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Spectral Preserver Problems in Uniform Algebras

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SPECTRAL PRESERVER PROBLEMS IN UNIFORM ALGEBRAS

by

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There has been much interest in characterizing maps between Banach algebras that preserve a certain equation or family of elements. There is a rich history in such problems that assume the map to be linear, so called linear preserver problems. More recently, there has been an interest in not assuming the map is linear a priori and instead to assume it preserves some equation involving the spectrum, a portion of the spectrum, or the norm.

After a brief introduction to uniform algebras, we give a rigorous development of the theory of boundaries. This includes a new alternative proof of the famous Shilov Theorem. Also a generalization of Bishop's Lemma is given and proved. Two spectral preserver problems are introduced and solved for the class of uniform algebras. One of these problems is given in terms of a portion of the spectrum called the peripheral spectrum. The other is given by a norm condition.

The first spectral preserver problem concerns weakly-peripherally multiplicative maps between uniform algebras. These are maps $T : A \to B$ such that $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in A$ where $\sigma_\pi(f)$ is the peripheral spectrum of f. It is proven that if T is a weakly-peripherally multiplicative map (not necessarily linear) that preserves the family of peak functions then it is an isometric algebra isomorphism.

The second of these preserver problems shows that if $T : A \to B$ is a map (not necessarily linear) between uniform algebras such that $\|TfTg + 1\| = \|fg + 1\|$ for all $f, g \in A$ then T is a weighted composition operator composed with a conjugation operator. In particular, if $T(1) = 1$ and $T(i) = i$ then T also is an isometric algebra isomorphism.

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Contents

Chapter 1

Overview

In this section we will outline the history and development of spectral preserver problems and describe the contributions made to the theory by this work.[1]

1.1 Preserver Problems

A preserver problem, loosely speaking, is an attempt to categorize all maps between objects of a category that preserve some property or class. Consider a map $\phi : G \to G$ between groups that preserves the product, i.e., $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in G$. Then, by definition, ϕ is a group homomorphism. This is a rather trivial example because the preservation of products is the defining characteristic of a group homomorphism. However other examples are more surprising. Consider the following theorem of Mazur-Ulam.

Theorem 1.1.1 (Mazur-Ulam Theorem). *Let $f : X \to Y$ be a surjective, zero pre-*

[1] For readers unfamiliar with the notation of uniform algebras, Chapter 2 contains the relevant definitions.

serving, distance preserving map between normed vector spaces over $\mathbb{R}$*, i.e.,* $\|x - y\| = \|f(x) - f(y)\|$ *for all* $x, y \in X$ *and* $f(0) = 0$*. Then* f *is an isometric linear transformation.*

As a consequence, $f(x + \alpha y) = f(x) + \alpha f(y)$ for all $x, y \in X$ and α in $\mathbb{R}$. It is not *a priori* clear that even $f(x - y) = f(x) - f(y)$ since we only assume that these are equal in norm. So the result is quite interesting. Note this result does not apply for complex-valued spaces to get a $\mathbb{C}$-linear map. Conjugation is a counter-example.

Banach spaces (i.e., complete normed vector spaces) are first and foremost vector spaces and some of the theory is inherited from the general algebraic study of vector spaces. Because of the additional norm condition, the important mappings between Banach spaces are continuous, linear transformations. To establish that a given map is a continuous, linear transformation, one typically shows first that it is linear and then shows that it is continuous using a norm condition. The Mazur-Ulam Theorem reverses the usual order of things. We verify a norm condition first and conclude the map is linear. This serves as an analogy of the spectral preserver problems to be discussed here. From a broad perspective we seek interesting analytic conditions that imply that a map automatically has some algebraic property, often to be an isomorphism.

Linear Preserver Problems

A common type of preserver problem is a linear preserver problem. In this case the maps are between algebras and the map is assumed to be linear . In classical linear presever problems from matrix theory the given map is from $M_n(\mathbb{F})$ to itself or $\mathbb{F}$ where $M_n(\mathbb{F})$ is the set of n by n matrices with entries in the field $\mathbb{F}$. Analogous results are explored for $\mathcal{B}(X)$, the set of bounded linear operators on the Banach space X.

An example of using analytic conditions to get algebraic properties in the context of Banach algebras is the famous Gleason-Kahane-Żelazko Theorem [18].

