pentagon”, “the sum of the three numbers”, and “is the same”, (10 sec)] The diagram below shows an arrangement with sum 16 [highlights 16]. Prove that the smallest possible value for the sum is 14. Oh boy. [pause, reads the question again silently, (25 sec)] Are you counting the, the sum of the numbers just on the sides or the vertices, as well?

R: So, it’ll be three numbers, so like 3, and 9, 4.

Amy: Okay.

R: Will add to 16 and 4 and 2 and 10 so along the whole edge there.

Amy: Okay. [pause, looking at question, draws pentagon, pause, tries numbers on the vertices, pause, crosses off numbers, (1 min 40 sec)]

R: Can I ask what you’re thinking about?

Amy: I’m just trying to arrange the numbers in my head. ‘Cause I’m figuring that it could be solved with a combinatorics proof but that isn’t gonna work [laughter] for me. So, I’d just rather try to solve it in a spatial form as much as possible.

R: Okay.

Amy: [pause, writes the numbers 1 through 13 to the side, crosses out 11, 12, and 13, pause, adds up numbers, writing 10, 15, 21, 28 above the list, (1 min 20 sec)] Hmmm – [writes 1+2 = 3+3 = 6+4 = 10, 15+6 = 21+7 = 28+8, 36+19 = 55, whispering while she adds, distracting noise from the classroom above during this time, (50 sec)]

R: So, what else are you trying?

Amy: Well, I just had an idea about adding, for whatever reason, adding up the numbers to see if I could do it that way. Honestly, at this point, I have no clue. [laughter, more noise from above, pause, (25 sec)]

R: Can you pursue the idea of putting them on the pentagon any further? So, just trying some more stuff.

Amy: Well, the problem is, is that even if I can arrange it somehow to you know to get the smallest possible value would be 14, it doesn’t seem like that would be a proof to me.

R: Okay. [someone opens the door to the interview room, but leaves again]
Amy: I mean to me a proof would be, you know, something that’s numerical in some form by showing it, er at the same time showing it as a picture but just showing it as one piece and not a whole just doesn’t make any sense to me.

R: Okay. [pause, more noise from above, (50 sec)]

R: What else are you looking at?

Amy: [pause (5 sec)] Just the numbers. Just hoping something would come out.

R: [laughter] Okay. Goodness sakes. [more noise from above, chairs and desks being moved quite a bit, very distracting, pause, (35 sec)]

Amy: [pause, writes numbers 1 through 10 on paper again, goes back to pentagon to put in values, unable to see what numbers are placed in video because her hair is in the way, scratches out numbers and tries them over again elsewhere, pause, (2 min 10 sec)]

R: So, you’re just trying out some things. What are you thinking?

Amy: Just trying [pause] part of the problem is I can’t remember exactly what I’ve written.

R: Oh.

Amy: And it’s kind of difficult when trying to arrange it, the digits, so that the sides are all 14. [pause]

R: There’s a lot more paper here if you’d like that, too.

Amy: Thank you. [pause, again cannot see what she is writing on video tape because of her hair, draws new pentagon, writes numbers along bottom, lists numbers 1 through 10 to the side, (60 sec)]

R: So, you’ve gone back and have rearranged that bottom –

Amy: Yeah –

R: Right?

Amy: It’s not working, it’s not working on that side. [pause, crosses off numbers in list that she has used in pentagon, (20 sec)] Okay. [pause, writes numbers along top edge, erases and tries again, working on all sides of the pentagon now, erasing and trying numbers all around, loses track of what she has tried at times, makes several addition errors, pause, (2 min 40 sec)]
R: Any other thoughts that are coming to mind as you’re doing this?
Amy: Well that there’s gotta be an easier way of doing this [laughter].
R: ‘Cause you’re just trying to find the one with 14, right?
Amy: Yeah.
R: So, you’ve been trying some different possibilities and –
Amy: Yeah, my real problem right now is that is seems like I can’t add. [laughter, pause, (10 sec)] Okay, do this. [works with pentagon still, erasing and trying new numbers on all sides, at this point she has 3 10 1 and 8 and 5 placed along the sides, (20 sec)]
R: So, you decided you needed to move the 3 with the 10.
Amy: Yeah, for some reason, just to see if it would work.
R: Okay.
Amy: [pause, looks over pentagon, (10 sec)] Okay. I have 4 left. 6 left, 7 – [writes 4, 6, and 7 under list of numbers, checking her pentagon with her list, continues to try new options, (40 sec)]
R: So, you’re just running into some problems. Once you get a couple of sides down, the rest don’t fit in, right?
Amy: Right. [pause, continues attempts at the pentagon, lots of erasing and trying different numbers, still has 3 10 1 and 1 8 5 kept on pentagon, working on other sides, pause, (1 min 30 sec)]
R: Any other thoughts you’ve had along the way?
Amy: Um, I wish I could use negative numbers. [laughter, pause, (10 sec)]
R: Any insights into the things that are causing you problems or – like which numbers might be difficult or anything like that?
Amy: It just seems like the 2 really isn’t treating me that well. [laughter] But then, other than mixing up 2 and 4, you know. [pause, writes 2, 4, 7, 3, (20 sec)] 2, 4, 7, 3 – [pause, draws new pentagon, begins placing numbers along the sides, again unable to see what numbers are being placed because of the blocked video shot, except occasionally, pause, rewrites a list of numbers 1 through 10, crosses off ones she has used, (2 min 15 sec)]
R: Can you think of any other ways to approach this guess and check system?
Amy: Well, what do you mean?

R: So, clearly we’re having some kind of circular issues or something here, right. Like putting them around this pentagon is not an easy process.

Amy: No.

R: Right? And randomly doing it, we’re gonna go a long time, right? Can you think of anything else, um, that you can do to try to think about how these numbers have to be placed. Like outside of just putting em on there, can you think of anything else?

Amy: [pause, puts pencil down, (10 sec)]

R: Any decisions that are forced, anything that needs to happen? Anything coming to mind?

Amy: Nothing.

R: Well you can tell me, would you like to pursue this problem or would you like to try something different?

Amy: I think I’ll try something different.

R: Okay. Sometimes we get stuck in our, in a loop in our brains, you know? [laughter]

Amy: Yeah.

R: And you just go, okay, I need to put this down. [researcher collects papers, gives student next question] Let’s try this question. A traditional chessboard consists of 64 squares, 8-by-8. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes. Does that make sense, what it says?

Amy: Yes.

R: Okay. Consider a generic chessboard of size $m$-by-$n$. Prove that this generic chessboard has a perfect cover if, and only if, at least one of $m$ or $n$ is even.

Amy: [pause, reads the problem, draws 8-by-8 chessboard, highlights “a traditional chessboard consists of 64 squares 8x8”, draws domino to the side, (1 min 45 sec)]

R: Can you tell me what you’re picturing in your head?

Amy: I know what the domino looks like. [laughter]
R: If nothing else, draw a good picture, right? [laughter, pause] Okay, that’s about right.

Amy: Let’s see, this is – okay, so it’s 8-by-8, 64 – [writes “64 squares”, pause, again her hair is blocking the video shot during her very quiet times because she leans over (30 sec)]

R: Any questions you have about what it says?

Amy: Well, it’s something I’m trying to remember ‘cause I remember doing this proof before.

R: Oh really?

Amy: And, from what I recall, is that, and I’m just trying to remember what the wording was before, but I found that you couldn’t do it, because, er, could be that, let’s see. Have to be if and only if at least one of m or n is even. [pause (15 sec)]

R: So, this looks at least similar to something you’ve done before?

Amy: Yeah.

R: Okay.

Amy: [pause, looking over chessboard, seems to be outlining with her fingers where dominoes would go, (35 sec)] Okay. So, let’s see – exactly two adjacent squares of the chessboard. And it looks like [points to her drawing of a domino] – so, you’d cover something like this. [outlines two squares of the chessboard]

R: Right.

Amy: [pause, outlining dominoes on chessboard, (5 sec)] Okay, so say you get to row 8. [still outlining dominoes, pause, (10 sec)] I mean if there are two, if there were two dominoes that looked like this [points to her drawing of a domino] and there’s 64 squares –

R: Mm, hmm.

Amy: – I mean 64 is an even number, I mean, if you divide, I mean you can divide an even by an even, so – that’s the same thing if I do it this way. Wait, covers two of them. [outlines dominoes on chessboard in a different way, pause, (30 sec)] So, say m is even and n is – [writes \( m \rightarrow \text{even}, n \rightarrow \text{odd} \), (10 sec)] Take off the – [pause, drawing on pentagon but cannot see past her hair, (40 sec)] When you say at least one of \( m \) or \( n \) is even, are you talking about a row or a column?

R: Yeah, either or.
Amy: Okay. [pause, writing but her hair is in the way again, (20 sec)] So – [pause, drawing] Doesn’t cover. [writes “\( n-1 \) rows (physically the last would be off the board), \( n \rightarrow \text{even}, m \rightarrow \text{odd}, m-1 \) rows (physically the last...”), (60 sec)]

R: So, you’ve written physically the last –

Amy: Well, yeah, would be off the board. So, if it looks something like that. [indicates coming off the board with her highlighter going over the edge of the table]

R: Okay. And that wouldn’t be called a perfect cover it that happened.

Amy: Okay. [crosses off what she had written]

R: ‘Cause it’s too big, so to speak.

Amy: [pause, goes back to chessboard and draws more, (35 sec)] Oh, okay, I can see how to do it. Alright, I did something really dumb. That last row, instead of having it trail off, turn it this way. [indicates being horizontal rather than vertical by turning her highlighter horizontally]

R: Okay.

Amy: And they would be all even. The same thing if you had it, I mean, if you had it \( m \) rows, you could just turn it that way [indicates horizontal using highlighter] if you had a \( n \) and you just turn it this way [indicates vertical with her highlighter], you’d just turn it so that the last remaining row of dominoes would be lying this way and then it works.

R: Okay.

Amy: And then if it were \( m \)-by-\( n \), you could do it both ways.

R: Okay. Okay, that makes sense. Um, what if anything is there left to do to prove that it has a perfect cover if and only if at least one of \( m \) or \( n \) is even? [pause] Or what would you do to write this down? You don’t have to necessarily write it down for me but you can describe verbally what you would want to say.

Amy: Um, I just, I’m still having trouble ‘cause I thought I did.

R: Okay. So, you said that, um, if at least one of them’s even, you’ve demonstrated how it would work, right?

Amy: Mm, hmm.

R: Okay. Um, have you ever seen anything like this before? You said something like it, right?
Amy: Yeah, yeah and I think it was just structured a little bit differently and that’s why I was having a little bit of trouble but I think this problem’s a little bit easier that I could actually pick things up and move – [picks up her highlighter and shows how she indicated the rows of dominoes with it]

R: Oh, okay.

Amy: – where I needed to go. So, I think it made it a little easier, I don’t know.

R: Okay. What strategies did you use when you were doing this problem?

Amy: Um, just, I mean I know what a chessboard looks like. I need, ah, it helps to know that, at least in my mind, that the chessboard has different colors –

R: Okay.

Amy: – so like and precise rows so I know where everything is supposed to be at one given point. Um, there are no numbers involved, it’s just um it’s the physicality of it. I guess I really don’t know how to explain it.

R: Okay. So, are you visualizing a chessboard in your head?

Amy: Yes.

R: Is that what’s going on?

Amy: Yes.

R: Okay, so you’re thinking about actually putting the dominoes on there?

Amy: Mm, hmm.

R: Okay. And just thinking about what happens so like you drew the picture to kinda put what was in here [researcher points to her head] down on the paper, right? And then cut off the row and see what happened differently, right?

Amy: Right.

R: Okay. Cool. One more. [researcher collects papers, gives student next question, student begins to read it right away]

Amy: [whispering while she reads the question, (10 sec)] Same digits as N but in reverse order.

R: This is probably nothing you’ve seen before, right?
Amy: No, no.

R: So, we have a positive integer, $N$. We have some number, Okay? We call it a 4-flip if when you multiply by 4, it flips it around. So, we get the same exact thing but in reversed order.

Amy: So, 14 becomes 41?

R: Exactly.

Amy: Okay.

R: Exactly. And so we want to prove that there are no two-digit 4-flips. So, whatever two-digit number you choose, when you multiply by 4, you’re not going to get what you want. Like when you multiply 14 by 4, you don’t get 41.

Amy: Okay.

R: How might you go about proving this?

Amy: [pause, reads the question again, pause, (40 sec)]

R: What are you thinking about?

Amy: Well, I just did it in my head but I multiplied but I did a palindrome and multiplied it by 4 and I didn’t get the original palindrome back. I guess I’m not sure, exactly sure, at this point, how to go about a formal proof, other than, I mean it would be kind of silly to list all the two-digit, two-digit numbers and flip ‘em and then multiply by 4. I guess it’s sort of, I don’t know, weird to me. Especially, if you can prove that a palindrome multiplied by 4, you don’t get the same number. I guess I’m sort of unsure how to say this in formal math, but if you flip, let’s say you have 14 and you flip it, becomes 41, it’s not the same number to begin with. So, why, I don’t know, why would thinking that 14 multiplied by 4 would somehow – I don’t know.

R: Well, would it help to know that there are bigger numbers for which this works? It just happens to not work in the two-digit case.

Amy: Okay.

R: So, when you get bigger, like longer numbers, it is possible. [pause (5 sec)] So, you said it would be possible to try all of ‘em and see what happens, right?

Amy: Mm, hmm.

R: What other thoughts do you have? What would you like to try to decide where to go?
Amy: [pause (15 sec)] Well, I mean, could it be – I mean, just throwin’ something out there would figure out which or divisible or multiples of 4, and start off that way. But I mean it would reduce the, the workload again, it would be kind of silly.

R: So, you’re thinking of the end result. So, 4N and seeing which of those goes back to it’s reverse?

Amy: Yeah. But the – [pause (10 sec)]

R: Do you have any desire to start trying some of em? [pause (5 sec)] Or does that seem like something you wanted to do?

Amy: I’m just still thinking.

R: Okay. Just throw out there whatever comes through your brain.

Amy: [laughter, pause, writes 10-99, 51 15, (50 sec)] Okay, I can’t remember, which one am I supposed to multiply by 4?

R: So, in that case it would be the 15 that you’re looking at, 15 is the N.

Amy: Okay. [writes 15*4=60, pause, writes list 11, 12, 13, 14, erases this list, (30 sec)]

R: What were you thinking?

Amy: Oh, I was thinking I could cut down on work, or time by saying okay, 11 becomes 11, 12 becomes 21, 13 becomes 31 [restarts her list, writes 11, 11, 12, 21, 13, 31], et cetera and if I were to multiply one of these by 4 [starts to multiply the 11, 21, and 31 by 4] 44, and 84, I think. [writes 44 and 84 next to 11 and 21, respectively]

R: And I think you’ve switched the N. N would be the 11, the 12, and the 13.

Amy: Okay, so I’m really confused, then. [labels column with 11, 12, and 13 as N]

R: ‘Cause that’s where you started with, right?

Amy: Right. [pause] And then I multiply this, this switch by 4, right?

R: {No}. You multiply the original by 4. I mean I guess it doesn’t matter because either way you’re trying some of the numbers, right, but –

Amy: Yeah.

R: – if those are your originals, that’s the idea.
Amy: [erases 44 and 84, begins to multiply her N column by 4, writes *4=44, 4=48, etc., (15 sec)] So, maybe the original idea was right. I’m okay. [writes 14*4=56, pause, (30 sec)] Um, can you read that, er again? [points to the statement of the question, puts down pencil and covers her eyes]

R: Read it again?

Amy: Yes, please.

R: Okay. We call a positive integer, N, a 4-flip if 4 times N has the same digits as N but in reverse order.

Amy: Okay, just one more time, it’s if I have original number and I multiply by 4 – [pause] ahhh.

R: So, you take a number, multiply it by 4. If it’s equal to its flip, then it’s called a 4-flip.

Amy: Okay. So if I were to multiply 14 times 4, it would have to be equal to 41.

R: Right.

Amy: [pause] Okay, so 44, 48, 52, 56, 62 – [writes 44, 48, 52, 56, 62, erases 62, writes “60, 64, 68, 72,…” (10 sec)] 64, 68, 72, dot, dot, dot. [pause, to the side writes “11, 21, 31, 41, 51, 61, 71, 81, 91, 01, 02, 03, 04, 05, 06, 07, 08, 09”, (1 min 10 sec)]

R: Can you describe to me what you’ve got there?

Amy: Um flipping – these are the ones I’m flipping over [points to list of 11, 21, 31, etc.] and just, I’m trying to remember well I’m trying to do this at the same I’m trying to remember, there’s a special rule for 4s that, other than that they’re even, if you look at say, the last digit if it’s divisible by 4, then it’s a multiple of something – That’s kind of what I’m doing right now.

R: Okay. So there’s something that you know about multiples by 4, right? Somewhere in the past?

Amy: Yeah.

R: Okay.

Amy: [pause, adds to her list to the right, 22, 32, 42, (10 sec)] Ow. [pencil breaks, keeps writing 52, 62, –]

R: And so, what you are writing are the flips?
Amy: Mm, hmm. Yeah. [finishes her list, pause, audiotape side one ends, researcher flips it over, student does work to the side, erases it, then puts her pencil down and her head in her hands, sighs, (50 sec)]

R: Can you describe to me what you’re thinking?

Amy: [pause (10 sec)] Well I had a thought and then –

R: As to where you were going with this?

Amy: Yeah. [pause (10 sec)]

R: We won’t spend too much more time, I won’t keep you too much later. But can you tell me about what you’re thinking, what you wanna try?

Amy: Well, sometimes it’s really hard because if I start off doing something, I loose it – 10 minutes into it, so.

R: So you have a path but then you kind of get caught up in the logistics and forget the path? [laughter] Is that the idea?

Amy: That’s the idea.

R: Okay, so let’s recap a little bit about what you’ve done. So, let’s see, first we tried to gain an understanding of the problem, right?

Amy: Yeah.

R: Going back to it a few times and making sure that you understood what was being asked, right?

Amy: Mm, hmm.

R: And we’ve gone back several times just to clarify and make sure – and then you thought, okay maybe I want some sort of a formula but I can’t think of one. So, maybe I just want to try the numbers 10 through 99 and you were kind of going through em’, right?

Amy: Mm, hmm. [pause]

R: And you tried some of em’ and saw, at least the first couple that you tried, didn’t work. Then you started thinking about multiples of 4, right?

Amy: Yeah. Well I would say that these [points to row with 11, 21, 31, etc.] would automatically be eliminated since multiples of 4 are even numbers. I know that much.
R: Okay.

Amy: And so, anything that there would be a flip, it would have to be an even number.

R: Okay.

Amy: Um, let’s see. And I’m thinking that it couldn’t be anything below a certain point, because if you flip it, the multiple, when you multiply it by 4 then it’s got, the flip would be less and you can’t have that.

R: Oh, okay.

Amy: For example, if you had like over here you have 11 times 4 which is 44 and if you flip it, it’s a palindrome, so if you flip it around, it’s still gonna be 11, 44 is greater than 11.

R: Okay.

Amy: So –

R: So, you can discount all of those.

Amy: Yeah. Beyond a certain point you’re not guessing um something um, let’s say 50 but at the same time, if you choose a higher number and you multiply it by 4, it’s gonna be something three-digit.

R: Hmm.

Amy: But, then again if you flip it around and yeah, 50 becomes 5 and you know, the three-digit number’s gonna out-number the 5.

R: Okay. So, you’re thinking you might be able to eliminate some because you’re gonna get too big.

Amy: Right.

R: You’re gonna get three digits and that’s too big. And you can eliminate all of the 10’s, for instance, because when flipped, they’re odd, right?

Amy: Yeah.

R: Same with like the 30s and the 50s you were saying basically, right?

Amy: Yeah. And that’s what I’m thinking that that would eliminate pretty much any number.
R: Almost any number, right?
Amy: Yeah.
R: ‘Cause we’re down to very few once we consider just those, when multiplied by 4 get you two digits. Okay, so I think we could probably get to a point where it would be reasonable, we’re narrowing it down, narrowing it down, and that’s where we’re headed, right?
Amy: Yeah.
R: To narrowing it so far that we have nothing left. Or just a few to check, right?
Amy: Right.
R: Okay. I think we could get there if we just worked a little further and just kind of formalized everything you just said. I think you had it down to probably just a few. Okay. Cool. Have you ever seen anything like this before?
Amy: This problem? No.
R: Okay. Um, anything in particular that you could say you used, that you’ve done before?
Amy: No. Other than that chessboard problem.
R: Okay.
Amy: We did it in combinatorics but the rest of it, no.
R: Okay.
Amy: It was just something new, kind of.
R: Okay. And when you approached the problems, like there was, that first one you said you wanted, you were thinking maybe you could do it combinatorially so you’re kind of thinking of what you’ve done before.
Amy: Mm, hmm.
R: And, and possibly trying it. You pushed it off for, for the moment, right?
Amy: Right.
R: – to try something else but um you were kind of thinking of the past. And the
chessboard, you thought of maybe there was a time you’d seen it before, that kind of
thing, right?
Amy: Right.
R: Is that a strategy you’d say you use sometimes?
Amy: Quite a bit, quite a bit. Um, the things that I’ve generally used, use in math or
physics is that if I can draw it and I can figure it out using tactile or visual methods, I’ll
do it or auditory. Because I don’t trust myself in numbers.
R: Oh!
Amy: I’m dyscalculiaic.
R: Okay.
Amy: So, um, it’s very difficult like with this flip, I can switch em [laughter]. And so if,
if it’s numerical then it’ll be much, much, much, much harder.
R: Okay.
Amy: But, if it’s something that I can easily visualize then I have an additional way of
solving a problem. I can get around the numbers.
R: Okay. That makes sense. Alright, great, thank you. *End of interview
Interview #14  (Total time = 1:01:35)

R: A traditional chessboard consists of 64 squares. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares. A perfect cover of the chessboard with dominoes covers every square without overlapping any of other dominoes. Does that make sense?

Vicki: Mm, hmm.

R: Prove a generic chessboard of size $m$-by-$n$ has a perfect cover, if and only if, at least one of $m$ or $n$ is even.

Vicki: [pause, reads question, pause, draws chessboard of size 8-by-7, writes $8 \times 7 = 56$ squares, pause, writes “If $m$ is even and $n$ is either even or odd, the number of squares in the chessboard is going to be even. Since the number of chessboard squares are even the dominos can cover all of the squares of the chessboard”, (3 min 50 sec)]

R: Okay. I was watching what you were writing.

Vicki: Um, because of the number of um, squares are always gonna be even on the chessboard um, all the dominoes are gonna be able to cover all of the squares on the chessboard because it’s always gonna be even.

R: Okay.

Vicki: So, you gonna be able to have an even number. Like since this is 8 rows this way and this is 7, just an example. Um, it can cover uh, 2s and 2s. [traces out where dominoes would go on her example chessboard, (10 sec)] No matter if it’s even or odd if you make it smaller. If this number’s even or odd, it’s always gonna be even. We’re always gonna have that an even number of pairs of dominoes that’ll be able to cover the whole set of, the whole chessboard.

R: Okay. Alright. Um, what, if anything at all, is left to prove in that statement or do you feel you’re done? [pause] I’ll just ask you that on every question so it’s not meant to be leading –

Vicki: Yeah, I know.

R: – I just want to make sure that you feel that you’re finished.

Vicki: Um, I guess I think I’m finished.

R: Okay. Um, what strategies did you use when you started working on this problem? What were you thinking about?
Vicki: Um, I looked at the number, well at first drawing a picture because it was kind of something you needed, I needed to see, sort of a picture of what I was drawing and understanding of what um, the question was asking.

R: Okay.

Vicki: So, I drew a picture first and then just kind of thought about no matter what, if one of the either $m$ or $n$ is even, then the number of squares on the chessboard are always gonna be even which means that if it’s even then there’s gonna be enough dominoes to be able to cover the whole thing.

R: Okay. Anything else that you were thinking about or strategies you used?

Vicki: Um – I guess somewhat of the proof is what I’ve learned from, um, abstract math, to writing a proof.

R: Okay. Have you ever seen anything like this proof before?

Vicki: Yeah.

R: Okay, where did you see it before?

Vicki: Abstract math.

R: Oh, did ya? Okay. Just like this one or similar or –?

Vicki: Similar.

R: Okay.

Vicki: Um, I guess probably maybe just properties of what one being, I mean it’s a lot different than the problems but –

R: Oh, okay.

Vicki: – using of $m$ and $n$ and being, one being even and the other one not, or could be may or may not be even or odd. Just looking at the properties of having even or odd and multiplying them together.

R: Okay.

Vicki: Um, besides I think not.

R: Alright, sounds good. Let’s go on to another one. [researcher collects paper, gives student next question] Kind of going backwards in the order, but trying to change it up a little bit. So, Question 2. We call a positive integer, $N$, a 4-flip is 4 times $N$ has the same
digits as $N$ but in reverse order. Prove that there are no two-digit 4-flips. So I’ll let you
read that and think about it for a second.

Vicki: [pause, reads the question, (30 sec)]

R: Do you have any question or need any clarifications on the problem?

Vicki: Um, right now, let me look at it first and then I might have a question.

R: Okay.

Vicki: [pause, writes $362 = N \times 4 = 1448$, pauses and looks at example, (35 sec)]

R: So, do you understand what you would want to see if it was a 4-flip?

Vicki: No.

R: Okay, so you took, the example 362 multiplied by 4. If 362 was a 4-flip then you
would expect that it would say 263 when you multiply it by 4.

Vicki: Oh, okay. [pause (10 sec)] So, are they asking like, prove that there are no two-
digit 4-flips, okay. [pause (20 sec)] I guess for me solving it or proving that it would be
not would probably take any multiple and multiply by 4 and try to get it in reverse order.

R: Okay.

Vicki: I meant multiple of 2, or two-digit number and multiply it by 4 to get the opposite.

R: Okay. [pause] You can proceed in that way to show me what that would be like.

Vicki: [writes $12 \times 4 = 48$, then writes 24 to the side, (20 sec)]

R: Can you tell me what you’re thinking about there?

Vicki: Um, no matter how, um any two-digit number that is multiplied by 4 is gonna be
um larger than any number that I can put in the 10 spot. So, even if I, like, let’s take, for
example 28 times 4. [writes $28 \times 4 =$] Well you know the opposite of that is gonna be 8
but that’s still gonna be too big, that’s gonna be too big for um, okay. So the, the 4-flip
of 28 would have to be 82 um, the number that you’re picking, the second digit has to be
large enough. Or at least twice as large, or 4 times as large, as the first digit in order for it
to work. Um, but because you’re taking 4 and multiplying it by a number that’s, that
when you multiply bigger, it um, if you multiply by 4 and 8 that’s larger than 10, so you
can’t do – can’t look at it that way.

R: I can see what you’re saying.
Vicki: Like you can’t, if you take, if you take 4, 4 times 2 is 8 so you know that could be
the first digit. By taking 4 times 8 and doing it that way, you can’t because this is larger
than 10, so it’s gonna carry over into the, the ah 10 spot, instead of the ones.

R: Okay.

Vicki: Um, so the next plan of attack would be to like take if I wanted [pause] say 27
still doesn’t work. Well, I guess it would because it – let’s see – 27 times 4 [writes 27*4
= 108, (5 sec)] and it’s still getting, we can’t – the ones place can’t be when 4 is
multiplied by the ones place, it can’t be larger than um, the, the leftover when – Okay,
when multiply – if you use the, the ah, multiplication rule from like, 4 times 7 equals 28
and it carries over 2, the number can’t be larger than – than – so it’s 2. So, 4 times 2 is
equal to 8 and add the 2 gives you 10. So, it’s gonna be larger, it’s gonna make a three-
digit number so you want something that is gonna be smaller than 10 when you multiply
these two and then add what you’re carrying over.

R: Okay.

Vicki: Um – oops – [pause, writes 26*4 = 104, 25*4 = 100, (10 sec)] So, for, using 20s
– um, if you went to, if in the 10s place the number was um, any larger than 2, it can’t
work because you’re automatically gonna get a three-digit number.

R: Okay.

Vicki: So this, so you know this number has to be smaller than 25.

R: Okay.

Vicki: Um – any – the 10 spot, or the excuse me, the 1 spot has to be – um, let’s see.
[pause, writes 24, (20 sec)] Can I just say it doesn’t work? [laughter]

R: Well, you’ve limited it.

Vicki: Well, okay, so but any number if you take, so the smallest two digit number you
can have is 10, which timesing it by 4 is 40. [writes 10*4 = 40] And – so [writes 11*4 =
44, pause, (5 sec)] – the your largest number is 25, your smallest number is 10 and any
number, in between those two numbers, so between 10 and 25 [writes 10 → 25] You
have 15 numbers and each one of those 15 numbers aren’t backwards or aren’t a 4-flip
because [pause] the number, you can’t, um – [pause (20 sec)] Um, let’s see. So, if you
go, [pause, starts writing list \( N = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23 \),
then writes below these 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92, (40 sec)] I guess,
these are all the numbers between 10 and well 24, which we know is 24 is 96. [writes 24,
96 below list] Um, and since um you have to have for the 4, well this just literally proves
it that there isn’t one. Um, but to do some more of a math, mathematical proof, I guess
you could say, is that the, the first digit, um in the 10’s place of what you’re multiplying
by so the actual \( N \) numbers are 1s or 2s and your not gonna get a 1 or a 2 other than if it’s
These are the only three numbers that you get a 1 or 2 in the second digit, the 1s digit in the actual 4-flip number. And um, none of them have a, a, the 10s digit of the 4-flip is too large for it to be a 1s digit in the $N$.

R: Okay.

Vicki: I guess, so um – I guess I can’t come up with a way of actually writing a real proof of what, of why we can’t do that. Um, other than that I just took the 15 – R: So, you limited it because any more than 25 and you’d have bigger than a two-digit number, right, when you multiply by 4 and then you just try all of those.

Vicki: Right. Yeah.

R: And you made some other observations which may help with the second part of the question that I’m about to give you but, um – good. What, if anything, left to do? Or do you feel you’re done?

Vicki: Um, the only thing would be if there was an actual proof to where I wouldn’t have had to write out these 15 –

R: Okay.

Vicki: – to figure it out, yeah, that’d be the only thing. Um.

R: What strategies did you use?

Vicki: Um, at first I used, just kind of picked a random number and then multiplied by 4 to see what it was, um, and understand what they were asking. And then, figuring out that any number above, any $N$ that, that symbolized $N$, any positive integer that was larger than 25, um is a three-digit number which isn’t gonna give me a two-digit 4-flip.

Um and then 10 is the smallest two-digit number and so I knew it had to be between 10 and 25 and then I just took those numbers because I couldn’t come up with some sort of, I mean I picked a couple numbers and tried to come up with a way to, um, see some sort of a, a pattern that may be able to tell me that there isn’t a two-digit 4-flip, but I just wrote them out.

R: Okay, sounds good. Okay, second part of the question. researcher moves work out of the way, gives student part b, but leaves part a within sight] Prove or disprove that there are no three-digit 4-flips.

Vicki: [pause, reads the question, (10 sec)] Well, I can tell you right now, there’s not.

R: Okay.
Vicki: Because the smallest number is 300 and if you multiply that 4, that’s gonna give you automatically 1200 [writes 300*4 = 1200], which means there isn’t gonna be a 4-flip for –

R: That’s the smallest three-digit number?

Vicki: Excuse me. [laughter] Hang on [erases the 3 from 300], okay, 100 [writes 100*4 = 400], so it has to be between 100 and 250. [writes 250*4 = 1000] Um, which if you look at that is the exact same thing that we have from the previous one, other than it is, one digit off. So, if you look at it to where it’s, you’re adding a 0 automatically.

R: Okay.

Vicki: Um, but, let’s see [pause, looking over work, pause, writes 125*4 = 500, (60 sec)]

R: Can you tell me a little bit about what you’re thinking?

Vicki: Um, well this one [points to 125] I just took a random in between the two and, um. [pause (5 sec)] I’m actually trying to think of, in the previous one I didn’t find a proof, um, there wasn’t any three-digit 4-flips. But, we’re dealing with more numbers between 100 and 250, so the 150 numbers, I’m not gonna write 150, so there’s a way of proving it without using, you know, writing out all 150 of ‘em.

