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SIMPLE TWO-SIDED RATIONAL VECTOR SPACES OF RANK TWO

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SIMPLE TWO-SIDED RATIONAL VECTOR SPACES OF RANK 2

by

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The purpose of this thesis is to find sufficient conditions under which a non-commutative version of the polynomial ring in two variables exists. The non-commutative rings we construct are non-commutative symmetric algebras over a two-sided vector space [VdB]. After reviewing the definition of a two-sided vector space and giving some examples, we briefly recall the theory of simple two-sided vector spaces [NP]. We then assume k is a field of characteristic zero and t is transcendental over k and we find sufficient conditions under which a simple k-central two-sided vector space V over $k(t)$ has left and right dimension two. Given such a V, and letting ${}^*V$ and V^* denote left and right duals we find conditions under which $(V^{i*}, V^{(i+1)*}, V^{(i+2)^*})$ has a simultaneous basis for all $i \in \mathbb{Z}$. This condition implies that the non-commutative symmetric algebra over V can be constructed. We conclude by exhibiting a five-dimensional family of simple k-central two-sided vector spaces over $k(t)$ of left and right dimension 2 whose non-commutative symmetric algebras exist.

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Contents

Chapter 1

Introduction

Let K be a perfect extension field of a field k.

The purpose of this thesis is to find sufficient conditions under which a non-commutative version of the polynomial ring in two variables exists. The significance of our construction of this non-commutative ring is that, in favorable circumstances, its corresponding division ring of fractions will be the division ring of fractions of a "non-commutative ruled surface" [VdB]. In addition, having a store of concrete examples of division rings of fractions of non-commutative ruled surfaces will give us more information regarding various conjectures, e.g., the relationship between "Zhang"-dimension and GK transcendence degree [A, p. 8] and Artin's conjecture on the birational classification of non-commutative surfaces [A]. For a survey of non-commutative curves and surfaces, including definitions of many of the above terms, see [SV]. Although similar questions to those we study were pursued in [P], we focus on the most complicated cases (not studied in [P]) and work in a more general context.

Recall that a commutative ruled surface over a (smooth projective) curve X (over k) is birational to $\operatorname{Proj} k(X)[x, y]$ where x and y are given degree 1, $k(X)$ is the function field of X and $k(X)[x, y]$ denotes the commutative polynomial ring in two variables with coefficients in

$k(X)$. It is reasonable, then, to expect the division ring of fractions of a non-commutative ruled surface to be the degree 0 part of the division ring of fractions of a non-commutative deformation of $k(X)[x, y]$. Van den Bergh proceeds using the following observation [VdB]: since $k(X)[x, y] = \mathrm{S}_{k(X)}(V)$, the symmetric algebra of a two-dimensional vector-space V over $k(X)$, one can obtain a non-commutative deformation of this ring by deforming V (instead of deforming the relations of the polynomial ring) and taking a *non-commutative* symmetric algebra of V. The appropriate deformation of V is a *rank 2 two-sided vector space over* $k(X)$, defined below:

Definition 1.1.1.

- A **two-sided vector space** over K is a k-central $K-K$-bimodule. It is **simple** if it is simple as a $K \otimes_k K$-module.
- A two-sided vector space over K is **rank** n if it is an n-dimensional vector space for each of the two K-actions induced by the canonical inclusions $K \otimes_k 1, 1 \otimes_k K \to K \otimes_k K$. We refer to these actions as the left and right actions of K respectively.
- Let V be a two-sided vector space over K of rank n. A **simultaneous basis** for V is a subset of V of size n which is a basis for V as a K-vector space via the left and right actions of K.

Van den Bergh also defines the concept of left and right dual of V, denoted V^{-*} and V^*, respectively [VdB] (we recall the definition in Section 3.5). These are typically not isomorphic to each other. We denote the ith iteration of taking the left dual or the right dual by V^{-i*} and V^{i*}, respectively.

The following definition is crucial to the construction of the non-commutative symmetric algebra of a two-sided vector space:

Definition 1.1.2. Suppose that V, V^{-*} and V^* are free rank n two-sided vector spaces. We say that (V^{-*}, V, V^*) **has a simultaneous basis** if there exists a simultaneous basis for V whose corresponding left dual is a simultaneous basis for V^{-*} and whose corresponding right dual is a simultaneous basis for V^*.

Theorem 1.1.3. *(Nyman [N_2]) If (V^{-*}, V, V^*) has a simultaneous basis then the functors*

$$(- \otimes_K V^{-*}, - \otimes_K V, - \otimes_K V^*)$$

from $\mathsf{Mod}K$ *to* $\mathsf{Mod}K$ *form an adjoint triple, i.e.*

$$(- \otimes_K V^{-*}, - \otimes_K V)$$

and

$$(- \otimes_K V, - \otimes_K V^*)$$

are canonically adjoint pairs.

Definition 1.1.4. Suppose V is a two-sided vector space such that

$$(- \otimes_K V^{(i-2)*}, - \otimes_K V^{(i-1)^*}, - \otimes_K V^{i*})$$

is an adjoint triple for all $i \in \mathbb{Z}$. The **non-commutative symmetric algebra**, $S_K^{n.c.}(V)$, is the $\mathbb{Z}$-algebra $\oplus_{i,j\in\mathbb{Z}} A_{ij}$ with $A_{ij} = 0$ if $i > j$, $A_{ii} = K$, $A_{i,i+1} := V^{i*}$, and, for $j > i+1$,

$$A_{ij} := V^{i*} \otimes \cdots \otimes V^{(j-1)*}/R_{ij}$$

where $\otimes$ denotes the bimodule tensor product over K, $R_{ij} \subset A_{i,i+1} \otimes \cdots \otimes A_{j-1,j}$ is the two-sided vector space

$$\sum_{k=i}^{j-2} A_{i,i+1} \otimes \cdots \otimes A_{k-1,k} \otimes Q_k \otimes A_{k+2,k+3} \otimes \cdots \otimes A_{j-1,j},$$

and Q_i is the image of the map $K \to V^{i*} \otimes V^{(i+1)*}$ induced by the unit of the adjoint pair $(- \otimes_K V^{i*}, - \otimes_K V^{(i+1)*})$. For the definition of multiplication see Definition 3.5.2.

In this thesis we find conditions on a simple two-sided vector space V of left-dimension 2 so that $S_K^{n.c.}(V)$ exists. By Theorem 1.1.3, $S_K^{n.c.}(V)$ will exist so long as $(V^{(i-2)*}, V^{(i-1)*}, V^{i*})$ has a simultaneous basis for all $i \in \mathbb{Z}$, so we attack this question.

More specifically, we study the following

Question 1. Let V be a simple two-sided vector space over K with left-dimension 2.

(a) What conditions on V ensure that V is rank 2? If it is rank 2, when does it have a simultaneous basis?

(b) Assume V is rank 2. What conditions on V ensure that V^{i*} is rank 2 for all i? What conditions on V ensure that $(V^{(i-2)*}, V^{(i-1)*}, V^{i*})$ has a simultaneous basis for all $i \in \mathbb{Z}$?

The construction of a division ring of fractions of $S_K^{n.c.}(V)$ (and the issue of when such a thing exists) is beyond the scope of this thesis.

In [P], Patrick answers special cases of parts (a) and (b) of Question 1 when V is *not* simple. In addition, Patrick makes, without proof, the following claim, pertinent to part (a) of Question 1: If V is rank n then the generic left basis for V is simultaneous [P]. This statement is more subtle than it appears and requires careful proof (assuming it is true). To prove our result Theorem 1.1.8, we work without this claim.

In case K/k is finite and V has left-dimension 2, $(V^{(i-2)*}, V^{(i-1)*}, V^{i*})$ form an adjoint triple for all $i \in \mathbb{Z}$ so that one can construct $S_K^{n.c.}(V)$ [N_2]. When K is transcendental of degree one over k and V is simple, parts (a) and (b) of Question 1 become much more subtle. Our strategy for tackling this case is to use the following Theorem 1.1.5, which relates the study of simple V to the study of certain field extensions. Before stating the theorem, we introduce some notation. We write $\mathrm{Emb}(K)$ for the set of k-linear embeddings of K into $\bar{K}$, and $G = Aut(\bar{K}/K)$. Now, G acts on $\mathrm{Emb}(K)$ by left composition. Given $\lambda \in \mathrm{Emb}(K)$, we denote the orbit of λ under this action by λ^G. We denote the set of finite orbits of $\mathrm{Emb}(K)$

under the action of G by $\Lambda(K)$.

Theorem 1.1.5. *(Nyman-Pappacena) [NP] There is a one-to-one correspondence between isomorphism classes of simple left finite-dimensional two-sided vector spaces and* $\Lambda(K)$*. Moreover, if* V *is a simple two-sided vector space corresponding to* $\lambda^G \in \Lambda(K)$*, then* $\dim_K V = |\lambda^G|$.

If M and N are subfields of L, let $M \vee N$ denote their composite in L. One can show using Theorem 1.1.5 that answering part (a) of Question 1 is equivalent to answering the following question:

Question 2. *Which k-linear embeddings $\lambda : K \to \overline{K}$ have the property that*

$$[K \vee \lambda(K) : K] = 2 = [K \vee \lambda(K) : \lambda(K)]?$$

Our main interest in this thesis is the case $k = \mathbb{C}$ and $K = \mathbb{C}(t)$ (where t is transcendental over $\mathbb{C}$), although all the proofs work, and are proven, if k is a field of characteristic zero, and t is transcendental over k. Suppose $\lambda : K \to \overline{K}$ is a k-linear embedding. In order for λ to satisfy the first equality in Question 2, we must have $\lambda(t) = \alpha + \sqrt{m}$ where $\alpha, m \in \mathbb{C}(t)$ but $\sqrt{m}$ is not in $\mathbb{C}(t)$.

We investigate conditions for when the second equality in Question 2 is satisfied. We prove the following in Theorem 3.3.1:

Theorem 1.1.6. *Suppose* $[K \vee \lambda(K) : K] = 2$*, that* $[K : \mathbb{C}(m)] = 2$ *and that* $\alpha \in \mathbb{C}(m)$*. Then* $[K \vee \lambda(K) : \lambda(K)] = 2$ *if and only if* $\alpha \in \mathbb{C}$*.*

Definition 1.1.7. Let $i \in \mathbb{Z}$. We say $(V^{(i-1)*}, V^{i*}, V^{(i+1)*})$ has a **simultaneous basis** if there is a simultaneous basis for $({}^*(V^{i*}), V^{i*}, V^{(i+1)*})$ in case $i \geq 0$ and for $(V^{(i-1)*}, V^{i*}, (V^{i*})^*)$ in case $i \leq 0$.

We then produce a five-parameter family of simple two-sided vector spaces of rank 2 over $\mathbb{C}(t)$

whose corresponding non-commutative symmetric algebras exist. Specifically, we prove the following in Theorem 3.4.13:

Theorem 1.1.8. *Let* $\lambda(t) = \alpha + \sqrt{\frac{at^2+bt+c}{dt^2+et+f}}$ *such that* $\alpha, a, b, c, d, e, f \in \mathbb{C}$, *and* $a, d \neq 0$, $ae = bd$, $af \neq cd$, $b^2 \neq 4ac$, $e^2 \neq 4df$, *then* $(V^{(i-2)*}, V^{(i-1)*}, V^{i*})$ *has a simultaneous basis for all* $i \in \mathbb{Z}$. *It follows that* $S_K^{n.c.}(V)$ *exists for such* V.

We may also ask if the classification is complete in the following sense:

Suppose V is a simple two-sided vector-space over $\mathbb{C}(t)$ which has left dimension 2. If V corresponds (via Theorem 1.1.5) to an embedding $\lambda : \mathbb{C}(t) \to \overline{\mathbb{C}(t)}$ with $\lambda(t) = \alpha + \sqrt{m}$ such that $\sqrt{m}$ is not in $\mathbb{C}(t)$, $\alpha \in \mathbb{C}$, and $S_K^{n.c.}(V)$ exists, must V be in the family described in Theorem 1.1.8?

Although we do not answer this question in this thesis, it provides direction for further study.

Chapter 2

Background on two-sided vector spaces

2.1 Two-sided vector spaces

The goal of this thesis is to construct a non-commutative analog of the symmetric algebra of a vector space. The role of the vector space will be played by a two-sided vector space. In this chapter we give examples of two-sided vector spaces and review the structure of simple two-sided vector spaces. The latter are especially relevant to us since in the next chapter we will restrict our attention to the study of simple two-sided vector spaces.