Theorem 1.1.2 (GKZ Theorem (1973))**.** *Let A be a unital commutative Banach algebra and B a uniform algebra. If $T : A \to B$ is a linear map with $\sigma(Tf) \subset \sigma(f)$ for all $f \in A$ then T is multiplicative, i.e., $T(fg) = T(f)T(g)$ for all $f, g \in A$.*

This theorem assumes some algebraic properties to start with (linearity) and concludes stronger algebraic properties (multiplicativity). The spectrum condition is considered to be analytic in nature since the existence of (complete) norms on these spaces guarantee some measure of invertibility that is inherent in the use of the spectrum, see Lemma 2.3.1 and Corollary 2.3.2. The Gleason-Kahane-Żelazko Theorem has inspired a great deal of research in this area.

The technique used in the proof of the GKŻ theorem is to first establish the result for $B = \mathbb{C}$ and apply a result from complex analysis. A classical linear preserver problem using a very different technique is given by the Banach-Stone Theorem.

Theorem 1.1.3 (Banach-Stone Theorem)**.** *Let X and Y be compact Hausdorff spaces and $T : C(X) \to C(Y)$ be a linear, surjective isometry. Then there exists $\kappa \in C(Y)$ with $|\kappa| = 1$ and a homeomorphism $\psi : Y \to X$ such that*

$$Tf = \kappa \cdot f \circ \psi$$

for all $f \in C(X)$. In particular, $\tilde{T} = \overline{\kappa}T$ is an isometric algebra isomorphism.

If $T(1) = 1$, then $Tf = f \circ \psi$ for all $f \in C(X)$. In particular, $T(fg)(y) = (fg)(\psi(y)) = f(\psi(y))\cdot g(\psi(y)) = Tf(y)\cdot Tg(y)$ for all $y \in Y$ and so $T(fg) = TfTg$ for all $f, g \in C(Y)$. Thus T is multiplicative, i.e., T is an isometric algebra isomorphism. The solution to this problem is expressed in a very important form. A map between uniform algebras of the form, $Tf = f \circ \psi$, for some homeomorphism ψ, is called a composition operator. It is automatically linear, multiplicative, and continuous. If T is of the form $Tf = \kappa \cdot f \circ \psi$ then we say it is a weighted composition operator. This result is the model for more general spectral preserver problems.

The Banach-Stone Theorem result follows for continuous real-valued[2] or complex-valued functions. In the real-valued case, note that it is unnecessary to assume that T is linear. By the Mazur-Ulam theorem we need only assume T is a surjective isometry (and $T(0) = 0$). By putting the two theorems together we can eliminate the hypothesis that T is linear.

Theorem 1.1.4. *Let X and Y be compact Hausdorff spaces and $T : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(Y)$ be a surjective map (not necessarily linear) such that $T(1) = 1$, $T(0) = 0$ and*

$$\|Tf - Tg\| = \|f - g\|$$

for all $f, g \in C_{\mathbb{R}}(X)$. Then there exists a homeomorphism $\psi : Y \to X$ such that $Tf = f \circ \psi$ for all $f \in C_{\mathbb{R}}(X)$. In particular, T is an isometric algebra isomorphism.

Spectral Preserver Problems

In the Gleason-Kahane-Żelazko Theorem, a condition involving the spectrum was used. Recently there has been a departure from assuming that the given map is linear to start with, and assuming only conditions involving the spectrum. We introduce the term *spectral* preserver problem to describe preserver problems of this nature, especially when

[2] We will rarely have occasion to discuss real-valued functions. All function spaces should be assumed to be complex-valued unless otherwise stated.

the map is not assumed to be linear. In a uniform algebra, the norm is the maximum modulus of the values in the spectrum so preserver problems expressed in terms of the norm conditions can be considered to be spectral in nature.

An early result that does not assume linearity is due to Kowalski and Słodkowski [10].

Theorem 1.1.5 (Kowalski-Słodkowski (1980)). *Let $T : A \to B$ be a surjective map between uniform algebras such that*

$$\sigma(Tf - Tg) = \sigma(f - g)$$

for all $f, g \in A$, then T is an isometric algebra isomorphism.

This result can be seen an analogy of Theorem 1.1.4 for complex-valued uniform algebras. Other spectral preserver problems involving an additive condition can be seen in [16].