R: Okay.

Vicki: Um, the, [pause] I can tell you the last digit of the $N$ number can’t be 0, or 1, or 2, or 3, or 4, um because all numbers, all of the 4-flip numbers are gonna be higher than 400. So, this digit here has to be 4 or greater [writes “4 or >” above the 0 of 250].

R: Oh, okay, I see what you’re saying.

Vicki: Um, well, in the end spot. So, if you have digit $a$, $b$, and $c$, the $c$ digit has to be larger than 4. [writes $abc$, and writes > 4 above the $c$]

R: Okay.

Vicki: Um – so that takes out every multiple of 4 between ah, the 1s place, between 100 and 250 um, but that’s only a couple numbers. [pause (20 sec)] Um, it can’t be a digit, it can’t okay, so I go 1, 2, 3 – [writes out the numbers 1 through 9 and 0 (5 sec)] And 0. So it can’t be 0, 1, 2, 3, 4, [crosses out 1, 2, 3, 4] oops I guess it can be 4, maybe. [erases mark on 4] but it can’t be 5 [crosses out 5 on her list] because when you multiply 4 times 5, you’re gonna get a 0 and you’re not gonna be able to have a 0 starting –

R: Okay.
Vicki: – number so 5 can’t be it. Um. [pause, writes 126*4 = 504, (30 sec)] Ah, if you look at 4 times 6 is 24, so you’re gonna get a 4 here. Um, you can’t have a number that’s greater than a 2 for the first one, so 6 doesn’t work. And 7 times 4 is – what’s 7 – yeah – 7 times 4 is 28 and the first number can’t be 8, so 7 doesn’t work [crosses out 7] and 8 doesn’t work [crosses out 8], 9 doesn’t work [crosses out 9], um, if you do it in the same pattern.

R: Did you check those?

Vicki: [writes 127, 508, and 128, 512, whispering while she works, (20 sec)] Um, so we’re alright with 8. [erases mark on 8] Um – [pause] 4 times 4 is 16, so 4 doesn’t work either. Um, if you go 124 and 4 [writes 124*4 = _ _ 6], that’s gonna give you 16, so that number doesn’t work either. [crosses out 4 on list]

R: Okay.

Vicki: 8 – um – 8 can work, so if we have 8 for the last digit, um 8 times 4 is going to be um 32, so you have to have a 2 [writes 8*4 = _ 2, puts in 2 8*4 = 8 2], so the first number has to be 2 – um – pause] 3, have to carry the 3, um. [pause (10 sec)] So, we know it has to be – um – it can’t be 208 [writes 208*4 = 832], and then 218 [writes 218*4 = 872, (5 sec)] It can’t be 218. 228 [writes 228*4 = _ _ 2, (10 sec)] so any um, [pause (10 sec)] um okay, so um it there is no three-digit 4-flips. The reason for that is, is because if you take 8, so you’re gonna have to get a 2 in your first digit and so it has to be, um, 8 has to be in the 1s place and a 2 has to be in the 100s place of your N number and so that means that your 2 has to be in your 1s place of the 4-flip number and the 8 has to be in the 100s place of your 4-flip number. Now, the in-between number is the one that’s in question and it’d either have to be a 0 or a 1 and neither one of those work because you’re carrying the 3 over which is going to um, like for instance this one is 7 in the middle [points to 218] which isn’t the opposite and this one, oops, and this one doesn’t work [points to 208] and anything higher than um 1 automatically changes the digit 8 to a 9, because you’re carrying it over.

R: Okay.

Vicki: Um, so, there isn’t any 4-flips, a three-digit 4-flip.

R: Okay. Um, can you describe the strategies you went through as you did this problem?

Vicki: I first found out where between the three digits, um, the three-digit numbers that it, a 4-flip could be. And then, ah, started to look at, ah the ones place, had to be. The one’s place of the 4-flip had to be ah smaller than um either a 2 or smaller because the hundred’s place of the N number had to be um a 2, 1, or a 0, had to smaller than that. So, I went through each number that could be in the one’s place and found out which ones, when multiplied by 4, gives me um, either 2, 1, or 0. And 8 was the only one that had done that. Ah, I also took out, well I said that there were less than 4 because anything, um, or excuse me, c had to be greater than, why did I say that? [pause]
R: So, I believe that you were looking at this equation, 100 times 4 equals 400 when you made that note – that that digit had to be – [pause]

Vicki: Oh, 4 or greater because that’s the biggest number that it could be or 4, 400 is the smallest number of a 4-flip that you can have so that’s why I said this number had to be greater than 4 because in order to get a 4-flip, this number [puts square around 4 of 400 in the calculation 100*4 = 400] had to be 4 or greater in the 4-flip. So, that’s why that had to be, why I said the 1’s place had to be greater than 4. So, I took out those numbers and then um kind of right here [underlines where she did calculations for 126, 127, and 128], did trial and error of ah, multiplying, um 4 by 7, 8, 6, 7, 8, 9 – and found out 8 was the only when multiplied by 4 that the 1s place was um 2 or less.

R: Okay.

Vicki: And then picking, uh, so I knew that it had to be a 2 in the 100s place and 8 in 1s place and then vice versa of the 4-flip and this is the N number, when multiplied by 4, um, I had to figure out what the 10’s place number would be and the only choices would be 0 and 1 without it changing this 8 in the 4-flip number. [pause] And 2 is too high and 0 and 1 didn’t work so those are the only numbers to look at.

R: Okay. Alright, have you ever seen anything like this problem before?

Vicki: Um. [pause (5 sec)] No, not that I can think of.

R: Okay. Any time you’ve seen anything in other classes that may have been, um, showed you how to do this problem or –

Vicki: I think just looking at numbers and trial and error and I, I think looking at this just by the numbers and picking it, I don’t think there was anything that I’ve learned in college or other, you know, higher-level math classes that helped me with it.

R: Okay.

Vicki: Just knowledge of numbers and things.

R: Alright, fair enough. Let’s do one more. [researcher collects papers] Let’s put this together. [gives student Question 1] Question 1, finally. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon, so that the sum of three numbers along each side is the same. The diagram below shows an arrangement with sum 16. Prove that the smallest possible value for the sum is 14.

Vicki: [pause, reads question, pause, (30 sec)] Um – the first thing I would look at is that in order to use all 10 of the numbers, two of them have to be um, put together. So, if you take and arrange every number in, every number that has to be used in any order, you’re gonna get – let’s see. [pause (20 sec)] Let’s um, take for example 13. So, if you have
the sum of 13. [writes “sum: 13”] And um – using all the numbers between 1 and 10, um
so let’s just say, let’s go 10 plus 2 plus 1 [writes 10+2+1], so out of, if the sum were 14,
that you’re adding together, this is one choice, or 9 plus 3 plus 2 [writes 9+3+2] or 9 plus
um, 4 plus 1. [writes 9+4+1]

R: Okay, so you’ve moved to looking at 14? Is that what you just said, that you moved
to looking at sum of 14?

Vicki: Um, no I think that was a mathematical error for me. Oops, let’s see, yeah.
Okay. I’m still looking at 13, I gotta go back.

R: Oh, okay.

Vicki: Um [pause, erases 9+3+2 and 9+4+1, (10 sec)] This is 13. [writes 9+3+1, (5
sec)] Okay, alright, if you take, um, if you’re taking all of the numbers that add um to
make 13, you have to look at ah the numbers that go together, so um, these numbers add
up to make 13 and 9 um, 9 plus 2 plus 2. [writes 9+2+2, researcher sneezes] Bless you.

R: Thank you.

Vicki: Um, you can’t use 2 twice, so 9 doesn’t work and if 1 has already worked with 10,
like, having the number 10 had, the only numbers that can go with 10 to add to make 13
is 2 and 1. So, you can’t use 9, the only suggestion, or the only solution for anything
with 3 digits to add to um 13, that are three separate numbers are 9, 3, and 1, which 1 was
already used, so that doesn’t work and 9 and 2 and 2 which that doesn’t work because
you can’t have a double, so 13 doesn’t work. [crosses out 9+3+1 and 9+2+2]

R: Can you look back at that again and look at your example here and confirm that no
number could be used twice? So, your 1 being used twice.

Vicki: Oh yes, it can be used twice. Okay.

R: And how does it, how did it work?

Vicki: Um, like for example between here, 10, 5, and 1 is used to make 16 and 1, 8 and
7. So, if we were to use, so let’s go for 13. So, we use 10, 2, and 1 [rewrites 10+2+1],
that could be here and then 9 plus 2 plus 3 [writes 9+2+3], oops, excuse me – 9 plus 3
plus 1 [erases 2+3 writes 3+1] could be used here to here [points at top two edges of
example pentagon].

R: Okay.

Vicki: So, that would work. So let’s, let me draw this out [draws pentagon, (5 sec)] So,
1 has to be on a corner. Um, oh well, here, let’s go back to that. 10 has to be used in two
different, two different equations um for here [points to where 10 is placed on the
example pentagon], 10, 10 plus 5 plus 1 is 16 and 10 plus 2 plus 4 is 16 and there are no other multiples for 10 to be equal to 13.

R: So, is there any other way to write that so that 10 doesn’t have to be used twice?

Vicki: Yes, if 10 is in the middle. Um – [pause] Okay, so, so then for here [points to list of sums of 13] we would put 10 in the middle [writes 10 in middle of one side], 1 has to be on the corner [writes 1 on corner next to 10 and 2 on other corner] because 1 has to be used twice for 9 because this is the only equation that works for 9.

R: Okay.

Vicki: Um, and so since 9 can’t be used twice, 9 has to go in the middle [writes 9 in middle of next side], so 3 is on the outside [writes 3 on corner next to 9].

R: Okay.

Vicki: Um, 8. [writes 8 in list of sums] 8 can be 4 plus 1 [writes +4+1] 3 plus 2. [writes 8+3+2] Um, [pause (10 sec)] For 8, these are the only two that can be used. Um, let’s see. [pause] This one [points to 8+4+1, crosses it off] can’t be used because 1 cannot be in it. Um, 1, 1 is already used up for two equations so it can’t be used in this one.

R: Okay.

Vicki: 3, uh so 2 would have to go with this one. [points to side labeled with 2 on the corner] 3 would have to go with this one because – 8 would have to be in the middle [writes 8 next to 2] and then if 2 is on this side, 3 would have to go on this side [writes 3 next to 8] and we don’t have duplicates so 13 doesn’t work.

R: Okay.

Vicki: [pause] Um, and I would say the same possible solution for anything under that. Um, it can’t be smaller than 10 because 10 um, so it can’t be smaller than 10 because you have to have 10 in it, in there.

R: Okay.

Vicki: 11, it can’t be 11 because you can’t have just 10 and 1 to add together to make a side.

R: Okay.

Vicki: And it can’t be 12 because you can’t have 10 and 1 and 1. And 13 was the only other one, so there can’t be any smaller possible value for the sum than 14.
R: Okay. [pause] Alright. Anything else that you want to do in that problem or are you finished?

Vicki: [pause, looks over work] Yeah, I’m finished.

R: Okay. What strategies did you use along the way?

Vicki: Um, well it had asked to prove the smallest possible value for the sum is 14, so it said that the smallest possible value is 14, so in order to prove, well I guess that there isn’t a smaller one than 14 would be to prove that the, there isn’t a 13. Um, I guess I didn’t really prove that the smallest sum is 14, so that would probably be something to do.

R: Okay and how would you do that?

Vicki: By doing the same way of what I did for 14 or for 13, for 14.

R: And trying to come up with a picture that works?

Vicki: Mm, hmm. Mm, hmm.

R: Okay.

Vicki: Um, I proved that 13 isn’t the smallest possible sum just by um picking numbers and starting with the highest one and working down.

R: Okay. Sounds good. Have you ever seen anything like this before? [pause] Or does it remind you of anything you’ve seen?

Vicki: Um, not that I can think of. I mean, I think all through my life, I’ve seen a lot of different, you know, problem, teasers sort of, but um, nothing that sticks out.

R: Okay. Alright, I think we probably have enough time for one more. [researcher collects papers and gives student Question 4] Let’s give this one a shot. So x and y, let them be two integers. We say that x divides y if there is an integer k, such that y equals \( kx \). Consider integers a, b, and c and prove that if a divides b and b divides c, then a divides c.

Vicki: [pause, reads question, writes \( x/y \) if \( y = kx \), (50 sec)] Um, I’m gonna take, um x divided by y [writes \( x/y \) again] and say it’s equal to sum integer z [writes \( z \)] So, if I multiply [pause (5 sec)] um, to get rid of the fraction, multiply both sides by y, I have x equals \( zy \). [writes \( x = zy \)]

R: And is that the same as the definition that they gave?
Vicki: No. Um [pause], the definition is \( y = k + k \times x \), which would be if \( z \) was over on the other side. So, [pause (5 sec)] okay. Um, if [pause (5 sec)] if you divide by both sides, by \( z \), you’re gonna get the \( z \) is rid of [writes \( 1/z = y \)]. And if you let \( k \) equal to 1 over \( z \) [writes \( k = 1/z \)], then \( kx \) equals \( y \) [writes \( kx = y \)] is the equation given.

R: And \( k \) is supposed to be an integer, right? [audiotape side one ends, researcher flips over tape]

Vicki: Right, so. [pause]

R: [tape pops back up, researcher has to work with it more] Oops, get in there, there you go. Okay.

Vicki: Um, since \( k \) is an integer then, \( z \) would have to not be an integer. ‘Cause if \( z \) was an integer then \( k \) couldn’t be an integer. Um. [pause, writes \( a/b \), pauses, rereads question and her work, pause, (40 sec)] Um, for this expression [points to the statement “if \( a \) divides \( b \) and \( b \) divides \( c \)”, etc] I want to take and give values to \( a \), \( b \), and \( c \) to show that this works and then I will relate it back to, um, variables.

R: Okay.

Vicki: Um, let’s go \( a=1 \), \( b=2 \), and \( c=3 \) [writes \( a = 1 \), \( b = 2 \), and \( c = 3 \)]. So, \( a \) divided by \( b \) is \( 1/2 \) [writes \( = 1/2 \) next to \( a/b \)]. And \( b \) divided by \( c \) is \( 2/3 \) [writes \( b/c = 2/3 \)] and \( a \) divided by \( c \) is \( 1/3 \) [writes \( a/c = 1/3 \), pause, (25 sec)].

R: What are you thinking?

Vicki: Well, I’m trying to, I mean if \( a \), \( b \), and \( c \) are integers, then you can take and divide them all you want. I’m just trying to find what, understanding what is [pause (5 sec)] the point, I guess. Or not really the point but um –

R: Should we, let me reread it. So if \( a \) divides \( b \) and \( b \) divides \( c \), then \( a \) divides \( c \). What are you looking back at to, so I see you rereading some of it.

Vicki: Um. [pause (5 sec)]

R: And reconsidering what you have written on the left, right?

Vicki: Well, yeah. Well, okay, so if \( a \), like \( a \), if these two – Okay, this is saying if these two divide each other, then these will divi – oh, okay, okay. Hold on. Let me think. Um. [pause] Right, so if [pause] okay, well \( z \), excuse me, \( z \) has to equal, has to be some sort um, has to be 1 over an integer [writes \( z = 1/integer \)]. Let’s see. [pause, looks back over work, pause, (25 sec)] Okay, no. Um, \( k \) is equal to 1 over \( z \) [writes \( k=1/z \)], so if \( z \) [pause (5 sec)] Let’s see. [pause, writes example at bottom of page \( \frac{1/2}{3/4} = \frac{4}{6} \), pause, (35 sec)]
R: So, what are you thinking?

Vicki: I'm trying to remember if this is the rule. It's invert and multiply?

R: Yeah.

Vicki: Yeah?

R: Yeah.

Vicki: Yeah, so that's right?

R: Mm, hmm.

Vicki: Okay. [laughter] Okay, so um. [pause, writes $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} = 2$, (45 sec)] Okay, so if $z$ equals to $1$ – ah excuse me, if $x$ equals to $1$ over $z$, then $z$ is equal to $1$ over $1$ over some integer [writes $z = \frac{1}{\text{integer}}$] which means that $k$ is equal to $1$ over $1$ over some integer [writes $k = \frac{1}{\text{integer}}$]. Which, um, is equal to $1$ divided by $1$ over $1$ over that integer [writes $\frac{1}{\text{integer}} = \text{integer}$]. So, which equals an integer. [pause, looks over work, (30 sec)] Um, I guess I'm trying to relate what I found here [points to what she just wrote with the $k$] to the $a$'s, $b$'s, $c$'s.

R: Okay. [pause] Can you go back to what you said on the right, there, with your example of $a$, $b$, and $c$ and consider that, given this new knowledge, that $z$ is $1$ over an integer?

Vicki: [pause (20 sec)] So, these – let's just state – this $1/2$ is what equals to $z$ [writes $z$ next to $1/2$ on the right hand side of page]. Um. [pause (10 sec)] Which would be right. Um. [pause]

R: So, that's $1$ over an integer, like you wanted, right?

Vicki: Right.

R: Okay.

Vicki: Um. [pause (5 sec)]

R: Can you keep going? What about the next one?
Vicki: Well, this isn’t 1 over an integer [points to 2/3]. But – this is [points to 1/3]. Um, as long as \( a \) is equal to 1, then it is equal to what I have said \( z \) has to equal. And any, so – [pause (5 sec)] This equation here [points to 2/3] is okay, um, [pause, looking at work, (25 sec)]

R: Any thoughts?

Vicki: Well, this, this one [puts square around \( b/c = 2/3 \)] isn’t 1 divided by an integer.

R: Okay.

Vicki: Um, um. [pause] So – in order for this to be 1 divided by the integer, \( b \) would have to be 1 which means \( a \) would have to be 1, \( b \) would have to be 1, and \( c \) would have to be 1. [pause] Um, in order for it to be 1 over something. So, everything would have to be 1 because this equation – if \( b \) and \( c \) were something greater than 1, then this equation [draws arrow next to \( b/c \)] isn’t 1 over an integer.

R: Can you think of any way to change \( c \) to make that 1 over an integer?

Vicki: Um, yes, by making it um, \( c \) as a fraction.

R: Any other way?

Vicki: [pause (10 sec)] So, we’re wanting to make this [points to \( b/c \)] 1 over?

R: Mm, hmm. So, I want to divided 2 by something and I want it to really be 1 over something.

Vicki: So, if \( c \) is 1. [pause (5 sec)] Oh, okay, so, 4 – so let’s let \( a = 1, b = 2, \) and \( c = 4 \) [writes let \( a = 1, b = 2, c = 4 \), which gives you 1 over 2 [writes \( ½ \)], oops, 1 over 2. And 1 over 2 [writes \( ½ \)] and 1 over 4. [writes \( ¼ \), pause, (10 sec)] So, \( a \) has to equal 1 and then \( b \) can be some integer and \( c \) has to be twice \( b \) [writes \( A=1, B = \text{some integer}, \) and \( C = 2B, \) pause].

R: Okay, in the interest of time, I’m going to stop you here and just ask for your insights and your strategies, what you did during this problem.

Vicki: Um, the first thing was to just look at this um [points to statement of question, where \( y = kx \)], equation and decide where to go with it. Um, finding out how \( k \) fits into it and I used a – a different value, \( z \), just to not get \( k \) and \( z \) confused to where we can figure out what that is. And then um, um, just looked at the integers or how \( k \) and \( z \) relate and what it needed to be and then put them back to \( a, b, \) and \( c \). Um, if I were to keep going with it, I would figure out how, you know, to where \( a \) can be a different number than 1.

R: Okay.
Vicki: Which, you know like, if it was 2 and \(b\) was, as long as it comes down to 1 over.

So, [pause] um, as long as when you, you simplify what \(a\) and \(b\) are, it equals 1 over a number.

R: Okay. [pause] Okay. Have you ever seen anything like this before?

Vicki: Um. [pause] Not necessarily like this, no.

R: Okay.

Vicki: Maybe pieces of it, but no.

R: Alrighty. *End of interview*
Interview #15  (Total Time 50:10)

R: A traditional chessboard consists of 64 squares, 8-by-8. Suppose dominoes are
created such that each domino covers exactly two adjacent squares of the chessboard.
a perfect cover of the chessboard with dominoes covers every square of the chessboard
without overlapping any of the dominoes. Okay, does that make sense?

Shaun: [pause] Yeah.

R: And the bottom says prove that a generic chessboard of size \( m \times n \) has a perfect
cover if and only if at least one of \( m \) or \( n \) is even.

Shaun: Okay. [pause, starts to draw chessboard at top of page, stops, (20 sec)] Let me
restart it. Mind if I work on this stuff? [referring to notebook paper]

R: Oh, nope, that’s why it’s there.

Shaun: This already has lines on it. [laughter] Okay. [pause, draws 8 x 8 chessboard,
(30 sec)] Okay. [pause, draws domino to the side, reads question, gets out another pen,
laugh at question, starts to draw where dominoes would go on chessboard, whispers while
rereading the question, (1 min 20 sec)] I wonder if this would work. I don’t know.
[pause, drawing in dominoes on chessboard, (20 sec)] I’m trying to just get kind of a
visual of what’s going on.

R: Okay.

Shaun: [continues filling in dominoes on chessboard, pause, looks back at question, puts
square around “perfect cover” in the statement of the question, writes “Perfect Cover: No
overlapping dominoes”, (1 min 30 sec)] It’s kind of funny because it kind of, in a way,
just because I started here on this side, it ended like that, too. I mean, this is probably
not the only way to draw them in, but it’s kind of interesting. They kind of position in a
way that they begin first – Because if you started like this, they’d be like this [indicating
horizontal versus vertical placement of dominoes, laughter]. I don’t know. So, you can –
Prove that – [whispering while reading the question] So, this is 8-by-8 [writes 8x8 to the
side of the chessboard, writes “\( m \times n \) is a perfect covering”, (30 sec)] Did you make these
up?

R: No, well to some extent, but they were all pulled from somewhere.

Shaun: Hmm, okay.

R: So, the wording is a bit different than their original wording.

Shaun: Yeah. [pause (20 sec)] Okay. [writes “\( \iff m,n: m=2l \text{ or } n=2p \)” pausing
several times while writing it and checking back to the statement of the question, pause,
(50 sec)]
R: Can you tell me a little bit about what you’re thinking?

Shaun: Yeah. I’m kind of, well first I just wanted to draw a chessboard.

R: Okay.

Shaun: I had a vision of what that is, but I kind of just wanted to see maybe how they’d lay out.

R: Okay.

Shaun: And then, kind of, I noticed that this [pointing to the second portion of the statement of the question] was kind of a highlighted piece of the problem here. So, I thought I’d write that down, just to cement that in my head. Um, and then I kind of wrote the, wrote the problem down. First step, well before I did that, I wrote that it was 8-by-8; that’s the dimension of it. And so, I tried to write the problem, restate the problem I guess. And now I’m thinking just to prove to myself what’s going on here, if this is true.

R: Okay.

Shaun: I may do another one different, with a different, uh, at least one of these being, maybe one odd.

R: Okay.

Shaun: So, then that would help me to see that visually. Um. Okay. Let’s go ahead and do that, I don’t know – [whispering (5 sec)] I’m sorry I’m not talking enough am I?

R: Oh, you’re fine.

Shaun: I need to be talking more here. [pause, whispering while he draws another chessboard, starts to draw dominoes on chessboard, (20 sec)]

R: Okay, so you have an 8-by-5 there?

Shaun: Mm, hmm, yeah.

R: Okay.

Shaun: [pause] I should write that down. [writes 8x5 to the side of the chessboard, keeps drawing dominoes in, gets another pen, whispers, (30 sec)]

R: So, what was it you, you just observed?
Shaun: Oh, I said, well ‘cause like in this one, we had a perfect covering and you could, well let’s see –

R: You’re kind of alternating how you started the –

Shaun: Yeah.

R: Okay.

Shaun: ‘Cause, see here you can’t, you can’t quite, ‘cause it says, um, in an 8-by-8. So, this doesn’t, this doesn’t hold for this thing that they can be, that they have to be adjacent, well I mean –

R: It still does.

Shaun: Oh, okay.

R: So, the domino still covers just like you have the picture of it covering, yeah.

Shaun: Yeah. But if I drew, see now, this is adjacent to this but, and I have to draw this one like this and this one isn’t – Well, I guess it’s kind of adjacent. I don’t know. I’m just thinking like –

R: Oh, those are totally fine.

Shaun: Okay. I was just kind of thinking, I don’t know, just to keep things uniform. I mean, I could do lots of different ways. But, I think this makes it easier.

R: Okay.

Shaun: To maybe, kind of keep it semi-organized.

R: Okay.

Shaun: Okay, so. That proves it to me that it works. I guess, maybe this will be an example. [writes “ex”. under chessboard and “non ex” to the side] I’m pretty sure that is right. But, just to see, I’ll do a non-example, I guess.

R: Okay.

Shaun: Just to, I don’t know. It may be a waste of time, I’m not sure.

R: No, we’ve got all the time in the world, no big deal.

Shaun: Let’s just do 3-by-5.
R: Okay.

Shaun: Oh, no, what a mess, excuse me. Um, okay. Let’s do 4-by-3. [whispering as he draws 4-by-3 chessboard, (5 sec)] On no, this wouldn’t be, no both of them have to be odd, okay, so this is, yeah. At least, so both have to be odd. [adds row to chessboard, (5 sec)] Okay, that works. [pause (5 sec)] Okay. [draws in dominoes on 5 x 3 chessboard, (30 sec)] But I got one left, keep that one – Okay. [Shades in one left over square]

R: So, that represents a square that wasn’t covered?

Shaun: Yeah.

R: Okay.

Shaun: I guess I should have – [writes “square not covered” with arrow to square, (5 sec)] Yeah. Square not covered. Okay. Well, let’s see. It’s an if and only if, so I need to prove it forwards and backwards. Which is going to be – easier? [pause] Oh, okay, so if it’s a perfect covering. [pause (5 sec)] Well, let’s do it forward. Since I have a non-example, [pause] maybe, maybe prove the [pause] Just right now I’m thinking that proving it directly isn’t kind of the way to go maybe.

R: Okay, the forward direction, you mean, proving that directly?

Shaun: Yeah. Proving the forward direction, I don’t know if that’s a good way to go. So maybe, I don’t know, maybe I’m thinking like contradiction here, possibly.

R: Okay.

Shaun: Let me write something down. I don’t know.

R: Okay.

Shaun: So, we’ll, we’ll call this $P$ [indicates that “$m \times n$ is a perfect covering” is labeled $P$ and that his there exists statement is labeled $Q$] and this $Q$. So, we’ll do $P$ and not $Q$. [on new sheet of paper writes “$ightarrow P \land \neg Q$”] Okay. So, we’ll assume we have a perfect covering. [writes “assume $m \times n$ is a perfect covering and” , (15 sec)] And, um. [pause (5 sec)] Not $Q$, okay. So, let’s, let’s see. What our –? [pause] $Q$ is for some, I think that’s right. This would be kind of sad if I couldn’t right up the statement. I think it’s right.

R: Okay.

Shaun: Can I ask you if it’s right?

R: You’re just writing $Q$?
Shaun: Yeah.

R: You’re just writing the statement that –?

Shaun: Yeah, is this okay? [points to his initial writing of the if and only if statement]

R: So in words, what do you want to say?

Shaun: So in words, we’re saying here – well, let’s see. In words, this says for some $m$

$n$, well there is some $m n$ that, yeah –

R: But, $m$ is what and $n$ is what?

Shaun: $m$ is even and $n$ is even.

R: Okay.

Shaun: Well, not and, but or.

R: Or, okay.

Shaun: Okay, so, is that right? Is that an okay translation?

R: That seems like it might work for you, right?

Shaun: Okay. Yeah. So –

R: So, $m$ or $n$ being even.

Shaun: Yeah. [writes “$Q: \exists mn, m = 2l \lor$”] So, I want to get it right, so I can negate it

and have it work out for me. $2 l$ –

R: So, I’m thinking maybe you don’t need the there exists, you just need the other part of

the statement.

Shaun: Okay, so you don’t need this?

R: Yeah, because we know $m$ and $n$ are something already and all we want is this –

Shaun: Okay.

R: So –

Shaun: Alright, so I guess – I won’t erase it I’ll just cross it off. [crosses off the “there

e exists” part of his statement]
R: There you go.


R: Or just say in words what you mean there.

Shaun: Okay. I need to tell you guys where like the $l$ and $p$ are, though.

R: Oh, okay.

Shaun: I think, yeah.

R: That’s fine.

Shaun: Yeah, okay, I need to say this. There is $l$ and $p$ such that, [adds in another there exists for $l$ and $p$] yeah, okay, that $m$ is even, $m$ or $n$ is even. Okay. So, I’m going to negate this, so it would be negation of $Q$. [writes “$\neg Q : \forall l, p : m = 2l \land n = 2p$”, (5 sec)] So, this would maybe have to be – okay. [pause (10 sec)] Since it’s a, okay, let’s see this is a, so proof by contradiction. [writes “proof by contradiction” at top of page] Okay, and I have, and all you need for proof by contradiction, is to show a non-example, or to show it doesn’t work.

R: Okay.

Shaun: So, um. I’ll have, okay, well let’s see. Okay, I lost my train of thought. Okay, so for all, for all – [pause (5 sec)] This doesn’t work. Hmm, I think I just confused myself. Hmm. Cause if it is, yeah maybe not – ‘cause if [pause] yeah. Yeah, if they are both even, then it’s going to be a perfect covering. Okay. So, maybe contradiction wasn’t a good, a good way.

R: So, you’re just checking over your negation?

Shaun: Yeah.

R: Okay. Tell me in words what that negation is.

Shaun: Okay.

R: So, the original statement is that one of $m$ or $n$ is even, right?

Shaun: Mm, hmm.

R: So, thinking outside the symbols for a second –

Shaun: Okay.
R: Tell me what the negation of that is.

Shaun: Okay, so the first part of it says that one of the, one of the, either $m$ or $n$ needs to be even.

R: Okay.

Shaun: The negation is going to tell me that both have to be even. ‘Cause when you negate an or, it becomes an and.

R: That’s true.

Shaun: So. [pause (20 sec)]

R: Any other thoughts on that?

Shaun: [pause (5 sec)] Okay, does it mean that? Yeah. $m$ and $n$, well yeah, okay. So, maybe, maybe this does – okay wait. So, it works so – yeah, let’s say, when does it work? So, when does the statement hold? [writes while talking “when does the statement hold”] It holds when – [writes “$m$ or $n$ is even”, (15 sec)] It holds when this and it holds – I’m writing cursive, now. And, and – [writes “$m$ and $n$ is even”, (10 sec)] Okay. The or is kind of a – you know, just from my reading in like the book, in the beginning, like a long time ago, it talked about or. The or is not something that is liked too much by mathematicians, is what I got from it.

R: You think so?

Shaun: Well, it kind of said that, kind of. Kind of said, or is kind of sometimes wishy-washy, kind of. I don’t know, I don’t know, that was in the very beginning.

R: I think they were talking about like the difference between or in mathematics and or in human language.

Shaun: Yeah, yeah.

R: Right?

Shaun: Yeah. They were saying it was kind of wishy-washy.

R: In human language.

Shaun: Yeah, in human language.

R: That’s right, is wishy-washy.

Shaun: Yeah, its wishy-washy.
R: In mathematics, we mean something precisely, right?

Shaun: Yeah.

R: And I think you’re getting at it there. So, you said the statement holds when $m$ or $n$ is even and when $m$ and $n$ is even.

Shaun: Mm, hmm.

R: And where would you go from there?

Shaun: And now we’ll say when doesn’t it hold. [writes “when doesn’t the statement hold”, (10 sec)] When doesn’t it hold? Okay. The only time that it doesn’t hold is when they’re both odd.

R: Okay.

Shaun: So, when $m$ and $n$ are odd. [writes “$m$ and $n$ odd”] Okay. [pause (5 sec)] So, maybe – [pause] Yeah, okay, so.

R: Can you tell me what just happened?

Shaun: Yeah, I was thinking the negation, the negation, I think I need to use the negation of $Q$. This is not, I think, this is not a good way to do it. [pointing to the top of the page] I just in my head, I thought, oh well the contrapositive.