We will follow the following conventions throughout the rest of the thesis:

1. k is of characteristic 0, and thus perfect.
2. $K = k(t)$ where t is transcendental over k, thus K is also of characteristic 0 and perfect.

A two-sided vector space is an ordinary vector space with an additional scalar multiplication.

More formally:

Definition 2.1.1. A **two-sided vector space** V is a K-bimodule where the left and right actions of K on V do not necessarily agree.

If we further fix a base field $k \subset K$ then we arrive at:

Definition 2.1.2. A **k-central** two-sided vector space, V, over K is a $K \bigotimes_k K$-module.

This is the same as a k-central $K - K$ bi-module. In the sequel, we will assume that every two-sided vector space is k-central. We will often say "two-sided vector space" omitting the "k-central" part. Given a two-sided vector space and a set of vectors $\{v_i : i \in I\}$, we write span$\{v_i\}$ to stand for the left span of the v_i. Span$\{v_i\}$ is not usually a two-sided subspace of V.

Let V be any two-sided vector space with finite left dimension, say n. Pick a left basis $\{a_1, \ldots, a_n\}$ of V, then as left K-modules $V \cong K^n$. Any element $\alpha \in K$ determines an endomorphism of ${}_KV$ via right multiplication, which we denote by $\phi(\alpha)$, i.e, given $\alpha \in K$ let $\phi(\alpha)(v) := v \cdot \alpha$ for all $\alpha \in K$ and $v \in {}_KV$. As ϕ gives an endomorphism of ${}_KV$ it gives an endomorphism of K^n, thus corresponds to multiplication of row vector on the right by a matrix. One can check that the assignment $\alpha \to \phi(\alpha)$ defines a homomorphism $\phi : K \to M_n(K)$.

Definition 2.1.3. Let $\phi : K \to M_n(K)$ be a nonzero k-algebra homomorphism. Then denote by ${}_{\mathbf{1}}\boldsymbol{K}^{\boldsymbol{n}}_{\boldsymbol{\phi}}$ the following two-sided vector space: as a set it is $\boldsymbol{K^n}$, with left action the usual one, and right action is via $\boldsymbol{\phi}$; that is,

$$\boldsymbol{x \cdot (v_1, \ldots, v_n) = (xv_1, \ldots, xv_n)\,, \ \ (v_1, \ldots, v_n) \cdot x = (v_1, \ldots, v_n)\,\phi\,(x)\,.}$$

Definition 2.1.4. If V is a two-sided vector space and $\{b_j : j \in J\} \subset V$ is a basis for both the left and right actions then we say that $\{b_j : j \in J\}$ is a **simultaneous basis**.

Example 2.1.5. Consider $\mathbb{C}$ over $\mathbb{C}$ with the usual multiplication on the left and $v \cdot c = \sigma(c)\, v$ for all $v, c \in \mathbb{C}$ where σ is complex conjugation and the multiplication on the right hand side is the usual complex multiplication. Then $\{1\}$ is both a left and right basis.

Definition 2.1.6. When the left and right basis have the same cardinality we define that cardinality to be the **rank** of the vector space.

Definition 2.1.7. A two-sided vector space is **simple** if it has no proper non-trivial subspaces.

The following example shows the contrast between the structure of, usual vector spaces and two-sided vector spaces.

Example 2.1.8. Let $k = \mathbb{C}, K = \mathbb{C}(t)$, where $t \notin \mathbb{C}$. Define a map $\phi : \mathbb{C}(t) \to M_2(\mathbb{C}(t))$ via $\phi(x) = \begin{pmatrix} x & \frac{d}{dt}x \\ 0 & x \end{pmatrix}$. We claim ϕ is a homomorphism, the set $\{(1,0),(0,1)\}$ is a simultaneous basis, and the two-sided vector space ${}_1\mathbb{C}(t)^2_\phi$ is not semi-simple. To prove the claim, let $a, b \in \mathbb{C}(t)$ and $c \in \mathbb{C}$. Then

$$\phi(ca+b) = \begin{pmatrix} ca+b & \frac{d}{dt}(ca+b) \\ 0 & ca+b \end{pmatrix} = \begin{pmatrix} ca & c\frac{d}{dt}(a) \\ 0 & ca \end{pmatrix} + \begin{pmatrix} b & \frac{d}{dt}(b) \\ 0 & b \end{pmatrix} = c\phi(a) + \phi(b),$$

so ϕ is $\mathbb{C}-$linear. Moreover,

$$\phi(a)\phi(b) = \begin{pmatrix} a & \frac{d}{dt}a \\ 0 & a \end{pmatrix}\begin{pmatrix} b & \frac{d}{dt}b \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & a\frac{d}{dt}b + b\frac{d}{dt}a \\ 0 & ab \end{pmatrix} = \begin{pmatrix} ab & \frac{d}{dt}ab \\ 0 & ab \end{pmatrix} = \phi(ab),$$

so ϕ is a ring homomorphism. We now show that $\{(1,0),(0,1)\}$ is a simultaneous basis.It is clear that the set forms a left basis. For the right we first show that they are linearly independent. Suppose there exists $a, b \in \mathbb{C}(t)$ such that

$$(1,0) \cdot a + (0,1) \cdot b = (0,0).$$

Then

$$(1,0)\begin{pmatrix} a & \frac{d}{dt}a \\ 0 & a \end{pmatrix} + (0,1)\begin{pmatrix} b & \frac{d}{dt}b \\ 0 & b \end{pmatrix} = (0,0)$$

so

$$(a, \frac{d}{dt}a) + (0,b) = (0,0).$$

Thus $a = 0 = b$ and they are linearly independent. We now show that they span on the right.

Let $(x,y) \in \mathbb{C}\,(t)^2$. We want to know if there exists $a, b \in \mathbb{C}(t)$ such that

$$(1,0)\cdot a + (0,1)\cdot b = (x,y)\,,$$

i.e.,

$$(1,0)\begin{pmatrix} a & \frac{d}{dt}a \\ 0 & a \end{pmatrix} + (0,1)\begin{pmatrix} b & \frac{d}{dt}b \\ 0 & b \end{pmatrix} = (x,y)$$

or

$$(a, \frac{d}{dt}a + b) = (x,y)\,.$$

The latter is true for $a = x$ and $b = y - \frac{d}{dt}x$. We conclude that $\{(1,0),(0,1)\}$ forms a right basis and thus is a simultaneous basis.

Now let's see that ${}_1\mathbb{C}\,(t)^2_\phi$ is not simple. We see that $\{(0,1)\}$ generates a proper two-sided subspace since for $a \in \mathbb{C}\,(t)$ we have

$$a(0,1) = (0,a) = (0,1)\cdot a\,.$$

Finally we show that ${}_1\mathbb{C}\,(t)^2_\phi$ is not the direct sum of two subspaces of rank 1. If ${}_1\mathbb{C}\,(t)^2_\phi$ is the sum of two proper subspaces they are each of rank one. To see this note that the sum of the left dimensions must be the sum of the right dimensions and both are two. As any non-trivial subspace must have left dimension at least one, we see that the right dimension must also be one. We next note that if L is any subspace of rank one it is generated by any non-zero vector. If $(a,b) \in L$ and $a \neq 0$ then $a^{-1}(a,b) = (1, a^{-1}b) \in L$ and thus generates L. So there is a generator of the form $(1,d)$ that generates L. We see that L contains both $r(1,d)$ and $(1,d) \cdot r = (r, rd + \frac{d}{dt}r)$ for all r in $\mathbb{C}(t)$. Now set $r = t$ and consider

$$(r, rd + \frac{d}{dt}r) - r(1,d) = (0,1) \in L\,.$$

Thus the only rank one subspace of ${}_1\mathbb{C}(t)^2_\phi$ is the one generated by $\{(0,1)\}$.

A question that arises naturally is, "Are the left and right dimensions always the same?" We now answer that.

If $[K : k]$ is infinite, it is no longer true that the finiteness of $\dim_K V$ implies the finiteness of $\dim V_K$, as witnessed by the following example [NP, Example 2.2].

Example 2.1.9. Let $K = k(x_1, x_2, \ldots)$, where the x_i are distinct indeterminants, and let $\phi : K \to K$ be the homomorphism defined by $\phi(x_i) = x_{i+1}$. Let $V = {}_1K_\phi$. Then the dimension of ${}_KV$ is 1, while the dimension of V_K is infinite. To see this note that K is an infinite dimensional extension field of $k(x_2, x_3, \ldots) = \phi(K)$. By taking direct sums of $V = {}_1K_\phi$ we can get two-sided vector spaces of left-dimension n and infinite right dimension. One could simply switch the actions to get an infinite left dimension and a right dimension of n.

Here are some other natural questions regarding two-sided vector spaces:

1. Can we find two-sided vector spaces of left dimension m and right dimension n, for positive integers m, n?

2. If V is a two-sided vector space of rank n when can we find a simultaneous basis?

3. For a fixed field K does a rank n two-sided vector space V always have a simultaneous basis? If so, can we classify all simultaneous basis?

We will answer Question 1 at the end of this chapter and special cases of 2 and 3 in the next chapter.

2.2 Simple two-sided vector spaces

In keeping with the notation of [NP, Theorem 3.2], we write $Emb\,(K)$ for the set of k-embeddings of K into $\bar{K}$, and $G = G\,(K)$ for the absolute Galois group $Aut\left(\bar{K}/K\right)$, $\bar{K}/K$ being Galois as K is perfect. If L is an intermediate field, then we write $G\,(L)$ for $Aut\left(\bar{K}/L\right)$. G acts on $Emb\,(K)$ by left composition. Given $\lambda \in Emb\,(K)$, we denote the orbit of λ under this action by λ^G, and we write $K\,(\lambda)$ for the composite field $K \vee \mathrm{im}\,(\lambda)$ which has a two-sided structure ${}_K K \vee \mathrm{im}\,(\lambda)_{\lambda(K)}$ which we denote $V(\lambda)$. The stabilizer of λ under this action is easy to calculate: $\sigma\lambda = \lambda$ if and only if σ fixes $Im(\lambda)$; since σ fixes K as well we have that the stabilizer is $G\,(K\,(\lambda))$.

Lemma 2.2.1. *[N_2, Lemma 3.1] $[K\,(\lambda) : K]$ is finite if and only if $|\lambda^G|$ is finite, and in this case $|\lambda^G| = [K\,(\lambda) : K]$.*

We will only be interested in those embeddings of λ with λ^G finite; we denote the set of finite orbits of Emb(K) under the action of G by $\Lambda\,(K)$. We denote the category of left finite-dimensional two-sided vector spaces by $\mathbf{Vect}(K)$. The following theorem due to [NP, Theorem 3.2], allows us to study simple two-sided vector spaces using field theory.

Theorem 2.2.2. *There is a one-to-one correspondence between isomorphism classes of simples in $\mathbf{Vect}(K)$ and $\Lambda(K)$. Moreover, if V is a simple two-sided vector space corresponding to $\lambda^G \in \Lambda\,(K)$, then $dim_K V = |\lambda^G|$ and $End(V) \cong K\,(\lambda)$.*

Example 2.2.3. Let us return to the question: Can we find two-sided vector spaces of left dimension m and right dimension n, for arbitrary positive integers m, n? We note that it is sufficient to find a two-sided vector space of left dimension one and right dimension n, for arbitrary n and to find a two-sided vector space of left dimension m and right dimension one, for arbitrary m. To see this we analyze three cases:

1. $n = m$: Form the direct sum of n copies of a rank 1 two-sided vector space.

2. $n < m$: Form the direct sum $n - 1$ copies of a rank 1 two-sided vector space with a left dimension $m - n + 1$, right dimension 1 two-sided vector space.

3. $n > m$: Form $m - 1$ copies of a rank 1 two-sided vector space with a left dimension 1, right dimension $n - m + 1$ two-sided vector space.

Now we show that we can find a two-sided vector space of left dimension one and right dimension n, and a two-sided vector space of left dimension m and right dimension one. Consider the following chain of fields: $\mathbb{C}(t^n) \subset \mathbb{C}(t) \subset \mathbb{C}\left(t^{\frac{1}{m}}\right)$. Let $\lambda_1 : \mathbb{C}(t) \to \mathbb{C}(t)$ be defined by $\lambda_1(t) = t^n$. Since $[\mathbb{C}(t) : \mathbb{C}(t)] = 1$ and $[\mathbb{C}(t) : \mathbb{C}(t^n)] = n$, $V(\lambda_1) = \mathbb{C}(t)$ has left dimension one and right dimension n. Let $\lambda_2 : \mathbb{C}(t) \to \mathbb{C}\left(t^{\frac{1}{m}}\right)$ via $\lambda_2(t) = t^{\frac{1}{m}}$. Since $[\mathbb{C}\left(t^{\frac{1}{m}}\right) : \mathbb{C}(t)] = m$ and $[\mathbb{C}\left(t^{\frac{1}{m}}\right) : \mathbb{C}\left(t^{\frac{1}{m}}\right)] = 1$, $V(\lambda_2)$ has left dimension m and right dimension one. Hence we can construct two-sided vector spaces of left dimension m and right dimension n for arbitrary positive integers m and n.