1.2 Multiplicative Spectral Preserver Problems

On might ask about problems involving a multiplicative spectral condition. In 2005 L. Molnár published a paper [14] addressing this case.[3]

Theorem 1.2.1 (Molnár (2005)). *Let X be a first countable compact Hausdorff space and $T : C(X) \to C(X)$ be a surjective map (not necessarily linear) such that*

$$\sigma(TfTg) = \sigma(fg) \tag{1.2.1}$$

[3]The paper also develops analogous, non-commutative results for $\mathcal{B}(X)$, bounded linear operators on a Banach space with special results if X is a Hilbert space.

for all $f, g \in A$. Then there exists a $\kappa \in B$ with $\kappa^2 = 1$ and a homemorphism $\psi : X \to X$ such that

$$Tf = \kappa \cdot f \circ \psi \tag{1.2.2}$$

for all $f \in A$. In particular, $\tilde{T} = \kappa T$ is an isometric algebra isomorphism.

The technique used was to directly establish the map ψ similar to the conclusion to Theorem 1.1.4. The result was extended to uniform algebras by Rao and Roy [15]. They showed that if A is a uniform algebra on its maximal ideal space, $\mathcal{M}_A$, and $T : A \to A$ is a surjective map satisfying (1.2.1) then there exists a homeomorphism $\psi : \mathcal{M}_A \to \mathcal{M}_A$ and $\kappa \in B$ such that T is as in (1.2.2).

Following this, Hatori, Miura, and Takagi [8] took $T : A \to B$ to be a mapping between uniform algebras on compact Hausdorff spaces X and Y and replaced the *multiplicatively spectral preserving* condition (1.2.1) with the weaker condition, *multiplicatively range preserving,* $\mathrm{Ran}(TfTg) = \mathrm{Ran}(fg)$. They also conclude that T must be a weighted composition operator (when the functions are extended on their maximal ideal spaces). Since $\mathrm{Ran}(f) \subset \sigma(f)$ this result is clearly an improvement.

Then Luttman and Tonev also considered uniform algebras A and B but further restricted the portion of the spectrum required to be preserved. The peripheral spectrum of f is given by $\sigma_\pi(f) = \{\lambda \in \sigma(f) : |\lambda| = \|f\|\}$. The homeomorphism they produce is between the Shilov boundaries ∂A and ∂B which can be considered to be a subset of the carrier spaces X and Y resp. They proved to following theorem.

Theorem 1.2.2 (Luttman-Tonev (2007)). *If $T : A \to B$ a surjective map (not necessarily linear) between uniform algebras such that*

$$\sigma_\pi(TfTg) = \sigma_\pi(fg) \tag{1.2.3}$$

for all $f, g \in A$*, then there exists a* $\kappa \in B$ *with* $\kappa^2 = 1$ *and a homemorphism* $\psi : \partial B \to \partial A$ *such that*

$$Tf = \kappa \cdot f \circ \psi$$

on ∂B *for all* $f \in A$*. In particular,* $\tilde{T} = \kappa T$ *is an isometric algebra isomorphism.*

One can see the hypotheses of the earlier result are obviously met by this theorem. Thus the result of Luttman and Tonev extended (and included) all earlier results.

The hypothesis of the Theorem 1.2.2 would be further weakened in [11] by considering a special class called *peaking functions*. A function $f \in A$ is called a peaking function if $\sigma_\pi(f) = \{1\}$. The class of peaking function of A is denoted by $\mathcal{F}(A)$. Thus if T is unital and $\sigma_\pi(TfTg) = \sigma_\pi(fg)$, then $\sigma_\pi(Tf) = \sigma_\pi(f)$, i.e., $T(\mathcal{F}(A)) = \mathcal{F}(B)$. In [11] we further improve the theorem by preserving this class, but requiring only that the peripheral spectra meet. We call this condition *weak peripheral multiplicativity*.

Theorem (4.2.5). *Let* $T \colon A \to B$ *be a mapping between uniform algebras. If* T *is weakly peripherally-multiplicative and preserves the peaking functions (i.e.,* $T(\mathcal{F}(A)) = \mathcal{F}(B)$*) and* $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \emptyset$ *for all* $f, g \in A$*, then* T *is an isometric algebra isomorphism.*

In fact this uses a similar composition operator technique. We produce a homeomorphism between the Choquet boundaries $\psi : \delta B \to \delta A$ such that $Tf = f \circ \psi$ on δB for all $f \in A$. Note the Choquet boundary δA may also be identified with a subset of the carrier space of A as is with the case of the Shilov boundary of Theorem 1.2.2.

For the non-unital case we established the following.