R: Okay.

Shaun: ‘Cause if it’s, if not $Q$ implies not $P$, then that implies that $P$ implies $Q$. [writes symbols for not $Q$ implies not $P$] They’re the same.

R: Okay.

Shaun: Well, not the symbols.

R: They’re logically equivalent. That’s true.

Shaun: Yeah, logically equivalent. Okay, so I have not $Q$, so not $P$ would mean that it’s not a perfect covering. Okay, so. [pause] But, this is my not $Q$. [points to his symbolic statement at top of page] Hmm. [pause (10 sec)] Maybe I’ll just, maybe this will be, when doesn’t it, when doesn’t it hold? [pause (5 sec)] Hmm. I guess with this problem, it’s kind of, I think the thing more is, well it’s kind of a little bit confusing just cause it kind of has cases, kind of, within it.

R: Okay. And you’re thinking of the cases as?
Shaun: I’m thinking of the cases as, well –
R: What you’ve outlined there?
Shaun: Yeah. I kind of – I’ve got a whole bunch of stuff in my head. And I’ve put most of it down. I think sometimes when I’m doing proofs, uh, it’s – you know after I’ve written a bunch of stuff and I don’t see anything, I’ll sit there and try and put it, put it all together in my head.
R: Okay.
Shaun: So, then I’m almost too scared to put anything else down until I feel like I’ve mastered what I’ve put down already.
R: Okay.
Shaun: You know, like I’m – ‘cause I don’t want to miss anything that I already put down, kind of.
R: Okay. So, can you describe to me how you’re putting these things together in your head?
Shaun: Well, I’m kind of –
R: Just let me know, just talk me through the process of what you’re doing.
Shaun: I’m kind of, I’m kind of when it comes to, I’m pretty, I guess I’m notation oriented when it comes to proof.
R: Okay.
Shaun: I’m okay with, I don’t know, if I’m kind of not really – that good at proofs. I’m good, I can do things like this last chapter that we were doing. It was kind of, I don’t know if was heavy notation. But, I can put, I can connect, like when we did that problem with the, hmm, what was it? It was something with the composition, anyway, my first –
R: So, you’re talking about functions and with the one-to-one and onto?
Shaun: Yeah. My first thing to say, my first thing I saw was that \( f \) of \( a \) was equal to \( b \). Or \( f \) of \( a_1 \) is equal to \( b_1 \).
R: Mm, hmm.
Shaun: So, I don’t know, I’m good with notation, kind of. And so, um –
R: So, that’s where you’re going first is notation?
Shaun: Yeah, notation. Yeah. That’s I think, that’s where some of my downfall comes in some problems.
R: Okay.
Shaun: Is kind of, if, if – ‘cause here not too much notation has been given to me. [points to the statement of the question]
R: Okay.
Shaun: More of the proof, or the problem I guess is more words. I have a harder time with words.
R: So, translating from those words to what you want to use notation-wise?
Shaun: Mm, hmm. I have a harder time with that.
R: Okay.
Shaun: ‘Cause I flip things around a lot in my head. With some, if it was, if more symbols were there, I would have a little bit easier time maybe.
R: Okay.
Shaun: But, anyways, now I guess I’m –
R: Do you want to pursue this further, or would you like to put it away and try another problem?
Shaun: [pause] Well, I’m kind of stubborn, too. [laughter]
R: Okay.
Shaun: I don’t know, let’s see. I know –
R: Let me know if at any point in time you would just prefer to stop and work on something different, okay?
Shaun: Okay.
R: But, you can pursue it as long as you want.
Shaun: Well, I have, I have an example here.
R: Okay.

Shaun: And I have this when doesn’t it hold. So, when doesn’t what hold? When doesn’t the whole statement hold. Okay. Well, the whole statement can’t hold if the beginning is false, I think, if I remember right.

R: Okay.

Shaun: Like in the truth tables, that’s the only time. Yeah, there’s only one false in the whole, in the whole table. The $P$ implies $Q$ one. [starts to draw truth table to the right]

R: Okay, so why don’t you make the truth table for that? Go ahead and finish that truth table.

Shaun: There’s only one time, yeah. So, it’s true and – [fills in truth table, pause, (20 sec)] Nope. Oh, I know what I did. We did true false, true, false. So, true true, false false, true false, false true. Okay, so. [laughter, pause] Yeah, okay. So, this is true. False. Can something false imply something else is false? Which, yeah this is – something true, yeah. Wait, I’m getting confused. This looks really bad. [laughter]

R: You’re confusing yourself in your notation, right?

Shaun: I’m confusing myself. Hmm. Uh. Let’s see. [pause (5 sec)] Can something – well, let’s just use this. If it’s not a perfect covering, then it can be, then it is true that $m$ or $n$ is even? No. Okay. I think this is the false one. Yeah, something false can’t imply something – I don’t know. Let’s make up a silly sentence, like, like we do.

R: Well, it’s the English language that’s getting you, right?

Shaun: Yeah.

R: Okay. So, why don’t I tell you that that true and the false on the bottom are interchanged?

Shaun: These are? [points to truth table]

R: Yeah, that last column.

Shaun: Okay, okay. Yeah, I knew. ‘Cause when I thought – oh yeah. So, when you usually write it it’s true true false false.

R: Yeah.

Shaun: This is always the same.

R: Yeah.
Shaun: I have a good memory, it’s always the second one from the top.

R: Yeah.

Shaun: This is the false, this is the true.

R: Yeah, so it’s in probably how you wrote the table most likely.

Shaun: Yeah, don’t tell (my professor) I did this. [laughter]

R: Okay.

Shaun: Okay, something true can’t imply – yeah that makes sense. Okay, where was I going with this? I don’t remember what I was talking about, like why I started doing this. Okay, oh no, I was saying when doesn’t this statement work. Okay, so maybe now I’m thinking proving it directly.

R: Okay.

Shaun: So. [pause (5 sec)] I can’t prove that directly. Okay, now, okay. So, now my thing is, is like if they’re odd, both have to be odd, then I need to make up, let’s say now, what does this mean? [points to where he wrote that both $m$ and $n$ were odd] So, I need – [pause (5 sec)] I don’t know if this is going to screw me up notation-wise, but – [writes $m = 2l+1$, $n = 2p+1$, (5 sec)]

R: Okay. So, you’ve just written down what it means for $m$ and $n$ to be odd?

Shaun: Mm, hmm.

R: Okay.

Shaun: [pause (25 sec)]

R: What are you thinking?

Shaun: Hmm, I think – Well, I’m thinking I’m kind of confused. I’m kind of – Now, I started off pretty confident in the contradiction part.

R: Okay.

Shaun: And then I went to this idea. And now I’m kind of reaching to my other bag of, okay would it be better to prove the contra, contrapositive direction – contra – what’s $Q$ implies $P$? What is that called?

R: Uh, converse?
Shaun: Converse. Okay. Does that lead, well that’s what I was just thinking, does that lead me anywhere? No, I don’t think so. ‘Cause that’s just saying that if one of these is true, or if one of these is even, if \( m \) or \( n \) is even, then that implies. See, now, and I’m, hmm. I’m being silly here.

R: So, what would you like to have happen here?

Shaun: Well –

R: Just tell me what, what your idea is, as to why that this is true, or what your, or where you’re going?

Shaun: Well, what I would like to have happened, or maybe it’s going to happen in the next 10 minutes or so, is that I would be able to use an example. Because, because I don’t have any symbols, I don’t have any notation really. [pause] Well yeah, I have, I have a non-example.

R: Okay.

Shaun: And if I’m even able to find a non-example, then that should, that should be able to help me. Well, in this situation, because I don’t have enough notation. This problem is not, I guess maybe it could be made – I don’t know what I’m trying to say. Hmm. Like. This is not really a, well, maybe it is a subject. But, so far as my knowledge, there’s no.

R: What do you mean?

Shaun: Well, there probably is the math of chessboards. I’m just thinking –

R: Oh, okay.

Shaun: You know what I’m saying. Like, this is not, like everything else I’ve been working on so far in this class has been like, okay, this is functions, okay that’s part of like, analysis or something.

R: Okay.

Shaun: You know. Sets are part of well, everything. Sets are kind of within all areas.

R: Okay.

Shaun: This is like, you know, someone was sitting down looking at a chessboard, playing around, and they noticed this. It probably is within some kind of –

R: So, you’re trying to connect it back to something in class?
Shaun: Yeah, yes.

R: And not finding anything?

Shaun: It’s like I don’t have really, I can’t relate this to any specific subject.

R: Okay.

Shaun: I think that’s another thing I’m having a hard time. This is more, I don’t know, I don’t know if this is like discrete math or, I don’t know. You know, I mean I see patterns here, you know, but –

R: So, let’s go back to this thought of this non-example. Okay?

Shaun: Okay.

R: So, this non-example is 3-by-5.

Shaun: Mm, hmm.

R: And you couldn’t find a perfect cover, right?

Shaun: Mm, hmm.

R: So, could you go into more detail, maybe about –

Shaun: Oh, oh, wait –

R: As to why there’s no perfect cover?

Shaun: Oh, well you know what’s funny is, oh that’s kind of funny you mentioned that. Cause each of these has a remainder one and so does, so does this. Has a remainder one.

R: So, there’s one left over square.

Shaun: Mm, hmm.

R: How are those related?

Shaun: [pause] Oh, yeah, okay. Well, like, so, each, each domino maybe is even.

[writes “each domino is even”] Each domino is even.

R: ‘Cause it covers two squares?

Shaun: Yeah. [writes “because it covers two spaces”]
R: Okay.

Shaun: [pause (10 sec)] Yeah, each domino is even. [pause (5 sec)] So, and this is, so I have – [pause, writes \(2l \times 2p\) in middle of page, crosses out odd in \(m\) and \(n\) odd and rewrites it then writes \(2l+1 \times 2p+1\), (15 sec)] Sorry I just –

R: Not a problem, you can scratch it out.

Shaun: I just wanted to clear it up here. Yeah. [pause (5 sec)]

R: So, what are you thinking?

Shaun: Here’s my problem, notation-wise. You know, you just kind of, I don’t know, I don’t remember quite what you said. You said let’s go back to the picture. And when you said that, I thought well, and look I even, I even –

R: You have one extra square.

Shaun: Yeah. And now I thought, just from (my professor) driving into me the cases of, like, you know, from those division problems. The division algorithm remainders and stuff, I just thought, well hey, look, there’s one remainder. And so, I wrote it out, what my remainder is.

R: Okay.

Shaun: So, this means, this means, uh. An even, a domino plus a half a domino, kind of. The domino is 2.

R: Okay.

Shaun: [pause] Well, let’s, let’s see, how many dominoes, whole dominoes I have. Dominoes, dominoes.. [laughter, counts the dominoes needed to cover the 5-by-3 chessboard] 1, 2, 3. Okay. 1, 2, 3, 4, 5, 6, I confused myself. 1, 2, 3, 4, 5, 6, 7. 7 whole dominoes, jeez, 7 dominoes. [to the side of the chessboard, writes “7 dominoes, remainder 1”, (5 sec)] Remainder 1. [pause] So, it means, okay. Unit wise, this thing, how many things, what is this unit wise. This is, um, 14, 14 plus an extra. [writes 14 +1 at bottom of page, (5 sec)] Hmm. [pause (10 sec)]

R: What else are you thinking about, about it?

Shaun: Hmm. Well, so I counted how many whole dominoes there were and I saw that this was an odd number. I saw that this is an odd number and I thought, well what does that mean?

R: Okay.
Shaun: So, if a domino is two units, so I tried to break it down unit-wise. Since, a one is a unit.

R: Okay.

Shaun: I don’t, well, because, because originally on the chessboard, you don’t have rectangle boxes. You have square boxes.

R: Okay.

Shaun: The square box is a unit of measure on the box, or on the chessboard.

R: Okay.

Shaun: So, that, this is not a good way to measure dimension.

R: I see.

Shaun: You have to measure the dimension of the chessboard unit-wise. Okay, so, and that means that when I do that, then I can see that it’s an even number –

R: Okay.

Shaun: – plus and odd number here. [pause (5 sec)] But – okay – [pause (5 sec)] Yeah, but see that, okay. [pause (10 sec)] Okay, so. This is [points to 5x3 chessboard, pause] this was 5 and this was 3. [writes \( m = 2l+1 \) and \( n = 2p+1 \), pause, (45 sec)] Hmm, okay, so this is, um. Okay, so I don’t know. [pause] So this is, 4 divided by 2 is 2. So, l is 2. [writes \( l = 2 \)] And \( p \) is 1. [writes \( p = 1 \) unit, pause (10 sec)] Yeah, so I don’t know, here, I don’t know what I’m thinking. Here \( p \) gives me the unit of measure on this chessboard. 2 gives me the domino, the uh, I guess [pause] the ruler, kind of.

R: Okay.

Shaun: It gives me the thing that we’re using here for the problem. I don’t know what I’m trying to say. [pause] And this thing also is in here. [points to chessboard] So, the unit is here, the \( p \) is here, the \( l \) is here. [pause, writes \( 15 = 7l+p \), with \( 2 \) and \( 1 \) above the \( l \) and \( p \), (45 sec)] I don’t know. Let’s go on to another problem.

R: Any other thoughts about it?

Shaun: Hmm.

R: Okay. If you want to leave it, we’ll leave it. But, we’re actually almost done for today, so I’ll ask you to reflect on this problem.
Shaun: Okay. Sorry, I couldn’t do more –

R: No, this is totally productive, watching you work is the point.

Shaun: Okay.

R: – of this. Okay? People who go through it quickly are actually less helpful. [laughter]

Shaun: Well, then I’m your man, because I’m slow.

R: You’re totally fine. So, can you just walk me through the process from the start of what happened, and kind of what things you looked at? For – So, when you first started you drew out the 8-by-8, so you looked at the specific example.

Shaun: Mm, hmm.

R: Put it together. What then?

Shaun: So, I drew the specific example that they gave me and then I looked at their statement, their general statement, for a general chessboard of m-by-n. And tried to show to myself, I thought okay let’s make another example, given their directions. Let’s make an example given their well, giving I guess a fault in one of their directions I guess. Going against a part of their directions –

R: You mean looking at the example and the non-example?

Shaun: Yeah, looking at the example and the non-example.

R: Okay.

Shaun: So, I drew, drew both of those.

R: And that was really to convince yourself that the statement was true?

Shaun: Yeah.

R: Okay.

Shaun: And then I thought well, okay, I have a non-example, so maybe I can use that to my advantage to prove the, the statement. [audiotape side one ends, researcher flips over tape, student pauses]

R: Okay.
Shaun: So, I looked at the non-example and thought that I could use that do to the proof in one direction. I never got to the second direction. So, I was looking at the forward direction of the if and only if.

R: Okay.

Shaun: So, I wrote down – tried to make some, tried to write down the notation, maybe extract a little more notation, um, to help me out. Then I, I uh, I wrote down again, then I worked on the proof by contradiction.

R: Still notation-wise and things like that?

Shaun: Still notation-wise, yeah.

R: Okay.

Shaun: And then I thought, well I’m not having too much luck with this, so let’s write down when this stuff works and when it doesn’t work.

R: Okay.

Shaun: And after that I was able to do a little bit more notation, kind of. I’m not sure what order that went in. Um.

R: And you noted that those were really your cases, and that one of the cases doesn’t work so you were kind of looking at that a little further?

Shaun: Yeah.

R: And looking back at how that related to your non-example?

Shaun: Yeah, and I saw that in the non-example I had one left over. So, maybe my problem is I’m not really [pause], I don’t know. I think I have more or less, it’s all there, kind of thing.

R: I think you’re almost, you’re getting there, right?

Shaun: But. [pause] There’s kind of –

R: Okay.

Shaun: There’s somewhere there’s an idea from the, I guess from the – sometimes it happens to me that I’m doing a problem and I get so into the specifics of the notation and stuff that I lose sight of the general picture.

R: Okay.
Shaun: I kind of think that’s what happened here. As I got more and more confused in
the ramblings in my head about what was going on, I lost sight of the original. So,
maybe I’ll read that again. A traditional chessboard – you know what I think that’s what
I do sometimes.

R: Okay, so then you would go back at this point, you’d go back and reread it and look
back at what you were doing and –

Shaun: Yeah.

R: – consider where you were at, and how that relates.

Shaun: Yeah.

R: And all that kind of stuff? Okay. And you made a note that you really haven’t seen
anything like this before, right?

Shaun: Yeah.

R: You weren’t sure where it fit into the picture.

Shaun: Yeah, yeah.

R: Uh, any, anything related to it that you might have seen at all? I’m just gaining an
idea of your experience.

Shaun: Mm. [pause]

R: Nothing you can think of?

Shaun: Not really.

R: Alright, well let’s stop there. **End of interview**
Interview #16  (Total time = 51:30)

R: We’ll start by reading this problem.

Beth: Okay.

R: Okay. We’ll call a positive integer, \(N\), a 4-flip if 4 times \(N\) has the same digits an \(N\) but in reverse order. So does that make sense, what that means?

Beth: We call a positive integer, \(N\), a 4-flip if 4 times \(N\) has the same digits – Okay, so like \(N\) is 1234 and then the 4-flip would be 4321 [writes 1234 4321].

R: Exactly. So when multiplied by 4, you would expect it to be 4321, if it was a 4-flip.

Beth: \(N\) times 4 equals [writes \(N\times4\) = then attempts a backwards \(N\)] – I’m not sure – [laughing] Okay, prove that there are no two-digit 4-flips. Okay, so a two-digit 4, two-digit 4-flip would be something that goes like that [writes 24 – 42].

R: Right.

Beth: Okay. [pause] So it’d be some number \(X\), that gets, goes to a new number where it’s just the opposite [writes \(4X\rightarrow\bar{Y}\)]. Okay, so – [pause (10 sec)] So, we have to prove that [writes \(4\times10\)], at the very least, that um this cannot be a 4-flip and then [writes \(4\times99\)] cause then there are no more in the row that I’m going to look for in between here [draws bracket from 10 to 99].

R: Okay.

Beth: And so – what could we do about this? Let’s just say the least is 40. Okay, what I’m thinking of is special cases where 4 times, like 4 times 22 [writes \(4\times22=88\)] equals 88. [pause] Alright, so why doesn’t this work for two-digit numbers? [pause, looks over question and then her work, (25 sec)] The lowest one is going to be 40. We need to find a two-digit number, \(N\), that’s gonna give us [writes \(4N=M\), then writes \(4(\text{ab})=\text{ba}\)] 4 Times \(ab\) is gonna give us \(ba\), that’s what we’re doing. Where this isn’t multiplication, this is just the order [referring to \(\text{ab}\)].

R: Okay.

Beth: [pause] Um. [pause (5 sec)] Well, I don’t know if you did multiply this number by 4, like, let’s say our \(ab\) is 24, it would, in order for 24 to be a \(N\)-flip, a 4-flip, we need 42 [writes 24=42 under \(4(\text{ab})=\text{ba}\)] but it’s 4 times 24 and that is not 42 [writes \(4\times24\) not equal to 42, pause, (10 sec)]. Alright, let’s look at why does it work for a three. Let’s see what we can, I want to see what we can – [pause, writes \(4(\text{cde})=\text{ecd}\) to the side, whispers while she writes, pause, (35 sec)] I guess I’m going to try to find examples. It’s kind of hard to find examples. [pause (20 sec)] Hmm.
R: So, what are you thinking about?

Beth: Right here, um, like what I’m thinking of a number to put here [points to cde] that when I multiply it by 4, like if I put um – a 2 here [writes 2 under c], like say this x num, x, x – I’m thinking if I had 4 times 2, it’s gonna have to be like this first number [pointing to e on other side, pause, (10 sec)]. Like 6 times 4 is 24 [writes 6*4  24]. So that would be the 4, cause I’m gonna be, cde times 4 is what I’m doing, you know if I’m – [pause (5 sec)] And what’s this going to have happen? [writes cde*4=(c4)(d4)(e4), pause, (25 sec)]

Maybe that idea was bad, okay. [pause (5 sec)] 234, let’s just see what would happen. 234 times 4 is 16, 936. [writes 234*4=936] So, this would have to be 639, and this would have to be – okay. So, 234 and 639 aren’t like this 1234, 4321. [points to 1234 written at the top of the page] So, that’s not an example. So –

R: And actually, you would want that other side to be 432. Because your original number was 234. You’d want that other side to be 432.

Beth: Oh, okay. Oh, okay, I did that one. I see. [writes 432 under ecd]

R: Does that make sense?

Beth: Mm, hmm.

R: And instead you’ve got 936.

Beth: Mm, hmm. [pause (5 sec)] Yeah, so what I was thinking in my head is, this number [circles the e and 4 of cde and 234, respectively] times 4, like it, it left a 6 cause it’s 16. [pause (5 sec)] So, then this number would have to be some sort of 6.

R: Hmmm.

Beth: Gosh, these, these problems make me feel dumb.

R: [laughter] They’re meant to be challenging, don’t forget. I mean the idea here is that I want you to have to work hard on them so that I can see what’s happening.

Beth: Yeah, well I guess I have no clue where to start though.

R: Do you want to go back to the 2 case?

Beth: Okay. [pause, draws line down page, (5 sec)]

R: So, you said we have the numbers 10 through 99 to test, right?

Beth: Mm, hmm.

R: And –
Beth: So, if I had all day, I could go out and um, do 4 times 10 and show that it does not equal 1. [writes 4*10 ≠ 01, 40 ≠ 01]
R: Okay.
Beth: And I would do that. 4 times 11 [writes 4*11 and 44 ≠ 11], okay that was 40 does not equal 01 and this would be a 44, does not equal 11 and so on. [pause, writes 4*12] Which would be 48 does not equal 21 [writes 48 ≠ 21]. Well, let me see if I can see a pattern or something.
R: Okay.
Beth: 4 times 13 [writes 4*13], which is [pause, writes 52 ≠ 31] 52, which does not, not equal 31. Okay, so [pause] our numbers are going 11, 21, 31. [pause, moves to new sheet of paper, (10 sec)] But, I don’t want to do ’em all. So let’s try the 20s. 4 times 20 [writes 4*20] is gonna be 80, does not equal 02 [writes 80 ≠ 02]. And this will follow – and so on, so 84 does not equal 12. [writes 4*21 and 84 ≠ 21] Okay, so it looks like –
R: Do a few more. [pause] Just because I’d really like to see, I think if you do a few more, you might see something, so.
Beth: Okay. [writes 4*22 and 88 ≠ 22]
R: I’ll encourage you not to stop there.
R: Okay.
Beth: And then I could go on to 30s next and maybe we can – we’ll see. 120 does not equal 03. [writes 4*30 and 120 ≠ 03, 4*31] 124 does not equal 13. [writes 124 ≠ 13, 4*32] 128 does not equal 23 [writes 128 ≠ 23, 4*33]. And 132 does not equal 33 [writes 132 ≠ 33].
R: Okay, any ideas formulating yet?
Beth: [pause (10 sec)] Well, I’m, let me just see what, let me do um the 90s because if I can see the, the largest that this would be [circles numbers 03, 13, 23, and 33], looks like it’s gonna be like 99 or something.
R: Okay.
Beth: And that [pause] Oh, um the smallest this side of the equation, this side can be is 40 [referring to 4N]. But that doesn’t really prove anything because –
R: Oh, I see what you’re saying, okay. The smallest 4 times something can be is 40.

Beth: Mm, hmm. [pause] That’s not really saying anything yet. [writes 4*90] 360 does not equal 09. [writes 360*09 and 91] 364 does not equal 19. [writes 364*19 and 4*92]. And 92 is, what would 92 be? 368 does not equal 29. [writes 368 ≠ 29, pause, (5 sec)]. Alright, so I’m looking for a pattern or something. [pause, looking over work, (20 sec)]

R: Can you tell me what you’re thinking, what you’re looking at?

Beth: Well, I don’t know. My first thought, I looked at this column [indicating column with 03, 13, 23, and 33] and I said oh, this will never, this one, the 4-30s would never match up cause these already have three numbers.

R: Okay.

Beth: And this only has, they’ll only have 2 or go up to 93.

R: Okay.

Beth: So, I know for sure, without going through all of them that this set, um, can never be a 4-flip. [indicates the 30 column]

R: Okay. Can you go further with that idea?

Beth: Maybe um, from 30 up.

R: Okay.

Beth: Because yeah, 4 times 40 would be next set of 10 and that is 160 and I just, 4 times any number, it’s getting bigger.

R: Keeps going up?

Beth: It’s gonna have three numbers, right. And this side, still’s just gonna have two numbers.

R: Okay.

Beth: So from, 30 to 90, all the way to 99, [writes 30-99] I know that can never happen just because more digits.

R: Okay.

Beth: So, I guess what I really need to look at is just the [pause] last 10, the first 10, 20 –
R: Okay.

Beth: Multiples. Let’s see what the 4-10s gonna go up to. [looks back to previous page of work] The last thing in this group’s gonna be um, 36, and 76, is this right? Yeah.

R: Okay. 

Beth: I just expanded my gap. [crosses out 30-99, writes 25-99]

R: Okay. 

Beth: Right, like here, here I could rule it out [pointing to lists of 30s and 90s].

R: Okay.

Beth: And then, this one’s gonna go all the way to um, 92 [on second page, writes 92 at bottom of left hand list]. And the last one here, that’s not going to equal on this side. 4 times 29. So I guess there’s a point here where I can, I can add cause 4 times – [writes 29*4] Okay. Oh wait, did I do this one wrong? [pointing to 4*19=76 on first page]

R: I don’t think so.

Beth: 9 times 4 is 36, okay. Oh, okay, 36. [finishes calculation of 29*4] And that’s 11. [writes 116 ≠ 92]

R: I’ll grab you a calculator if you’re interested. You don’t have to, but if you want it, there it is.

Beth: Thank you. I see here that I have another three-digit. [pointing to 116]

R: Okay.

Beth: So I could, I could expand this interval.

R: Okay.

Beth: So I guess that I can work backwards, actually, this is gonna be what number, this is just 29, cause this is going by 4s. I know that 28 would be at 112, 27 would be at 108. 26 would be at 104 and 25 would be 100 [writes 28 with 112, 17 with 108, 26 with 104 and 25 with 100]. So, from 25 –

R: Okay.

Beth: I just expanded my gap. [crosses out 30-99, writes 25-99]

R: Okay.
Beth: That I have to account for, ‘cause 25 does not equal 50, 100 does not equal 52, 104 does not equal 62 and onward all the way [writes 100≠52, 104≠62]. 72, 82. [writes 72, 82 under 62].

R: Okay.

Beth: That’s messy, but – [draws line between 24 line and 25 line in list, pause] Okay, why could it never match up? [pause] Alright, so my next thought would be maybe to look at like the 4-table – like multiples of 4. Why 4 times something, like for any between this gap. Okay, yeah. Cause these are all end in um a 1 [pointing to 10 through 19 list].

R: Okay.

Beth: So, I have to think to myself, 4 times anything will it ever give a 1 as like –

R: Okay.

Beth: – in the one’s position? So, if I just look up the um up til 10 because then it repeats. 16, 20, 24. [writes 4, 8, 12, 16, … , 40 on first page] I’m writing out the 4’s table.

R: Oh, okay.

Beth: This is um 4 times 1, all the way up to 4 times 10 [under table writes 4*1…. 4*10].

R: Okay.

Beth: And I notice that it never [pause (5 sec)] um results with a 1 in this spot [underlines ones place of numbers in table].

R: Okay.

Beth: And then, so that, that covers 4 through 10 but 4 times 10. Okay. So I have to see if that’s true for, you know, 4 times 10 through 4 times 19.

R: Okay.

Beth: Which it is, because you know, this really, 4-19s really 19 is 4 and we already covered 9 times 4 is 36, so –

R: Okay.

Beth: I mean, I’m not saying it eloquently, but since – how do you say it? Sort of like divisibility by 2 you know –
R: Mm, hmm.

Beth: – there’s a integer, k, such that 2k equals the number you’re dealing with.

R: So, you’re saying that none of the 10 through 19 can work because the flip would end
in 1 –

Beth: Mm, hmm.

R: – but no multiple of 4 ends in 1.

Beth: Right.

R: Okay, so that covers those guys.

Beth: Alright, so 10 through 19, we’re good [writes 10-19 under 25-99 on bottom of
second page]. So now we’ll have to deal with 20 to 24. Let me just write this, um that
would be 16, 96 does not equal 42 [writes 4*24, 96≠42 in left hand list on second page].

R: Okay.

Beth: So why? [pause] I mean I guess with that small amount left, I can just say because
I –

R: It doesn’t work for the rest?

Beth: – because I did it out.

R: Mm, hmm.

Beth: I mean let’s see if um, yeah, what I used here in this section. [puts box around 10
through 19 list on first page]

R: Mm, hmm.

Beth: It doesn’t apply here because this is all ending in 2 [points to 20-24 list].

R: Okay.

Beth: So, it could, actually it stops right here [draws another line in middle of 20s list].
So it could be that um, 4 times something is equal to [pause] Yes, this is what I think this
is what I was trying to do, not exactly –

R: Okay.
Beth: – but 4 times 21, for instance [writes calculation for 4*21], from right here [underlines 4 of 84] we get a 4.

R: Mm, hmm.

Beth: And so, therefore, [pause] we would have to have something here with a 4 in, because the flip, I don’t know exactly how to say this. So, 4N of 84, the flip would be 48. [pause (5 sec)] Okay. [crosses off 21*4 calculation] That’s just confusing me.

R: So let’s go over what you’ve done so far. So, you reduced it down to saying 25 through 99 can’t work –

Beth: Mm, hmm.

R: – because they all get two three-digit numbers when multiplied by 4. And then you said 10 through 19 can’t work because they all end in 1 when flipped and that’s not possible. So, then we’re just left with 20 through 24.

Beth: Mm, hmm.

R: Right? And you have them all written; you can see that it doesn’t work for any of them.

Beth: Mm, hmm.

R: Right? So, barring maybe looking for something more elegant, looks like that’s what you’re kinda looking for is maybe something more elegant –

Beth: Mm, hmm.

R: – but does this prove the statement that there’s no two-digit 4-flips?

Beth: Prove that there are no two-digit 4-flips – yes.

R: Okay.

Beth: I believe so.

R: Can you tell me anything about the strategies you used while you did this?

Beth: Strategies – well, sorta like last time I just, well first what I, I understood the problem. And then I, I realized okay, two-digits, the one’s we’re dealing with, are 10 through 99. And then I understood what it meant to be a 4-flip. And then I tried saying okay – well let’s see why it works from maybe 3 on –

R: Okay.
Beth: – to start with three-digits. But that didn’t really work because it was sort of like a guess and check –

R: Okay.

Beth: – and I was just getting confused. So then, what helped, I guess is I wrote out the first and divided it up into um, a few segments of 10 and I just wrote the first few up til 90 and then that, that enabled me to see a pattern, oh look for 30 on um, on the left hand side of the equation is always gonna be three digits and three digits can never be equal to um, two digits.

R: Okay.

Beth: And then I narrowed it down and then I looked, so at the beginning I guess, and I saw that there was a 1 right here [points to 10 through 19 list] for every one in that no multiple of something times 4 can never give a 1 in the one’s column.

R: Okay.

Beth: So, I was left with maybe five that I just did by hand.

R: Okay.

Beth: So, it’s lots of different strategies, I think.

R: Mm, hmm. Good. Alright, let’s go on to the next one. [puts papers for part a aside, researcher gives student part b of question 2] Okay, this is going along the same lines. Prove or disprove the following statement. There are no three-digit 4-flips.

Beth: Oh, wow. So, I guess it was kind of good that I didn’t try to –

R: That’s why I pulled you back to the two-digit, I thought, well we’re gonna get there.

Beth: [laughter] Can I write on this? [referring to page with statement of question on it]

R: Yep.