Chapter 3

Results

3.1 Rational two-sided vectors spaces of rank two, Part I

Let k be an arbitrary field of characteristic zero, and let $K = k(t)$ be a function field in one variable.

Definition 3.1.1. By a **rational two-sided vector space of rank two**, we mean a k-central two-sided vector space V over $k(t)$ with simultaneous basis.

Let $\bar{K}$ be a fixed algebraic closure of K. Our goal in the next few sections is to study simple rational two-sided vector spaces, and in particular to characterize those V that have rank 2. By the following lemma such V correspond to k-linear field embeddings $\lambda : k(t) \to \bar{K}$ such that

$$[K(\lambda(t)) : K] = 2 = [K(\lambda(t)) : k(\lambda(t))] \, ,$$

i.e.,

$$[k(t, \lambda(t)) : k(t)] = 2 = [k(t, \lambda(t)) : k(\lambda(t))]$$

Recall that by Lemma 2.2.1 and Theorem 2.2.2, the left dimension of V is 2 if and only if $[K(\lambda(t)) : K] = 2$.

Lemma 3.1.2. *If $\lambda \in Emb(K)$ with $\mid \lambda^G \mid = 2$ then $K(\lambda) = K(\sqrt{m})$ for some $m \in K$ by Lemma 2.2.1. Using $\{1, \sqrt{m}\}$ as a K-basis for $K(\sqrt{m})$, we write $\lambda(x) = \lambda_1(x) + \lambda_2(x)\sqrt{m}$ with $\lambda_i(x) \in K$. Define $\phi : K \to M_2(K)$ as follows:*

$$\phi(x) = \begin{pmatrix} \lambda_1(x) & m\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) \end{pmatrix}$$

Then $V := {}_1K^2_\phi$ is the simple two-sided K-vector space corresponding to λ in Theorem 2.2.2. Moreover, $V \cong {}_1K(\sqrt{m})_\lambda$. Consequently the right dimension of V is

$$[k(t, \sqrt{m}) : k(\lambda(t))].$$

Proof. We first show that $\phi : K \to M_2(K)$ is a k-linear ring homomorphism. Let $a \in k$, $b, c \in K$.

ϕ is clearly k-linear additive as λ_1, λ_2 are k-linear. To see that ϕ is multiplicative we follow the proof of [NP, Proposition 3.5]. First we calculate an eigenvalue, r, of ϕ: Calculating the determinant of $\phi - rI$ yields $(\lambda_1(x) - r)^2 - m\lambda_2(x)^2$. Setting this equal to zero leads to $r = \lambda(x)$ as an eigenvalue of ϕ, with corresponding eigenvector $v = (1, \sqrt{m})$.

Consider $\{\sigma_1, \sigma_2\}$ where σ_1 is the identity and σ_2 is conjugation. Note that the automorphisms σ_i of $\bar{K}$ extend componentwise to automorphisms of $\bar{K}^2$ and $M_2(\bar{K})$. Moreover, $\sigma_i(\phi(x)) = \phi(x)$ since the entries of $\phi(x)$ lie in K. Then $\{\sigma_1(v), \sigma_2(v)\}$ is a basis for $\bar{K}^2$, and for all $x, y \in K$, we have:

$$\sigma_i(v)\phi(x)\phi(y) = \sigma_i\lambda(x)\sigma_i(v)\phi(y) = \sigma_i\lambda(x)\sigma_i\lambda(y)\sigma_i(v) = \sigma_i\lambda(xy)\sigma_i(v) = \sigma_i(v)\phi(xy).$$

This shows that $\phi(x)\phi(y)$ and $\phi(xy)$ act as the same linear transformation on each $\sigma_i(v)$. As the $\sigma_i(v)$ form a basis for $\bar{K}^2$, we have $\phi(x)\phi(y) = \phi(xy)$ for all $x, y \in K$. Thus ϕ is multiplicative, and thus a k-linear ring homomorphism.

Since $v = (1, \sqrt{m})$ is for all $x \in K$, an eigenvector of $\phi(x)$ with eigenvalue $\lambda(x)$, the proof of [NP, Proposition 3.5] shows that $V := {}_1K^2_\phi$ is the simple two-sided vector space corresponding to λ.

To complete the proof of the lemma, we show that $\varphi : {}_1K^2_\phi \cong {}_1V(\sqrt{m})_\lambda$.

Define $\varphi : {}_1K^2_\phi \to {}_1K(\sqrt{m})_\lambda$ by $\varphi((a,b)) = b + a\sqrt{m}$.

We show that φ is surjective: Let $c + d\sqrt{m} \in {}_1K(\sqrt{m})_\lambda$. Then $\varphi((d,c)) = c + d\sqrt{m}$.

We now show that φ is injective: If $\varphi((a,b)) = \varphi((c,d))$ then $b + a\sqrt{m} = d + c\sqrt{m}$ so $a = c, b = d$.

We now show that φ is additive:

$$\varphi((a,b) + (c,d)) = b + d + (a+c)\sqrt{m} = (b + a\sqrt{m}) + (d + c\sqrt{m}) = \varphi((a,b)) + \varphi((c,d)).$$

We now show that φ is left K-linear: Let $x \in K$. Then

$$\varphi(x \cdot (a,b)) = \varphi((xa, xb)) = xb + xa\sqrt{m} = x \cdot (b + a\sqrt{m}) = x \cdot \varphi((a,b)).$$

Finally we show that φ is right K-linear: Let $x \in K$. Then

$$\begin{aligned}
\varphi((a,b)\cdot x) = \varphi((a,b)\phi(x)) &= \varphi\left((a,b)\begin{pmatrix} \lambda_1(x) & m\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) \end{pmatrix}\right) \\
&= \varphi((a\lambda_1 + b\lambda_2, am\lambda_2 + b\lambda_1)) = (am\lambda_2 + b\lambda_1) + (a\lambda_1 + b\lambda_2)\sqrt{m} \\
&= (b + a\sqrt{m})(\lambda_1 + \lambda_2\sqrt{m}) = \varphi((a,b))(\lambda_1 + \lambda_2\sqrt{m}) = \varphi((a,b))\lambda(x) \\
&= \varphi((a,b)) \cdot x\,.
\end{aligned}$$

□

Remark 3.1.3. Note that if instead we define $\phi(x) := \begin{pmatrix} \lambda_1(x) & \lambda_2(x) \\ m\lambda_2(x) & \lambda_1(x) \end{pmatrix}$ we get ${}_\phi K_1^2 \cong {}_\lambda K(\sqrt{m})_1$.

Now we assume $[K(\lambda(t)) : K] = 2$. Then $\lambda(t)$ satisfies a monic polynomial of degree 2 in $K[x]$. As we are in characteristic zero, we can complete the square, and can write

$$\lambda(t) = \alpha + \sqrt{m},$$

where $\alpha, m \in k(t)$ but $\sqrt{m} \notin k(t)$. We are now interested in conditions on m and α which assure that $[k(t, \sqrt{m}), k(\alpha + \sqrt{m})] = 2$.

For future reference, we state a well-known result, whose proof can be found, e.g., in [M, p. 8].

Lemma 3.1.4. *Let k be a field and $k\,(t)$ be the field of rational functions in t over k. Then if $u \in k\,(t) \setminus k$ where $u = \frac{f}{g}$ with $f, g \in k[t]$ and gcd$(f,g) = 1$ then $[k\,(t) : k\,(u)] =$ max $\{r, s\}$ where the degree of f and g respectively are r and s.*

Lemma 3.1.5. *$k\,(t) \cap k\,(\sqrt{m}) = k\,(m)$.*

Proof. Clearly $k(m) \subset k(t) \cap k(\sqrt{m}) \subset k(\sqrt{m})$. Since $\sqrt{m} \notin k(t)$, $k(t) \cap k(\sqrt{m}) \subsetneq k(\sqrt{m})$. Since $[k(\sqrt{m}) : k(m)] = 2$, we obtain $k\,(t) \cap k\,(\sqrt{m}) = k\,(m)$. □

As $\alpha \in k(t)$, Lemma 3.1.5 implies $\alpha \in k(\sqrt{m})$ if and only if $\alpha \in k(m)$.

3.2 Rational two-sided vectors spaces of rank two, Part II

We now study three special cases:

1. $m \in k$.

2. $k(m) = k(t)$.

3. $\alpha \in k$.

Case 1. Assume first that $\alpha \notin k$. In this case, we have the following diagram of field extensions:

$$\begin{tikzcd}
 & k(t,\sqrt{m}) \\
k(t) \arrow[ur, dash, "2"] & k(\alpha,\sqrt{m}) \arrow[u, dash, "v"] \\
k(\alpha) \arrow[u, dash, "u"] \arrow[ur, dash, "d"] & k(\alpha+\sqrt{m}) \arrow[u, dash, "e"]
\end{tikzcd}$$

Note that $d, e \leq 2$ as the larger field is generated over the smaller fields by adjoining $\sqrt{m}$. Now $k(t)$ is isomorphic to $k(\alpha+\sqrt{m})$ (via λ). Hence k is algebraically closed in $k(\alpha+\sqrt{m})$. But k is clearly not algebraically closed in $k(\alpha, \sqrt{m})$. Hence $e = 2$. As $\sqrt{m} \notin k(t)$, $d = 2$. Consequently, $u = v$. Then

$$[k(t, \sqrt{m}) : k(\alpha+\sqrt{m})] = 2$$

if and only if $u = v = 1$ if and only if $k(t) = k(\alpha)$ if and only if (by Lemma 3.1.4)

$$\alpha = \tfrac{at+b}{ct+d} \text{ with } a, b, c, d \in k \text{ and } ad - bc \neq 0.$$

This proves the following theorem in the case $\alpha \notin k$.

Theorem 3.2.1. *If $m \in k$ then $[k(t,\sqrt{m}) : k(\alpha+\sqrt{m})] = 2$ if and only if $\alpha = \frac{at+b}{ct+d}$ with $a, b, c, d \in k$ and $ad - bc \neq 0$.*

Proof. It remains to deal with the case $\alpha \in k$. But in this case $[k(t,\sqrt{m}) : k(\alpha+\sqrt{m})] = \infty$. And for $a, b, c, d \in k$ with $ad - bc \neq 0$, $\frac{at+b}{ct+d} \notin k$, $\alpha \neq \frac{at+b}{ct+d}$. □

Case 2. In this case $m \notin k$ and $\alpha \in k(t) = k(m) \subset k(\sqrt{m})$. In addition we assume that $[k(t,\sqrt{m}) : k(\alpha+\sqrt{m})] = 2$. Under all these assumptions, we have the following diagram of field extensions:

$$\begin{tikzcd}
 & k(t,\sqrt{m}) = k(\sqrt{m}) \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] & \\
k(t) = k(m) & & k(\alpha+\sqrt{m})
\end{tikzcd}$$

Consider $u = \sqrt{m}$ as a new indeterminate over k. Since $k(m) = k(u^2)$ and since $\alpha \in k(m)$, there are polynomials $f(x), g(x) \in k[x]$ such that $\gcd_{k[x]}(f(x), g(x)) = 1$ and $\alpha = \frac{f(u^2)}{g(u^2)}$. Now

$$\alpha + \sqrt{m} = \frac{f(u^2)}{g(u^2)} + u = \frac{f(u^2) + ug(u^2)}{g(u^2)}$$

Also as $\gcd_{k[x]}(f(x), g(x)) = 1$ there exists $v(x), w(x) \in k[x]$ such that $1 = v(x)f(x) + w(x)g(x)$. Substituting in u^2 yields, $1 = v(u^2)f(u^2) + w(u^2)g(u^2)$ which holds in $k[u]$. If $d(u)$ is the gcd of $f(u^2), g(u^2)$ in $k[u]$ then $d(u)$ divides 1. As $d(u)$ is monic, $d(u) = 1$, so that $\gcd_{k[u]}(f(u^2), g(u^2)) = 1$. Hence, if $\deg_x$ and $\deg_u$ denote the degree with respect to x and u, respectively, we have

$$\begin{aligned}
2 =& [k(t,\sqrt{m}) : k(\alpha+\sqrt{m})] = [k(\sqrt{m}) : k(\alpha+\sqrt{m})] \\
=& max(deg_u(f(u^2) + ug(u^2)), deg_u g(u^2)) \\
=& max(2deg_x(f(x)), 2deg_x(g(x)) + 1)\,.
\end{aligned}$$

The third equality holds by Lemma 3.1.4. As $2deg_x(g(x))+1$ is odd we must have $deg_x(f(x)) = 1$ and thus $deg_x(g(x)) = 0$. Hence $\alpha = am + b$ with $a, b \in k$ and $a \neq 0$.