Theorem (4.3.6). *Let* $T \colon A \to B$ *be a weakly peripherally-multiplicative mapping, not necessarily linear, between uniform algebras such that (a)* $\mathcal{F}(B) \subset (T1) \cdot T(\mathcal{F}(A))$*, or,* *(b)* $(T1) \cdot T(\mathcal{F}(A)) \subset \mathcal{F}(B) \subset (T1) \cdot T(A)$*. Then there exists a* $\kappa \in B$ *with* $\kappa^2 = 1$ *and a homemorphism* $\psi : \delta B \to \delta A$ *such that*

$$Tf = \kappa \cdot f \circ \psi$$

on δB *for all* $f \in A$*. In particular,* $\tilde{T} = \kappa T$ *is an isometric algebra isomorphism.*

Note whenever $T : A \to B$ and satisfies (1.2.3) we have, $\{1\} = \sigma_\pi(1h) = \sigma_\pi(T1Th)$ for all $h \in \mathcal{F}(A)$, i.e., $T1 \cdot T(\mathcal{F}(A)) \subset \mathcal{F}(B)$. Thus this theorem applies whenever Theorem 1.2.2, does and this is the strongest known version of this type of spectral preserver problem. Also an alternative proof of Shilov's theorem is given in [11] and by Theorem 3.2.17.

One improvement contained here is the introduction of multiplicatively isolating families (m.i.f.) of functions. We prove $\mathcal{F}(A)$ is a m.i.f. of A as is A^{-1}. We establish the following.

Theorem (4.1.9)**.** *Let* $T : A \to B$ *be a mapping between uniform algebras. If there exists a multiplicatively isolating set* $\mathcal{A}$ *such that* $T(\mathcal{A})$ *is a multiplicatively isolating set and*

$$\|TfTg\| = \|fg\|$$

for all $f \in A$ *and* $g \in \mathcal{A}$*, then there exists a homeomorphism* $\psi : \delta B \to \delta A$ *such that*

$$|Tf| = |f \circ \psi| \text{ on } \delta B.$$

In [11] this result is only shown for the case $\mathcal{A} = \mathcal{F}(A)$ and $T(\mathcal{A}) = \mathcal{F}(B)$. In proving the Hatori conjecture of Chapter 5, it is useful to have this stronger version since we can immediately establish invertibles are preserved but not all peaking functions. In considering other cases in the future, one needs only show a m.i.f. is preserved to get similar results.

1.3 Hatori's Conjecture

In private communication in 2005, O. Hatori proposed the following conjecture. A surjective map $T : A \to B$ between uniform algebras satisfying

$$\|TfTg + 1\| = \|fg + 1\| \tag{H}$$

for all $f, g \in A$ is an isometric algebra isomorphism. It is "clearly" not true, since, if T is either negation or conjugation of the function then (H), is satisfied, but negation is not multiplicative and conjugation is not linear. The problem then becomes to characterize such maps. If it is assumed that T is homogeneous, it was shown that T is indeed an isometric isomorphism [11]. Further investigations led to results completely characterizing such maps [12].

Theorem (5.1.4). *Let $T\colon A \to B$ be a surjective map that satisfies $\|TfTg + 1\| = \|fg + 1\|$ for all $f, g \in A$. Then there exist an idempotent $e \in B$ and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that*

$$T(f) = T(1)\left(e\tilde{T}f + e'\overline{\tilde{T}(f)}\right)$$

for all $f \in A$, where $e' = 1 - e$ and $(T1)^2 = 1$.

The interpretation of this result is that T is an "almost" isomorphism. The map T can be thought of an isomorphism composed with a map that negates the values of functions on a connectedness component of the carrier space and conjugates on another. (We also get a result if we replace (H) with $\|TfTg + \lambda\| = \|fg + \lambda\|$ for any fixed non-zero constant.) This is an interesting form to result from a spectral preserver problem since it implies T is, in general, neither linear nor multiplicative but clearly implies the algebras have the same "structure". We have demonstrated spectral preserver problems that show the map T is a composition operator (when $T(1) = 1$), a weighted composition operator, and now a form that is a weighed composition operator composed with a conjugation operator.

Publication

The major results of Chapter 4 were were presented at the Fifth International Conference on Function Spaces in Edwardsville, IL in May 2006 and at a conference at University Cork in Cork, Ireland in October 2006 and published in the Proceedings of the American Mathematical Society in 2007. The results of Chapter 5 will appear in the Central European Journal of Mathematics in 2008. All major results were presented in the Analysis Seminar at University of Montana in April 2008.