Beth: Alright. I feel wasteful. So, [reads the question] prove or disprove the following statement. There are no three-digit 4-flips. So, it could be true or could be false. [pause] So, three-digits means I’m going from 100 to 999 [writes 100–999]. And I am guessing this is where I’m gonna have to use strategy because there’s no way I would want to write, if I divide this into um the same way I did this [points to part 1] like nine groupings –

R: Okay.
Beth: – of 100 in each. There’s no way I’d want to, um, write out 100 –
R: Okay.
Beth: – different equations or not equal equations. What would you call these? Not
equations? [referring to her lists of $N$ flipped $\neq 4N$]
R: Um, I don’t know. I don’t know what you’d call that. It’s not an inequality, it’s a
non-equality, I don’t know.
Beth: Yeah. So, okay, I’m just, I’m gonna maybe write down the first few. Hopefully,
um, 4 times a certain interval in here will give me like four digits.
R: Oh, okay.
Beth: – equal to, sort of. We’re going along kind of like 30 through 99, maybe I’ll get
four digits on this side and it can’t be equal to three digits.
R: Okay.
Beth: So, I’m hoping for that.
R: So where might that be? Can you investigate where that point might come?
Beth: 1000 divided by 4. [writes 1000/4] Because – [takes calculator, does calculation,
pause, (10 sec)] Okay, so, after 250, cause 4 times 250, is it equal to 052? [writes 4*250
= 052 with question mark over equals sign] And no it can’t be, because 1000 does not
equal to that. [writes 1000 $\neq$ 052] So, I’m guessing, I know from there on out, their gonna
have four digits. Oh, it’s just like this one, right? [points to part a of the question]
R: In the 25 through 99?
Beth: Mm, hmm.
R: Okay.
Beth: So, 250 through 999, um [pause] we have no three-digit 4-flips. [writes 250-999
“no 3 dig 4F”]
R: Okay.
Beth: Because um [pause, writes b/c $\neq$] 4, this is like third grade. What do you call it,
like – this is called the 1s, the 10s and 20s up –
R: Yeah, 100s and 1000s.
Beth: – what do you call those? Place, not place holder?

R: I’m not sure what you’re going for.

Beth: Like if you have the number 253 and 5 is –

R: That would be the 10s digit.

Beth: Okay, so these digits –

R: You’re looking for something else.

Beth: – the 4, four digits can never be equal to three digits [writes “4 digits ≠ 3 digits”].

R: Okay.

Beth: That’s what I’m trying to say. [pause] Okay, so now maybe going off of here [points to part a], I’ll look at um, 4 times 100 all the way down to 4 times 199 [writes 4*100 … 4*199].

R: Okay.

Beth: And I know it’s gonna be 4 times 101, 102, 103 [writes 4*101, 102, 103 in list]. And since I’m in the 100s column, it’s gonna be 001 something. This is all question mark, plausible. Is it equal? [writes = 001, = 101, with question marks above equals signs, =201, =301 in list with 100 through 103, respectively] And we’re gonna find that it’s not cause this is all the same thing, it’s all ending in a one’s.

R: Okay.

Beth: And like I proved before there’s nothing you can multiply 4 by and get a 1 in the one’s column.

R: Okay.

Beth: So, 100 to 199, we’re good [writes 100-199 in list of not 4-flips to the right]. And hopefully, we can see something else. So now we’re just investigating um 4 times 200 all the way down to um 4 times 250, that’s not bad [writes 4*200… 4*250]. Okay, so, maybe let me start writing the first few of what we’re gonna get over here and – [writes 4*201, 4*202, 4*203, writes 002, 102, 202, 302 to the side, (10 sec)].

R: Um, look back at those again and just make sure you got what you want. So, 201 becomes 102, right?
Beth: Okay, we’re gonna go all the way down to 249 equal 942 [writes 249 = 942, = 052 next to 250] 052. Okay, so these are all ending in 2s [pause (10 sec)] so why can’t –? [pause (5 sec)] Okay, up until a certain point, I should have left room here. So, this is um 800 does not equal 202 [writes 800≠, 804≠, etc. as she is talking]. 804 does not equal 202, 808 does not equal 202, 812 does not equal that. So, we can go up until I get to the 800s, I can block off another interval since um, you know, this is, this sides too low, it’s not even hit the 800s yet.

R: Okay.

Beth: So, it’s going to be 402, 502. [writes 402, 502, to 902 below 202, 302, (5 sec)] Oh yeah, this is way down [draws line above 249 = 942]. So, let me just, [writes 816≠, 820≠, 824≠, 828≠, 832≠ next to 402, 502, etc., (15 sec)] Well I guess really up until here [draws line below 828≠702], you know for sure that um [pause] 700 can never, these can never be equal.

R: Okay.

Beth: I mean, I guess if I’m doing it this way, then I might as well just look at all the 50, which is why I think, there has to be another way of looking at this.

R: Okay.

Beth: Of why there can be no. Okay, cause I guess I’m trying to disprove right now cause it says either prove or disprove.

R: So, you’re thinking you’re probably gonna disprove?

Beth: Yes.

R: Okay. [pause] So, you’ve discounted everything from 250 to 999 and everything from 100 to 199 –

Beth: Mm, hmm.

R: – and now from, I guess 200 to 207.

Beth: Mm, hmm. [writes 200-207 in not a 4-flip list to the right]

R: For sure. Okay, now what?

Beth: [pause (10 sec)] I don’t know. [pause (15 sec)]

R: Any ideas?
Beth: No, I guess the place I would go next is I would break it down into, um, smaller groups –

R: Okay.

Beth: – into 10s. And so –

R: Can you say any more about um, you were talking about the fact that they ended in 2, right, and what that might mean for the original number. Can you pursue that idea any further?

Beth: Well, maybe I could look at, cause 200 to 249. Okay, if you read backwards, I’m in the 2-0-0s. And I now it’s gonna go to 210s and then 220s, 230s, 240s, so yeah, [writes _12, _22, _32, _42] actually I should ask can 4 times something ever result in 12? And it can. You know what I’m saying?

R: Mm, hmm.

Beth: I guess 4 times something cannot, the last things there would be 22. How would I check that? [pause (10 sec)] Sure it can, it can be [pause] could have a 3 here and [writes _ _ 3*4= _ _ 2, researcher sneezes] Bless you.

R: Excuse me, thank you.

Beth: [pause (5 sec)] 4 times what plus 1 ends in a 2? [writes _ _ 3 *4 = _ _ 22]. That’s what I’m thinking right now.

R: Okay.

Beth: [pause] Cause, if there’s nothing, 4 times something, if 22’s not the last two digits of anything times 4 then I can cross off this, the whole um, 220 to 229.

R: Okay.

Beth: 4 times – let me look back to my little chart [looks back to part a work]. And I don’t think you can cause if you add, cause I have a remainder –

R: Mm, hmm.

Beth: I just picked random um, times by 3 so we have a 12. 4 times what plus 1 will end as a 2 and I don’t think anything will.

R: Okay. So is there anything else you can multiply by 4 and get the 2?

Beth: Um, yes [pause] 4 times 8, hold on [erases 3 writes 8 in _ _ 8*4=_22], what’d I do? 3, 4, 5, 6, 7, 8. So yes, 8’s the next one.
R: Okay.

Beth: Next and only, and then you’d have a 3. Okay, and I think we’re gonna have the same thing. 4 times something plus 3 ends in a 2, it’s gonna be all odd numbers. Like 20 plus 3 is 23. [pause (10 sec)]

R: Okay.

Beth: So, and then 9, yeah, so since we put, since I tried all the numbers 1 through 9 here [circles last digit] and found out 4 times that will never yield a 22 here, I can add 220 to 229 to my group [writes 220-229 to list of not 4-flips on the right].

R: Okay.

Beth: Since no matter what number is here, if I were to write them out, I can never find a number, x, multiply by 4 and get a 22 right here.

R: Okay.

Beth: So, I guess I’ll do it for 32s. 4 times what equals 32 [writes _ _ _ *4=32]. And so, again, since 3 and 8 were the only numbers that yielded a 2 right here, those are the ones we’re gonna check. Okay, I need more room. [rewrites _ _ _ *4=32 to the left, pause, writes in 3 in the one’s place, carries a 1, (5 sec)]. Okay, so um, gonna be 12, so 4 times something plus 1 equals 3. [pause (5 sec)] Okay, hold on, I’m confusing myself. [pause (5 sec)] Okay, yeah, you can have anything. [pause (10 sec)] Or could you? [pause] 1, 2 – Yeah, I mean I could put a 3 here and that’d make 12, 13. So yeah, it can end in a 32. Yeah, so this method does not say anything about 230 to 239.

R: Okay.

Beth: And 42 [writes _ _ _ *4=42], I’m just trying to see if I can think about a number for sure. That won’t work. [pause] So, I guess right now I’m checking divisibility by 4. That’s more or less what I’m doing. So, we want to see if ends in 42, like I said the only two numbers are gonna be 3 and 8 again so those are the only two I have to check.

R: Okay.

Beth: 3 carry the 1. [writes 3 in one’s place, carries the 1] 4 times something plus 1, give me a 4. Alright and that’s nothing cause I would need 4 times something to end in a 3 to add a 1, we have nothing ending in a 3.

R: Okay.

Beth: So the 3 doesn’t work [erases 3, writes 8 in one’s place]. 8, it’s gonna be 32
[writes 3 as carry over] so 4 times something plus 3 ends in 4, that means we would have
to have 4 times something ending in a 1 and just for this whole reason, why I did this
[points to her “4-table” from part a], nothing ends in a 1.

R: Mm, hmm.

Beth: So that when I add 3, I get 4.

R: Okay.

Beth: So, this is, now I can add this group, 240 to 249 [writes 240-249 in list of not 4-
flips]. So the groups I’m looking at are [pause, puts stars next to _ 12 and _32]

R: There’s more paper, too, if you want.

Beth: [switches to new paper] So I’m looking at, I guess that’s a weird interval but 208
through 219 and then 230 to 239 [writes 208-219 and 230-239 on second page]. So, that
pretty much is [pause (5 sec)] 208 through 219, okay that one was – [pause] Okay, so
going on my theory I want to show that there’s nothing. Cause I, um, I think I’m gonna
disprove it.

R: Okay.

Beth: [pause] Which makes me wonder if there is such a thing as a 4-flip.

R: [laughter] There is, yeah, there is.

Beth: But, yeah, I wonder how you’d find it? I mean, I’m not gonna find a formula right
now on how you do find it but I bet you there is a formula and I bet you it discludes
three-digit numbers. So, how can we look at – okay, I’m gonna deal with this group right
here, [draws arrow to 230-239] just cause, it’s only 9 numbers and it’s a nice interval.

R: Okay.

Beth: So, I’m just gonna write them out so I know what I’m dealing with. Is this equal
to 032? [writes 4*230 = 032 with question mark] And that is 920 I think. [pause] 920
does not equal 032 [writes 920 ≠ 032, writes 4*231 = 132] Not equal to 132. [writes 924
≠ 132, 4*232 = 232] That does not equal 232. [writes 928 ≠ 232] Um. [pause (15 sec)]
Okay, so, sort of like my weird theory for 200s to 207s, but since the lowest number is
um 4 times 230 equals 920, and they’re 900s from here on out [circles 920, 924, etc.] and
I know the biggest number over here’s gonna be 239 backwards, which is 932. [writes
4*239, 932] So, oh look, I think this next one is it – 233 um. [writes 4*233 = 332] No,
don’t go off from what I just said.

R: [laughter] So, you’re just noticing that the 932 is a multiple of 4 –
Beth: Right.

R: – but not the right one for the job, right?

Beth: Right, because this is 932 but it’s not equal to 332 [writes 932 ≠ 332]. Okay, I thought we were gonna have one, which would be kind of random if we only had one. You know. Okay, so what was I thinking? [pause (5 sec)] Alright, since this is going to go out since 932’s the last number here, and that we start off on this left hand side, starting at 900, and we’ll go to whatever, we don’t care, but as long as I know that this [points to 032, 132, 232, etc.] won’t catch up to 900 until the end, where this will be, well let me just show this to be certain. 239. 4 times 239 [writes 4*239] Yeah, 956 does not equal 9 [writes 956 ≠ 932], oh okay, well – okay, so there could be something in between here [points to 956 and 932], so I have to be careful. [pause (10 sec)] Oh, but this is going to be at 832 [writes 832]. Yeah and 732. [writes 732] Okay, yeah, so because um we don’t get to the 900s til the very end, the last one we have to check and that’s, they’re not equal.

R: Okay.

Beth: Then I don’t need to worry about any of these cause I know anything from 32 to 832, since it doesn’t have a 9 in front, it’s not 900 something, which you need to have like a 9 to start with over here, in order to try to even be equal. I can be sure this group doesn’t have a 4-flip.

R: Okay.

Beth: So, let me just add this 230 to 239 [writes 230-239 to her not a 4-flip list on other page]. So, now we’re just left with 208 to 219, which [pause] I will write out, just so that I understand what’s going on. Sometimes I feel silly, like if other people don’t need to write it out to like see something, like right now, I’m thinking oh my gosh, like what if it’s so obvious what the next thing is and I have to write it out. So sometimes, like I feel silly.

R: None of these problems should be obvious, so you shouldn’t feel silly about that.

Beth: Yeah.

R: But, I’m trying to discover the technique and one of your techniques is to write things out and look at examples and look at lots of examples –

Beth: Mm, hmm.

R: – and that’s just your technique, that’s your style.

Beth: Mm, hmm.
R: It’s totally okay. It doesn’t make you any less smart than anybody else.

Beth: Mm, hmm.

R: It’s just your technique.

Beth: That’s true. I have to not care what others think so [laughter] cause who cares? So, 208 is that equal to 802? We’re going to start there. [writes 4*208 = 802] 832. [pause] Oh, let’s just see if I got that right. [checks calculation with calculator] Yeah, I did. So, that’s not equal to 802. [writes 832 ≠ 802] Okay, so I’m going from 208 to 219 so that means somewhere all the way down here we’re going to have 912 [leaves space after 832, then writes 912 below]. And how many numbers is in here? 11? I want to say – So, that means that, at most this is gonna be 11 times 4, cause we’re multiplying by 4. Um, 44 plus 32 is 876. [writes 876] Let’s just see if I did that right. 219 [writes 4*219 above 876, uses calculator, (5 sec)]. 876, okay, yeah, so there could be the chance that they meet up because this is going from 208, 209, 210, [writes 902, 012, 112, 212, 312, 412, 512, 612, 712, 812 in list, audiotape side one ends, researcher flips tape, (10 sec)]

R: Come on. [referring to tape player, pause (5 sec)] I think you lost your way somewhere there. Oh, nope, you’re okay, 211, Okay.

Beth: 210, 211, 212, 213, 214, 215, oops, 218 [finishes listing numbers] Hmm, we have too many lines. So, since I do have – I, I noticed that this is gonna to be interval just like this was an interval of 900s.

R: Okay.

Beth: I’m going to only have numbers between 832 and 876 and since I checked this one, these two [points to 208], just because I was trying to find a pattern, I’m gonna check this one which is 4 times 218 [writes 4*218] to see, cause this is the only one that has a possibility of matching up.

R: Okay.

Beth: And um, that would just be 872 [writes 872 ≠ 812]. So, since these don’t, I can say then, therefore, none are gonna match up in this interval. So, we checked –

R: And that’s everything, right?

Beth: Yep.

R: Okay, so just to recap, you went through and you used some of the strategies you used in the first one, right?

Beth: Mm, hmm.
R: To limit it to 250 through 999 don’t work. The 100s don’t work.

Beth: Mm, hmm.

R: Um, you started thinking a little bit more, which I think was a new strategy to think about um, what numbers they could equal. And so you were thinking about the 800 and the 900 range –

Beth: Mm, hmm.

R: – and, and that sort of thing.

Beth: Right, because I had more to deal with.

R: Mm, hmm.

Beth: I mean, in between the 200 to 250, 249, which would be equivalent to like the 20 through the 24 here. [points to part a]

R: Yeah.

Beth: So I could just check those five, but I still didn’t want to have to go through all, which I didn’t, I didn’t write out a lot of ’em, so that’s good. But I just had to break it up into smaller intervals. And yeah, instead of checking the divisibility, because I did that to eliminate, ah, a few, at least two intervals, the 12 or the 120s, 210s and 230s. No, the 220s and the 240s, just like the last problem but I had to come up with a new way because there were some that could end in a 32. There were some that could end in a 12.

R: Mm, hmm.

Beth: So –

R: Okay, so you used a lot of the same strategies but some new ones and some good techniques happening, things like that. Right? Have you ever seen a problem like this before?

Beth: {No}, never.

R: Okay. Anything that you’ve done, um, in your math career that helped you with what you did here or is this all new stuff?

Beth: Probably one of the key things is in my abstract math class, we talk about divisibility by 3 –

R: Okay.
Beth: – divisibility by 2, or any number because we have certain proofs, I forget which ones but, prove that 2n is always divisible by 4 times something. Something like that so, that got me thinking about the last digit cause I noticed these are all 21, 31, 41, so then I, well, can 4 times anything ever equal 1?

R: Okay.

Beth: Like that’s really the only thing I can think of but from my math experience, I mean obviously you know.

R: Knowing how to multiply and stuff.

Beth: Yeah [laughter] *End of interview
[student is given part a of Question 2 and begins reading silently]

Jill: Oh, do I go?

R: Yeah. So, we call a positive integer, $N$, a 4-flip is 4 times $N$ has the same digits as $N$ but in reverse order. Prove there are no two-digit 4-flips.

Jill: Okay. Well, can I?

R: Go ahead.

Jill: The first thing that I did was that I reread the, um, the sentence slowly.

R: Okay.

Jill: To make sure that I understand every, um, aspect of what they’re saying. So, that’s what I’m going to do.

R: Okay.

Jill: [reads question] So, I know what a positive integer $N$ is. [laughter] A 4-flip – okay, if 4 times the positive integer has the same digits as the integer but in reverse order. Okay, so now I’m going to try and do an example.

R: Okay.

Jill: Um, to see if I really know what it is. So, I’m going to pick, um, a random positive integer. I’ll start with three-digits, just because, I don’t know.

R: Okay.

Jill: Um, so 312 [writes 312]. Okay, positive integer – A 4-flip, okay – so 4 times $N$, so I’ll see what 4 times $N$ is.

R: Okay. [gets calculator out for student]

Jill: Do I get to use a calculator?

R: Yes.

Jill: Ahh, yeah. [laughter] Okay. [pause, computes 4*312] So, 4 times that equals 1248. [writes 4(312)=1248]. Okay. So, has the same digits as $N$, and this is my $N$, but in reverse order. So, if it was – it would be 213 [writes 213].
R: Correct.

Jill: Okay. [pause] Prove that there are no two-digit 4-flips. Okay, first of all, I want to find an actual 4-flip in three-digits. I’m going to try. Okay, so, if I multiply 300 times 4, I’m getting something that’s above a three-digit and I need a three-digit, if I started with three-digits.

R: Okay.

Jill: I’m going to go down and try something in the lower range and see like what that does. [pause, using calculator, writes 4(105)=420] Okay, so that’s 420, so that doesn’t work, but I’ve got my three-digits. So, let’s see, hopefully I can find one. Um. [pause] So, um. [pause, writes 4(_ _ _)= _ _ _] Okay, so we need to have that last digit here be the first digit here. [underlines last digit of 4N and first digit of N, pause] Right?

R: Mm, hmm.

Jill: [laughter] Okay. [pause] Um, okay, so – [pause, researcher checks audiotape and noticed it was not working, meanwhile, student looks over work, (15 sec)] This is bad. [noting her work] Do you want me to stop for a second?

R: Yeah, unfortunately.

Jill: Was it not recording?

R: I’m picking up – But, I will pick up most of it on the videotape. So, that’s a backup. But, I’m wondering if the batteries are dead. I’m thinking they might be. Hmm. [plays with tape player more] There we go. We played with it enough, we got it. [audio recording begins here] Okay, go ahead

Jill: Okay, so um, I still really don’t know how to pick the numbers. Um. [pause (5 sec)] Okay, so I’m gonna see what doing – um – No, I don’t want 3. So, if I pick a one here [writes 1 in first digit of N] then that’s gonna be a 4 [writes 4 in first digit of 4N]. So, we wanna pick – [pause (15 sec)] Okay, so I’m gonna try that [writes 104 for N] and see what happens. It’s 416 [writes 416] so that definitely doesn’t work, cause I need it to be 401. [writes 401 below 416, pause, (5 sec)] Okay, so maybe I should go back. Now I’m starting to get, wonder if there’s even a three digit or if I’m going about it the right away. And I know that I’m proving that there are no two-digits, so I’m assuming that there’s no two-digits, um, so that’s why I’m choosing three-digits and sticking with the three digits because I don’t want to go up to anything higher.

R: Okay.

Jill: Um, at least at the moment. I’ll try a few more things before I give up and try something else. Um. [pause] Okay, so we’re gonna do –? I chose a 1 here. [pause]
That’s 4. 4 here and I need to change something here [points to last digit of 4N, pause] to give me a 1. And it’s not possible. Right? [pause (5 sec)]

R: So, can you explain what you’re thinking?

Jill: Okay, well I’m thinking that’s not possible because um, okay, let me – [pause] Okay, so 4 times 1, 4, so I need something in here times 4 that would give me a 1. Well, since we’re working with integers, that’s not possible. Or it could be 11, but that doesn’t work either [writes 11, crosses out].

R: Okay.

Jill: So, obviously my 100s is not gonna work. I really hope a 200 will work, let’s see. So I’m gonna take 4 times any 200 [writes 4(222)], just to see um. Oh wait, we’ll see. [pause, writes =888] Yeah, okay. Okay, so I’m still trying to find digits that are less than –

R: Okay.

Jill: –1000 [pause] but greater than 100, obviously. Okay, so let’s see if anything works. I really hope something works, [laughter, writes 4(____) = ____] because I’m assuming that if I can figure out why it works or what it takes to work in a three-digit, then maybe I can see why it wouldn’t work in a two-digit?

R: Okay.

Jill: I don’t know if that will work. But, I guess I don’t even know what a 4-flip, I guess I need – I figure I need to find what a 4-flip – find a 4-flip number.

R: Okay.

Jill: I guess, I don’t know. Um, okay. So, let’s see, okay, – so if I pick um – Okay, so let’s say I pick a 2 here [writes 2 in as first digit of N]. So, that’s gonna maybe give me an 8 or a 9. Oh. [pause (5 sec)] But that means I want an 8 or a 9 here [points to last digit of 4N, pause, (5 sec)]. Okay, let me see. So, I need to pick [pause (10 sec)] Huh. [pause (10 sec)] Oh, wait, duh, never mind I got it wrong. Right? [pause] Okay, so if I pick a 2, that gives me an 8 [writes 8 as first digit of 4N]. So, I want an 8 here. [writes 8 as last digit of N, pause, (5 sec)] It’s hard to write all this down, too, cause a lot of times you can punch stuff really quickly in the calculator but I’m trying to – write.

R: You can punch away, as long as you say it out loud, I’ll get it. [laughter]

Jill: Oh, okay. Okay. Um – [pause (20 sec)] Okay, so now I’m going to try – oh, I gave up on the 3s [crosses out 4(312)=1248, pause, (5 sec)].

R: So, what were you thinking?
Jill: Well, okay, what I was thinking was um, if you find um – what was I thinking? [pause] I wanna find um, two numbers here that are closer together [circles the 2 and 8 she wrote down for N]. Because if one of em’s larger, much larger than the other, then you’re gonna have a um – it carries over.

R: Okay.

Jill: So, that’s kind of what I was thinking. So, I think I won’t do that [crosses off work on three-digits].

R: Okay. [laughter]

Jill: Ummm, okay. So, that’s not getting me anywhere so I wanna try doing this. Prove that there are no two-digit, digit, flips. So, um, a two-digit flip, here’s my two-digit [writes _ _]. Um, so I have this number and I multiply it by 4 [writes 4(_ _)]. Let’s call this x₁ and x₂, [writes in x₁ and x₂] meaning that I’d have to get x₂ there, um, and x₁ there [writes = x₂ x₁]. Um – [pause (25 sec)] Okay, so I’m thinking it’s gotta – it has something to do with multiplying this by 4. [pause] And then how can I, um, use that for my proof? [pause (10 sec)] Okay, so I still don’t really have any direction so what I would do is um – I would – I’m guessing I would just try um, a proof by contradiction would probably be my first approach.

R: Okay.

Jill: Am I supposed to eventually be able to get this?

R: If you do, great and if you don’t, that’s fine, too. The part that I care about is your process and working through it. So, some people have, some people haven’t, so.

Jill: [laughter] Okay, okay, so um, assuming to the contrary but I’m not gonna write that down because –

R: Okay.

Jill: – um, not until I actually get the proof. Um, that um, I can take 4 times um – [pause] I’m just say a two-digit number – so I’ll say xy equals yx [writes 4(xy) = yx]. And then, now at this point, I’m gonna make sure that I’m still using the definition right.

R: Okay.

Jill: Um, [reads question] positive integer N, 4-flip, if 4 times N has the same digits as N but in reverse order. Okay. Okay. Sooo, um – [pause (5 sec)] Okay, so – [pause (15 sec)] Okay, so I could maybe think about going – okay, so this is – I’m assuming – [writes assume before 4(xy) = yx]. So, from here I could divide by 4 and see if that gets me anywhere [writes /4 under both sides of equation lightly, pause]. Um, and this is –
we’ve gotta make sure that that has to be an integer, I think. Wait a minute. [pause]
Yes, that’d have to be an integer, cause this is an integer. Um, but I don’t know if that’s
gonna get me anywhere, so I might come back to that later. [pause] What I’m gonna try
and do is um – so the number, this number [circles xy] can’t be bigger than – [pause (5
sec)] let’s see. [pause (5 sec)] Okay. So I’m just trying to find numbers cause it’s gotta
stay in the two-digit, right?

R: Okay.

Jill: So, I’m finding numbers that um when you multiply it by 4, is not greater than a
two-digit.

R: Okay.

Jill: Just to see where that falls. So that’s what I’m doing. Um. [pause] Oh, duh, okay.
So, um if you multiply 4 by anything great [writes 4(2, crosses out], greater than – So,
okay, so I’ve gotta have um – xy’s gotta be less than, strictly less than 25 [writes xy < 25].
Okay. [pause] Okay. So, then if I really got stuck, I would – I could always do it by
brute force.

R: Okay.

Jill: That seems to work sometimes with number theory when it’s small enough so that,
obviously that’s not something I wanna do. So I um, now I’m gonna try and find cases.

R: Okay.

Jill: To where I can – oh wait, assume that this – Okay, yeah. Okay, so now I’m gonna
try and find cases. Um. [pause] It has to be less than 25. [pause] Ummm, so what kind
of cases would I wanna use? [pause (10 sec)] Well –two-digit, I’m assuming you can’t
use 0 as a placeholder.

R: Right.

Jill: Okay. Um, so, now we’ve got um, [pause, writes 10 ≤ in front of xy < 25, (5 sec)]
the parameters set at that.

R: Okay.

Jill: So, then you could really use brute force, which wouldn’t be bad. [laughter]

R: If that’s what you’d like to do, go ahead. Anything you wanna do is fine.

Jill: Well, ahhh, I’d rather, um, try and get a little bit further.

R: Okay.
Jill: But then, eventually, I’ll give up after a certain amount of time and just go by brute force. And, this also depends on how much time I have to spend on things –

R: Okay.

Jill: – too. Um, okay, so, obviously 10 doesn’t work. I mean, that’s obvious cause 4 times 10 is 40 [writes 4(10)=40], which does not work, so, go like that [crosses out 4(10)]. Um – so, I guess you could always – Okay, so you have 4 times 20 doesn’t work [writes 4(20)=x] I’ll just go like that to represent that doesn’t work. [crosses out 4(20)]

[writes _ _ _ _ _ _ _ _ _ _ _]. Okay, so – if – okay, so, I’m gonna look at y first [writes y in as second digit of N]. Cause multiplying 4 times this comes first. Um – and, okay. So, um, if you take – [pause (5 sec)] Okay, so if you multiply, well I kind of am doing it by brute force. So if you multiply um – you have y, is that y? [writes y = 1] y equals 1, you’re gonna get 4. [writes 4 next to this] And if y equals 2, you’re gonna get 8 [writes y = 2, 8], y equals 3, you’re gonna get 2 [writes y = 3, 2]. If y equals 4, you’re gonna get a 6 [writes y = 4, 6].

And obviously we know that they’re all gonna be even [writes even to the side]. Because 4 is even. And so you’re multiplying everything by 4, it’s gonna be even.

R: Okay.

Jill: Right?

R: Mm, hmm.

Jill: Okay. So, that, well that makes that even easier. [pause] So, um – my xy is less than or equal to 24 [writes 10 ≤ xy ≤ 24]. Um – where xy’s gotta be an element of [writes xy element of Z’] well, Z plus, but even I guess [writes even below xy], is what I’m trying to say. So, um – [pause (10 sec)] well, really you just look at multiples of 4. [pause] Okay, so um, what do we have? So really we’re looking at 12, 16, 18, oh 20, 24 [writes 12, 16,20, 24].

R: Okay.

Jill: Right? [laughter] Oh, no, you can’t confirm, okay.

R: I really can’t.’

Jill: Yeah, so I’ll just – let me go. Um – [pause] oh, okay. So, the number’s gotta be between here [circles 10 ≤ xy ≤ 24]. Doesn’t mean it has to be even, though. Cause you could take 4 times and odd number duh – um, okay so – [pause] 4 times 11 is 44 [writes 11 = 44] that obviously doesn’t work. What’s 4 times 12 – 36? Let’s see what it is. [pause, writes 12 = 48] 48, so I’m starting with 44, 48. Um – [pause, writes 44, 48, uses calculator, (5 sec)] And this is 11, 12. [writes 11, 12 above] 13. [pause] So, I gotta get a number like [pause (5 sec)]. Okay. So, 48, 52, 56 – [pause, uses calculator, (5 sec)] 60,
Okay. So, this is actually one that is the closest I found. [circles 16 and 64]

R: Okay.

Jill: [laughter] Um – [pause (30 sec)] 1, 2, 3, 4, 5, 6 – [counting numbers in list, writes n = 6 below 44 through 64, pause, (15 sec)] I’m just writing down a few more. 17, 18, 19, 20, 21, 22 – [writes 17, 18, etc. above] so [pause] 23, 24. [writes 23, 24 with 92, 96] Okay, so here’s all the numbers. I did it by brute force.

R: [laughter] And they don’t work.

Jill: But, I would still, I’m still gonna see if I can figure out something else. [pause] So, [pause (10 sec)] here’s one that’s partially right [circles 23 with 92]. And it looks like those are the only two. [sigh, pause, (5 sec)] Yeah, those are the only two, so – if I take 4 times 23, I get 92. [writes 4(23) = 92, pause, (5 sec)]. That’s 12. [pause (5 sec)] Okay, so – now, I’m starting to think. Well, at least by looking at this, um, you can’t have a – oh wait. What did I just do? Oh, 4 times 3 is 12. 4 times 4, 16. [pause (10 sec)] Okay, so now I’m taking 4 times 16 equals 64 [writes 4(16) = 64]. Okay, so 4 times 3 is 12. [pause (20 sec)] Um – [pause (5 sec)] so, [pause] Okay, I think I’m tired of looking at it.

R: Okay. So we got it brute force, though.

Jill: But this is, this is – yeah. [puts work to the side]

R: Okay, well the good new for you –

Jill: And this is like my homework –

R: – is that there is one more part, there is one more part. Well, actually two more parts, but so the next part –

Jill: Oh – no!

R: – is thinking about three-digits. So, prove or disprove that there are no three-digit 4-flips.

Jill: – okay, so then I’m gonna take this back over here [pulls her work from part a back].

R: Okay.

Jill: And at least look at what I was doing with three-digits.

R: Okay.
Jill: Now um, okay, so this one is prove that there are no two-digits and this one says prove or disprove. So, my thinking would be that just from part a to part b, now they’re saying prove or disprove it, I would conjecture that um, [pause] there is, oh wait, there are no 3 – I’m thinking there is or otherwise, I would assume that would follow more of the format of a.

R: Okay.

Jill: So, that’s why I would think that. So, um, so by – to disprove this statement you just gotta show one counterexample. [pause] So – if I had a better reasoning of why they didn’t work for the two-digit, it might be easier to come up with a three-digit. But, I’m moving past that still.