Conversely, if m and α are of this form and $k(m) = k(t)$, then $k(m) = k(\alpha)$ and $k(t, \sqrt{m}) = k(\sqrt{m})$. Again we let $u = \sqrt{m}$, so that $\alpha = au^2+b$. We are interested in $[k(u) : k(au^2+b+u)]$, which is two by Lemma 3.1.4, thus $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2$.

This proves the following:

Theorem 3.2.2. *If $k(m) = k(t)$, then $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2$ if and only if $\alpha = am + b$ with $a, b \in k$, $a \neq 0$.*

Case 3. If $\alpha \in k$ then $k(\alpha + \sqrt{m}) = k(\sqrt{m})$. Hence $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2$ if and only if $[k(t, \sqrt{m}) : k(\sqrt{m})] = 2$ if and only if $[k(t) : k(m)] = 2$. The last equivalence follows immediately from the diagram:

$$\begin{tikzcd}
 & k(t, \sqrt{m}) \\
k(t) \arrow[ur, dash, "2"] \arrow[d, dash] & k(\sqrt{m}) \arrow[u, dash] \\
k(m) \arrow[ur, dash, "2"] &
\end{tikzcd}$$

Using Lemma 3.1.4, we deduce the following result.

Theorem 3.2.3. *Let $\alpha \in k$. Then $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2$ if and only if $m = \frac{f}{g}$ with $f, g \in k[t]$, $gcd(f, g) = 1$, and $max(deg(f), deg(g)) = 2$.*

3.3 Rational two-sided vectors spaces of rank two, Part III

Our goal is to find conditions on α and m to ensure that

$$[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2\,.$$

In light of Theorem 3.2.1 and Theorem 3.2.2, we may assume $m \notin k$ and $k(m) \subsetneq k(t)$. In this section, we additionally assume $\alpha \in k(\sqrt{m})$, so $k(\alpha + \sqrt{m}) \subset k(\sqrt{m})$.

Assume for the moment that $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2$. We get the following diagram of field extensions:

$$\begin{tikzcd}
 & k(t,\sqrt{m}) \arrow[dl, dash, "2"'] \arrow[d, dash, "e"'] \arrow[ddr, dash, "2"] & \\
k(t) \arrow[d, dash, "d"'] & k(\sqrt{m}) \arrow[dl, dash, "2"'] \arrow[dr, dash] & \\
k(m) & & k(\alpha+\sqrt{m})
\end{tikzcd}$$

Clearly $d = e \leq 2$, and $d \neq 1$, so $d = e = 2$. That is

$$[k(t) : k(m)] = 2$$

i.e., $m = \frac{f}{g}$ where $f, g \in k[t]$, and $\gcd(f, g) = 1$, and $\max(deg(f), deg(g)) = 2$, see Lemma 3.1.4. Since under the assumptions of this section $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2$ implies $[k(t) : k(m)] = 2$, it makes sense to assume the latter.

Theorem 3.3.1. *Assume $\alpha \in k(\sqrt{m})$ and $[k(t) : k(m)] = 2$. Then $[k\,(t, \sqrt{m}) : k\,(\alpha + \sqrt{m})] = 2$ if and only if $\alpha \in k$.*

Note that by Lemma 3.1.5, $\alpha \in k(\sqrt{m})$ is equivalent to $\alpha \in k(m)$. After the proof of the theorem, we will present an example which shows that if $\alpha \notin k(\sqrt{m})$, $[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})]$

need not be 2, even if $[k(t) : k(m)] = 2$.

Proof. If $\alpha \in k$ we have the result by the discussion before Theorem 3.2.3.

We now prove the other direction. We work under the following assumptions:

$$\alpha \in k(\sqrt{m}),\ [k(t) : k(m)] = 2 = \left[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})\right].$$

From Lemma 3.1.5 we get that $\alpha \in k\,(m)$. Note also that as $\alpha \in k(\sqrt{m})$ we have $k\,(\alpha + \sqrt{m}) \subset k\,(\sqrt{m})$:

$$\begin{tikzcd}
k(t, \sqrt{m}) \arrow[d, dash, "2"] \\
k(\sqrt{m}) \arrow[d, dash] \\
k(\alpha + \sqrt{m})
\end{tikzcd}$$

Since $[k\,(t, \sqrt{m}) : k\,(\sqrt{m})] = 2$ (see above) we get $k\,(\sqrt{m}) = k\,(\alpha + \sqrt{m})$.

Now let $u = \sqrt{m}$, then $\alpha \in k\left(u^2\right)$ and $k\,(\alpha + u) = k\,(u)$. We write $\alpha = \frac{f(u^2)}{g(u^2)}$ with $f(u^2), g(u^2) \in k[u^2]$ and $(f(u^2), g(u^2))_{k[u^2]} = 1$. As $(f(u^2), g(u^2))_{k[u^2]} = 1$, there exist $w(u^2),\ z(u^2) \in k[u^2]$ such that $f(u^2)w(u^2) + g(u^2)z(u^2) = 1$, note that this holds in $k[u]$ also. Let $d(u) = (f(u^2), g(u^2))_{k[u]}$. As $d(u)$ divides both $f(u^2), g(u^2)$ then $d(u)$ divides 1. As $d(u)$ is monic, $d(u) = 1$. Since $k(u) = k(\alpha + u)$, we can write

$$u = \frac{h\,(\alpha + u)}{l\,(\alpha + u)} = \frac{h\left(\frac{f\left(u^2\right) + ug\left(u^2\right)}{g\left(u^2\right)}\right)}{l\left(\frac{f(u^2) + ug(u^2)}{g(u^2)}\right)}$$

where $h\,(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ and $l\,(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_0$; $a_i, b_j \in k$ and allowing coefficients to be zero but not both a_n, b_n zero. Now substituting and cross multiplying we get:

$$u\left(b_n\left(\frac{f(u^2)+ug(u^2)}{g(u^2)}\right)^n+b_{n-1}\left(\frac{f(u^2)+ug(u^2)}{g(u^2)}\right)^{n-1}+\ldots+b_0\right)$$

$$=a_n\left(\frac{f(u^2)+ug(u^2)}{g(u^2)}\right)^n+a_{n-1}\left(\frac{f(u^2)+ug(u^2)}{g(u^2)}\right)^{n-1}+\ldots+a_0\ .$$

Next we multiply both sides by $g\left(u^2\right)^n$ we get:

$$(2)\ u\left(b_n\left(f\left(u^2\right)+ug\left(u^2\right)\right)^n+b_{n-1}\left(f\left(u^2\right)+ug\left(u^2\right)\right)^{n-1}g\left(u^2\right)+\ldots+b_0 g\left(u^2\right)^n\right)$$

$$=a_n\left(f\left(u^2\right)+ug\left(u^2\right)\right)^n+a_{n-1}\left(f\left(u^2\right)+ug\left(u^2\right)\right)^{n-1}g\left(u^2\right)+\ldots+a_0 g\left(u^2\right)^n\ .$$

First note that $f\left(u^2\right)$ has all even exponents and that $ug\left(u^2\right)$ has all odd exponents and a 0 constant term. If $b_n \neq 0$ then the degree of the LHS of (2) is greater than the degree of the RHS of (2), a contradiction. Thus $b_n = 0$ and $a_n \neq 0$. As $b_n = 0$, $g\left(u^2\right)$ divides the LHS of (2) and thus the RHS of (2). Each summand of the RHS of (2) has a power of $g\left(u^2\right)$ in it except $a_n f\left(u^2\right)^n$ and therefore $g\left(u^2\right) \in k$. Let the degree of f be r. If $f\left(u^2\right)$ is not in k then the LHS of (2) has at most degree $2r\left(n-1\right)+1 = 2rn-2r+1$ while the RHS of (2) has degree $2rn$, a contradiction. Thus we get that $\alpha \in k$. □

Theorem 3.3.1 shows that many rational rank-two simple two-sided vector spaces are of the form with $\alpha \in k$.

Example 3.3.2. Let $\alpha = t$ and $m = t^2+1$. We now show that $k(\alpha+\sqrt{m}) = k(t,\sqrt{m})$.

As $t+\sqrt{t^2+1} \in k(t+\sqrt{t^2+1}), (t+\sqrt{t^2+1})^2 = 2t^2+2t\sqrt{t^2+1}+1 \in k(t+\sqrt{t^2+1})$. As $1 \in k, 2t^2+2t\sqrt{t^2+1} \in k(t+\sqrt{t^2+1})$ but then $t^2+t\sqrt{t^2+1} = t(t+\sqrt{t^2+1}) \in k(t+\sqrt{t^2+1})$ which leads to $t \in k(t+\sqrt{t^2+1})$ implying $\sqrt{t^2+1} \in k(t+\sqrt{t^2+1})$ and thus $k(\alpha+\sqrt{m}) = k(t,\sqrt{m})$.

A natural question that we hope to answer in the future is: "Do all rational rank-two simple two-sided vector spaces have $\alpha \in k$, provided $[k(t) : k(m)] = 2$?"

3.4 Simultaneous Bases

In this section we first ask when two-sided vector spaces of rank 2 have simultaneous bases under the conditions:

$$[k(t, \sqrt{m}) : k(\alpha + \sqrt{m})] = 2\,,$$

$m \notin k$ and $k(m) \subsetneq k(t)$. Additionally we assume $\alpha \in k(\sqrt{m})$, so $k(\alpha + \sqrt{m}) \subset k(\sqrt{m})$. Theorem 3.4.1 answers that question.

Recall from Lemma 3.1.2 that ${}_1k(t, \sqrt{m})_\lambda$ is a simple two-sided vector space of rank two over $k(t)$. We now study when this has a simultaneous basis.

Theorem 3.4.1. *If $\beta \neq 0$, then the set $\{\gamma, \beta\}$ is a simultaneous basis for $k\,(t, \sqrt{m})$ over $k\,(t)$ if and only if $\frac{\gamma}{\beta} \notin k\,(t) \cup k\,(\alpha + \sqrt{m})$.*

Proof. Here $k(t, \sqrt{m})$ can be either ${}_1k(t, \sqrt{m}))_\lambda$ or ${}_\lambda k(t, \sqrt{m})_1$. We prove the theorem for the case ${}_1k(t, \sqrt{m}))_\lambda$. The proof for ${}_\lambda k(t, \sqrt{m})_1$ is nearly identical. Let $\{\gamma, \beta\}$ be a simultaneous basis for, $k\,(t, \sqrt{m})$ over $k\,(t)$. If $\frac{\gamma}{\beta} \in k\,(t)$ then $\gamma = \beta\theta$ for some $\theta \in k\,(t)$. This yields a contradiction in $\{\gamma, \beta\}$ being a left basis. Similarly if $\frac{\gamma}{\beta} \in k\,(\alpha + \sqrt{m})$ then $\gamma = \beta r$ with $r \in k\,(\alpha + \sqrt{m})$. This yields a contradiction that $\{\gamma, \beta\}$ is a right basis.

For the other direction suppose $\frac{\gamma}{\beta} \notin k\,(t) \cup k\,(\alpha + \sqrt{m})$. It is sufficient to show that γ, β are linearly independent on the left and right. Suppose that there exist non-zero $a, b \in k\,(t)$ such that $a\gamma + b\beta = 0$. Then we obtain $\frac{\gamma}{\beta} \in k\,(t)$ a contradiction. Similarly for the right, if there exist non-zero $a, b \in k\,(t)$ such that $\gamma a + \beta b = 0$, i.e. $\gamma\lambda\,(a) + \beta\lambda\,(b) = 0$ yielding $\frac{\gamma}{\beta} \in k\,(\alpha + \sqrt{m})$, a contradiction. □

Definition 3.4.2. *[VdB] [N_2]* Let V be a two-sided vector space. The **right dual** of V, denoted by V^*, is the set $Hom_K(V_K, K)$ with action $(a \cdot \phi \cdot b)(x) = a\phi(bx)$ for all $\phi \in Hom_K(V_K, K)$ and $a, b \in K$. The **left dual** of V, denoted $^*V$, is the set $Hom_K({}_KV, K)$ with action $(a \cdot \phi \cdot b)(x) = b\phi(xa)$ for all $\phi \in Hom_K({}_KV, K)$ and $a, b \in K$.