Chapter 2

Basics of Uniform Algebras

2.1 Definitions and Examples

Definition 2.1.1. We say A is a *Banach algebra* if A is a Banach space over $\mathbb{C}$ with a multiplication also making A into a ring such that

$$f(\alpha g) = (\alpha f)g = \alpha fg \tag{2.1.1}$$

and $\|fg\| \leq \|f\|\|g\|$ for all $f, g \in A$ and $\alpha \in \mathbb{C}$.

We say A is *unital* if there exists an identity element, $1_A \in A$ such that $1_A \cdot f = f \cdot 1_A = f$ for all $f \in A$ and $\|1_A\| = 1$. We say A is *commutative* if the multiplication is commutative.

It is customary to write both scalar action and multiplication as juxtaposition, relying on context to distinguish between these when necessary. If A is a unital Banach algebra, we may consider $j : \mathbb{C} \to A$ by $j(\lambda) = \lambda \cdot 1_A$ (scalar action of λ on the vector 1_A). Clearly j is linear and multiplicative. Also, j is isometric since $\|j(\lambda)\| = \|\lambda \cdot 1_A\| = |\lambda| \, \|1_A\| = |\lambda|$, where we use the homogeneous property of the norm. Thus j embeds $\mathbb{C}$ into A, and we may identify $\mathbb{C}$ with its image, i.e., we can assume $\mathbb{C} \subset A$. With this perspective the scalar $\alpha \in \mathbb{C}$ acting on the vector $f \in A$ is simply the product of α and f as elements of A, and there is no need to distinguish between scalar action and multiplication. Thus, in the unital case, (2.1.1) simply mandates this perspective.

Definition 2.1.2. Let A be a unital Banach algebra and $f \in A$. The *invertible* elements of A are the members of the set $A^{-1} = \{f \in A : fg = 1 \text{ for some } g \in A\}$. The *spectrum* of f is defined by, $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda \notin A^{-1}\}$.

Example 2.1.3. Clearly $\mathbb{C}$ itself is a unital commutative Banach algebra. The norm, of course, is the modulus, i.e., $\|z\| = |z|$ for every $z \in \mathbb{C}$.

Example 2.1.4. Let X be a Banach space. The set of bounded linear operators from X to itself is denoted by $\mathcal{B}(X)$. This is well known to be a Banach space (with the operator norm) and, in fact, is a Banach algebra with multiplication defined as composition. This is an example of a non-commutative, unital Banach algebra. The identity element of $\mathcal{B}(X)$ is id_X, the identity operator on X. In fact $\mathcal{B}(\mathbb{C}^n)$ can be thought of as the usual ring of $n \times n$ matrices. With this perspective it is clear from the definition that the spectrum is the set of eigenvalues. There are matrices over $\mathbb{R}$ with no eigenvalues, i.e., empty spectrum. This is undesirable which is why we consider Banach algebras over $\mathbb{C}$ only.

Example 2.1.5. The Banach space of bounded sequences, l^∞, can be given coordinate-wise multiplication that makes it into a unital commutative Banach algebra. Clearly $e = (1, 1, \ldots)$ is the multiplicative identity of the algebra. The algebra of zero convergent sequences c_0 is also a commutative Banach subalgebra of l^∞, but it does not have an identity and certainly does not contain e, the identity of l^∞. Each element $x \in c_0$

has a spectrum defined for it as an element of l^∞. However the spectrum of x as an element of the (non-unital) Banach algebra c_0 is undefined. Even if $A \subset B$ are both unital Banach algebras, a given element of A need not have the same spectrum when considered as an element of B.

Example 2.1.6. Let X be a compact Hausdorff space. Then the set of complex-valued, continuous functions on X, denoted $C(X)$, with pointwise addition and multiplication and endowed with the sup norm, is a commutative Banach algebra. Since continuous functions on a compact set attain their maximum modulus at some point of the domain, we have for any $f, g \in C(X)$, $|f(x)g(x)| = \|fg\|$ for some $x \in X$. Thus $\|fg\| = |f(x)g(x)| = |f(x)||g(x)| \leq \|f\|\|g\|$. The identity is the constant function 1 and the invertibles are precisely the functions which do not take the value zero. Thus $\lambda \in \sigma(f)$ if and only $f - \lambda$ takes the value zero, i.e., f takes the value λ. Therefore $\sigma(f) = \mathrm{Ran}(f)$.