R: [laughter] And anytime you wanna give up this problem altogether, just say so, and we’ll go on to something else, okay?

Jill: Something completely different?

R: Yeah.

Jill: Okay.

R: Okay?

Jill: Well, how much time do I have left?

R: Um, half an hour.

Jill: Okay.

R: Why don’t you go back to your reasoning about the three-digits and see if you can remember what you were doing there and get yourself back into it?

Jill: Okay, so with the three-digits, um, 4 times 3 in this place value is 12 [points to 3 in 312 on previous work]. So, I was saying that it definitely would have to be smaller than the 300 [writes < 300]. Cause I still need a three-digit number. So, 4 times 2 is 8. So, um, that gives me – [pause] a digit that’s three-digits. So, now let’s just see what 299 is times 4. [pause] Okay, so that’s greater than a three-digit, so I’m going back and trying to find my parameters.

R: Okay.

Jill: Um – cause I don’t think I’m gonna do this one by brute force.

R: A few more numbers here, right?
Jill: [laughter] Um, okay so, I’m not gonna try and think too hard about this, I’m just gonna plug and chug some numbers in to find something that is – [writes 999] Okay. Um, oh – [pause] Okay, so 299 times 4 is. [pause] Okay, I’m sure I can take some kind of ceiling function. But, whatever, that takes up too much time, so. [laughter, pause, working on calculator, (25 sec)] Duh – I do this stuff all the time.

R: So, you just discovered that the top, the top number is –?

Jill: Um, 249 times – oh – 249 times 4 equals um [writes 4(249) = 996], 996 and um – so I would have taken something like [pause, writes $\left\lfloor \frac{1000}{4} \right\rfloor =, (5$ sec)] Yes. And that is actually much quicker. [pause] Yes.

R: Okay.

Jill: So, that was junk. Okay, so this might actually come in handy further down the road. [pause] Okay, so I have got to pick a number – obviously I know now that doesn’t work. Um, let’s go xyz. Um [writes $100 \leq xyz \leq 249$] is less than 249. Okay – so 4 times $xyz$ [writes $4(\text{xyz}) = \text{zyx}$] equals $\text{zyx}$. Okay. [pause] Soooo – um. [pause (5 sec)] Yeah, three-digit. Okay [pause (5 sec)] so, that [writes $4(\_\_\_) = \_]$ equals [writes \_\_\_]. Okay, so if 4 times something here – I almost wanna go back and look at this [goes to part a work, bottom of page]. So, 4 times something here – 4 times 6 is 24. Okay, let’s see – [writes $4(6) = 24$] and my number’s 16. [pause (15 sec)] Oh wait, 4 times 6 is 24. [pause (5 sec)] Okay, so – [pause (15 sec)] Okay, so now, I’m still not coming up with this. Something’s still not clicking like I feel like it should be. So, just now, um I was just thinking – well, I was kind of looking at parities, um, between the odds and the evens to see if I could come up with some type of conjecture and um, I didn’t really come up with anything because – [pause (5 sec)] um, yeah, that wasn’t working. But that’s what I was just thinking.

R: Okay, an odd $N$ versus an even $N$?

Jill: Mm, hmm.

R: Okay.

Jill: Yeah. [pause (5 sec)] Yeah. So now I’m just, I’m still just trying to figure out patterns, I guess.

R: Okay.

Jill: [sigh, pause (30 sec)] Okay, so I’m just gonna try stuff.

R: Okay.
Jill: Okay, so if I put a 2 here – [writes 2 for last digit of N], or wait, 4 times 3 is 12, 4 –
Okay, so at least what I’m seeing here [points to work on part a], I don’t know if this is
applicable, but I’m gonna try it. Um, okay so I was going to use something small, which
would give me an 8, and you don’t have to carry anything over. But over here, when I’m
taking 4 times 3, is 12, so I’m carrying over. 4 times 6 is 24 and I’m carrying over. I
don’t know if that had, is parallel to anything but um it’s a start.

R: Okay.

Jill: So then, I erased, sorry [erased 2 from N].

R: That’s okay.

Jill: So um, I’m no longer choosing 2.

R: Okay.

Jill: I’m gonna try um – okay well so – 6, 6 kind of seems to work [writes 6 in for last
digit of N]. 6 and the 3, actually kinda seemed to work, so, um – [pause]. So, 4 times 6
is what? 24. So then that’s gonna be 4 [writes 4 for last digit of 4N]. So – I need a 6
here [writes 6 above first digit of 4N], right? And that means I want a 4 here [writes 4
above first digit of N]. Well, not necessarily but um – [pause (5 sec)] Okay, so 6 times 4
is 24. [pause, writes _ _ 6*4= _ _ 4 in column format, (5 sec)] So, it’d be 24, so now I
have a 2 there. [carries over 2, pause, (5 sec)] So, obviously this number’s gonna have to
be the same [pointing to center number of both N and 4N]. Um – so – [sigh, pause, (10
sec)] is that possible? [pause (10 sec)] Okay – ugh. [pause] Sorry, now I’m starting to
go back and not finishing my thoughts. But I’m starting to go back and wonder um –
[pause (5 sec)] thinking still about maybe dividing 4 through and looking at those two
cases [writes /4 under _ _ 4 middle of page]. I don’t know if that will help. [pause]

R: What two cases do you mean?

Jill: [pause] Um – oh, just looking at 4 times this number [points to N] and, and looking
at um, this number divided by 4 [points to 4N], I guess that’s what I’m thinking.

R: Okay.

Jill: Um – so, [pause (10 sec)] So, I pick a number here. You’re times-ing that by 4. So
then this number right here has got to be a multiple [pause] obviously of 4, but it’s got to
be a multiple of [pause (5 sec)] this number, really. Because, so, even if I start going
back to really simple algebra, we can take 4 times x equals, okay [writes 4(x) = 4x, pause,
(5 sec)] so [pause (10 sec)] So, um [writes 4(1)=?, pause, (5 sec)] Obviously, 4 times 1
equals 1 or 4 but if I would go – so equals 4, so if I have this number [writes 4x/4=4/4]
and I don’t know what – [pause (10 sec)] Alright, okay. [pause (25 sec)] So, I guess I’m
just wondering if it’s gonna be easier to find the number on this side [points to 4N].
R: Okay.

Jill: And then see if you can go back this way.

R: Okay.

Jill: But, I’m just not sure if that’s fruitful. [pause (5 sec)] And then, obviously, this number’s gotta be divisible by 4. [pause (20 sec)]

R: What else are you thinking?

Jill: Um – I’m trying to think if there’s a, a nice abstract way to um, show the numbers here that are divisible by 4.

R: Okay.

Jill: Obviously without, I mean you could go, um, 996 [writes 996, 992, next to spaces for 4N]. You can go 992, but I don’t wanna do that. I just want to see if there’s – [pause (15 sec)] Hmm. [pause (10 sec)]

R: Any other thoughts?

Jill: No, not really. [pause (5 sec)]

R: Do you want to leave this one?

Jill: Now, I’m just trying to decide if I should just start kinda plugging away with the numbers and seeing what happens to ‘em.

R: Okay.

Jill: But, I don’t know if I wanna do that. So that’s kinda what I’m thinking about right now. I’m not really thinking about the problem as much. [laughter] Um – Okay, so well let’s go back to here. Okay, so I am taking 6 times 4, um – oh so I started thinking, before I strayed, um – what number [underlines the middle number of N, pause] here – Okay. So [pause (10 sec)]. It’s gotta be the same number, so – [pause (35 sec)]

R: What else are you thinking? I see you’re looking back at some of the work?

Jill: Yeah, I know, I know. [laughter]

R: Trying to make some connections there?

Jill: Yeah, um, so [pause] if I have a number here, let’s say 2, 3 and then I want this to be 3, 2. [writes 23 = 32 on part a top of page] Um. [pause (20 sec)] So –64, so I would need, um, 46 and 46 is not a multiple of 4 [looking at bottom of part a page, writes 46 =
64]. Um, in this case I would need a 29 [writes 29 above 23] and 29’s not a multiple of 4. [sigh, pause (10 sec)]

R: So, you’re thinking of going backwards?

Jill: Yeah, I don’t know why I keep doing that. I feel like it. [pause (5 sec)] Hmm – [pause (5 sec)] Okay, so, I guess what I really need to do is figure out if um, I can have the same number in both these slots [circling the middle numbers of \(N\) and \(4N\)] and if so, how can I do that? Or if not, then that could be prove – Okay, um [pause (10 sec)].

R: So, do you want me to keep going on this?

Jill: That’s totally up to you.

R: Oh, okay.

Jill: But if you want me to, but if that’s not like the purpose, obviously then um I can keep going.

R: Well, it’s up to you. I think we probably don’t have time to look at another problem.

Jill: Okay.

R: So, you can take maybe 5 more minutes to look at it and then we’ll stop.

Jill: Okay, that sounds good.

R: And get back to your paper. [laughter]

Jill: Yeah. Okay, so –

R: Any other thoughts? So, let’s look back at what you’re doing and see if we can redirect you or something, right?

Jill: [laughter] Are you starting to feel bad for me?

R: No, I just – I, I, I think you’ve got lots of good ideas down on this paper and somehow they’re not connecting for you, right? [student laughing] So, go back again and look at what you started doing with three digits.

Jill: Over here? [points to work on part a with three-digits]

R: Yeah.
Jill: Okay.

R: Cause you eliminated some stuff that you haven’t eliminated this time around.

Jill: Okay. Well, okay – I guess if I wanna break it up into cases, I can, er um, obviously I can eliminate – where’s – okay, so I can eliminate 111, cause that does not equal 444 [writes 111 ≠ 444], 222 does not equal [writes 222 ≠ 888] or not equal is not the correct sign, but –

R: Yeah, I understand. Well, that’s also true, they’re not equal. [laughter] But, I get what you mean.

Jill: [laughter] Well, yeah, but that’s not the whole point of this. Okay, so you get rid of these cases [circles 111 and 222]. Um, you can also get rid of the cases where [pause] um [pause] Okay, so if I get rid of the cases where it ends in a 5, right? That means I’d have to have a 0. Okay, so if I get rid of – okay, the cases where 4 times 105 is 420. [writes 4(105) = 420] That doesn’t work because 4 times 5 is always gonna give me a 0 here [underlines 0 in 420], that means I’d have to have a 0 there [underlines 1 of 105] and then that doesn’t work because then it’s no longer considered a three, three-digit –

R: Okay.

Jill: – number. So I can get rid of um, all of the numbers that, er, multiples of 5.

R: Okay.

Jill: Um, yeah cause the ones that end in 0 would work, too, right? Because 4 times 0 is 0 and that means I’d have to have a 0. Okay, so I can get rid of everything that’s a multiple of 5 [writes multiple 5x]. So, everything that’s a multiple of 5. Um [pause] um – let’s see. So, even if I did something like over here I did 4 times 202 equals 808 [writes 4(202) = 808]. Um [pause] so and because my parameter’s this [points to 100 to 249 equation, pause]. Okay, so um, I’m also gonna look at – okay. Ummm. [pause (5 sec)] Okay, so – and I’m, I’m just gonna try and focus on something smaller. Okay, so if I, um, 4 times 2 is 8. So then we’re going to need um an 8 here. And that’s not possible?

R: Okay.

Jill: So um, I can eliminate [pause (5 sec)] um, everything where 2’s in the first digit placeholder.

R: Okay.

Jill: Um –

R: So you’ve eliminated 0, 5 and 2.
Jill: Mm, hmm.
R: Okay.
Jill: 0, 2, 5. [writes 0, 2, 5 to the side] Um, okay. Um – [pause (5 sec)] Okay, that would
be no – okay so um, [pause] 4, okay, um – so if I put a [pause, writes 4(____)=____, (10
sec)] Okay, if I put a 4 in here [writes 4 in last place of N], I’m going to get a 16. Right
and that means I’d have to have a 6 [writes 6 in last digit of 4N]. And I can’t do – oh,
yep I can’t do that [points to 100 to 249 restriction].
R: Okay.
Jill: So, – okay if I put a 7 in here [crosses out 4, puts in 7], what happens is 4 times 7 is
28. [pause (5 sec)] Oh, wait, now I lost my thinking. What have I been doing? Oh, so if
I have a 7 here that means I’ve gotta have a 7 here [writes 7 in first digit of N and last
digit of 4N]. Okay. Well that pretty much takes care of –
R: Well, look, look back at what you just did though. Compare your two case with what
you just did with 7.
Jill: Oh, okay, so um, oh – you mean with my 4 or with the 7?
R: With both.
Jill: Oh okay, so I was plugging in the 7 here, is what I was doing, right?
R: Yeah.
Jill: So 4 times 7. Okay, so that doesn’t really matter because if I have the 7 here, that
means I’d have to have a 7 here [pause] and I can’t do that.
R: Okay.
Jill: So, if I had a 2 here [writes new line ___ = __ 2], um that means I’d have to have a
2 there [puts 2 in first digit of N].
R: Okay.
Jill: Which I can’t. [pause (10 sec)] Right? Yes. And it’s, okay [laughter].
R: But I think this is the backwards thought of what you were doing above, when you
said blank blank 2 to start with.
Jill: Ohhh –.
R: That was different than what you’re doing now. Okay?
Jill: Yes, yes. Okay, then in the same sense, if I had a 1 [writes _ _ _ = _ _ 1], I’d definitely have a 1 there [puts 1 in first digit of N, pause]. Um, so I can’t have a 3, so that leaves me with, what have I ruled out, I already ruled out a 0. So that leaves me with the cases to where I can have a 1 or a 2 there. Right. [writes _ ___ with “1 or 2” above first digit]

R: And you’re thinking about after you multiply by 4? Right, is that what you’re thinking about?

Jill: I don’t know. No, let me go back. [pause] Okay, um – right okay, so, yeah. Okay, so now – um, so 4 times this can be 1 or 2, right? And – [writes _ ___ with 1 or 2 in last blank, audio tape side one ends, researcher flips over tape] I wonder what my range is here? Are we done?

R: Nope, we’ve got just a couple more minutes.

Jill: Okay, so now I’ve got to figure out why I have these parameters, so why do I have these parameters? [circles the 100 to 249 restriction, pause] Okay, cause I multiplied 4 times xyz. And it’s gotta – okay so xyz’s gotta be, can’t be any bigger than that. So that’s why I only have a 1 or a 2 – no that’s not why I only have a 1 or a 2 here. Um [pause]

because multiplying 4 with anything bigger than this gives me something bigger than this. So um –okay – 1, it’s gotta be bigger than 100. [pause (15 sec)] So, if I take – [pause (5 sec)] so, okay, if I can only have a 1 or a 2 here [points to last digit of 4N] that means um, I can’t have a 1 there. Because 4 times [pause] any digit over here’s not gonna give me a 1. Okay, so I think that this has to be a 2 [erases 1 or from _ ___ at bottom of page, making it say ___ 2], if anything. Um – Okay, so what here – [points to last digit of N, pause] Okay, so that’s gotta be a 2. So what here, um – produces that. So it’s gotta be a 3. It’s the only way I can get a 2, multiplying it by 4.

R: There’s one other one.

Jill: Oh, is there?

R: Mm, hmm.

Jill: Umm, oh 32? Right?

R: Yeah, so 8 –

Jill: Yeah, thanks. Okay, so, I’m gonna come over here [switches to another sheet of paper].

R: I’ll let you finish up kind of this thought that you’re going on, cause I think you’re, you’re just about there.
Jill: Okay. So, I’m saying that this has to be a 2 and this is a 8 or a 3 um, okay [writes \(4(x_1 + x_2) = x_3\), writes 8 or 3 above last digit of \(N\)]. And then I’m saying this has to be a 1 or a 2 [writes 1 or 2 above first digit of \(N\)]. Okay so, if I take 4 times 3, I’m just gonna see, it gives me 12, right? [writes \(3 	imes 4 = 2\), carries 1] Okay. Um –[pause] what’d I say, 4 times 3 gives me 12. Okay. [pause (5 sec)] Um, I could spend a little bit of time with this stuff here and then –

R: Whatever you want to do because I know you’re –

Jill: Yeah.

R: – you’ve gotta get back to stuff.

Jill: Yeah.

R: But I’ll stop you here –

Jill: Okay.

R: – and we’ll maybe talk about it real quick together when we’re done but let me just um, recap what you did here. So first, you reread the question, right?

Jill: Mm, hmm.

R: To make sure you understood it?

Jill: Mm, hmm.

R: Um, you considered – okay, if I want to prove there’s no two-digit, maybe I’ll search for 1, a three-digit, right?

Jill: Mm, hmm.

R: And so you did some eliminating, some trying to find that?

Jill: Just to see what the properties would be.

R: Okay, and then when that didn’t go [student laughter] quickly, really, you went back to looking at two-digits, you considered capping it, um – considered some cases, looked at some examples. Um, eventually, you ended up doing the two-digit in brute force because you only had 15 to do.

Jill: Mm, hmm.
R: So you did that. You moved over to the three-digit, you kind of recapped it. Um, and it looks, basically, like what’s going on here is you’re looking for something maybe a little bit more—I don’t know elegant, than just brute force.

Jill: Yes.

R: Right?

Jill: You know there’s always little tricks.

R: That’s right. So you’re looking for the trick or maybe the –

Jill: And I know there has to be one.

R: And the thing that goes there, right? Okay, so have you ever seen anything like this before?

Jill: No.

R: Okay. Um, you mentioned that maybe number theory was leading you in some directions here, right?

Jill: Mm, hmm, yeah.

R: Any other class that you’ve taken that was kind of guiding what you were doing?

Jill: Um, I would say [pause (5 sec)] probably number theory.

R: Okay.

Jill: And, you know I have –

R: Just thinking about playing with the digits and that kind of thing?

Jill: Yeah, um I mean my class still hasn’t done that. But yeah, number theory. I mean, obviously a little bit of 305 goes into this cause you kinda learn a little bit more about proofs, but um, yeah, basically it’s number theory.

R: Okay. Alright. *End of Interview
Interview #18  (Total time = 44:35)

R: Okay. A traditional chessboard consists of 64 squares, 8-by-8. Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square without overlapping any of the dominoes.

Andy:  [pause] Okay.

R: Consider a generic chessboard. Prove that the generic chessboard of size m-by-n has a perfect cover, if and only if, at least one of m or n is even. Okay, I’ll give you a second to think about that.

Andy:  [pause (35 sec)] Okay.  [pause (15 sec)] Well – [pause (10 sec)] I’m thinking about all the different proof types that we’ve gone over in 305.

R: Okay.

Andy: And I’m thinking about cases; doing either you know, a case where one is even or none are even.

R: Okay.

Andy: I’m trying to formulate in my mind how I’m gonna do that. [pause (25 sec)]

R: What are you thinking?

Andy: [pause (5 sec)] I’m trying to put my mind into an abstract mode.

R: [laughter, pause, (10 sec)] Can you tell me what’s running through your head, what you’re thinking about, what you’re considering?

Andy: I’m trying to come up with a generic case. I mean we’ve got, I have this case here, the traditional chessboard, [pause (5 sec)] which is an m-by-n, I guess we could say. [pause (20 sec)] I’m just unsure of how (throat clearing) as how to start I guess, is what I’m trying to figure out. [pause (30 sec)] ‘Cause in my mind, I can see what I’m trying to prove, I know what I want to do.

R: Okay. Tell me about what you’re drawing in your mind.

Andy: Well, I’m just picturing the fact that, ah, [pause (5 sec)] I’m picturing the traditional chessboard and how it’s covered with each, uh, two dominoes takes, or a domino takes up two squares.

R: Okay.
Andy: And um, I’m seeing in my mind how it obviously has to be even, because of that fact.

R: Can you draw me what you’re thinking?

Andy: [draws chessboard size 6-by-7, (25 sec)]

R: Okay.

Andy: Okay, so I have a chessboard and one domino covers exactly two adjacent squares so I’m saying this dom, one domino is covering that right there [fills in chessboard].

R: Okay.

Andy: The second domino is gonna cover that way. Then the third.

R: Okay.

Andy: So I know it has to be even, either direction, so I don’t have any overlap here.

R: Okay.

Andy: And if it’s even one way, it doesn’t really matter the other direction ‘cause [pause], it’d be the same like this.

R: Okay. [pause (55 sec)] What other thoughts?

Andy: I’m thinking about the definition of even and odd [pause] integers.

R: Okay. And what are you thinking about them?

Andy: Well, I’m trying to think if there’s, ah, some sort of a formula that I can come up with. Or if I need to come up with a generic matrix. [pause (30 sec)]

R: Can you put some more of your ideas down? Are you still pursuing in your mind the even and odd or have you moved on to something else or what?

Andy: No, I’m still thinking about that.

R: Okay.

Andy: [pause, writes Case 1, \(m = 2k\), for some \(k\) in the integers, \(n = 2l+1\), for some \(l\) in the integers, Case 2, \(m = 2k+1\) for some \(k\) in the integers, (60 sec)] I’m doing cases. I’m gonna go with odd or even, or I mean odd, just the different cases to consider.

R: Okay.
Andy: [pause, writes $n = 2l+1$, for some $l$ in the integers, (15 sec)] So, now we need to look at if they’re both even because of this base case here, the traditional chessboard.

R: Okay.

Andy: [pause] ‘Cause I’m assuming that to be true.

R: Okay. That the traditional chessboard has a perfect cover?

Andy: Mm, hmm.

R: Okay. [pause (5 sec)] Where would you like to go with these cases?

Andy: Um. [pause (50 sec)] Or maybe I wanna do [pause] a contrapositive of this. When you consider the fact that you try looking at the fact that $m$ and $n$ both odd, if $m$ and $n$ are both odd, then it does not have a perfect covering. It’s another idea, I don’t know exactly what to do.

R: Okay, then what would that look like? How would that flesh out for you?

Andy: [pause (10 sec)] I always want to quantify things. That’s where I – that’s why I had troubles in this class in the first place anyways.

R: Okay, so you wanna put like for alls and there exists in there?

Andy: Uh, yeah, I’d like to see equations and always end up doing, wanting to do, ah, chasing arguments –

R: Okay.

Andy: – proving existence.

R: Okay.

Andy: And it’s not necessary all the time. It’s a long, hard way to do it, I think – but, so.

R: So, you’re fighting the urge to want to do that.

Andy: I am, I’m trying to think outside that box.

R: [laughter] Okay.
Andy: But even thinking about that, looking at this [points to cases], I just don’t – [pause (10 sec)] I’m not sure how I would prove the existence of [pause] ah, this is just some small definitions, it doesn’t really give me anything that I want to be able to see.

R: So you’re trying to prove the existence of? [pause] A perfect cover? Is that what you mean?

Andy: Yeah, I want to prove that.

R: And you said, somehow, you want Case 2 to lead to not having a perfect cover?

Andy: Right.

R: That was the idea? Okay. [pause (15 sec)] Can you pursue those ideas further?

Andy: Let’s see – [pause (25 sec)] I’m just trying to think here ‘cause I know it’s [pause, draws new chessboard] it’s simple to see it like this.

R: Okay. [pause, student finishes drawing chessboard, (10 sec)] So that’s a 3-by-9?

Andy: It’s a 3-by-9 [writes 3x9].

R: Okay, so it’s an odd-by-odd. Okay.

Andy: [pause, fills in dominoes on chessboard, whispering while placing dominoes, pause, (45 sec)] Hmm. [pause (25 sec)]

R: What are you thinking about?

Andy: Going back to try and think of [pause (5 sec)] my existence that I want to try to prove here. [pause] It isn’t what I want to use because that’s best used for like sets, comparing sets I think.

R: Mm, hmm. You’re thinking, like you said, the element chasing stuff?

Andy: Yeah.

R: Okay.

Andy: Um, [pause (10 sec)] I just don’t know how to do, go about this like mathematically other than just like a picture, you know, writing it in short, concise sentences or –

R: Okay.

Andy: – as a –
R: So, describe to me what your picture tells me. So, we have an odd by an odd.

Andy: An odd-by-odd. We can even do an odd by, but, yeah, the definition of a perfect cover is there are every square is covered without any overlapping of the dominoes and we can have that except there’s gonna be overlap probably well, depending on how I lay these. But there’s gonna be overlap on an entire column or row, depending on –

R: Can you show me again how that works?

Andy: I have a domino here, here [draws dominoes on odd-by-odd chessboard] and the other dominoes’ gonna have to lay on top of this one.

R: Okay.

Andy: And then I can either continue to lay ‘em like this again if I wanted to and have overlap again there for both these rows. [pause (5 sec)] So, I can say this column right here has overlap.

R: Okay.

Andy: Or if I went this way [indicates vertically, pause, (10 sec)] everything would be okay but this one area right here that had overlap.

R: Okay. So you did your best –

Andy: So, I know –

R: The best that you can do is having one overlapping.

Andy: Right. So, I know there’s gonna occur at least one place that we’re gonna have an overlap.

R: Okay. [pause (15 sec)] Any other thoughts on how you could pursue that, in general?

Andy: [pause (50 sec)] Ah, let’s see here. [pause (5 sec)] There’s gonna have to be an even number of squares on the board. [pause (15 sec)]

R: Can you describe to me what you mean? So, there’s gonna have to be an even number of squares on the board.

Andy: Um, like with this one, there’s 64 squares, that’s 8-by-8.

R: Okay.
Andy: And if I have, [pause] ah, you know, say one even, one odd, if I had those two numbers multiplied together to see what I would get [writes \((2k)(2l+1) = 2(k2l+1)\)]. I can pull out my 2, ’cause this is just gonna be some integer \(k\) times \(l\) plus 1 is just gonna be some whole number.

R: Okay.

Andy: And I know that, so I can say that’s like 2 times \(z\) [writes \(= 2z\)]. So I know it’s gonna be even.

R: Okay.

Andy: [pause (15 sec)] Can I do that?

R: Can you do what?

Andy: [pause (20 sec)] Now I’m just looking at this arithmetically, I think I did that incorrectly. So, that’d be, 2 times \(kl\) plus 1 [writes \(2(kl)+1\), pause, (5 sec)] Yeah, that’s true. [pause (5 sec)] Anyways, I’m pretty sure what I’m saying here.

R: So what you’re saying is that in Case 1, you have an even times an odd and you were trying to prove with symbols that what you did is always even.

Andy: Yeah.

R: Right? Okay. What does that mean for Case 2?

Andy: For Case 2, an odd times an odd is gonna be an odd number.

R: What does that tell us? If it’s odd?

Andy: It means there’s a remainder. [pause (15 sec)]

R: So describe to me, in words, what you, what you’re wanting to say here. Without needing to write it down.

Andy: Well –

R: You’re trying to say that Case 2 doesn’t work, right?

Andy: Mm, hmm, yeah. Because [pause (10 sec)] because I know that a domino covers an even number of squares [pause], I want an even number of squares to cover.

R: Okay.
Andy: Because I know if there’s an odd number, there’s gonna be a remainder or I’m gonna have to have an overlap because a domino covers two squares.

R: Okay.

Andy: It can’t just cover one.

R: Okay. [pause (5 sec)] And so the writing it down part, it doesn’t matter to me. You’ve described to me what you would want to write down, right?

Andy: Mm, hmm.

R: Okay. Um, anything else you want to say about the problem? [pause (5 sec)] That you would want to do or any other things you’d pursue at all?

Andy: Um, [pause (10 sec)] I, I don’t know just, you know, with problems like this I just try to always, like I said, try to put an equation or something, I would like to see an equation just so I can work with that or –

R: Okay.

Andy: – yeah.

R: Have you ever seen anything like this question before?

Andy: Um –

R: Or does it remind you, in any part, of anything else you’ve seen?

Andy: [pause (5 sec)] Well, let’s see. [pause (5 sec)] It just makes me think about um, matrices because of the \( m \)-by-\( n \) –

R: Okay.

Andy: – thing and –

R: And the fact that you’re in linear algebra, right? [laughter] Okay. Can you walk me through, um, the process you were going through in your head as you looked at this problem? So we read it and what was going through, in your head and what were you doing in that little chalkboard in your head that you were working on there?

Andy: Um, well I first tried to visualize what it’d look like, what this chessboard looked like and then what each, what it looked like to have the dominoes covering the chessboard.

R: Okay.
Andy: And then I tried to think about an arbitrary, you know, just an arbitrary chessboard that was not necessarily even, I mean not even but ah, \textit{m-by-m} or you know, the same size.


Andy: And, ah, how I could go about proving that generic case. And that’s where I ran into my brick wall. I wanted to started thinking about, I just started thinking about the different methods that we’ve used in 305 for proving things. You know, proving contrapositive or using induction and all those other things and ah, proving by cases and that’s what popped, what came to mind was the ca – there’s a couple of different cases or where it could be even or odd, both odd.

R: So is that something that you, you go through, so when you say you go through those, do you look at this and say would induction work? And you look at it and say yes or no? Or do you just kind of consider those and if one comes out that it might work for you and you pick that up, is that kinda the idea?

Andy: Yeah. Kind of, if I think something might work, then I’ll ah try it out a little bit and then if nothing comes about then, maybe I’ll try something else.

R: Okay. Alright. Sounds good. Let’s do another question. Okay. Let’s do Question 1. [researcher collects papers, gives student Question 1] The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum of 16, for an example. Prove the smallest possible value for the sum is 14.

Andy: [pause] Okay. [pause, reads the question, (40 sec)]

R: Any questions about the problem at all or is it clear?

Andy: I think it’s pretty clear.

R: Okay.

Andy: [pause (20 sec)] So what I want to look at is [pause] proving that [pause] no matter how I arrange the numbers around here, the smallest sum I can get from the three all the way around would have to be the same, and is 14?

R: Mm, hmm.

Andy: Okay. [pause (20 sec)] The first thing that comes to mind is doing trial and error kind of things.
Andy: Um – [pause (5 sec)] making one side equal to 14. [pause (10 sec)]
R: And working from that one?
Andy: And working from that one side and trying to rearrange the numbers so that they
do work out.
R: Okay.
Andy: So, [pause (10 sec)] in looking at this, I can tell that 10 is, cannot be on the corner
anymore, ‘cause it can’t be used two ways.
R: Okay.
Andy: So I know that 10 would have to be in the middle. [pause (40 sec)]
R: And just going through some of the numbers in your head?
Andy: Yeah, I’m just thinking that it’s probably these three, 8, 9, and 10, that most like
gonna wanna keep ‘em in the middle [pause (5 sec)] and build around them instead of
sticking ‘em on a corner.
R: Okay.
Andy: ‘Cause, there’s not that many small numbers that it’s gonna give me a sum of 14
with a 9, actually there’s not –
R: Can you write down a little bit about what you’re trying? Give me a visual peak into
your head?
Andy: Like redraw the diagram?
R: Mm, hmm.
Andy: [pause, draws pentagon] So, I’m thinking about having the larger numbers in the
middle ‘cause I don’t really [pause, puts 10, 9, 8 on edges, (5 sec)] let’s see [pause (10
sec)] 2 and 3. [writes 2+3+9 = 14] That’s going to be 14, and 4. [writes 1+3+9 = 14] I
know I need 1 and 3 for 10. [writes 1+3+10 = 14] So, that 9’s gonna have to be on the
line adjacent to the 10. I think what I would probably do is gonna write the different
ways to get 14, using the larger numbers first.
R: Okay. To kind of get an idea of where they might need to be?
Andy: Mm, hmm.
R: Okay.

Andy: [writes $1+5+8 = 14$, $2+4+8 = 14$, $1+6+7$, $2+5+7$, $3+4+7$, $3+5+6 = 14$, pause, (1 min 20 sec)] I could, I mean I could sit here and do this and figure this out but I’m not [pause] that’s all I’m thinking about. I’m not, I’m not thinking about how to try to prove that that is a smallest value that is possible. Which is what the question was. [laughter]

R: Well you can do whatever you like. If you want to pursue making that diagram, feel free to do that.

Andy: [pause (10 sec)] Maybe I’d think about [pause] the smallest possible sum for 10 would be a 13 [writes $10+1+2 = 13$. Since it can’t be the sum of three numbers, 10 with the two smallest number’s gonna be 13.

R: Okay.

Andy: [pause (5 sec)] So, then I would want to think about [pause (5 sec)] the other numbers and seeing how possible it is with those numbers.

R: Okay.