Note that V^* and $^*V$ are a k-central $K-K-$bimodules since V is. We remind the reader that we denote the ith iteration of taking the left dual or the right dual by V^{-i*} and V^{i*}, respectively. In particular $^*V = V^{-*}$.

Example 3.4.3. Let K be a field of characteristic zero, let σ be an automorphism of K, and let $V = {}_1K_\sigma$. Then $V^* \cong {}_1K_{\sigma^{-1}}$ and $^*V \cong {}_1K_{\sigma^{-1}}$.

Proof. We first note that $\{1\}$ is both a left and right basis of V.

We first show that $V^* \cong {}_1K_{\sigma^{-1}}$. Let $\phi \in V^* = Hom_K(K_K, K)$, and $a \in K$. As ϕ is $K-$linear on the right we have $\phi(1a) = \phi(1)a$ and therefore ϕ is determined by where it sends 1. We claim that $V^* \cong {}_1K_{\sigma^{-1}}$. To this end we define $\tau : Hom_K(K_K, K) \to {}_1K_{\sigma^{-1}}$ by,

$$\tau(\phi) = \phi(1).$$

We first show that τ is bijective:

τ is injective: Let $\phi, \delta \in Hom_K(K_K, K)$, if $\tau(\phi) = \tau(\delta)$ then $\phi(1) = \delta(1)$ and therefore $\phi = \delta$, which shows that τ is injective.

τ is surjective: Let $v, a \in K$, and define $\phi_v(a) = v \cdot a$.

We now show that $\phi_v \in Hom_K(K_K, K)$.

ϕ_v is linear: If $v_1, v_2 \in K_K$, then $\phi_v(v_1 + v_2) = v \cdot (v_1 + v_2) = v \cdot v_1 + v \cdot v_2 = \phi_v(v_1) + \phi_v(v_2)$.

ϕ_v is right $K-$linear: If $x, a \in K$ then $\phi_v(xa) = v \cdot xa = \phi_v(x)a$.

Note that $\phi_v(1) = v$ so that $\tau(\phi_v) = v$ making τ surjective.

We now show that τ is linear: Let $\phi_1, \phi_2 \in Hom_K(K_K, K)$ then,

$$\tau(\phi_1 + \phi_2) = (\phi_1 + \phi_2)(1) = \phi_1(1) + \phi_2(1) = \tau(\phi_1) + \tau(\phi_2)\,.$$

Finally let $\phi \in Hom_K(K_K, K)$. Then

$$\tau(a \cdot \phi \cdot b) = (a \cdot \phi \cdot b)(1) = a \cdot \phi(b1) = a \cdot \phi(1 \cdot \sigma^{-1}(b)) = a \cdot \phi(1)\sigma^{-1}(b) = a \cdot \tau(\phi) \cdot b.$$

Now we show that ${}^*V \cong {}_1K_{\sigma^{-1}}$. As the proof is similar to the above we give an outline of it. Let $\phi \in {}^*V$. Define $\tau : {}^*V \to {}_\sigma K_1$ by

$$\tau(\phi) = \phi(1).$$

As above, one checks that ϕ is bijective and respects addition. Finally,

$$\tau(a \cdot \phi \cdot b) = (a \cdot \phi \cdot b)(1) = b\phi(1a) = b\phi(\sigma(a) \cdot 1) = b\phi(1)\sigma(a) = a \cdot \tau(\phi) \cdot b.$$

Now we show that $\alpha : {}_\sigma K_1 \to {}_1K_{\sigma^{-1}}$ defined by

$$\alpha(x) = \sigma^{-1}(x)$$

is an isomorphism of two-sided vector spaces. As σ is an automorphism of K, so is α. Now let $a, b, x \in {}_\sigma K_1$, then

$$\alpha(a \cdot x \cdot b) = \alpha(\sigma(a)xb) = \alpha(\sigma(a))\alpha(x)\alpha(b) = a\alpha(x)\sigma^{-1}(b) = a \cdot \alpha(x) \cdot b\,. \qquad \square$$

Definition 3.4.4. Let V be a simple two-sided vector space. By a **simultaneous basis for**

$({}^*V, V, V^*)$ we mean there is a simultaneous basis for V, such that the canonical dual bases of ${}^*V$ and V^* are simultaneous bases.

Definition 3.4.5. Let $i \in \mathbb{Z}$. We say $(V^{(i-1)*}, V^{i*}, V^{(i+1)*})$ has a **simultaneous basis** if there is a simultaneous basis for $({}^*(V^{i*}), V^{i*}, V^{(i+1)*})$ in case $i \geq 0$ and for $(V^{(i-1)*}, V^{i*}, (V^{i*})^*)$ in case $i \leq 0$.

The following proposition from [N_2, p. 9] tells us when certain pairs of functors are adjoint and this will let us construct, in Section 3.5, the non-commutative analog of the symmetric algebra of a vector space.

We call the reader's attention to Theorem 1.1.3 due to [N_2]:

Theorem 3.4.6. *If (V^{-*}, V, V^*) has a simultaneous basis then the functors*

$$(- \otimes_K V^{-*}, - \otimes_K V, - \otimes_K V^*)$$

from $\mathsf{Mod}K$ *to* $\mathsf{Mod}K$ *form an adjoint triple.*

Due to this theorem, if we can find simultaneous bases for $(V^{i*}, V^{(i+1)*}, V^{(i+2)*})$, for all $i \in \mathbb{Z}$, then

$$(- \otimes_K V^{(i)*}, - \otimes_K V^{(i+1)^*}, - \otimes_K V^{(i+2)*})$$

is an adjoint triple for all $i \in \mathbb{Z}$, so that one can construct $S_K^{n.c.}(V)$.

We let $\lambda : K \to \overline{K}$ be a k-linear embedding and we let $V(\lambda)$ denote the two-sided vector space ${}_{\mathrm{id}}K \vee \lambda(K)_\lambda$. Let $\overline{\lambda} : \overline{K} \to \overline{K}$ denote an extension of λ and let μ denote the restriction of $\overline{\lambda}^{-1}$ to K.

Lemma 3.4.7. *If $V(\lambda)$ and $V(\mu)$ have rank 2 then*

$$\overline{\lambda} : {}_\mu\mu(K) \vee K_{\mathrm{id}} \to {}_{\mathrm{id}}K \vee \overline{\lambda}(K)_\lambda = V(\lambda)$$

is an isomorphism of two-sided vector spaces.

Proof. Note that $\bar{\lambda}(\bar{\lambda}^{-1}(K)) = K$ and $\bar{\lambda}(K) = \lambda(K)$ and therefore id $\bar{\lambda}$ equals $K \vee \bar{\lambda}(K)$. Note that $\bar{\lambda}$ is clearly injective, surjective and k-linear. We now show that $\bar{\lambda}$ is compatible with the left and right action respectively. Let $a \in \bar{\lambda}^{-1}(K) \vee K, b \in K$. For the right action:

$$\bar{\lambda}(a \cdot b) = \bar{\lambda}(a)\bar{\lambda}(b) = \bar{\lambda}(a) \cdot b\,.$$

For the left action:

$$\bar{\lambda}(b \cdot a) = \bar{\lambda}(\bar{\lambda}^{-1}(b)a) = \bar{\lambda}(\bar{\lambda}^{-1}(b))\bar{\lambda}(a) = b\bar{\lambda}(a) = b \cdot \bar{\lambda}(a)\,,$$

thus proving the claim. $\square$

For the next theorem we make the following hypothesis: $\sqrt{m} \notin K$, $[K : k(m)] = 2$, and $\lambda(t) = \alpha + \sqrt{m}$. We also assume $\mu(t) = \beta + \sqrt{n}$ where $\beta \in k$, $n \in K$, $\sqrt{n}$ not in K and $V(\mu)$ has rank 2.

Theorem 3.4.8. *Let $V = V(\lambda)$. Then ${}^*V \cong V^* \cong V(\mu)$.*

Proof. Part I. We show $V^* \cong V(\mu)$ in the following three steps: Step 1: Let $\tau : K(\sqrt{n}) \to K(\sqrt{n})$ be conjugation. Consider μ as a map from V to $K(\mu)$. We prove $\mu + \tau\mu$, $\frac{1}{\sqrt{n}}(\mu - \tau\mu) \in V^*$. We first show that the images of these two maps are in K.

We use that fact that $V(\lambda) = {}_1K \vee \lambda(K)_\lambda = {}_1K(\sqrt{m})_\lambda$.

Since $[K(\sqrt{n}) : K] = 2$, in order to show $\mu + \tau\mu$, $\frac{1}{\sqrt{n}}(\mu - \tau\mu)$ are functions from $V(\lambda)$ to K, it suffices to show they send K to K and $\lambda(K)$ to K and are right K-linear. We define functions $\mu_1, \mu_2 : K \to K$ by $\mu(a) = \mu_1(a) + \mu_2(a)\sqrt{n}$.

If $a \in K$ then:

$$\begin{aligned}
(\mu+\tau\mu)(a) =& \mu_1(a)+\sqrt{n}\mu_2(a)+\tau\mu_1(a)+\tau\sqrt{n}\mu_2(a) = 2\mu_1(a) \in K \\
\frac{1}{\sqrt{n}}(\mu-\tau\mu)(a) =& \frac{1}{\sqrt{n}}\left(\mu_1(a)+\sqrt{n}\mu_2(a)-\tau\mu_1(a)-\sqrt{n}\tau\mu_2(a)\right) = 2\mu_2(a) \in K \\
(\mu+\tau\mu)(\lambda(a)) =& \mu\lambda(a)+\tau\mu\lambda(a) = 2a \\
\frac{1}{\sqrt{n}}(\mu-\tau\mu)(\lambda(a)) =& \frac{1}{\sqrt{n}}(\mu\lambda(a)-\tau\mu\lambda(a)) = 0
\end{aligned}$$

so $\mu+\tau\mu$, $\frac{1}{\sqrt{n}}(\mu-\tau\mu)$ are functions from $V \to K$. Next we show that $\mu+\tau\mu$, $\frac{1}{\sqrt{n}}(\mu-\tau\mu)$ are compatible with right multiplication by $b \in K$. It is sufficient to check this on K and $\lambda(K)$ as they generate ${}_{Id}K \vee \lambda(K)_\lambda$. Let $a \in K$.

$$\begin{aligned}
(\mu+\tau\mu)(a\cdot b) =& (\mu+\tau\mu)(a\lambda(b)) = \mu(a)b+\tau\mu(a)b = (\mu(a)+\tau\mu(a))b \\
\frac{1}{\sqrt{n}}(\mu-\tau\mu)(a\cdot b) =& \frac{1}{\sqrt{n}}(\mu-\tau\mu)(a\lambda(b)) = \frac{1}{\sqrt{n}}(\mu(a)b-\tau\mu(a)\tau(b)) = \left(\frac{1}{\sqrt{n}}(\mu(a)-\tau\mu(a))\right)b \\
(\mu+\tau\mu)(\lambda(a)\cdot b) =& (\mu+\tau\mu)(\lambda(ab)) = 2ab = (\mu+\tau\mu)(\lambda(a))b.
\end{aligned}$$

Finally

$$\frac{1}{\sqrt{n}}(\mu-\tau\mu)(\lambda(a)\cdot b) = 0 = \left(\frac{1}{\sqrt{n}}(\mu-\tau\mu)(\lambda(a))\right)b$$

whence we proved $\mu+\tau\mu$, $\frac{1}{\sqrt{n}}(\mu-\tau\mu) \in V^*$.

Step 2: We prove $\{\mu+\tau\mu, \frac{1}{\sqrt{n}}(\mu-\tau\mu)\}$ is left linearly independent. Therefore, since V^* has left-dimension 2, we have V^* equals the left span of $\{\mu+\tau\mu, \frac{1}{\sqrt{n}}(\mu-\tau\mu)\}$.

Suppose not. Then $\mu+\tau\mu = \frac{b}{\sqrt{n}}(\mu-\tau\mu)$ for some $b \in K \setminus \{0\}$. Let $a \in k$. Then $2\mu_1(a) = b\cdot 2\mu_2(a)$, i.e. $\mu_1(a) = b\mu_2(a)$ for all $a \in k$. But $\mu_1 = id$ on k, $\mu_2 = 0$ on k, a contradiction.

Step 3: We show that with respect to the left basis $\mu+\tau\mu$, $\frac{1}{\sqrt{n}}(\mu-\tau\mu)$, V^* has right-action matrix given by

$$\begin{pmatrix} \mu_1 & n\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}$$

where $\mu = \mu_1 + \sqrt{n}\mu_2$.

We compute the right structure of the right subspace of $V(\lambda)^*$ spanned by $\mu+\tau\mu$, $\frac{1}{\sqrt{n}}(\mu-\tau\mu)$.

First, let $a, b \in K$. We compute:

$$\begin{aligned}((\mu+\tau\mu)\cdot b)(a) =& (\mu+\tau\mu)(ba) = 2\mu_1(ba) = 2(\mu_1(b)\mu_1(a) + n\mu_2(b)\mu_2(a)) \\ =& \mu_1(b)(\mu+\tau\mu)(a) + n\mu_2(b)\frac{1}{\sqrt{n}}(\mu-\tau\mu)(a)\end{aligned}$$

Similarly:

$$\begin{aligned}((\mu+\tau\mu)\cdot b)(\lambda(a)) =& (\mu+\tau\mu)(b\lambda(a)) = \mu(b)a + \tau\mu(b)a = a(\mu(b)+\tau\mu(b)) \\ =& 2\mu_1(b)\cdot a = 2\mu_1(b)\frac{1}{2}(\mu+\tau\mu)(\lambda(a)) \\ =& \mu_1(b)\left((\mu+\tau\mu)(\lambda(a))\right) + n\mu_2(b)\left(\frac{1}{\sqrt{n}}(\mu-\tau\mu)(\lambda(a))\right)\end{aligned}$$

Next, we compute:

$$\begin{aligned}(\frac{1}{\sqrt{n}}(\mu-\tau\mu)\cdot b)(a) =& \frac{1}{\sqrt{n}}(\mu-\tau\mu)(ba) = 2\mu_2(ba) = 2\mu_1(b)\mu_2(a) + 2\mu_1(a)\mu_2(b) \\ =& \mu_1(b)\frac{1}{\sqrt{n}}(\mu-\tau\mu)(a) + \mu_2(b)(\mu+\tau\mu)(a).\end{aligned}$$

Finally:

$$\begin{aligned}\left(\frac{1}{\sqrt{n}}(\mu-\tau\mu)\cdot b\right)(\lambda(a)) =&\frac{1}{\sqrt{n}}(\mu-\tau\mu)(b\lambda(a))\\ =&\frac{1}{\sqrt{n}}(\mu(b)\mu\lambda(a)-\tau\mu(b)\tau\mu(\lambda(a)))\\ =&\frac{1}{\sqrt{n}}(a(\mu(b)-\tau\mu(b)))=a\cdot 2\mu_2(b)\\ =&\mu_2(b)(\mu+\tau\mu)(\lambda(a))+\mu_1(b)\frac{1}{\sqrt{n}}(\mu-\tau\mu)(\lambda(a)).\end{aligned}$$

Next let $v_1=\mu+\tau\mu, v_2=\frac{1}{\sqrt{n}}(\mu-\tau\mu)$. We proved:

$$\begin{aligned}v_1\cdot b =&\mu_1(b)v_1+n\mu_2(b)v_2\\ v_2\cdot b =&\mu_2(b)v_1+\mu_1(b)v_2\end{aligned}$$

That is, with respect to the left basis v_1, v_2, the right action by b is given by

$$(\alpha,\beta)\cdot b=(\alpha,\beta)\begin{pmatrix}\mu_1(b) & n\mu_2(b)\\ \mu_2(b) & \mu_1(b)\end{pmatrix}$$

It follows from Lemma 3.1.2 that $V^*\cong V(\mu)$ as desired. This concludes Part I.

Part II: We complete the proof by showing ${}^*V\cong V(\mu)$. First let $W:={}_\mu K(\mu)_1={}_\mu\mu(K)\vee K_1$. By Lemma 3.4.7, $W\cong V(\lambda)$. Hence it suffices to show ${}^*W\cong V(\mu)$. We proceed in three steps.

Step 1: Let $\sigma: K(\sqrt{m})\to K(\sqrt{m})$ denote conjugation. Consider λ as a map from W to $K(\lambda)$. We prove $\lambda+\sigma\lambda$, $\frac{1}{\sqrt{m}}(\lambda-\sigma\lambda)$ are in ${}^*W$. We first show that the images of these two maps are in K.

We use that fact that $W = K \vee \mu(K) = K(\sqrt{n})$. Since $[K(\sqrt{n}) : K] = 2$, in order to show that $\lambda + \sigma\lambda$, $\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)$ are functions from W to K, it suffices to show they send K to K and $\mu(K)$ to K and are left K-linear. We define functions λ_1, $\lambda_2 : K \to K$ by $\lambda(a) = \lambda_1(a) + \lambda_2(a)\sqrt{m}$.

If $a \in K$ then:

$$\begin{aligned}
(\lambda + \sigma\lambda)(a) =& \lambda_1(a) + \sqrt{m}\lambda_2(a) + \sigma\lambda_1(a) + \sigma(\sqrt{m})\lambda_2(a) = 2\lambda_1(a) \in K \\
\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(a) =& \frac{1}{\sqrt{m}}(\lambda_1(a) + \sqrt{m}\lambda_2(a) - \lambda_1(a) + \sqrt{m}\lambda_2(a)) = 2\lambda_2(a) \in K \\
(\lambda + \sigma\lambda)(\mu(a)) =& \lambda\mu(a) + \sigma\lambda\mu = 2a \\
\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(\mu(a)) =& \frac{1}{\sqrt{m}}(\lambda\mu(a) - \sigma\lambda\mu(a)) = 0.
\end{aligned}$$

So we can think of $\lambda + \sigma\lambda$ and $\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)$ as maps from $W \to K$. Next, we show that $\lambda + \sigma\lambda$, $\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)$ are compatible with left multiplication by $b \in K$. It is sufficient to check this on K and $\mu(K)$ as they generate ${}_{\mu}\mu(K) \vee K_{Id}$. Let $a, b \in K$.

$$\begin{aligned}
(\lambda + \sigma\lambda)(b \cdot a) =& (\lambda + \sigma\lambda)(\mu(b)a) = b\lambda(a) + b(\sigma\lambda(a)) = b(\lambda + \sigma\lambda)(a) \\
\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(b \cdot a) =& \frac{1}{\sqrt{m}}(\lambda(\mu(b)a) - \sigma\lambda(\mu(b)a)) \\
=& \frac{1}{\sqrt{m}}(b\lambda(a) - b\sigma\lambda(a)) = b\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(a) \\
(\lambda + \sigma\lambda)(b \cdot \mu(a)) =& (\lambda + \sigma\lambda)(\mu(ba)) = 2ba = b(2a) = b(\lambda + \sigma\lambda)(\mu(a)) \\
\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(b \cdot \mu(a)) =& 0 = b \cdot \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(\mu(a)).
\end{aligned}$$

Therefore $\lambda + \sigma\lambda, \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda) \in {}^*W$.

Step 2: We prove $\{\lambda + \sigma\lambda, \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)\}$ are right linear independent in ${}^*W$. Therefore, since ${}^*W$ has right-dimension 2, we have ${}^*W$ equals the right span of $\{\lambda + \sigma\lambda, \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)\}$.

Suppose not. Then $\lambda + \sigma\lambda = \frac{b}{\sqrt{m}}(\lambda - \sigma\lambda)$ for some $b \in K \backslash \{0\}$. Let $a \in K$. Then $\lambda_1(a) = b\lambda_2(a)$ for all $a \in K$. But $\lambda_1 = \mathrm{id}$ on k, $\lambda_2 = 0$ on k, a contradiction.

Step 3: Next, we compute the left structure of the left subspace of $^*W$ spanned by $\lambda + \sigma\lambda$, $\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)$:

Let $a, b \in K$.

$$\begin{aligned}(b \cdot (\lambda + \sigma\lambda))(a) =& (\lambda + \sigma\lambda)(a \cdot b) = (\lambda + \sigma\lambda)(ab) = 2\lambda_1(ab) \\ =& 2(\lambda_1(a)\lambda_1(b) + m\lambda_2(a)\lambda_2(b)) \\ =& (\lambda + \sigma\lambda)(a)\lambda_1(b) + \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(a)m\lambda_2(b).\end{aligned}$$

Similarly,

$$\begin{aligned}(b \cdot (\lambda + \sigma\lambda))(\mu(a)) =& (\lambda + \sigma\lambda)(\mu(a)b) = a\lambda(b) + a\sigma\lambda(b) \\ =& a(\lambda(b) + \sigma\lambda(b)) = a2\lambda_1(b) \\ =& \lambda_1(b)(\lambda + \sigma\lambda)(\mu(a)) + m\lambda_2(b)(\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda))(\mu(a)).\end{aligned}$$

Next, we compute:

$$\begin{aligned}b \cdot (\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda))(a) =& \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(ab) = 2\lambda_2(ab) \\ =& 2\lambda_1(a)\lambda_2(b) + 2\lambda_1(b)\lambda_2(a) \\ =& \lambda_2(b)(\lambda + \sigma\lambda)(a) + \lambda_1(b)\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(a).\end{aligned}$$

Finally,

$$\begin{aligned}(b \cdot \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda))(\mu(a)) =& \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(\mu(a)b) = \frac{1}{\sqrt{m}}(a\lambda(b) - a\sigma\lambda(b)) \\ =& a\frac{1}{\sqrt{m}}(\lambda(b) - \sigma\lambda(b)) = a2\lambda_2(b) \\ =& \lambda_2(b)(\lambda + \sigma\lambda)(\mu(a)) + \lambda_1(a)\frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)(\mu(a)).\end{aligned}$$

Let $v_1 = \lambda + \sigma\lambda$ and let $v_2 = \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda)$. Write $\lambda = \lambda_1 + \sqrt{m}\lambda_2$. Then for $b \in K$ we proved:

$$b \cdot v_1 = \lambda_1(b)v_1 + m\lambda_2(b)v_2$$

and

$$b \cdot v_2 = \lambda_2(b)v_1 + \lambda_2(b)v_2.$$

That is, with respect to the right basis v_1, v_2, the left action by b is given by

$$b \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \lambda_1(b) & \lambda_2(b) \\ m\lambda_2(b) & \lambda_1(b) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

From Remark 3.1.3, ${}^*W \cong {}_\lambda\lambda(K) \vee K_{\mathrm{id}}$. From Lemma 3.4.7,

$$\mu : {}_\lambda\lambda(K) \vee K_{\mathrm{id}} \to {}_{\mathrm{id}}K \vee \mu(K)_\mu = V(\mu)$$

is an isomorphism and Part II follows. $\square$

We continue with the hypotheses of the previous theorem: $\alpha \in k$, $\sqrt{m}$ not in K, $[K : k(m)] = 2$, and $\lambda(t) = \alpha + \sqrt{m}$. We also assume $\alpha \in k$, $\mu(t) = \beta + \sqrt{n}$ where $\beta \in k$, $n \in K$, $\sqrt{n}$ not in K and $V(\mu)$ has rank 2.

Theorem 3.4.9. *$({}^*V, V, V^*)$ has a simultaneous basis.*

Proof. We use the notation from the proof of Theorem 3.4.8. We first prove that V and V^* have a simultaneous basis. By Theorem 3.4.1, $(\lambda(t), t)$ is a simultaneous basis for V. We claim that the functions γ_1 and γ_2 defined by

$$\gamma_1 = \frac{1}{2t}(\mu + \tau\mu) - \frac{\beta}{t}\frac{1}{2\sqrt{n}}(\mu - \tau\mu)$$

and

$$\gamma_2 = \frac{1}{2\sqrt{n}}(\mu - \tau\mu)$$

are in V^*, are right dual to $(\lambda(t), t)$ and are a simultaneous basis for V^*. To prove they are a simultaneous basis, we use Part I, Step 3 to get an isomorphism $V^* \to V(\mu)$. Then we see what the images of γ_1 and γ_2 are in $V(\mu)$ and use Theorem 3.4.1 to check that they are indeed simultaneous.

First we claim that the set $\{t, \lambda(t)\}$ is a simultaneous basis for $K(\lambda)/K$. By Theorem 3.4.1 it is sufficient to show that $\frac{\lambda(t)}{t} \notin k(t) \cup k(\alpha + \sqrt{m})$.

1. If $\frac{\lambda(t)}{t} \in k(t)$ then $\lambda(t) = \alpha + \sqrt{m} \in k(t)$ which leads to $\sqrt{m} \in k(t)$ a contradiction.
2. If $\frac{\lambda(t)}{t} \in k(\alpha + \sqrt{m})$ then $\frac{1}{t} \in k(\alpha + \sqrt{m})$ leading to $t \in k(\lambda(t))$ a contradiction.