Andy: [pause, writes $9+3+1 = 13$, $8+2+3 = 13$, $8+1+4 = 13$, (45 sec)] Looking at this, I’m only seeing one way to get 13 with 9 and 10 both.

R: Okay.

Andy: And that’s having them on a common side and then 1 would have to go there. [puts 1 between 9 and 10 on pentagon] which would force 2 and 3 here [writes 2 and 3 on pentagon]

R: Okay.

Andy: [pause (15 sec)] That’s where it falls apart for 8.

R: How so? Describe to me what you’re seeing.

Andy: Well [pause (5 sec)] the only combinations for 8 that I can see that add up to 13 are 2 and 3 and 1 and 4. ‘Cause there’s no 0.

R: Okay.

Andy: So I can’t use 5. And, which would mean 8 would have to have 2 and 3 along the same line or 1 and 4 along the same line. 2 and 3 are on opposite sides and 1 is already used in there twice. So there’s nowhere I can put this 8 to get a sum of 13.

R: Okay. And so we can conclude that –
Andy: So I can conclude that 13’s not possible.

R: Okay.

Andy: Which would be the, generally, it’s gonna be the smallest value.

R: Okay.

Andy: So then, if I look at the sum of 14 [draws new pentagon], which is next, which I was working on here – [pause, writes 1 10 3 on one side, (10 sec)] – same case as up there, there’s only be one possible thing for the 14. So, [pause (5 sec)] just by, just start going by trial and error [pause (5 sec)] this would be – [pause, writes 3+5+6 = 14, 6, (40 sec)] Must be 6. [pause (15 sec)] This one, and this one. [circles combinations 1+6+7 and 3+5+6] So, 6 has to be with 1 or with 3. [pause, puts 9 and 4 on pentagon, then 8 and 2, and fills in the rest of the numbers, (1 min 5 sec)]

R: So, through a little work, we found 14?

Andy: Yeah.

R: Alright. So, can you sum up what you’ve got? You’ve got that it is possible for 14.

Andy: Mm, hmm. I, by looking at the combination, the sums of three numbers to equal 13, the smallest possible sum, I figured is, is 13. Taking the greatest and the two smallest.

R: Okay, and so you’re saying you couldn’t have 12, or 11 –?

Andy: There’s no way to make 12 out of, using the 10.

R: Okay, got it.

Andy: So, I know that that can’t be. So, this is my smallest. And then I looked at the 9 and what is, how do I get a sum of 9 or of 13 with the 9 and there I found that there’s only one possible way there, too. And then I looked at the 8s and only found two. And then by looking at the numbers that I’d have to use for 8 and the numbers that are already used, with the 9 with the 10, I concluded that there’s no way that I could get 13 using the 8.

R: Okay. So, kind of just process of elimination –

Andy: Mm, hmm.

R: – of going through it. That kinda led you to finding your solution for 14, right?
Andy: Yeah.
R: And then once you did that, you’re satisfied we have proof, right?
Andy: Pictorially, yeah.
R: Okay, we could write it out. Orally and pictorially you have a proof, right?
Andy: Yeah.
R: Okay. Um, have you ever seen a question like this before?
Andy: Um, not that I can remember.
R: Okay.
Andy: I’ve probably run into something along the lines over the years, but nothing that pops out at me.
R: Okay. Um, any strategies or processes that you used that we didn’t mention. I mean you, you were kind of running through your head when you started and just trying to move some numbers around on, on the 16 example, right?
Andy: Mm, hmm.
R: And trying do it?
Andy: Yeah, um – [pause]
R: And I think that kinda lead to putting the big numbers on the si – on the middles, right?
Andy: Right. That’s how I came, ‘cause I looked at the number 10 first and how there was only one way to get this, the number that I wanted.
R: Mm, hmm.
Andy: Like the then, if then, I wanted the 14.
R: Okay.
Andy: I saw that there was only one way to use it to get 14. If it’s on the corner, it’s gonna get used twice and I don’t wanna use it twice, I just wanna use it once.
Andy: So I knew I needed to put it in the middle. And then I thought well if 10’s that way, it’s probably gonna be the same for 9 and then possibly 8.

R: Okay. And then from there, just kind of what we already described right?

Andy: Mm, hmm.

R: Process of elimination, looking at some examples and things like that?

Andy: Right.

R: Okay. Alright. *End of Interview
R: We call a positive integer N a 4-flip if 4 times N has the same digits as N but in reverse order, prove that there are no two-digit 4-flips. I'll give you a second to read through that.

Katy: [pause (5 sec)] Oh, okay, so it's the same like, frontwards and backwards?

R: Well, when you multiply by 4, it flips it around.

Katy: Oh, okay, okay I see what you mean. [pause (30 sec)] Huh, that's kind of cool. I don't really – [laughter] So, if you have like a number abcd and you multiply by 4, you're going to get dcba? [writes abcd*4 = dcba]

R: Right.

Katy: [pause (35 sec)] So, maybe I would do it by counterexample? I really don't know what I'm doing but... or not counterexample but contradiction, maybe. Hmm. [pause, writes “Proof by contradiction: assume that there are 2 digit 4-flips”, pause, writes ab*2 = ba off to the side, then writes “then 2*ab = ba”, (1 min 40 sec)]

R: So, let me just make a clarification, so that I know that you understand the problem. So, a 4-flip means it's always multiplied by 4, no matter how many digits it is.

Katy: Oh, it doesn't have to be 4? Oh, so like this could be anything? Any, as long a number as it wanted to be.

R: No, I mean, you're trying to prove it for two-digits, but it's still a 4-flip. So, it's still that it would be multiplied by 4.

Katy: By 4, oh. Okay. Thanks. [pause, changes 2 to 4 throughout, writes , 4*b = a or c+a, 4*a+0 or 4*a+c = b”, then to the left writes 4b = a or 4b = c+a, (1 min 50 sec)]

R: Can you describe to me what you're doing?

Katy: Oh, I don't really know – Well, I was just thinking that, so if you take, maybe I'm wrong yet, a two-digit number and multiply it by 4 like if I'm assuming that there are two-digit 4-flips, then um, so you have the number ab and these are numbers and you multiply it by 4, then 4 times – like to get b and a as your answer, 4 times b has to either be a, or something like, either has to be, how like, 2 like 20 plus a or like uh, another digit. That's what I couldn't say, but you know what I mean? So, it either has to be 10 or 20, or 30, so you still get a like in this column when you –

R: Oh, I see.

Katy: – when you go to add them. I don't really know, that's what I'm thinking though.
[laughter] and then, so then $a$, when you multiply $a$ by 4, it either has to be $b$, or then 4 times $a$ plus whatever this $c$ was before, has to be $b$ to get that.

R: Okay.

Katy: I don't really know if it's going to help that. I don't know [pause, writes $4a=b$ or $4a+c=b$, pause, (50 sec)]. Now, I'm just wanting kind of an example to maybe look and see or somehow -- so. [pause] If $b$ is 5, oh no, I don't want that. If $b$ is 2 then $a$ is 8 [writes $b=2$, $a=8$], so that would be – [writes 28*4, erases, writes 82*4, (5 sec)] oops 82 I mean. Oops, sorry I wasn't supposed to erase.

R: It's okay.

Katy: [laughter, writes = 328, (10 sec)] Okay. Hmm. [pause]

R: So, what did you just notice?

Katy: Well, that –

R: What did it give you?

Katy: A three-digit number, not a two-digit number, so it can't -- I mean like, you start with a two-digit number and you just want it to flip it, but then it gives you a three-digit number, so obviously it can't.

R: Okay.

Katy: But, you get the 8 back. Well, I was just using this, that 4 times $b$ should equal $a$.

So, then –

R: Oh, okay.

Katy: That's where I just came up with those numbers. But -- so then it's too big, so then you'd have to pick something smaller, like, then $b$ would have to equal 1, then $a$ would have to equal 4 [writes $b=1$, $a=4$], so then you'd have 41 [writes 41]. But, I guess you could go the other way too, so you could also have 28 and then 14 [writes 28 and 14]. But, then, yeah, I don't know. [pause, circles 41, points to 82, (15 sec)] But, these are all too big to be multiplied by 4 and to get a two-digit number. Is that what we want? Is that helpful? I don't know. [pause, writes 14*4 = 56, (5 sec)] Then, that's just not one at all. Oh, so then maybe this isn't true? [points to $4b=a$, etc., pause, (30 sec)] Yeah, so.

[pause (20 sec)] So, I guess mainly I just think that it's going to be too big, when you multiply a two-digit number by 4. That would work for this, I don't know. [pause (10 sec)]

R: What do you mean by too big?
Katy: Well, I – 'cause – like this, like 82 is just too big, because then it gives you a three-digit number. But, does 28? Or, I don't know. [writes 28*4 = 112, pause, (10 sec)] Or whatever I was thinking then – [pause, writes 4b = a ⇒ a = c+a ⇒ 2a = c, 4a+c = b, 4a+2a = b, 6a = b, a = 1/6 b, (1 min 35 sec)] Hmm. I don't think what I have makes sense. Oh, I bet my hair was in the way the whole time.

R: Oh, that's fine.

Katy: I'm sorry.

R: I'll keep that paper, so it's more important that I just know when you wrote what you wrote.

Katy: Oh, because it's so messy. I don't think I can do this.

R: So, you mean working with the equations now?

Katy: Yeah, well, maybe because I don't know if they're really right. Like, I guess, 'cause this one [points to 4b = c+a, pause] So, maybe I won't use this then. I don't know. [pause, puts box around 4b = a, 4a = b, and a = a/6 b, (25 sec)] Well, I guess if I was working with the equations this – wouldn't make sense. I don't know, maybe it just doesn't make sense anyway.

R: Can you describe that a little more to me?

Katy: Well, 'cause if I take what I have, whatever these are, see I don't think it really works, because I just wanted to call c whatever – um. Well, I don't know, maybe. So, I just said that 4b equals a, so then just using these equations and like substitution, then I get that a equals 1/6 of b. [pause] Which doesn't make sense, but then it does because you could have it either way. Like, it doesn't matter really if you have b then a or a then b to start out with. So then, that really didn't say anything I think. [pause (25 sec)]

R: So, what else would you like to try?

Katy: Hmm. [pause, laughter, pause, (20 sec)] Maybe going backwards, I don't know.

Like instead of having, saying that 4 divides b, no I need to times it. I guess I just really don't know how to write this number [writes 4 | ba]. You know? 4 divides this two-digit number ba, um, that's good, to write like that. So, then that means that 4 times something is going to equal b, no times a, shoot! ba as a number [writes ⇒ 4k = ba], and then you want k to equal the number ab, not a times b [writes k = ab]. Can we then show that that's wrong? [pause (20 sec)] I don't know, I just don't really – is this like a normal way to go about this? I don't even know. [laughter]

R: So, if this were assigned to you, what else would you do? What would you do next?

Katy: Well, I'd probably talk to someone and see if they understand [laughter] if they
understood it. And then we'd talk about it. I don't know, I never usually write – like
sometimes I write proofs by myself entirely, but not really all the time. Like usually it's
like a group of people and we're like, we each contribute something. And like, oh, no
that didn't make sense. Or, you know?

R: Okay. Would you like to do something else?

Katy: [pause] Something, what?

R: Try another problem?

Katy: Oh, yeah. [laughter]

R: Okay, I have more problems that go along with this, you can read it and um let me
know if you want to work on it or not, okay? Either way is fine. So, the next stage in
this problem would be proving or disproving that there are three-digit 4-flips.

Katy: [researcher gives student part b] Oh, well I couldn't do the first one, so – [laughter]

R: Alright, well let me ask you a few questions about the process that you did here.

Katy: Okay.

R: Um, can you describe any of the things that you tried while you were doing this?

Um, you started by saying proof by contradiction, right?

Katy: Well, that's what I immediately thought, since to prove that there are none, then,
um that you would assume that there are some and then get a contradiction, so then there
are none. And then, well then I just wanted to work with equations because that seemed
right. I don't know, it seems easier to see equations for me I guess. Like to show that
things aren't equal, or something, I don't know. Um, so then I was trying, like, to make
equations of how you would get the flip or whatever, but – I don't – what else did you
want to know?

R: Well, that was most of it. Um. Have you ever seen anything like this before?

Katy: Not really. Well, I mean, like, no, [laughter] I don't think so.

R: Okay, alright. Let's move on to a different one. [researcher collects papers, gives
student Question 3] A traditional chessboard consists of 64 squares, 8-by-8. Suppose
dominoes are constructed so that each domino covers exactly two adjacent squares of the
chessboard. A perfect cover of the chessboard with dominoes covers every square of the
chessboard without overlapping any of the dominoes. Consider a generic chessboard.
Prove that the generic chessboard of size $m$-by-$n$ has a perfect cover if and only if at least
one of $m$ or $n$ is even. [pause (20 sec)]
Katy: So, you're like laying dominoes on the chessboard to cover two of the squares? Like these are the squares? [draws two squares of a chessboard]

R: Yeah.

Katy: And then you put the domino over this?

R: Exactly, mm hmm.

Katy: And then, so then they're saying that it can only have a perfect cover if the length of \( m \) or \( n \) is even?

R: Mm, hmm.

Katy: Okay. [pause (25 sec)] I just need a picture.

R: Okay.

Katy: [pause, draws chessboard of size 8x8, (30 sec)]

R: So, you're drawing a regular chessboard there, 8-by-8? Is that what I see?

Katy: Yeah. [pause (15 sec)] Okay. [pause, labels chessboard \( m \)-by-\( n \), draws in one domino, writes “1 domino covers \( 2/m \) of the row and \( 1/n \) of the column”, pause, (65 sec)]

R: Can you read to me what you wrote? I can't quite see it.

Katy: Oh sorry.

R: It's okay.

Katy: I was just saying that like if you have one, if you put one domino down, it's gonna, if this, if \( m \) is the row, they'll make this row, that it's gonna cover two over \( m \) of the row. Right? [pause] Well, shoot. Well, and then like one over \( n \). [pause] But then, that would be assuming that this is – [pause, draws box around row of chessboard, (10 sec)]

R: What do you mean?

Katy: ‘Cause if I'm saying this, is this saying that I have to know the length? Or how long the square is, each square?

R: Oh, yeah, a domino covers two squares exactly, like that. Like it's two long and one tall.
Katy: But then, can I say that? That one domino would cover two over $m$ of this row?

R: You mean, two out of the $m$ squares, is that what you mean?

Katy: Yeah, I guess. But then if I say that this length is $m$, then this is $m$ squares. Oh yeah, okay thanks. Sorry. See, that's what happens to me when I'm writing, when I'm writing proofs, too. [laughter] Then, I'm like I don't know what I meant by that. Then, I just get confused and give up. Okay. [pause] So, then if, so then, do you want me to really write this as a proof? ‘Cause then that makes so much sense to me. So, then obviously if it covers two of these squares, then $m$, the number of squares in this row has to be even and it doesn't matter if $n$ is even or odd, because you can have as many dominoes as you want, stacked on top of each other.

R: I see.

Katy: But then, if you just, if you put it the other way, then it wouldn't matter if the row is even, you know?

R: Okay.

Katy: Does that make sense?

R: Yeah.

Katy: Do you want me to write that?

R: So, you're saying the either or thing?

Katy: Yeah.

R: Okay, and that's what you would — so you've described verbally what you would write down in a proof?

Katy: Mm, hmm.

R: Okay.

Katy: Well, ‘cause, yeah, and that makes sense because you would want this to be like mathematically, wouldn't you just want this, since, if $m$ is even, it would fill up the write amount of squares or whatever.

R: Okay, I see.

Katy: Right? I don't know. [laughter] Do you want me to —?

R: No, you don't have to write it down.
Katy: Okay.

R: But, you're good at that point? Like, that's what you would write down for a proof?

Katy: Yeah, like two cases, right? ‘Cause either \( n \) is even, or well which it said to do in the first place, I guess. But, – yeah, that's probably what I would write down.

R: Okay. Um. So, let's see, let's talk about the strategies you used this time. So, I saw you, you reread the problem. Right?

Katy: Mm, hmm.

R: Just to make sure you understood it. And then you drew an example –

Katy: Mm, hmm.

R: – of the generic. And then you kind of analyzed what from there? Kind of looked at where a domino would go?

Katy: Yeah, I have to see it visually. Or else, like, I don't know, I really realized this year – ‘cause I'm only a sophomore but um – so I've actually taken a lot of classes right now that you have to write proofs for, but um – Which, I don't know – a lot of the people in my classes have taken a lot more and are older than me. But, um. What I really have realized this year is that it's hard to write a proof if you don't believe it.

R: Oh, okay.

Katy: So, like I have to sometimes really think about it and like believe it before I can, like – then it makes sense to me.

R: Okay.

Katy: Like because it's so abstract I like to be able to be like, Oh I can see it in a small case, that makes more sense.

R: Okay.

Katy: So, that's why I just draw the picture.

R: I see, convince yourself that it's true before you can write it?

Katy: Mm, hmm.

R: Okay. That totally makes sense. Anything else that you can add to the things you were thinking or doing while you went through the problem?
Katy: Um. Well, I just had to make it make sense. That, like, of what was going on with the dominoes. But then, [pause] Um. I don't think so. ‘Cause I think once you understand something it's way more simple than when you don't get it. Then you're like uh – [laughter] I don't know.

R: Okay. Have you ever seen anything like this before?

Katy: Well, like, not the chessboard thing, but I guess like proofs with cases, and evens and odds and whatever.

R: Okay. And considering those possibilities?

Katy: Mm, hmm.

R: Okay. Alright. Let's do another one. [researcher collects papers, gives student next question] I've just got to decide which one we should do. Let's go, if I can find Question 1. [laughter] There you go.

Katy: Oh, geometry, shoot.

R: The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum equal to 16, for an example. Prove that the smallest possible value for the sum is 14.

Katy: [pause, reads the question, (20 sec)] Oh, and you only put each number once?

R: Right.

Katy: [pause] Oh. [pause, circles 10 on given pentagon, (60 sec)]

R: Can you tell me what you're thinking about? What you're doing in your head?

Katy: [laughter] Oh, um, I'm thinking that I would start by just like starting with the 10, to make sense of like the smallest value for this along a side. ‘Cause the smallest thing, if you start with the 10, ‘cause you're always going to have to have the 10 – I don't know, that just seems like a good starting spot for me. And then the two smallest numbers that you can put are 1 and 2, so that would be 13. And then I was thinking of how then that's obviously not valid, because it says prove that the smallest possible value is 14. So, then you'd have to put a 3 and a 1. That's all I was thinking. [laughter]

R: Okay.

Katy: [pause (5 sec)] Oh, but then that makes sense, too, because you couldn't make 14 going this way. [points to edge of pentagon, where it says 10 2 4 now, pause, (5 sec)] I
mean 13. [pause, writes 10, 3, 1 on pentagon on top left, (40 sec)] I guess now I just
want to figure it out.

R: For 14 you mean?

Katy: Yeah, like how, where you'd put – [pause] the numbers to make it make sense.
Jeez, people are smart [laughter] to think about this stuff. I don't know.

R: To come up with a problem in the first place?

Katy: Yeah, that's so funny to me. [laughter]

R: Well, I didn't come up with them, so.

Katy: No, you're probably – [laughter]

R: They're not my original ideas.

Katy: [pause (15 sec)] Well. [pause, writes 10 3 1 to the side, (10 sec)] Well, how
would you? Am I really just missing something? I don't know, it's so ready for school to
be done. My brain is fried, I'm ready to give up. [laughers]

R: So, what are you thinking?

Katy: Well, for it to be 14, you'd have to have a 3 and a 1. [points to where she wrote
10, 3, 1, writes 10, 2] And a 2 and a 2, but you can't have 2 twice. For like this to be 14,
and this to be 14. Alright, what am I missing? Can I not add today? 'Cause you would
have to use – [pause (5 sec)] 'Cause there's always three numbers, right?

R: Right, there are always three numbers on each side.

Katy: [pause (10 sec)] And so, [pause (15 sec)] and they're always positive, you can't
make them negative? [laughter]

R: Right. Just 1 through 10. No 0, no negatives. [pause] So, is there anything else you
could think to try, that would make it work?

Katy: Well, I guess I wouldn't have to start at 10. But, that just is the biggest, so it would
make sense to me to do that, but I guess I don't have to start there. But – I should
probably try to think about how to write it as a proof, so I'm not just sitting here forever.
[pause, starts to erase 10 3 1 on pentagon, (30 sec)] Sorry. [laughter]

R: There's plenty more papers, so you can draw whatever you want.

Katy: I know, I just want to erase.
R: Natural inclination, not a problem. [pause (40 sec)] What else are you thinking of?

Katy: [laughter] Well, I just don't, I'm thinking still of how to get 14 on this side. I just can't get past it, 'cause then I'm like well – if you started with this – Oh, you don't have to put 10 in that spot! Shoot! I get it. [laughter] So, maybe I should do it differently. 'Cause that was what was hanging me up. [erases 3 1 on top of pentagon] I'm not erasing, I'm putting those back. [laughter, writes 3 1 back in, crosses out] You're going to be annoyed to watch me erase all this.

R: No. You're totally okay. You can redraw, you can cross off, whatever you want.

Katy: Okay. Oh shoot!

R: So, now your thought is move the 10?

Katy: Yeah. Oh, that took me a long time! It's just nice out I want to go play. [laughter, writes 3 10 1 on top of pentagon] So, if I did that. [pause, writes 5 6 on side with 3, (10 sec)] Okay, so then, okay I get it. I just couldn't do anything until I had understood that, I think, mainly. Oh, which I think that's what makes people give up in proofs, when you don't get what – that's what makes me give up. [laughter] I would've given up, if I wasn't in here. [laughter]

R: And gone and sought someone else, right?

Katy: [laughter] Most definitely. [pause (15 sec)] So, I just don't. [pause (10 sec)] So, to prove that the smallest possible value is 14 I really don’t – [pause (20 sec)] Okay, I can’t look at this row. [crosses off 10 3 1 at top again, laughter, pause, (30 sec)]

R: What work are you doing in your head?

Katy: Oh, I’m thinking of the odds and evens.

R: Oh, okay.

Katy: Um. So, you have to have an even and two odds every time, for the sum to be even. [pause] And then, 'cause I feel like – Okay, 'cause the first thing I think of is the smallest you could actually do is 13, but then that's not going to work because – that's what I'm trying to figure out.

R: Okay. Why 13 doesn't work?

Katy: ‘Cause to have an odd sum, you would have to have one even, opps – I mean two evens and an odd, or three odds? [writes 2 even, 1 odd, 3 odd to the side] Yeah. And, [pause (5 sec)] there's only 5 odds that you could have. 5 sides. Okay, so then, I'm just thinking of it that way, then you'd have every case is this [circles 2 even 1 odd]. But then, that can't work because there's not 10 evens between 1 and 10, so that's why 13
doesn't work. Does that make sense? ‘Cause, if you have to have 10 on one side, then, I already said that, that you’d have to have the smallest numbers you could pair there are 2 and 1. So then, um, yeah, you'd have to have distributed. This is, like you can't have three odds on every side. But, to have the smallest number of odds, which there's 5 odds in 1 through 10, you put them all here, but then that'd mean you have to have 10 other even numbers to go along, but you only have 10 numbers. So, that's why 13 doesn't work. So, then 14. [pause (5 sec)] You could have three evens. [pause, writes 3 evens, 2 odd 1 even] Two odds and an even. [pause (5 sec)] To equal 14, the sum. [writes summation symbol before 3 evens and before 2 odd, 1 even, then writes = 14 by each] So, wait so I was thinking completely wrong? Oh, yeah, but you share the vertices, so I don't know. I'll just accept that 13 doesn't work. [laughter, pause, circles vertices of pentagon, pause, writes 6 8 2 along a side of pentagon, (1 min 45 sec)]

R: So, can you describe to me what you're trying?

Katy: [laughter] Oh, I don't know. I guess I'm just trying to think of how to write – I don't know. I guess I'm just trying to think of how I would write it as a proof, which I just don't really know. [pause, writes $2l+2k+2m = 14$, and $(2a+1)+(2b+1)+2c = 14$, (65 sec)] So, now I'm just thinking of the sums, for these to equal 14. Then, whatever, ‘cause I'm just writing them as even numbers, or, and odd numbers. So, 2 times some integer, and for each one, so I'm just thinking that whatever these variables that I put here have to be less than or equal to 5 [writes “where $l,k,m,a,b,c \leq 5$”]. Since it's 2 times it and we only have up to 10. So, I don't know, maybe that's helpful. [pause, finishes writing, hits camera with her foot accidentally, (35 sec)] Sorry. I hope that didn’t mess it up.

R: No, that's fine.

Katy: [pause (10 sec)] Then, okay then I don't know if I'm doing this right, because this just means – well I think it might be – ‘cause you want it, the sum to be even and the smallest would be 14. So, then, [pause, audiotape side one ends, researcher flips over tape, pause, (20 sec)] Oh, you tape record and videotape it?

R: Yeah.

Katy: Wow.

R: This is the best way to capture your voice. It's easier to transcribe off of a tape than it is off of the videotape.

Katy: Oh, cool.

R: So, any other ideas you have?

Katy: Oh, I don't know. I feel like maybe you could get – I don't know I guess I would just do something like this. Whatever, I don't really know, I don't really have any other ideas. [pause (10 sec)] I guess just try it maybe [laughter]. I don't know, just say that
these numbers. I think I would just have to plug them in. Plug in for each of them. Oh, I
guess then you would have to have that these are all distinct. These are not equal [circles
variables \( l, k, m \), etc.]. I don't know. I don't have any other ideas. [laughter *videotape
ends]

R: Okay. Well, why don't we go back and discuss what you've used while you were
working through the problem. Um. So, again, you reread the problem and you kind of
looked at what you had, right? In the example?

Katy: Mm, hmm.

R: I think it was pretty soon that you realized that 13 was the smallest –

Katy: Yeah. Mm, hmm.

R: With 10 and 1 and 2. Right? And so then you kind of worked with 14 for a while, to
see if it was possible, right?

Katy: Mm, hmm.

R: And then what?

Katy: Okay, so then, then I was just confused that I could move the 10 [laughter]. That
took a long time. Then I was just trying to look at like, even, 'cause if 14 is the smallest
— I was trying to look at why 13 wasn't going to work. I then I thought it made sense.
But, then whatever I said didn't make sense it was wrong, I realized that. [laughter] And
then, so then I was just trying to look at evens and odds and the sums of evens and odds.
'Cause you want it to be 14, so the sums have to be even, and the possibilities of the sums
of three numbers being even. And then writing them as even and odd integers.

R: So, again, you are going back to equations.

Katy: Yeah.

R: It that something that you typically do when you write proofs?

Katy: Yeah.

R: You prefer working with equations?

Katy: Mm, hmm. I think so. Like proof by induction is probably my favorite.

R: Oh okay.

Katy: 'Cause you can use equations [laughter] and it's like set up really good for you.
You don't have to like go crazy. Yeah, I like equations better. It's easier for me to
balance the equations and think through things, I don't know.

R: Okay. Um, have you ever seen anything like this before?

Katy: Mmm, I don't think so, not really. I mean the principles are the same, like you know, odd and even integers and like, stuff like that. I don't really –

R: So, some of the things you were working with were things that you would have seen before maybe, thinking about the evens and the odds and thinking about the sums and things like that.

Katy: Yeah.

Interview #20  (Total time = 46:55)

R: So, here's the first question. We call a positive integer $N$ a 4-flip if 4 times $N$ has the same digits as $N$ but in reverse order. [pause, researcher shuts door to room] Prove that there are no two-digit 4-flips.

Rick: Yeah, okay. So, we call a positive integer a 4-flip if $N$, if 4 times $N$ has the same digits as $N$ but they're in reverse order. So, that would – same digits, okay. So, like 12 and 21? [writes 12 and 21]

R: Mm, hmm. Yeah, if we multiplied 12 by 4, if it was a 4-flip, we would get 21.

Rick: [pause, writes 48 and 84, pause, (15 sec)] Oh, like that? Well, I don't know what I'm doing. Okay. Um. [pause, reads question, whispering while he reads, (10 sec)] Okay. So, for this one, 12 when we multiply it by 2 and got 24 [writes 12*2=24], it's still two-digits, but it's not in reverse order because they're different digits.

R: So, it's a 4-flip, it's two digits, but it's a 4-flip.

Rick: Oh, 4 times $N$.

R: Yeah.

Rick: Okay. Um, okay. So, we’ll take 10. [writes 10] There's a positive integer. [pause (5 sec)] We will let $N$ equal 2. [writes $N = 2$] So, [pause (5 sec)] 4 times $N$, that would equal 8 [writes 4*2 = 8], well I guess I'm confused.

R: Okay, any questions you can ask to clarify the question?

Rick: Okay. Prove that there are no two-digit 4-flips. Okay. So, let's try a three-digit. [pause] Cause I was going to just –

R: Is there any questions you have though, that I could answer at all? Or do you?

Rick: Um.

R: Or, are you good?

Rick: Am I going the right way so far?

R: I just want to make sure that you understand the definition of it, I guess.

Rick: Well, I'm trying – that's what I'm trying to figure out.

R: Okay.
Rick: So, there are no two-digit 4-flips. So, we'll try three-digits and see if works. So, 1 1 1, three digits. [writes 111] Ah. 1 2 3. [writes 123] I don’t like that one necessarily either, but okay. [pause] Wait a minute, wait, okay, a positive integer – So, that's our N.

[writes =N by 123] Times 4. [writes x4] Um. So, oh I can do it in my head, but –

R: That's fine, use your calculator, your totally fine.

Rick: It's all a matter of time. [pause, does calculation, writes = 492, (5 sec)] Okay. So, that clearly doesn't work. [pause (10 sec)] Okay. So, what I'm doing, I was just trying to figure out the definition. I'm trying to get an example to see if I can get it to work. Pick 123, times it by 4, I got 492, which is not even the same digits. So, I might spend an hour just trying to figure this out. [laughter, pause, (10 sec)] 4-flip. Alright, so we'll try N equal to 100. [writes N=100] Times it by 4. N is going to be 4 times 100 equals 400.

[writes 4xN = 4x100 = 400] Ooo. Alright. [pause (5 sec)] So, I can't necessarily ask you questions. What kind of questions can I ask you?

R: Anything clarifying.

Rick: [laughter] Um. Okay.

R: I think you understand what a 4-flip is now. So, you plugged in N being 100 and said when I multiply by 4, I get 400. But if it was a 4-flip, what would you expect?

Rick: That it's got the same digits as, same digits but in reverse order. But, by multiplying it by 4, I’m going to get different digits. So, I had a 1 0 0, now I've got a 4 0 0. So, according to the definition, it would just be 0 0 1. [writes 001]

R: Right.

Rick: Which doesn't work, clearly. Um. Okay. So, we'll try 222. [writes 222] Multiply that by 4, it's going to equal 44— I don’t know. Um, I love it. I think it was 888. [writes x4 = 888] I think so. I’ll prove that. [finds 222x4 in calculator, (5 sec)] Um. Oh, once again, different digits. [pause] So, I'm proving that there are no two-digit 4-flips at all. Oh! I'm proving there's no three-digits. Uh. [pause] Hmm. [pause]

R: So, is that what you're thinking, that there's probably no –? I mean you’re searching for three-digits?

Rick: I'm sure there is. I'm just searching for one example to prove that this definition works for at least something.

R: Okay.

Rick: [pause (5 sec)] Cause I like concrete examples, obviously. [pause, sigh, pause, (25 sec)] I’m assuming it doesn’t work for two-digits flips, either. [pause (10 sec)]
R: So, how can you prove that it doesn’t work for two-digits?

Rick: [pause (10 sec)] Okay. [pause, writes 5, (25 sec)] So, I’m thinking 5 is the lowest one we’ll get. Yeah. [pause (15 sec)] I’m thinking of one-digit. I’m thinking– [pause]

Well, it clearly doesn’t work for one-digit. [pause] So, two-digits. [pause (15 sec)]

Hm. [whispering, pause, writes 22, 33, 44, 55, 66, 77, 88, 99, (50 sec)]

R: Can you tell me what you just wrote? What you’re thinking?