Therefore $\{t, \lambda(t)\}$ is a simultaneous basis for $K(\lambda)/K$.

Define $\delta_1 := \frac{1}{2t}(\mu + \tau\mu) = \frac{\mu_1}{t}$ and $\delta_2 := \frac{1}{2\sqrt{n}}(\mu - \tau\mu) = \mu_2$. Then $\delta_1(\lambda(t)) = 1, \delta_1(t) = \frac{\beta}{t}$ and $\delta_2(\lambda(t)) = 0, \delta_2(t) = 1$. Now define, $\gamma_1 := \delta_1 - \frac{\beta}{t}\delta_2$ and $\gamma_2 := \delta_2$.

From Part I, Step 1 we have $\{\gamma_1, \gamma_2\} \subset V^*$. We now check that $\{\gamma_1, \gamma_2\}$ is a simultaneous basis for V^* by using Theorem 3.4.1.

As $\gamma_1(\lambda(t)) = 1,\ \gamma_1(t) = 0$ and $\gamma_2(\lambda(t)) = 0,\ \gamma_2(t) = 1$ thus $\{\gamma_1, \gamma_2\}$ are right dual to $\{\lambda(t), t\}$.

Recall that for $\phi = \begin{pmatrix} \mu_1 & n\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix}$ we showed in the proof of Theorem 3.4.8, Part I, Step 3, that V^* is isomorphic to ${}_1K^2_\phi$, and that

$$\begin{aligned} v_1 &= \mu + \tau\mu \to (1, 0) \\ v_2 &= \frac{1}{\sqrt{n}}(\mu - \tau\mu) \to (0, 1) \end{aligned}$$

By Lemma 3.1.2, ${}_1K^2_\phi$ is isomorphic to $V(\mu) = {}_1K(\mu(t))_\mu$, via

$$(a,b) \to b + a\sqrt{n}\,.$$

The composition is an isomorphism $V^* \to V(\mu)$ such that

$$v_1 \to \sqrt{n},\ v_2 \to 1\,.$$

Since

$$\gamma_2 = \delta_2 = \frac{1}{2}v_2$$

and

$$\gamma_1 = \delta_1 - \frac{\beta}{t}\delta_2 = \frac{1}{2t}v_1 - \frac{\beta}{2t}v_2\,,$$

we see that

$$\begin{aligned}\gamma_1 \to \tilde{\gamma_1} :=& \frac{1}{2t}\sqrt{n} - \frac{\beta}{2t} = -\frac{1}{2t}(\beta - \sqrt{n})\\ \gamma_2 \to \tilde{\gamma_2} :=& \frac{1}{2}\end{aligned}$$

Hence

$$\frac{\tilde{\gamma_1}}{\tilde{\gamma_2}} = -\left(\frac{\beta - \sqrt{n}}{t}\right)\,.$$

If $\frac{\beta-\sqrt{n}}{t} \in k(t) \cup k(\beta + \sqrt{n})$ then we are in one of the following cases:

1. $\frac{\beta-\sqrt{n}}{t} \in k(t)$, but this leads to $\sqrt{n} \in k(t)$ a contradiction.
2. $\frac{\beta-\sqrt{n}}{t} \in k(\beta+\sqrt{n}) = k(\sqrt{n})$ then $\frac{\beta-\sqrt{n}}{t}(\beta+\sqrt{n}) = \frac{\beta^2-n}{t} \in k(\sqrt{n})$ implying $t \in k(\sqrt{n})$, a contradiction.

Thus the set $\{\gamma_1, \gamma_2\}$ is a simultaneous basis for V^*.

Now we claim that the set $\{t, \mu(t)\}$ is a simultaneous basis for $K(\mu) = {}_{\mu}K(\mu)_1 = W$(from Step II) over K. By Theorem 3.4.1 it is sufficient to show that $\frac{\mu(t)}{t} \notin k(t) \cup k(\alpha + \sqrt{n})$.

1. If $\frac{\mu(t)}{t} \in k(t)$ then $\mu(t) = \alpha + \sqrt{n} \in k(t)$ which leads to $\sqrt{n} \in k(t)$ a contradiction.

2. If $\frac{\mu(t)}{t} \in k(\alpha + \sqrt{n})$ then $\frac{1}{t} \in k(\alpha + \sqrt{n})$ leading to $t \in k(\alpha + \sqrt{n})$ a contradiction.

Therefore $\{t, \mu(t)\}$ is a simultaneous basis for $K(\mu)/K$.

Define:

1. $\delta_1 := \frac{1}{2t}(\lambda + \sigma\lambda) = \frac{\lambda_1}{t}$

2. $\delta_2 := \frac{1}{2\sqrt{m}}(\lambda - \sigma\lambda) = \lambda_2$

1. $\delta_1(\mu(t)) = 1, \delta_1(t) = \frac{\alpha}{t}$

2. $\delta_2(\mu(t)) = 0, \delta_2(t) = 1$

Now define the following:

1. $\eta_1 := \delta_1 - \frac{\alpha}{t}\delta_2$

2. $\eta_2 := \delta_2$

From Part II, Step 1 we have $\{\eta_1, \eta_2\} \subset {}^*W$. We now check that $\{\eta_1, \eta_2\}$ is a simultaneous basis for ${}^*W$.

Now $\eta_1(\mu(t)) = 1$, $\eta_1(t) = 0$ and $\eta_2(\mu(t)) = 0$, $\eta_2(t) = 1$. Hence $\{\eta_1, \eta_2\}$ are left dual to $\{\mu(t), t\}$.

We now show that $\{\eta_1, \eta_2\}$ are a simultaneous basis for ${}^*W$ by using Theorem 3.4.1. Recall that for $\phi = \begin{pmatrix} \lambda_1 & \lambda_2 \\ m\lambda_2 & \lambda_1 \end{pmatrix}$ we showed in the proof of Theorem 3.4.8, Part II, Step 3, that ${}^*W$ is isomorphic to ${}_{\phi}K_1^2$, and that

$$\begin{aligned} v_1 =& \lambda + \sigma\lambda \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ v_2 =& \frac{1}{\sqrt{m}}(\lambda - \sigma\lambda) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

By remark after Lemma 3.1.2, ${}_{\phi}K_1^2$ is isomorphic to ${}_{\lambda}K(\lambda(t))_1$, via

$$\begin{pmatrix} a \\ b \end{pmatrix} \to b + a\sqrt{m}\,.$$

The composition is an isomorphism ${}^*W \to {}_{\lambda}K(\lambda(t))_1$ such that

$$v_1 \to \sqrt{m},\ v_2 \to 1\,.$$

Since

$$\eta_2 = \delta_2 = \frac{1}{2}v_2$$

and

$$\eta_1 = \delta_1 - \frac{\alpha}{t}\delta_2 = \frac{1}{2t}v_1 - \frac{\alpha}{2t}v_2\,,$$

we see that

$$\begin{aligned} \eta_1 \to \tilde{\eta_1} :=& \frac{1}{2t}\sqrt{m} - \frac{\alpha}{2t} = -\frac{1}{2t}(\alpha - \sqrt{m}) \\ \eta_2 \to \tilde{\eta_2} :=& \frac{1}{2} \end{aligned}$$

Hence

$$\frac{\tilde{\eta_1}}{\tilde{\eta_2}} = -\left(\frac{\alpha - \sqrt{m}}{t}\right).$$

If $\frac{\alpha-\sqrt{m}}{t} \in k(t) \cup k(\alpha+\sqrt{m})$ then we are in one of the following cases:

1. $\frac{\alpha-\sqrt{m}}{t} \in k(t)$, but this leads to $\sqrt{m} \in k(t)$ a contradiction.

2. $\frac{\alpha-\sqrt{m}}{t} \in k(\alpha+\sqrt{m}) = k(\sqrt{m})$ leading to $\frac{\alpha^2-m}{t} \in k(\sqrt{m})$ yielding $t \in k(\sqrt{m})$ a contradiction.

Thus the set $\{\eta_1, \eta_2\}$ is a simultaneous basis for ${}^*W \cong {}_\lambda K(\lambda(t))_1$ by above. By Lemma 3.4.7, we have the isomorphism

$$\mu = \bar{\lambda}^{-1} : V = {}_1K(\lambda)_\lambda \to {}_\mu K(\mu)_1 = W\,.$$

We now define

$${}^*\mu : {}^*W \to {}^*V$$

by ${}^*\mu(\phi) = \phi \circ \mu$. As $\{\eta_1, \eta_2\}$ is a simultaneous basis for ${}^*W$ and ${}^*\mu$ is an isomorphism the set $\{{}^*\mu(\eta_1), {}^*\mu(\eta_2)\}$ is a simultaneous basis for ${}^*V$. We now claim that $\{{}^*\mu(\eta_1), {}^*\mu(\eta_2)\}$ is dual to $\{t, \lambda(t)\}$:

$$\begin{aligned}
(\eta_1 \circ \mu)(t) &= \eta_1(\mu(t)) = 1\\
(\eta_1 \circ \mu)(\lambda(t)) &= \eta_1(t) = 0\\
(\eta_2 \circ \mu)(t) &= \eta_2(\mu(t)) = 0\\
(\eta_2 \circ \mu)(\lambda(t)) &= \eta_2(t) = 1
\end{aligned}$$

□

We continue with the hypotheses of the previous theorem for the next two corollaries: $\alpha \in$

$k, \sqrt{m} \notin K, [K : k(m)] = 2$, and $\lambda(t) = \alpha + \sqrt{m}$. We also assume $\mu(t) = \beta + \sqrt{n}$ where $\beta \in k$, $n \in K$, $\sqrt{n}$ not in K and $V(\mu)$ has rank 2.

Corollary 3.4.10. *If $W \cong V = V(\lambda)$ then $({}^*W, W, W^*)$ has a simultaneous basis.*

Proof. Let $\phi : V \to W$ be a two-sided K-vector space isomorphism. We note that if $W \cong V$ then ${}^*W \cong {}^*V$ and $W^* \cong V^*$, we prove the latter and the first follows by a similar argument. Define $\tau : V^* \to W^*$ by $\tau(\sigma) = \sigma \circ \phi^{-1}$, which is clearly in W^* as ϕ^{-1} is an isomorphism from W to V.

τ is injective: If $\sigma_1 \circ \phi^{-1} = \sigma_2 \circ \phi^{-1}$ then $\sigma_1 = \sigma_2$ as ϕ is an isomorphism.

τ is surjective: Let $\gamma \in W^*$. Then $\gamma \circ \phi \in V^*$, and $\tau(\gamma \circ \phi) = \gamma$.

τ is clearly additive so finally we note that $\tau(a \cdot \sigma \cdot b)(x) = a\sigma(b\phi^{-1}(x)) = a\sigma(\phi^{-1}(bx)) = (a \cdot \tau(\sigma) \cdot b)(x)$, thus τ is linear.

Also if $(\lambda(t), t)$ is a simultaneous basis for V then $(\phi(\lambda(t)), \phi(t))$ is a simultaneous basis for W. This implies that $\{\tau(\gamma_i)\}$, γ_i as defined in Theorem 3.4.9, are right dual to $\{\phi(\lambda(t)), \phi(t)\}$ and are a simultaneous basis for W^*. Similarly if $\psi : {}^*V \to {}^*W$ is the two-sided vector space isomorphism defined by $\psi(\sigma) = \sigma \circ \phi^{-1}$, then ψ maps the dual basis of $\{\lambda(t), t\}$ in ${}^*V$ to a simultaneous basis for ${}^*W$ that is dual to $\{\phi(\lambda(t)), \phi(t)\}$. $\square$

Corollary 3.4.11. *Let $V = V(\lambda)$. For all $i \in \mathbb{Z}$, $(V^{i*}, V^{(i+1)*}, V^{(i+2)*})$ has a simultaneous basis.*

Proof. For all $i \in \mathbb{Z}$, $V^{i*}, (V^{i*})^*$, and ${}^*(V^{i*})$ are isomorphic to $V(\mu)$ or $V(\lambda)$, and so the hypotheses of Theorem 3.4.9 are satisfied for $W = V^{(i+1)*}$. $\square$

Theorem 3.4.12. *Let $m = \left(\frac{at^2+bt+c}{dt^2+et+f}\right)^{\frac{1}{2}}$ with $a, b, c, d, e, f, \in k$, a, d not both zero, and $ae = bd, af \neq cd, b^2 \neq 4ac, e^2 \neq 4df$. Then $[k(t) : k(m)] = 2$ and $\sqrt{m} \notin k(t)$.*

Proof. If $m \in k$, then $a = dm, c = fm$, so $af = (dm)f = d(fm) = cd$, a contradiction. Hence $m \notin k$.