Rick: Well, okay, I’m thinking of two-digit – two-digit numbers, where it doesn’t matter because the forward order is the same as the reverse order. Just thought I’d throw it out there, I don’t know why.

R: What do you mean it doesn’t matter?

Rick: [pause] If I multiply – Okay, so 4, so how would I get, okay so 11. [pause (10 sec)] So, 4 times 11 is 44 [writes 4x11=44 at top of page], but 44 and 11 are totally different digits. Same digits – I’m trying to narrow this down.

R: Okay.

Rick: [pause (10 sec)] I obviously like concrete examples, but this is kind of need an abstract thinking to solve. Um. [pause (5 sec)] Oh. I love it.

R: What other thoughts do you have? What else would you try?

Rick: Um. [pause (20 sec)] I’m not sure what else to try. Let’s see. 1 5, 5 1. [writes 15 and 51] I don’t know, let’s try 2 4 and 4 2. [writes 24 and 42] I’m obviously just getting examples of reverse order. [pause (25 sec)] It’s hard to imagine that there are anything, that it works for anything. I’m sure it does. [pause (15 sec)] Okay. Hm. [pause, whispering, writes 12345 and 54321, (30 sec)] This goes smaller, hmm bigger. [writes an arrow and “bigger” between numbers] I’m erasing, sorry.

R: It’s okay.

Rick: Let’s see, we start from here to here and we get smaller. Inverse? No. Hmm. [pause (5 sec)] Oh, wow. What’s unfortunate is that I don’t know how useful this would be to you, because I’m kind of at a big, huge roadblock.

R: Okay. Can you think of anything else that you’d like to try?

Rick: Hmm. [pause, whispering, switches to new piece of paper, (30 sec)] Uh, let’s see here. [pause (10 sec)] So, we have two-digits, so whenever you multiply that by 4, you still have to stay two-digits. [writes N at top of page, writes xx and 4 xx] So, that means 4 times 25 is 100. [writes 4x25=100] So, 4 times 24.999 et cetera [writes 4x24.999] is the largest possible number without going into three-digits. So, smallest number would be,
well it’s 10. So, 4x equals 10, [writes 4x = 10, x = 2.5] so x equals 2.5. What did I just
do there? Uh. [pause] Okay, so that’s not going to work like that. [pause (5 sec)] No,
oh, oh yuck. [pause (10 sec)]
R: What just happened there?
Rick: Um, I lost my train of thought, that’s what happened there I think.
R: Okay. [pause] So, let’s go back to what you were saying. 4 times 25 is 100.
Rick: Yeah.
R: And so, you were saying the biggest we could use is –
Rick: The biggest two-digit number I could use is 20 – well, okay 24.999 etcetera, that’s
the largest two-digit number I could use to multiply by 4 and get a two-digit number.
R: Okay.
Rick: Well, are we talking whole numbers here? [pause (10 sec)] I’m going to simplify
it, I’m going to say yes, we’re talking about whole numbers here.
R: Okay.
Rick: I don’t know why, I don’t know if that’s good or not. Um. [pause, working on
calculator, (5 sec)] 98, that would – [pause] okay, yeah. I think the largest whole number
I could use is – I wonder if it’s 88. Cause I know it works for 88. [pause] That’s just 22,
right? [writes 22x4 = 88] Okay. So, that’s the largest number I could think of off the bat
that I still get a two-digit, two-digit number. [pause (10 sec)] Hmm. So, the smallest
whole number would be [pause] 5, no 10 times 4, it would be 40. [pause (5 sec)] I think.
10 times 4 equals 40. [writes 10x4 = 40] So, anything below 10 – This is N. [writes N
above 22x4 and 10x4] Um. [pause, works on calculator, writes N and draws line, (35
sec)] So, 22. 4. [writes 22 and 4, then erases 4 and writes 10 instead] Oh, 10. Sorry.
[pause (5 sec)] Times 4 is 88. [writes 4 and 88 in line with 22, then 4 and 40 in line with
10] 40. So, if I went to 23, I got 92. [writes 23, 4 and 92] So, it’s clearly – [pause (5
sec)] This is the exhaustive, exhaustive method I think.
R: Okay.
Rick: Just plug and chug and chug.
R: So, what would you need to try then?
Rick: Well, I’ve got to figure out my goal first.
R: If you pursue this method –
Rick: Yeah.

R: Um –

Rick: Well, I could ultimately try every two-digit number between 10, because that’s my lowest one, and I’m trying to figure out my upper bound. I think it’s 22, but I’m trying to verify that. I’m not sure, as far as what technique I’m using. [pause (5 sec)] Well, I’m just you know, well with 24, [writes 24, 4 and 96] Well it still works for 24, then at 25 it doesn’t work. [writes 25, 4 and 100, pause] Um. So, I’m just trying to figure out my limits. [pause (10 sec)]

R: Can you keep going on that then? On trying those?

Rick: Well. [pause (5 sec)] I could, but let me think about the problem.

R: Okay.

Rick: Cause what I’m trying to prove is – [pause (5 sec)] Okay, what I’m trying to prove is there’s a positive integer N – [pause (5 sec)] Okay, I’m trying to prove there are no two-digit 4-flips. So, the question was, we call a positive integer N a 4-flip if 4 times N has the same digits as N but in reverse order. [pause] And now I’m just trying to prove that there are no two-digit, two-digit 4-flips. Hmm. So, I’m going through two-digit numbers and I’m basically eliminating choices. Starting with 10, because [pause] 10 multiplied by 4 gets 40. See the thing is, this is very – [pause]

R: What were you going to say?

Rick: It’s a tough problem. [laughter, pause, (10 sec)] Cause it just – it’s a hard problem. So, you’re multiplying a number by a number, you’re multiplying a number N by 4. Yet, when you reverse the order of the digits, you’re going to get a smaller number. [pause (5 sec)] So, I’m just trying to grasp that. How there exists a number that when you multiply it by 4, you flip the result, the order in the result, but you’ll still have the same digits? It’s a mind-breaker. At least my mind-breaker. [pause, writes “Elimination of 2 digit #’s” at tope of page, starts new page and writes “Trying to find one” on top of new page, (55 sec)]

R: What does that one say?

Rick: Trying to find one.

R: Okay.

Rick: So, I just divided this into two things. I initially tried this, with this. [referring to trying to find one in the beginning] So, I’m just trying to think of what that means. So, I’ll just start with – I started with a five-digit there. [writes 12345] That’s going to equal 5
Okay, that makes sense. Makes just a little more sense. Um.

[pause, writes 123 = 321, and 1234 = 4321, works on calculator, (40 sec)] Okay, so I’m going to get actual answer of 4936. [writes (4936)] Hmm. [pause, sigh, writes 444 = , (25 sec)] Hmm, well yeah. Oh. [pause (15 sec)] Oh, that’s right. [looks back at 4936 and confirms that] That, hmm. [pause (5 sec)] Okay. So, 144 =, does calculation, writes 576, works more on calculator, pause, (25 sec)] Hmm.

R: So, you’re just trying some three-digits, and you’re looking for?

Rick: I’m trying to get it to work once.

R: Okay.

Rick: I’m trying to find one. [points to title of his page]

R: Any thoughts about the process or what’s going on?

Rick: Guessing and checking. I’m trying to get, trying to find one that feels like I might be heading the right way. And I’m not really feeling it yet. [pause (10 sec)]

R: You seem to be hitting another roadblock. Would you like to keep thinking about this one, or would you like to try a different problem?

Rick: Hmm, that’s a big question, because I want the answer. Um.

R: It’s up to you.

Rick: Hmm. Confident that you will tell me the answer in the end, new problem.

R: Okay. [researcher collects papers] So, if I could just grab all those. [gives student Question 3] Alright, this is a whole different problem.

Rick: (inaudible, referring to previous question)

R: We have a – do you want to keep going on this one? It’s totally fine.

Rick: Hmm. [pause]

R: You’re struggling with the decision.

Rick: I know because I want the answer, I want the answer. I want to find it myself, but I want the answer to this one too. I don’t know what this is, but I want the answer to this one, too.

R: Okay, so you want to do this one?
Rick: Yeah. I want to do this one.

R: Okay, if you change your mind and want to go back, just let me know. So, a traditional chessboard consists of 64 squares, 8-by-8. Suppose that dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes.

Rick: Hmm. [pause, reads question, (5 sec)] Okay. [gets out new piece of paper, (5 sec)] Okay. [draws chessboard, (5 sec)] I guess I’m doing the full thing.

R: Okay.

Rick: 8-by-8’s not so bad. [continues drawing chessboard, (40 sec)] You know what? Working problems like this is actually tiring. [laughter] Or actually, I can’t say whether this is tiring, or the fact that I really have not had much sleep in the past two weeks.

R: That doesn’t help.

Rick: What’s that?

R: That doesn’t help.

Rick: Yeah, so I can’t really make that base – that call that doing math problems like this is tiring. Cause I do find it very interesting actually. [finishes last of drawing of chessboard, (5 sec)] Okay, so we have our traditional chessboard. [reads question] Covers exactly two adjacent squares of the chessboard. [pause (12 sec)]

R: And the rest of the question – consider if it was generic m-by-n, the proof is to prove that a generic chessboard of size m-by-n has a perfect cover if and only if at least one of m or n is even.

Rick: Oh. [pause] Okay, so m, n. [labels his chessboard with m and n, pause, (15 sec)] So, I’m guessing what they mean is like [draws domino with dots] that’s your domino.

R: Yeah.

Rick: So, domino, domino, domino. So, so, 4 times 8 dominoes would cover this thing. [writes 4x8 = 32] I think? 32? That’s probably irrelevant because the thing is the actual question. [whispering while he reads question, (10 sec)] Oh. Hmm. This is kind of funny because it makes perfect sense in my mind. Which actually a lot of the problems in like chapter 21 and 22 [referring to his MATH 305 course] make perfect sense in my mind, but now I have to write it down, I have to prove it.

R: So, can you tell me what you’re thinking in your mind about the problem?
Rick: Okay. Well, okay, each domino covers over two of the squares, yeah two adjacent squares, that’s actually said in the problem. [points to the question] So, [pause] that means $m$ or $n$ has to be even, or a multiple of 2. Is that the right way to put it? Definition of even is like, even equals $2k$. [laughter, writes even $= 2k$] Or $x$ equals $2k$ or something like that. Umm. [pause (10 sec)] Yeah, where am I going to go with this? I could use a little contradiction. So, a number can only be even or odd. I’m trying to – [pause (20 sec)] So, we have $m$, $n$, integers, [writes $n$, $m$ are in $\mathbb{Z}$] or actually natural numbers would be, okay. Okay, um. Hmm. [pause (30 sec)] Okay. Yeah. Oh wow.

[audio tape side one ends, researcher flips it over, but it does not record, which was discovered later, the remainder of the transcription has been taken from the videotape]

Rick: Okay, no I can think some more. Huh?

R: Yeah.

Rick: Don’t want to miss my thinking. [pause (5 sec)] Oh. [sigh] You know I bet those who, no I’m not –

R: So, where were you going with it?

Rick: Well, this is 305, this is so 305. And I’m doing bad in that class. This is actually probably an easy problem, too. It seems very simple. [pause] Okay, so you have your chess, you have your domino, not your chess – So, you have your domino. That $d$ equals $2k$? [laughter, writes $d = 2k$] Um, okay, such that, so [pause] $m$ times, $m$ times $n$ equals $2k$. [writes $mn = 2k$] So, that means that will cover all those. [pause, reads question, sigh, pause, (25 sec)] Okay, so. [pause (5 sec)] Okay. So, $m$ equal $2k$ and $n$ equal $2k + 1$.

[writes $m = 2k$ and $n = 2m + 1$] Hmm. So, that would equal, or it’s suppose to equal –

[writes $(2k) \times (2m + 1)$, pause (5 sec)] Oh, uh, oh finish. [writes $= 2p$].

R: So, can you tell me your thought there?

Rick: [pause] Um. I’m trying to do a contradiction. I’m letting one of them be odd, and then, no I won’t go there. And seeing if it’s still even. An even times an odd will give you an even. [pause] Which it will. [pause] Oh wait, yeah, that’s the case where it works. So, $2m$ plus 1, [writes $2m + 1$, crosses out $m$, writes $n$] or $2n$, oh, I love it. [crosses out $n$] $2x$ plus $2y$ equals $2z$. [writes $(2x + 1)(2y + 1) = 2z$] Okay. [pause (5 sec)] So, this is the case where neither one of them are odd. [researcher sneezes] Bless you.

R: Thank you. So, are you stating that, or are you checking that? Or what is your thought there?

Rick: Well, I’m, okay. [pause] Contradiction still, um. I’m trying to show that this doesn’t equal that. [points to left and right hand side of previous equation]

R: Okay.
Rick: So, I could go ahead and complete this maybe. [writes $4xy+2x+2y+1 = $, (5 sec)]
Ha. [puts parentheses around $4xy+2x+2y$, (5 sec)] It’s gonna be odd. Okay. [pause (5 sec)] So, I mean, what I need to do. I have a contradiction here, so. First I have to establish the fact that – [pause (5 sec)] when $m$ and $n$ [pause (5 sec)] Okay. What do I have to establish? I have to establish that – [pause, looks back at question, (15 sec)] Okay, so when both those are odd, well not even, then that can’t possibly, then you’ll always have –What am I trying to say? [laughter] Um. Ah. Oh, I love it. [sigh, pause, (15 sec)] Hmm, I’m sorry, I’m dying here.

R: No, I think you, just explain what you’ve done.

Rick: Yeah, that’s why I’m dying, I can’t explain it. Okay. So, we have our chess, dominoes, right? Dominoes. We have our game board. Um. And we have a piece that’s two, that’s um, two areas. So, we have a game board of $x$ amount of areas and we have a piece of two areas. So, we’re trying to prove that [pause] if we have $m$ is equal to $n$, no, ah! I can’t think. What I am trying to do? Um. [pause (5 sec)] Wow. [laughter]

R: So, tell me what you did observe?

Rick: Ah, I know. I can’t really do that. Let’s see. [pause] How do I do this? Wow. I’ve hit a roadblock, I’m so sorry.

R: Just tell me, describe to me what you’ve done on your page.

Rick: Okay, in the beginning. [pause] So, what I’ve done in the beginning is, uh, the problem states this [points to question] I basically drew a picture of what the problem stated.

R: Okay.

Rick: I have a game board. 8-by-8. And I have, it happens to be 64 spots, um.

R: And at some point, you even visualized putting those dominoes on, right?

Rick: Yes.

R: I mean I saw you counting down, putting 4 vertically.

Rick: Well, I had to establish, um, we had a domino. And I first figured out that it covered exactly two areas, um, spots. So, then I just, uh, okay how many dominoes will it take to cover, to cover up a whole board? And that is, uh, 48? Have I established that? I’m not even sure. Anyway, that’s what I wrote down I think. No, it’s not. [pause] Anyway, um. Oh, gosh. Anyway, so after that, I established that. Then, I went back to the question and it says that [pause] I had one of $m$ or $n$ had to be even. So, I figured okay, I would get a contradiction where I would say nether one of them, they’re both odd.
So, then I went here. And then, I figured out an equation where both of them were odd. To equal that even number. And so, um, basically I figured out, that hey, when I factor this out, I'll still get a plus 1. Therefore making it odd. So, odd essentially does not equal even. [writes odd ≠ even] Um. Therefore, it goes up that it doesn't work.

R: Okay. Well, let’s stop there and ask a few other questions about it? So, what other strategies did you use? You drew a picture.

Rick: Pictures.

R: And you were kind of thinking about even and odd and expressing them as $2k$, was that coming from your 305 class, probably? Your experience there? Is that?

Rick: Yeah, oh yeah, no doubt.

R: Cause you mentioned during the problem that this is a 305 question.

Rick: Yeah. Yeah.

R: Is that what you meant by it being a 305 question?

Rick: Yeah, this is all 305, but yes.

R: Okay. Have you seen anything like this, before this exact question at all?

Rick: Well, even and odd questions in 305, yeah, something, yeah.

R: Okay. *End of Interview
Interview #21  (Total time = 22:50)

R: Here you go, oh sorry pencil.

Jon: I got it.

R: So, a traditional chessboard consists of 64 squares, 8-by-8, suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square without overlapping any of the dominoes. [pause, student reading problem, (5 sec)] Consider a generic one, prove that the generic chessboard has a perfect cover if and only if at least one of \(m\) or \(n\) is even.

Jon: Hmm. So, let's see. So, I'm imagining [pause, draws rectangle] adjacent is, [draws domino with another at an angle] so a domino goes like that?

R: {No}

Jon: Domino goes – [redraws so two dominoes are in same direction]

R: Like that, yeah, it can't go diagonally. It actually covers the squares.

Jon: The whole square?

R: Mm, hmm.

Jon: Okay, so each domino is exactly the size of, like, the chess square? You know what I mean? [draws two squares of chessboard while he talks]

R: Right, exactly.

Jon: And, okay, so the domino is equal to 2. Okay. [writes \(m \times n\)] So, an \(m\)-by-\(n\) would be – an \(m\)-by-\(n\). Hmm. Well I can already imagine in my mind that it has to be even [draw 2x4 chessboard], to, because the dominoes – [adds column to chessboard, pause, (5 sec)] one of them is still even. [draws 3x3 chessboard, draws lines where dominoes will go, draws circle in square that cannot be covered, pause, (25 sec)] Hmm. [sigh, pause, (10 sec)] So, I guess, again, just like last time, I’m, I am wanting to put it into an equation, but I, I can't. [pause (5 sec)] Hmm. [reading problem] At least one of \(m\) or \(n\) must be even. So, that would mean, for \(n\) equals 2 times something. [writes \(n = 2^3\times\)] It's gotta be that for the dominoes – [pause, writes \(m \times (2^3\times)\), (10 sec)] Hmm. Well, I guess I'm going to try to prove the contradiction. So, try to show that it can work, when they're both odd. Hmm. [pause, writes \((2n+1)*(2n+1) = 4n^2 + 4n + 1\), pause, writes \(n = 1 \rightarrow 4 + 4 + 1 = 9\), (60 sec)]

R: Can you explain to me what you're doing there?
Jon: Um. I'm just making $m$ and $n$ both odd. [labels $2n+1$ and other $2n+1$ with $m$ and $n$]

R: Okay.

Jon: And then, and then multiplying together would be this. [underlines $4n^2+4n+1$] And then I was just trying to – when $n$ equals 1, this is going to be odd, when $n$ equals 2, [writes $8+8+1$] It’s gonna be odd. Or, 16. [crosses out first 8, writes 16, pause (20 sec)] But then the dominoes have to cover exactly 2. [draws domino over two squares in 3x4 chessboard, (5 sec)] Okay. So, to be perfect $m$ times $n$ divided by $2x$ equals the number of dominoes. [writes while talking “to be perfect $(mxn)/2x$=# of dominoes”, pause, (5 sec)] It's perfect if no remainder. [writes “no remainder”, pause, (35 sec)] I don’t know how to show that. [pause]

R: What are you considering?

Jon: Hmm. I don't know. [pause (10 sec)] Because I was thinking of doing the contradiction. Show that when they're both odd, that it can work, but it can't. But I don't – but I'm not really writing a proof. [pause (35 sec)] Hmm. [pause, crosses out $x$ from last line written, (15 sec)]

R: What other thoughts do you have?

Jon: I don't know. [pause (10 sec)]

R: Can you describe to me what you'd like to be able to say?

Jon: I want to try to say that since this, [points to $4n^2+4n+1$] since this can't be divisible by 2 that means it won't work. But I don't really know how to – then in a proof. I don't – then obviously I’m, plus 1, and 9. [pause (10 sec)] Okay. So, maybe I could do it in a thing by cases. [writes “1 $m$ is even, $n$ is even, product even, 2 $m$ is odd, $n$ is odd, product odd, $m$ is even, $n$” on the side of the paper, runs out of room, (25 sec)]

R: If you need more space, you can use that.

Jon: Sure. [switches to other sheet, back of other question, writes 3 $m$ is even, $n$ is odd, product even, 4 $m$ is odd, $n$ is even, product even, pause, (30 sec)]

R: [researcher sneezes] Excuse me.

Jon: [writes 1 $mxn = 2x$, $2mxn = 2x+1$, 3 $mxn = 2x$, 4 $mxn = 2x$, pause, writes “but only these can be evenly divisible by 2 and $x$ is the # of dominoes” to the side, sigh, (60 sec)] So, I think this is what I want to do, but I guess I don't. [pause] I don't really know how to finish it I guess. [pause]

R: If you'd like to move on, we can. It's up to you.
Jon: Do we have another question to go through?
R: Mm, hmm. It's on the other side of that one.
Jon: Yeah, we probably should then.
R: [student turns over sheet he was working on, researcher collects other papers] Let $x$ and $y$ be two integers. We say that $x$ divides $y$ if there is an integer $k$ such that $y$ equals $kx$. Consider three integers $a$, $b$, and $c$. Prove the following: If $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$.
Jon: Okay. So, if $a$ divides $b$ then $a$ equals $bk$, I guess. [writes $a = bk$, $b = ck$, pause, writes $b = a/k$, pause, writes $k = a/b$, pause, writes $b = c(a/b)$, pause, writes $b/c = a/b$, $b^2 = ca$, pause, (1 min 30 sec)]
R: What are you thinking? What are you considering?
Jon: Hmm. [pause] I guess I was just thinking in general, nothing concrete. [pause, writes $a = b^2/c$, pause, (60 sec)]
R: What other thoughts do you have on it?
Jon: Well, I just, because I wanted to make it so it looks somewhat like this, where it'd be like, $b$ squared is equal to $a$ divided by $c$. [writes $b^2 = a/c$] $a$ divides $c$? No other way around. [crosses out $c$ and $a$, writes $c/a$, pause, (10 sec)] Do it to the other side. [writes $a = b^2/c$, (5 sec)] Oh, same stuff. [pause (25 sec)]
R: What would you try next?
Jon: Hmm. [pause] I don't know. [pause (60 sec)]
R: What's running through your head?
Jon: Well, I just think I got the thing backwards. [pause, writes $b = ak$, $c = bk$, $k = b/a$, (15 sec)] Same stuff. [pause] I want -- [writes want $c = ak$, pause, (50 sec)]
R: What else are you thinking?
Jon: Well, I was just thinking of maybe putting this into that [points to last equations he has written] so it would be -- [writes $bk = ak$, $b = a$, (5 sec)] So, $b$ equals $a$ then -- since $b$ divides $c$ -- [writes “then since $b + c$, and $a = b$ then $a + c$”, (15 sec)]
R: Okay.
Jon: Maybe? I don't know. [pause]
R: Any other thoughts on it?

Jon: No, I'm uncertain, but this is probably what I would hand in, I guess.

R: Okay. Well, why don't we go back and talk about the problems and the strategies you used on them? Anything you can, well have you ever seen either of these before?

Jon: No. Well, this seemed a little more familiar [points to Question 3] than this one [points to Question 4] I guess.

R: Okay.

Jon: But, maybe just because I think I understand it a little more.

R: Okay.

Jon: So, maybe that's why it seems like I've seen it before.

R: Okay. Well, looking at this chessboard one, anything you can think of that you tried? I mean, so, you went straight to kind of drawing a picture to understand the problem, right?

Jon: Mm, hmm.

R: And then what?

Jon: Well, then I was trying to think of – that if both of them are odd, see what that would equal up to and try to show that it can't be divisible by 2. So, then I wasn't necessarily sure how to do this, so then I wanted to show it by cases. And then, by cases I see that only three ways can it be divisible by 2 [points to his second sheet of cases], and the only one it can't is when they're both odd.

R: Okay.

Jon: Which is –

R: What it says?

Jon: Mm, hmm.

R: Okay.

Jon: So, I don't know if that's necessarily right, but –

R: What about any experiences you've had that led you to helping in this, in that proof?
Jon: Well, [pause (5 sec)] I guess that like showing cases, or cases, or solving something by cases, it's almost like drawing the picture –

R: Okay.

Jon: – because you get to see the pattern, and see how the pattern works. So, I guess that's why I draw the pictures and that's why by cases. [pause] Not to say that either of them work necessarily very well for me, but I'm able to visualize a little better.

R: Okay.

Jon: So, I guess if that, I mean if that's what the answer is then it's the, that is the reason that I got it because I went in by cases so that I could see what it looks like.

R: Okay, how about on the second one? So, this time you got to do some equation manipulating, right?

Jon: Mmm.

R: And that makes you happy?

Jon: I think so. [laughter] Well, at least I can start writing stuff instead of just brainstorming.

R: Okay.

Jon: So, that's why I just prefer to try to just get right into it. But then I don't, [pause] maybe the way that I'm doing it. I finally came down to, it's like an if then. So, then I was just saying that this is already proven, or we already know it, [circles \( b = ak \) and \( c = bk \) at bottom of page] so that's why I was thinking we might as well insert the \( bk \) for \( c \) and find that \( b \) equals \( a \), and if that's the case, then \( a \) does divide into \( c \).

R: Okay. So, it helped when you considered what it was you wanted to show, it gave you a little more direction?

Jon: Mm, hmm.

R: Okay, Alright. Well, that's enough for today.

*End of interview.
Interview #22    (Total time = 47:50)

R: Do you need a pencil?
Shaun: No, I've got stuff.

R: [gives student Question 2, pause while researcher gathers up extra papers on table]
So, we call a positive integer \( N \) a 4-flip if 4 times \( N \) has the same digits as \( N \) but in reverse order. Prove there are no two-digit 4-flips.

Shaun: [pause (5 sec)] Gotta take the hat off for this. [laughter] Alright. Got to have the hand on the head.

R: Does that help to hold the thoughts in?
Shaun: [laughter] Yeah. Don't go out, don't go out. Stay in. [pause] So, we have a positive integer \( N \) – sometimes I do this, sometimes I don't, just to remind myself, you know. [writes out \( N = \{1,2,3,4,\ldots\} \)]

R: What the positive integers are?
Shaun: Just so I can see a visual.
R: Okay.
Shaun: So, I'll just do that. [pause (5 sec)] So, it's a 4-flip if any, 4 times \( N \) has the same digits as \( N \) but in reverse order.

R: Here, I'm just going to take the paper out, for the next questions. [researcher takes other questions from under student sheet, and gives him blank paper to use] If you need it, there's plenty of paper there.
Shaun: Okay. I like a little padding. [puts all the extra paper under sheet he is working on]
R: Okay. [laughter]
Shaun: Okay. Um. [pause (15 sec)] Well, so, umm. Oh, what am I saying positive integers? Oh yeah, it is the positive integers. Well, we can – okay yeah, yeah, alright. I skipped down – I just went naturally – I just did the natural numbers. I didn't really even think about that.
R: Oh, okay.
Shaun: I probably just thought it said integers, or the natural numbers. So, we're not including 0.
R: Yeah.

Shaun: Well, I guess I don't know what the flip has to do with anything. Okay, so,
[pause] So, I don't know. I just kind of have to rewrite what, what's been put down –

R: Okay.

Shaun: Just to kind of get it in my head. [writes “A 4-flip integer” crosses out integer,
(25 sec)] Maybe just call it a number, just to generalize it in my head.

R: Okay.

Shaun: Okay, so a 4-flip number. [writes “number”, then writes “is a number that when
multiplied by 4 has the same amount of digits as $N$ but in reverse order.”, pause, writes in
$(N)$, and $(4N)$ in definition, (1 min 20 sec)] Okay. So, I guess. The plan – [writes “Plan:
Do some examples”] is I'm gonna do some examples. [pause (10 sec)] Maybe with
certain– maybe with uh – well, I mean, for in the problem it says no two-digit flips, or no,
two-digit 4-flips. Okay. Bear with me. Well, so if there's no two – I'm assuming that's
the only kind – that there aren't any of – well that's like the lowest because you can't
have, you know, 1 – it doesn't work.

R: Okay.

Shaun: 1, reverse order – no it's just that, that's what it is. There's no digit to – well wait,
it is a digit, singular, singular digit there, uh. Okay. I'll just do an example maybe of, uh,
what they're talking about here. [writes “example:”]

R: Okay.

Shaun: I kind of, you know, I kind of have it in my head. But, maybe just to see – so, I
guess we'll let $N$ equal, uh, how about uh, 13? [writes “Let $N = 13”] Just so that it doesn't
have a 2 in it, and I get confused.

R: Okay.

Shaun: So, let $N$ equal 13. And, so we have 4 times, obviously – So, $4N$ equals – [writes
$4N=$]

R: Calculator if you want it.

Shaun: Thanks.

R: Yeah.

Shaun: 26. 54, 52? Which is it? It's 52, yeah. Thanks, though. [writes 52]
R: Yeah, it'll be there.

Shaun: It'll be there later – I may need it later. Okay, she's warning me. So, $4N$ is 52. So, alright, we call a positive integer [whispering while reading the question, pause, (45 sec)]

R: I see you looking perplexed at the statement. Is there any clarifying questions I can answer?

Shaun: Umm. I don't know. I don't think so. I'm just kind of [pause]. I think I kind of got off track in my head a little bit.

R: Okay.

Shaun: So, okay, this – so let $N$ equal to 13, okay. So, uh, here $N$ has two digits. [writes “Here $N$ has 2 digits” under $N = 13$] What was I saying? Oh geez. I was trying to say – I was going to do – I think I even said that didn't I? I was going to do an example with three –

R: Okay.

Shaun: But I didn't do that, I did it with two. Well, let's just stick with that. Here $N$ has two digits. Okay. $4N$ has two digits. [writes “Here $4N$ has two digits”, pause, (20 sec)] So, I have a question. [writes “Question: Is 13 a 4 flip number?”, (5 sec)] Is 13 a 4-flip number? Alright. Well, it's a 4-flip if $4N$ has the same number of digits. So, we have the first part of this answer is, uh, [writes “i) $4N$ has equal number of digits as $N$”, (25 sec)] Check, yeah.

R: Okay.

Shaun: That works out nicely. Can't really prove anything, but working through it I guess. Uh. Second part we need to look at here, uh, [pause] in reverse order. So, [pause (20 sec)] Well, okay so, [writes “ii) “$4N$ is $N$” but in reverse”, (15 sec)] Okay, so – Okay, so, $4N$ is $N$ but in reverse. Okay. No. [writes “No, 13 $\neq 25$”] 13 is not equal to 25. [pause] Okay. Okay. I think I am going to need to do an example of one that does work.

R: Okay.

Shaun: Like I was going to originally. Okay, so we'll call this, uh, example for the proof I guess, that makes it true. [writes “for the proof to be true” next to current example, then on new sheet writes “example for the def to be true” while he talks] Example for the, I guess the definition. Yeah, I guess we can call that a definition. [finishes writing last statement, labels the statement of the question with “Def:” for definition, (10 sec)] Alright, so, I'll try and go faster.
R: You're totally fine. Remember, it's the process, not necessarily the results that I'm looking at.

Shaun: Okay. So, we'll let $N$ this time equal 1 2 3. [writes “Let $N = 123$$] So, $4N$, yeah, I don’t know, I’m gonna – [writes $4N =$, then picks up calculator]

R: That's okay. No shame in that, not a problem.

Shaun: No shame, okay.

R: The point is not to check your arithmetic. [laughter]

Shaun: I could figure it out by hand –

R: I'm sure you could, I have faith that you could.

Shaun: I still retain those skills, because one day there might not be calculators or batteries to run them.

R: There you go. [laughter]

Shaun: Not enough lead in the world. Do you know most of the lead in the world is caught up in car batteries? [writes 492 for $4N$]

R: I did not know that.

Shaun: Yep. Lead batteries. Okay. That's not what we're talking about. [laughter] So, $N$ and $4N$ – now let's reverse this. [points to $4N$] Well, let's reverse this first. [points to $N$] So, equal number – we'll do, we'll ask the same question. We'll just, same as previous. [writes “Question: Same as previous”] Okay. Um. [pause (10 sec)] Maybe I will actually write that. These are different numbers after all. [crosses out “same as previous”, writes “Is 123 a 4 flip number?”] Is 1 2 3 a 4-flip? Alright. So, $4N$ does have the same number of digits as $N$. [writes “(i) $4N$ has same amount of digits as $N$”, (5 sec)]

Same amount. I'm going to change this [crosses out number in first page and changes to amount], because I changed my wording of the definition and that needs to be kept consistent.