If $\gcd(at^2 + bt + c, dt^2 + et + f) \neq 1$, then these two polynomials have a common root u in some extension field of k. So

$$au^2 + bu + c = 0 = du^2 + eu + f\,.$$

Hence

$$adu^2 + bdu + cd = 0 = adu^2 + aeu + af\,.$$

Since $bd = ae$, it follows that $af = cd$, a contradiction. Since a or d is nonzero, it follows that

$$[k(t) : k(m)] = 2\,.$$

Now suppose $\sqrt{m} \in k(t)$. Then there exist polynomials $p, q \in k[t]$ such that $m = \frac{p^2}{q^2}$. We may assume $\gcd(p, q) = 1$. Now

$$(*)\ p^2(dt^2 + et + f) = q^2(at^2 + bt + c)\,.$$

Since $\gcd(p, q) = 1$,

$$p^2 | at^2 + bt + c,\ \ q^2 | dt^2 + et + f\,.$$

Hence $\deg p, q \leq 1$.

Suppose $a = 0$. Then $d \neq 0$, so the LHS of $(*)$ has even degree. Hence the RHS of $(*)$ has even degree, implying $b = 0$. Hence

$$b^2 = 0 = 4ac\,,$$

a contradiction. Thus $a \neq 0$. Similarly, if $d = 0$ then $e = 0$, so that $e^2 = 0 = 4df$, a contradiction. Consequently both $a \neq 0$ and $d \neq 0$.

It now follows from $(*)$ that deg$p$ =deg$q$. If deg$p$ =deg$q = 0$, then $m = \frac{p^2}{q^2} \in k$, a contradiction. Hence deg$p$ =deg$q = 1$. Now $at^2 + bt + c = u_1 p^2$, for some $u_1 \in k$. If $t + x$ $(x \in k)$ is the monic polynomial associated to p, then

$$at^2 + bt + c = u_2(t + x)^2\,,$$

for some $u_2 \in k$. Clearly $u_2 = a$. We obtain

$$at^2 + bt + c = at^2 + 2axt + ax^2\,.$$

Hence $b = 2ax, c = ax^2$, and $b^2 = 4a(ax^2) = 4ac$, a contradiction. This final contradiction concludes the proof. $\square$

Theorem 3.4.13. *Let k be a field of characteristic 0 and let t be transcendental over k. Let $\lambda(t) = \alpha + \sqrt{m}$ where $\alpha, a, b, c, d, e, f \in k$, and $m = \frac{at^2+bt+c}{dt^2+et+f}$ such that a, d not both zero and $ae = bd, af \neq cd, b^2 \neq 4ac, e^2 \neq 4df$. Then $\sqrt{m} \notin k(t), [k(t) : k(m)] = 2$, and λ corresponds to a simple two-sided vector space of rank two. Extend λ to $\bar{K}$ and call this extension $\bar{\lambda} : \bar{K} \to \bar{K}$. Define $\gamma = \bar{\lambda}^{-1}|_{k(t)}$. Then γ corresponds to a simple two-sided vector space of rank 2 such that $\gamma(t) = \beta + \sqrt{n}$, where $\beta \in k, n \in k(t), \sqrt{n} \notin k(t)$.*

Proof. By Theorem 3.4.12 $[k(t) : k(m)] = 2$ and $\sqrt{m} \notin k(t)$.

Since $\lambda(t) = \alpha + \left(\frac{at^2+bt+c}{dt^2+et+f}\right)^{\frac{1}{2}}$ we obtain

$$(d(t-\alpha)^2 - a)\gamma^2(t) + (e(t-\alpha)^2 - b)\gamma(t) + f(t-\alpha)^2 - c = 0.$$

Suppose $d(t-\alpha)^2 - a = 0$. If $d = 0$, then $a = 0$, so that $ae = bd = 0$, a contradiction. So $d \neq 0$, and $(t-\alpha)^2 = \frac{a}{d}$ which leads to $t^2 - 2t\alpha + \alpha^2 - \frac{a}{d} = 0$ implying $[k(t) : k] \leq 2$ a contradiction to t being transcendental over k.

Solving $(d(t-\alpha)^2-a)\gamma^2(t)+(e(t-\alpha)^2-b)\gamma(t)+f(t-\alpha)^2-c=0$ for $\gamma(t)$ yields:

$$\gamma(t)=-\frac{1}{2}\frac{e(t-\alpha)^2-b}{d(t-\alpha)^2-a}\pm\left(\frac{(e(t-\alpha)^2-b)^2-4(f(t-\alpha)^2-c)(d(t-\alpha)^2-a)}{4(d(t-\alpha)^2-a)^2}\right)^{\frac{1}{2}}.$$

It suffices to show $\frac{e(t-\alpha)^2-b}{d(t-\alpha)^2-a}\in k$, $\frac{(e(t-\alpha)^2-b)^2-4(f(t-\alpha)^2-c)(d(t-\alpha)^2-a)}{4(d(t-\alpha)^2-a)^2}=\frac{g(t)}{h(t)}$ where the maximum degree of g,h is 2 and at least one of them is of degree 2, and finally that $\gamma(t)\notin k(t)$.

If $\gamma(t)\in k(t)$, then λ is defined at $\gamma(t)$, and $\lambda(\gamma(t))=t$. Then $t\in\lambda(k(t))=k(\lambda(t))$, so that $k(t,\lambda(t))$ is not a quadratic extension of $k(\lambda(t))$. Hence $\gamma(t)\notin k(t)$.

Recall that we assume $ae=bd$ and $af\neq cd$.

Case 1. $a=0$. Then $d\neq 0$ so $b=0$. We obtain $\frac{e(t-\alpha)^2-b}{d(t-\alpha)^2-a}=\frac{e}{d}\in k$ which implies $e=ds$ with $s\in k$. Also

$$\frac{(e(t-\alpha)^2-b)^2-4(f(t-\alpha)^2-c)(d(t-\alpha)^2-a)}{4(d(t-\alpha)^2-a)^2}=\frac{s^2d(t-\alpha)^2-4(f(t-\alpha)^2-c)}{4d(t-\alpha)^2}.$$

If

$$(s^2d(t-\alpha)^2-4(f(t-\alpha)^2-c),4(d(t-\alpha)^2))\neq 1$$

then $(t-\alpha)^2\in k$ a contradiction, so that the above conditions are met.

Case 2. $a\neq 0$. Then $e=\frac{bd}{a}$. So $\frac{e(t-\alpha)^2-b}{d(t-\alpha)^2-a}=\frac{b}{a}\in k$ and

$$\begin{aligned}&\frac{(e(t-\alpha)^2-b)^2-4(f(t-\alpha)^2-c)(d(t-\alpha)^2-a)}{4(d(t-\alpha)^2-a)^2}\\&=\frac{((\frac{b}{a})^2d(t-\alpha)^2-a)-4(f(t-\alpha)^2-c)}{4(d(t-\alpha)^2-a)}.\end{aligned}$$

We note either the numerator or the denominator has degree 2. If $d\neq 0$, it is clear. If $d=0$, then $f\neq 0$, else $af=cd$ a contradiction.

If

$$((\frac{b}{a})^2(d(t-\alpha)^2-a)-4(f(t-\alpha)^2-c),4(d(t-\alpha)^2-a))\neq 1$$

then $f(t-\alpha)^2-c$ and $d(t-\alpha)^2-a$ have a common root u in some extension field of k. Then

$$f(u-\alpha)^2=c,\ d(u-\alpha)^2=a$$

implying

$$af=cd$$

a contradiction. Thus $((\frac{b}{a})^2(d(t-\alpha)^2-a)-4(f(t-\alpha)^2-c),4(d(t-\alpha)^2-a))=1$. □

3.5 Non-commutative symmetric algebras

Recall that if V is a vector space over K of dimension two, $K[x,y]=\mathrm{Sym}_K(V)$, where $K[x,y]$ is a $\mathbb{Z}$-graded ring with deg$x$ =deg$y=1$. Now we consider an analog for the non-commutative case. The notion of a $\mathbb{Z}$-algebra will be necessary for our construction. This is not to be confused with a ring R which is graded by $\mathbb{Z}$.

Definition 3.5.1. *[PP, p. 96]* A $\mathbb{Z}-$**algebra** A over K is a collection of $K-K-$bimodules A_{ij} with $i,j\in\mathbb{Z}$ such that $A_{ii}=K, A_{ij}=0$ for $i>j$, equipped with associative product maps $A_{ij}\bigotimes_K A_{jk}\to A_{ik}$.

Now we use this concept to construct $S_K^{n.c.}(V)$.

Definition 3.5.2. Let V be a finite rank, two-sided vector space with simultaneous bases over K, such that $(V^{(i-1)*},V^{i*},V^{(i+1)*})$ are adjoint triples for all $i\in\mathbb{N}$. The **non-commutative symmetric algebra** generated by V, denoted $S_K^{n.c.}(V)$ is the $\mathbb{Z}-$algebra $\bigoplus_{i,j\in\mathbb{Z}}A_{ij}$ with components:

1. $A_{ii} = K$

2. $A_{i,i+1} = V^{i*}$

3. For $j > i+1$, $A_{ij} = A_{i,i+1} \bigotimes \ldots \bigotimes A_{j-1,j}/R_{ij}$ where $R_{ij} \subset A_{i,i+1} \bigotimes \ldots \bigotimes A_{j-1,j}$ is the two-sided vector space $\sum_{k=i}^{j-2} A_{i,i+1} \bigotimes \ldots \bigotimes A_{k-1,k} \bigotimes Q_k \bigotimes A_{k+2,k+3} \bigotimes \ldots \bigotimes A_{j-1,j}$. Here Q_i is the image of the inclusion $K \to V^{i*} \bigotimes V^{(i+1)*}$ induced by the unit of the adjoint pair $(- \bigotimes V^{i*}, - \bigotimes V^{(i+1)*})$.

4. $A_{ij} = 0$ if $i > j$ Multiplication is defined as follows for $i < j < k$:(we define all other multiplications to be 0)

$$
\begin{aligned}
A_{ij} \bigotimes A_{jk} = & \frac{A_{i,i+1} \bigotimes \ldots \bigotimes A_{j-1,j}}{R_{ij}} \bigotimes \frac{A_{j,j+1} \bigotimes \ldots \bigotimes A_{k-1,k}}{R_{jk}} \\
& \xrightarrow{\cong} \frac{A_{i,i+1} \bigotimes \ldots \bigotimes A_{k-1,k}}{R_{ij} \bigotimes A_{j,j+1} \bigotimes \ldots \bigotimes A_{k-1,k} + A_{i,i+1} \bigotimes \ldots \bigotimes A_{j-1,j} \bigotimes R_{jk}} \\
& \longrightarrow \frac{A_{i,i+1} \bigotimes \ldots \bigotimes A_{k-1,k}}{R_{ik}} = A_{ik}.
\end{aligned}
$$

The isomorphism is shown in [N_1, Corollary 3.18,p. 41].

Theorem 3.5.3. *Let k be a field of characteristic 0 and let t be transcendental over k. Let $\lambda(t) = \alpha + \sqrt{m}$ where $\alpha, a, b, c, d, e, f \in k$, and $m = \frac{at^2+bt+c}{dt^2+et+f}$ such that a, d not both zero and $ae = bd, af \neq cd, b^2 \neq 4ac, e^2 \neq 4df$. Then λ and μ correspond to simple two-sided vector spaces of rank two over $k(t)$, and $S_K^{n.c.}(V(\lambda))$ exists.*

Proof. Theorem 3.4.12 implies $\sqrt{m} \notin k(t)$ and $[k(t) : k(m)] = 2$. Then Theorem 3.3.1 implies λ corresponds to a simple two-sided vector space, V of rank two. As $ae = bd, af \neq cd, b^2 \neq 4ac, e^2 \neq 4df$ Theorem 3.4.13 tells us that μ corresponds to a simple two-sided vector space, W, of rank two. As λ and μ have satisfied the hypothesis of Theorem 3.4.13, thus $V^* \cong {}^*V \cong V(\mu)$. Hence by Corollary 3.4.11, $(V^{i*}, V^{(i+1)*}, V^{(i+2)*})$ has a simultaneous basis for all $i \in \mathbb{Z}$. The result follows now from a result of Nyman (Theorem 1.1.3). □

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