R: Okay.

Shaun: So, we want the same amount of digits as $N$. [finishes writing previous statement] Check, got that. [writes check mark] Now, so, check to see that $4N$ is $N$ but in reverse. [writes “(ii) $4N$ is $N$” in reverse, (5 sec)] Okay. Well. Uh. So, here's $N$ [writes $N=123$]. Here is [writes $4N =$ ] 293? 294. [writes 294] So, to find $N$ divide by 4. [writes $N = 123$, laughter] Okay. Alright, so we flip it. I wonder if I did that right. [pause (5 sec)] Huh, yeah. [pause (10 sec)] Okay. What was I doing? This is $4N$, [points to 492 above] so 4, oh yeah, okay. 492. Right? Right. Okay. Okay. [laughter] Okay, so – I'm assuming
that that's going to go ahead and equal. [writes = 492/4] Well, let's just do it, I'm not assuming anything. I can't even think today. 492 divided by 4 – alrighty that's good. [writes = 123]

R: Can you tell me what your thought process is here?
Shaun: Well, you mean like making the questions up?
R: Mm, hmm.
Shaun: Well, I originally meant to do this one first. [points to second example]
R: Okay.
Shaun: To see that the definition holds.
R: Okay.
Shaun: After all, it is a definition so it should hold. But, I needed – I needed to see it.
R: For something, okay.
Shaun: Yeah. And uh – but I did it the other way around. But, so now I'm – I've seen here first off, okay, it doesn't work here.
R: Okay.
Shaun: Here, I've seen that it does work. Uh. Because in this case, the – it works with three and I'm assuming every, every digit order greater than two.
R: Okay.
Shaun: I think it's gonna – it'll probably work.
R: Okay.
Shaun: I don't know that for sure, but maybe. Um. But, it works because – I have no clue. But, it divides. [laughter] \(4N\), I mean, yeah. This is pretty interesting. I've never sat down and ever looked at this. Yeah, okay – so my thought process. So, I made up the, kind of the two quantifier parts of the definition. Two things it must satisfy. They both satisfied the first. That's pretty hard not to satisfy.
R: Okay.
Shaun: Um. Cause multiplying, you know, any, you know – well, [pause (5 sec)] Uh. That's a weird question. So, if you have a number, let's say it's four-digits, and you
multiply it by a one digit, is it still going to be a four-digit always?

R: How would you investigate that question?

Shaun: Try some?

R: Okay.

Shaun: Okay, do you want me to write them down? The ones that I try? [picks up calculator]

R: No, it's fine.

Shaun: Okay.

R: I've got two tapes to get ya, so –

Shaun: Okay, let's try it. Um. Times –

R: If you just tell me what you're trying, that's good enough.

Shaun: [pause, uses calculator] Okay, that's obviously – [laughter] I don't know why I thought of that.

R: What'd you try?

Shaun: I tried 3625 times something. Yeah, I don't know why I thought of that.

R: Okay.

Shaun: Um, okay. Well, I have an example here. I don't know, I was just kind of – that would seem like kind of a crazy answer though. Or a crazy question to kind of think of. Any number times – any, for digit wise, you know – well it's going to depend on what you're multiplying by – what single-digit you're multiplying it by.

R: Okay.

Shaun: If you times, you know, 1,000,000 by 2, you're just going to get – you're going to have the same number of digits there.

R: Okay.

Shaun: But if you – there's a number to where – a certain number that it's going to spike over that digit boundary or whatever.

R: Okay.
Shaun: Uh.

R: So, can we go back to the question of proving that there are no two-digit 4-flips?

Shaun: Yeah.

R: Can we head back there?

Shaun: Yeah, sorry.

R: How might – no you're totally okay. This is your process. How might you go about proving that there are no two-digit ones?

Shaun: Hmm. Well, [pause (20 sec)] Well, let's see, uh. Maybe try and uh, make an if-then statement out of this definition –

R: Okay.

Shaun: – and then possibly use some logic uh, to guide me to see how to do it. You know, because, I mean, I could – I can sit here all day and make up examples to show that this is true. You know, but that doesn't really get us anywhere.

R: Okay.

Shaun: Because I can't – I can't write infinitely many answers down. That doesn't prove anything, because you can always do one more. Um. Okay, so let's do that then. Let's try to make this into a – so, new plan. I really didn't have a plan before, did I? New plan. Make an – [on new sheet writes “New Plan: Make an if then statement out of the def.”, (20 sec)] Okay. [pause (10 sec)] So, if this, then this. Okay. So, there are two things it has to have. So, that's going to be the first part, and – [pause (5 sec)] Okay, so. Uh. It's kind of – it's kind of actually written there, but in reverse kind of.

R: Okay.

Shaun: I mean here seems to be – what is it called? Antecedent [sic]?

R: Antecedent?

Shaun: Antecedent. This is the conclusion.

R: Or hypothesis, it's sometimes called, too.

Shaun: Yeah, that's the second part here. So, then – okay, then N is a – [writes “then N is a 4 flip integer”, (15 sec)] Alright, so, um. Oh, I'll just write that down. If 4N has – [writes “If 4N has same digits as N and is in reverse order”, whispering while he writes,
checks over statement, (55 sec)] It's funny. A problem that uses the word reverse, kind of gets me reversed in my head. [laughter]

R: Really?

Shaun: Yeah.

R: That's interesting.

Shaun: Yeah, I don't know, I'm kind of reversed around. Okay, so now I have maybe something to work with a little better. Um. Well, that's all nice and dandy, but not really – not too much – not too many symbols there again. [laughter] Not a whole lot of notation, but a lot more words. Okay, so, so we have – [pause] um, these and this – [points to the hypothesis] Okay. [writes curly bracket over top of hypothesis and writes P and over conclusion and writes Q] Well, I learned how to do that in class, too. Before, they always looked like that; they were always like clouds. [writes several versions of curly brackets at top of page] That's what [another student] told me.

R: That's what who told you?

Shaun: [Another student] She taught me how to do them.

R: She told you your brackets looked like clouds?

Shaun: Yeah. She told me – [laughter]

R: I remember that from 221, that your brackets were clouds.

Shaun: Oh, do you?

R: Mm, hmm.

Shaun: Yeah, cause you graded.

R: Mm, hmm.

Shaun: Yeah, I know – man, I'd be like – [writes more curly brackets at top]

R: So, I had to tell what the difference was between a bracket and the starting of something you were saying. Whether that was an E, or if it was a bracket? [laughter]

Shaun: Now look at me go!

R: There you go. Those are very nice.

Shaun: I did that a lot in the PDE's too.
R: Very pretty.
Shaun: Yeah, cause we use a lot of the – well I don't know – Um, okay. So, we need to prove an if then, so maybe we'll try and give uh – so assume. [writes “*assume”] Yeah, that was real coherent. Sorry.
R: What?
Shaun: I was like – and then I just went in my head. [laughter] So, assume $P$ and not $Q$. [writes $P \land \neg Q$] So, what would that give us? Uh. Don’t know. [pause] Well, sort of if $4N$ has the same digits as $N$ and is in reversed order then $N$ is – oh wait, let me see here. Oh, no I can't do that. Oopsies, this is wrong. [writes “$\neg Q$: $N$ is not a 4 flip integer”, (10 sec)]
R: What are you thinking? You're shaking your head.
Shaun: Uh. I'm getting confused again. Kind of like last time. Well, it's not the same problem, but – So, I'm trying to work with the if then part of it, and so I – well so this is – Assuming this [points to $P \land \neg Q$], you want – you want this to lead to a contradiction. [draws arrow and writes “this to lead to a contradiction”, pause, (35 sec)] These things – these things usually tend to get me, until I see it, you know, written.
R: What do you mean, these things?
Shaun: This problem in general, uh, if there's a contradiction or something like that. That’s, there again, not so much seen from the point of view of – I mean I'm calling this a definition, but, you know, definitions that I've worked with quite a bit before, uh. I, you know, in our class where we talked about like the first time, in one of the uh boxed problems, there was an answer to a question in there that said, is ah, is $\mathbb{Z}$ countable, I think. No, is $\mathbb{N}$ countable, I don't remember.
R: Okay.
Shaun: Anyways, what they did is they took it and they said, okay, well, you know, we're taking – Maybe that’s not a good example. I guess it's kind of like twisting the words around in a different manner to make logic out of that. You know, kind of –
R: Okay. Do you mean the process of taking the words and putting them into the logical statements? Is that what you mean?
Shaun: Yeah. That, but – it's the – I think maybe I get lost and then I take the definition that I have and I try to twist it around to get the contradiction I'm looking for.
Shaun: I think I – I think I get – my mind gets so kind of wrapped up in that twisting of
the definition that I don't know anymore what the heck is going on. You know?
R: Okay. You lose yourself somewhere in the notation?
Shaun: Yeah, yeah.
R: Okay.
Shaun: The thing is because it's, uh – Well, I think – I don't know because I haven't done
that many proofs in my life, so far yet. But, the majority – the majority of times that I
experience that is with problems more like this. You know?
R: You mean more words?
Shaun: Yeah, more words, you know. I don't keep track of the words as much, you
know. Or, as easily, or maybe not even easily at all.
R: Well, I would love to let you look at one different problem before we end today.
Would you be willing to put this one down?
Shaun: Okay, yeah.
R: And pick up a different one?
Shaun: Yeah, sure. Hope that you got some kind of, something out of that.
R: Oh yeah. [researcher gathers papers, gives student Question 1] This one. This is
finally to problem number one, because I started you in kind of backwards order here.
[laughter] The numbers 1 through 10 can be arranged along the vertices and sides of a
pentagon so that the sum of the three numbers along each side is the same. The diagram
below shows an example, shows an arrangement with sum 16, for an example. Prove that
the smallest possible value for the sum is 14.
Shaun: Huh. This is neat. These are all neat little problems. [pause] See, I don't know
– this is really cool like that people think of doing stuff like this.
R: Mm, hmm.
Shaun: Okay. So. The numbers 1 through 10 can be arranged along the vertices and
sides of a pentagon. [reading question] Okay, so, so – [pause] A pentagon – we’ll start
with the shape, what we know about it maybe.
R: Okay.
Shaun: Uh, so, it has five sides. [writes “Pentagon: 5 sides”] I did a quick little check
there to make sure. [laughter] Five sides and um, five vertices. [writes “and 5 vertices”, (10 sec)] Okay. The sum of the three numbers along each side is the same. Okay. The diagram below shows an arrangement with sum 16. [reading the question] Okay. [writes curly brackets by each side of the pentagon and writes 16 next to each, (10 sec)] New clouds. So, this is 16, 16 – alright. Prove that the smallest possible value of the sum is 14. Alright, let's just go ahead and do that.

R: Okay.

Shaun: I'll try and figure that out.

R: Try to figure out one with 14?

Shaun: Yeah, I'm trying to figure it out. This might require more than one try. [draws pentagon]

R: Lots of paper, not a problem.

Shaun: Yeah. Okay, so we want um – [pause, draws line across paper, (20 sec)] 14 – [pause, writes differences between numbers around example pentagon, (35 sec)]

R: Can you describe to me what you're doing?

Shaun: I took, well I – okay. Well, I don’t know. Well, I can – I can start just throwing down numbers.

R: Okay.

Shaun: But, that's not a really great way to get something done. So, I tried to say, okay, well, I have an example given to me. Somebody's already worked that out. Let me see if I can try and find maybe a pattern between the numbers on here.

R: Okay.

Shaun: So, I took the differences –

R: Okay.

Shaun: – between the numbers next to each other. Um, and – [pause] at first, like when I did – I think I started here, I think that's where I started. Maybe I'll just put start here [writes start and arrow on top vertex of pentagon] and I went this way. I took the difference of these two, I saw it was 7, and I said like, oh that's great, because 7's like – 7's a vertex [sic] over here. Over here I took the difference, it was 1, and I was like, well jeez great again, it's the other vertex [sic] on the other side of this number.

R: Okay, I see what you're saying.
Shaun: Um. And then, I got to here, and I was like, well it's not the number over here and it can't be because it was up there, and you can't use them twice. Well, can you?
R: No.
Shaun: No, no, you can't use them twice. So, that wasn't too great. Um, then yeah, I – so this difference here is not really – [points to difference between 7 and 6] um, anything special. But, now I notice that these two – it's nice on the – on the two sides of the vertex [sic] you have same numbers so far, that may be something. And then I got to there and that wasn't true. Um.
R: You're searching for patterns that are not appearing, right?
Shaun: Not appearing, that's right.
R: What else might you try?
Shaun: [pause (5 sec)] Well – [pause (15 sec)] You know, I don't know, this – I don't know. I guess a part of the problem is, I mean – you can just answer this for me – I mean, is this – can you do this for – like, what's the biggest number you could do this for?
R: Mmm, I could probably sit and figure it out, but [pause] I don't know right off the top of my head.
Shaun: Okay. But –
R: But, it could be figured out, yeah.
Shaun: Okay. So, there's going to be a definite – okay this is –
R: Yeah. [pause] Yeah, I could have equally asked the question, to prove whatever number that is, is the largest instead of asking for the smallest there is such a bound, yeah.
Shaun: Is that kind of like a combinatorics thing? Or, I don’t know –
R: Um, it can be approached in several ways, that's why I like the problems. What else would you like to try?
Shaun: Well, I don't really have anything else but to try to write some stuff down.
R: Okay. Can you give me any ideas you've got? Try some stuff.
Shaun: Yeah, since I didn't notice a pattern really. I wish there would've been one.
[laughter]
R: So, what are you thinking is the next option?

Shaun: Just start –

R: Okay.

Shaun: – putting some numbers down.

R: Go for it.

Shaun: So, 1 through 10, we want to get – so here at least we have – maybe there is something. Maybe, I don't know. So, here we have – probably not – so you have the biggest number – so here along the same side you have both the biggest and largest, or smallest and largest number – [points to 10, 2, 4 side]

R: Okay.

Shaun: – with the middle, right – Or the middle number right in the middle of the side.

R: Okay.

Shaun: Um. [pause (10 sec)] Here they took – I don't – I guess I'm still trying to – I'm still kind of interested in what the pattern is for this one. Is that okay to talk about?

R: You can do whatever you want, not a problem.

Shaun: Okay, so here, so here they take – they take the biggest number and the largest number and they put it on the side. And then they put the middle, the number that's in between, uh, in one of 'em, or in the middle of that side.

R: Okay.

Shaun: And then maybe you think, well, maybe they'll take the next biggest and the next smallest, so maybe like 2 and 9 put that on a side and the middle of that.

R: Okay.

Shaun: But they don't. [pause] They take – they instead take the second largest number, put that in the middle, in the middle of a side. They put, they put the next number down in the middle of a side. They put that next number down in a corner, then the next one down in the middle, then the next one down in the middle. Huh. The next one down is in the corner. Next one down is – [pause] well that kind of breaks it, doesn't it? I was kind of thinking, well maybe it's kind of like middle, middle, corner, –

R: Okay.
Shaun: – middle, middle, corner. But, they don't do that. Do they? Let me check that real quick. So, they have middle, middle, middle, corner, middle, middle, corner, yeah they don't do that. [pause (35 sec)] So, our number here is 2 less than the number they have. [pause (15 sec)] Maybe – that's probably silly. Okay, here's my silly idea. Let's take the numbers below – from the example up here, take the numbers that are less than 5 and add 2 more. Take the numbers that are 5 and greater and subtract.

R: Okay.

Shaun: I don't know what that will do.

R: And still trying to use the 16, leaving them in same places?

Shaun: Yeah, leaving them in the same positions. I don't know. That's probably not going to do anything, but that's all I have. I guess I don't feel comfortable just writing –

R: Well, let me stop you here, just in the interest of time.

Shaun: Okay.

R: Okay? And ask you to reflect on your strategies when you approached this problem.

Shaun: Okay, um. This problem was kind of, the strategy was trying to notice a pattern. I'd need a lot more time to play with it.

R: Okay.

Shaun: I think it's a really neat problem. Uh.

R: Have you seen anything like either of the problems that you worked on today before?

Shaun: Unh, uh.

R: Or any, um, experience you've ever had that led you to what you tried and what you used?

Shaun: Uh. Maybe the only thing that was, kind of the experience from that last one – just the fact that it was some number times a defined number, 4, like division –

R: Oh, okay.

Shaun: –just from, from class.

R: I saw some of what [your teacher] has done, you know considering an example or a non-example.
Shaun: Oh yeah.

R: Formulating a plan, those kind of things I saw coming out.

Shaun: Yeah.

R: And those are from 305, aren’t they?

Shaun: Yeah, but previous to that, no. With this – [points to Question 1] previous experience would probably be elementary school. When I was playing with these blocks, I mean – just trying to, you know, fit sides together. And, yeah, this problem, I mean – I, I don't have enough, I guess – Well, it seems like both of these are kind of – maybe not so much that one [points to Question 2], this one definitely, [points to Question 1] I don't know, just more counting kind of issues. I don't really – I've always thought they were pretty interesting, but I've never, never had any experience with them really.

R: Okay, like number theory –?

Shaun: Yeah.


Shaun: Yeah.

R: Those kind of things are what you mean?

Shaun: Yeah, I've never really had any experience with any of that really.

R: Okay.

Shaun: Well, I mean, kind of, not with, not with uh, with the combinatorics really. [audiotape ends, videotape continues] I guess besides the binomial theorem, but even that, not even that much.

R: Okay.

Shaun: Just pretty much what I had from proving, using mathematical induction. Like I’ve never used it to – well, I’ve – I guess that’s not true, I did in pre-calculus.

R: Oh, yeah, a little bit.

Shaun: Yeah, I used it, but – um.

R: Okay. *End of interview.
Interview #23  (Total time = 16:15)

**This interview was transcribed from the videotape, the audio tape recorder malfunctioned and never recorded the session.**

R: A traditional chessboard consists of 64 squares, 8-by-8. Suppose dominoes are constructed so that each domino covers exactly two squares of the chessboard. [pause] A perfect cover of the chessboard with dominoes is covering every square, without overlapping any of the dominoes. Consider a generic chessboard. That’s just \( m \) by \( n \). And the proof is to prove that the chessboard of size \( m \)-by-\( n \) has a perfect cover if and only if at least one of \( m \) or \( n \) is even.

Lisa: [pause, draws chessboard size 8x8, draws in dominoes to cover, pause (1min 20 sec)]

R: Can you describe a bit of what you were seeing, and what you were trying?

Lisa: So, I looked at what an 8-by-8 chessboard would look like. And then I thought about the dimensions of a domino. And a domino is 1-by-2.

R: Okay.

Lisa: And so, I can see where the proof would be correct, because if one was even, then it’d be divisible by 2, so say \( m \) is even. It would equal \( 2k \) [writes \( m = 2k \)], which is some number, which would work, because the dominoes are 2 long and only 1 wide. So, \( n \) could be whatever number, and that would just equal the number of dominoes. [writes \( n = \# \) dominoes, pause] That are side-by-side.

R: Okay.

Lisa: [pause (10 sec)] I don’t know how to go about proving it. [pause, writes “\( mxn \) perfect cover \( \Leftrightarrow m=2k \) or \( n=2k \)”, pause, writes “dominoes = 1x2”, pause, draws domino labels sides with 1 and 2, then labels individual boxes with 1, pause, writes “\( nxm=1n \times 2k \)” (2 min 30 sec)

R: Can you tell me what you’re thinking and considering?

Lisa: Well, I’m thinking about the ratios of 1 to 2 on both. So, if \( m \) was even, then it would be \( 1n \) for every \( 2k \)’s. And then for dominoes, they’re 1-by-2 so it would be 1 for every 2.

R: Okay.

Lisa: [pause (40 sec)] That clock is really loud.

R: Yes. [laughter, pause (10 sec)] What else are you considering for the proof?
Lisa: [pause] I’m not really sure how to go about proving it. [pause, draws 3x5 chessboard, fills in dominoes, notes a leftover square by shading it in (55 sec)] If both were odd, it wouldn’t work.

R: Okay.

Lisa: [pause (30 sec)]

R: Anything else that you can think of?

Lisa: Not really.

R: That you’d like to try?

Lisa: [pause] Um.

R: Can you keep going with this idea that both are odd and it doesn’t work, why? Can you explain why that is true?

Lisa: Because there’s always one left over, and in order for it to work, two would have to be left over.

R: Okay.

Lisa: Which wouldn’t be the case anyway with odd.

R: And that’s true in general, right? Not just for your 3-by-5 example?

Lisa: Right. [pause, writes \((2k+1)(2l+1)\) and \(4kl+2k+2l+1\), then writes \((2k)(2k+1)\) and \(4k^2+2k\) and \(2k(k+1)\), pause (1 min 10 sec)]

R: And so, what is it that you’re noticing there?

Lisa: That there’s, um, [pause] the product would be divisible by 2.

R: In the case where at least one of them’s even?

Lisa: Mm, hmm. And that’d be the same if they were both even. They would both be divisible by 2.

R: Okay.

Lisa: [pause] And since the domino fills up two squares, that’s the reason why it works.

R: Okay. And that doesn’t work in the odd-by-odd, you already said.
Lisa: Right. Because there’s a remainder of one and that would require half a domino, which isn’t possible.

R: Okay. So, what would you want to do to complete the proof, in terms of turning it in and things like that?

Lisa: I think I would just write it all out, what I found and why it works in the even-by-even case and the even-by-odd case, but not in the odd-by-odd case.

R: Okay. Have you ever seen anything like this problem before? Or does it remind you of anything?

Lisa: Not necessarily of this sort, but of the proving that the number is divisible by 24.

R: Okay. The 305 proof?

Lisa: Yeah.

R: So, thinking about the cases of evens and odds and what to consider?

Lisa: Yeah.

R: Okay. Um, any strategies you used? So, the first thing you did was look at the 8-by-8 chessboard and drew that in?

Lisa: Mm, hmm. To see what the pattern was.

R: And then what?

Lisa: And then I thought of, um, [pause] why it worked that way.

R: Okay.

Lisa: And saw that, um, since a domino was 1-by-2 it would have to be, one side would have to be even, at least.

R: Okay. And you kind of considered what you would need for the proof then, and looked at another example of why the odd didn’t work?

Lisa: Yeah.

R: And you kind of went through that process?

Lisa: Yeah.
R: And again, looking back through that process, basically, because you were thinking about the cases of evens and odds?

Lisa: Right.

R: – and other experiences? Okay. Let’s try one more. [researcher collects papers, gives student question 4] Last question. Let x and y be two integers. We say that x divides y if there is an integer k such that y=kx. Consider three integers a, b, and c. Prove that if a divides b and b divides c, then a divides c.

Lisa: [pause, writes “y/x → y=kx” and “b/a → b=qa” and “c/b → c=lb”, pause, then writes “b=qa, c=lb → c=lqa → c/a = lq” (1 min 30 sec)]

R: Okay, can you describe to me what you have?

Lisa: Well, I looked at the example. [points to y/x] And I used that to show that if a can divide b, then b equals something times a. And if b can divide c, then c will equal something times b. And then, with the two equations, I can plug in b, so I get c equals something times a, making c divisible by a.

R: Okay. [pause] So, that was pretty quick, right?

Lisa: Yeah.

R: Um, have you ever seen anything like it before?

Lisa: Probably.

R: Does it remind you of anything you did in 305 or anything like that?

Lisa: [pause] Not really.

R: Is there any, um, clues that gave you what to do? So, you picked apart that definition pretty well, and that led you straight to doing the proof pretty quickly, right?

Lisa: Mm, hmm.

R: And I noticed you were real careful choosing what to multiply by a, like that q and that l. You stopped and thought about, okay let’s make that a different letter, right?

Lisa: Right, because then it just causes confusion if they were the same thing.

R: Is that something that you picked up in 305, was being careful about that?

Lisa: Mm. Yes and no. I’ve always been really careful about stuff like that.
R: Oh, okay.

Lisa: Because, if these are the same thing, then that’s the same thing as saying that $a$ would be equal to $c$, I think. And that’s not necessarily the case.

R: Okay. [pause] And so, just being careful was important, right? Okay. Great.

Perfect. All done. *End of interview.*
Interview #24     (Total time = 16:29)

R: We call a positive integer N a 4-flip if 4 times N has the same digits as N but in reverse order. Prove that there are no two-digit 4-flips.

Lily: [pause (1 min)]

R: So, what are you thinking about?

Lily: I'm kind of right now, just trying to, you know, just read the definition. I guess.

[laughter]

R: To digest it?

Lily: Yeah.

R: Okay.

Lily: Because it's just kind of confusing to me.

R: Is there any questions I can answer for you?

Lily: Well, [pause] so basically it's saying that you can take a positive integer N, and just multiply it by 4, right?

R: Mm, hmm.

Lily: And then it has the same number of digits as N? But in reverse order?

R: The same exact digits but in reverse order.

Lily: Okay. [pause]

R: So, instead of N in this way, it flips it. [researcher makes motions with her hand of having N in one direction and flipping it over]

Lily: Oh, okay. [pause] So, sorry I'm just really – my mind's not even processing – [laughter] So, basically, if we just say, like, N equals 100, then we're just doing, like, 4 times 100 equals 400? [writes 4(100)=400]

R: Yep.

Lily: And that's – it basically just says that the same digits – there are three digits in 100 that's how many are in 400? Is that what the definition says?

R: No, you would expect that 4 times 100, if it was a 4-flip, that you would get 0-0-1.
Lily: Oh.

R: So, you would get those exact digits, but in reverse order.

Lily: Oh, okay. [pause (25 sec)] Basically now, since I guess I understand the definition more. I'm just trying to think of, like, two-digit ones, even though –

R: Okay. Would you write down what you're thinking? What ones you're considering?

Lily: Yeah, so – [pause]

R: And I have a calculator if you'd like it.

Lily: Oh, okay. So, if we just take 4 times 10, if that was a 4-flip, then that would be 0-1, correct? [writes 4(10) → 01]

R: Yes, mm, hmm.

Lily: Okay. So, that's not true.

R: Obviously, right? That one's not true. [laughter, pause]

Lily: 4 times 25 that'd have to be 52 [writes 4(25)→52], but we obviously know that's not true. And it's not the same number of digits, anyways. Is that what it means, too?

R: Yeah.

Lily: It has to have the same number of digits, so that wouldn't even –

R: Right. Exactly.

Lily: [pause (15 sec)] So, basically too I know that, um, if it's going to be over – if it's a positive integer 25 or greater, then it automatically can't be a 4-flip because it's going to have more, um, more digits than –

R: Okay. So, 25 and up are out, for the two-digits?

Lily: Yeah. Do you want me to write that down?

R: Sure.

Lily: [pause, writes “N cannot be ≥ 25 because it would result in an integer w/ 3 digits” (35 sec)] So, basically then I know that all that's left is like 10 through 24. [writes “10-24”]
R: Okay. And how would you like to proceed from there?

Lily: Um. [pause (10 sec)] I guess – I mean – I'm trying to think of an easier way, like just to prove it in general.

R: Oh, okay.

Lily: But I mean – I guess if you wanted to, you could just take the digits 10 through 24 and list them and show that –

R: Okay, if you'd like to do that, that's perfectly okay. [student starts to write out numbers (8 sec)] Not too many to check.

Lily: [writes 10 through 24 in column (10 sec)] So, we'd expect this to be 0-1, but that doesn't equal 40 obviously [writes \( \rightarrow 01 \neq 40 \) next to 10], which is 10 times 4. [writes remainder of list, \( \rightarrow 11 \neq 44, \rightarrow 21 \neq 48 \ldots \) down to \( \rightarrow 42 \neq 96 \) (55 sec)]

R: It's harder to videotape in this room. [laughter] But, that's okay, we'll get it.

Lily: [finishes list, labels columns \( N \) and \( 4N \) (40 sec)]

R: Alright.

Lily: So, I guess that proves that none of those are equal.

R: Okay, sounds good. Alright. [researcher puts paper to the side, gives student next question] Now, we'll challenge you a little bit and go on to three-digits.

Lily: Okay.

R: Prove or disprove the following statement: There are no three-digit 4-flips.

Lily: [pause, lots of noise in the room from a printer, writes 100 -999 (40 sec)]

R: Goodness, that's a lot. [referring to papers printing off]

Lily: Yeah. [laughter, pause (10 sec)] So, basically now I'm just kind of trying to think of – is there any integers like 25 that's going to make all those over it?

R: Okay.

Lily: [pause, (20 sec)] So, I guess our biggest is actually – since we can take 1000 and divide that by 4, [writes 1000 ÷ 4] because you know that that is going to have one more digits than all those numbers.

R: Okay.
Lily: Which would equal 500, er – [writes =500, crosses out, writes 250] I can tell I’ve been thinking too long. [laughter]
R: 250, okay.
Lily: That sounds a little more reasonable. So then, from that point I know that we're only dealing with these integers for sure. [writes 100–250]
R: Okay.
Lily: Because these ones have more digits automatically. But then, that's a lot of numbers to check. [laughter]
R: It is. So, what else could you try?
Lily: [pause (15 sec)] I'm just looking at this [points to her work on part a] to see if there's any pattern between them –
R: Okay.
Lily: – that might help. [pause] I mean, basically, kind of what I'm looking for is like the numbers that are closest to each other.
R: Oh, I see.
Lily: Because these ones are so drastically different. But, I don't know if that really makes a difference anyways. [pause (15 sec)] Maybe – can I use this calculator?
R: Definitely, that's why I went and got it.
Lily: [pause, works on calculator (10 sec)] Basically, what I'm thinking is – like, just from this example [points to part a again], which doesn't obviously have to work for all of them. But, it seems that the integers in the middle – those are, what appear to be, to have the closest.
R: Okay.
Lily: So, I'm just going to try a few just to see what happens. So, let's start with 75, er, 175? [writes 175 →571] So, we would think that would be 571. [works on calculator] Which does not equal 700. [writes ≠700, pause, writes 176→671 ≠ 704, pause, writes 177→771 ≠ 708, using calculator to find 4N throughout, pause (40 sec)]
R: So, what are you considering? What are you thinking of?
Lily: Well, I think I'm kind of just looking at the pattern. Because, this column [draws
arrow to $4N$ column] is always going to – it's always going to be plus or minus 4 from the
previous one [writes ± 4], depending on which way you go. [pause, points to $N$ flipped
column] But then this one is changing by 100s – or that place value? [pause (15 sec)]
And then I'm just kind of figuring out with this example to see – [points to part a again]
R: Okay.
Lily: Like, basically the pattern between the two. [pause, checks number in calculator,
(50 sec)] I guess I'm just kind of stuck, like, on a way of checking these without going
through the numbers 100 through 250.
R: Okay, is there anything else you can think of? I won't keep you for too much longer,
but if you have maybe one more idea that you might like to pursue.
Lily: [pause (10 sec)] I don't really know a way to go about this one without having to
check them all.
R: So, you feel kind of stuck?
Lily: Yeah.
R: Okay. So, we'll leave that there. But, let me ask if you've seen anything like this
before?
Lily: No.
R: Okay. Any –?
Lily: I think the reason I was kind of, well where I first got stuck on it, too, was I've
never seen anything like it before and the definition kind of just threw me off from the
beginning.
R: It's a weird definition. So, you started by kind of picking apart the definition and –
Lily: Yeah.
R: – thinking about it, making sure you understood it.
Lily: Like figuring out what the definition meant and then kind of then trying to figure
out what the question meant for sure, too.
R: Okay, and then you tried a few examples to try to see what would happen.
Lily: And that's kind of how I figured I've always kind of started stuff, just finding
examples just to see how it works out.
231  R: Okay.
232  
233  Lily: Whether it proves it or disproves it, just so I know in my mind how it's flowing together.
234  
235  R: Okay. And then you proceeded a little bit more with a proof – so you discounted the 25 and above. And then you just proceeded with brute force with the rest of them, right?
236  
237  Lily: Yeah.
238  
239  R: And on the other one, kind of looked back and said well, this is similar to this other problem –
240  
241  Lily: Yeah.
242  
243  R: – so I'm going to look for some patterns.
244  
245  Lily: Yeah.
246  
247  R: Look for a way that I can use what I did here to kind of work on that. Right?
248  
249  Lily: Yeah.
250  
251  R: Okay, well great, thank you very much. *End of interview.*