Subalgebras of $C(X)$ form a very import class of Banach algebras.

Definition 2.1.7. Let X be a compact Hausdorff space. We say A is a *uniform algebra* on X if

1. the elements of A are complex-valued continuous functions on X, i.e., $A \subset C(X)$,
2. the constant functions are contained in A,
3. the operations are pointwise addition and multiplication,
4. the set A is (topologically) closed in $C(X)$ with the sup norm, and
5. the functions of A *separate the points of* X, i.e., for every $x \neq y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$.

Clearly $C(X)$ itself is a uniform algebra with separation of points given by Urysohn's Lemma. Also since X is a compact Hausdorff space, the supremum is attained on some point $x \in X$. Thus

$$\|f\| = \max_{x \in X} |f(x)|$$

for all $f \in A$. This is also called the uniform norm, hence the term.

Within this class, one of the most important examples is the, so-called, *disk algebra*: $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic}\}$.

Example 2.1.8. Clearly $A(\mathbb{D})$ is a linearly and multiplicatively closed subset of continuous functions. To show it is topologically closed in the uniform norm we recall that uniform limits of analytic functions are analytic. We claim the invertible functions are simply functions that do not take the value zero on $\overline{\mathbb{D}}$. If f is invertible in $A(\mathbb{D})$ then there exists $g \in A(\mathbb{D})$ such that $fg = 1$ then neither f nor g can take the value zero. Conversely, if f is never zero we know $g = \dfrac{1}{f}$ is a analytic function and f is invertible. Thus, as in Example 2.1.6, $\sigma(f) = \mathrm{Ran}(f)$.

2.2 Mappings Between Banach Algebras

Since Banach algebras are simultaneously vector spaces, rings, and metric spaces, the natural mappings in each of these theories (linear transformations, ring homomorphisms, and isometries, respectively), are important mappings for the theory of Banach algebras.

Definition 2.2.1. Let $T : A \to B$ be a mapping between Banach algebras.

1. If $T(fg) = TfTg$ for all $f, g \in A$, then we say T is *multiplicative*.

2. If T is multiplicative and a linear transformation we say T is an *algebraic homomorphism.*

3. If T is a bijective, algebraic homomorphism that preserves the norm, we say it is an *isometric algebraic isomorphism* and A and B are *isometrically, algebraically isomorphic.*

4. Also, if A and B are both unital and $T(1) = 1$ we say T is *unital.*

These terms are standard in functional analysis, although some care is needed. In mathematics, an isomorphism is usually defined so that isomorphic objects are equivalent for the theory being developed. The unqualified term *isomorphism* is generally avoided in the theory of Banach algebras (and Banach spaces) in favor of a more specific description in order to avoid confusion with (weaker) isomorphisms of vector spaces or rings.

2.3 The Spectrum of an Element in an Algebra

In the specific examples examined so far, we characterized the spectrum of elements in the algebra. Here we show that, for any commutative Banach algebra, the spectrum of an element is a non-empty compact set.[1]

Lemma 2.3.1. *Let B be a unital, commutative Banach algebra and $f \in B$.*

(a) If $\|f\| < 1$ then $(1-f) \in B^{-1}$ and $(1-f)^{-1} = \sum_{n=0}^{\infty} f^n$ (with the interpretation $f^0 = 1$).

(b) If $\lambda \in \mathbb{C}$ with $|\lambda| > \|f\|$ we have $(\lambda - f) \in B^{-1}$ and $(\lambda - f)^{-1} = \sum_{n=0}^{\infty} \frac{f^n}{\lambda^{n+1}}$.

(c) If $f \in B^{-1}$ and $g \in B$ such that $\|f-g\| < \frac{1}{\|f^{-1}\|}$ then $g \in B^{-1}$ and $g^{-1} = \sum_{n=0}^{\infty} f^{-n-1}(f-g)^n$.

[1] Much of the development in this section may be done (with care) in the non-commutative case.

Proof. If $\|f\| < 1$ then $\sum_{n=0}^{\infty} \|f^n\| \leq \sum_{n=0}^{\infty} \|f\|^n = \frac{1}{1-\|f\|} < \infty$. Thus the sequence of partial sums is a Cauchy sequence in B, and since B is a Banach space, we have $\sum_{n=0}^{\infty} f^n \in B$. Hence,

$$(1-f)\left(\sum_{n=0}^{\infty} f^n\right) = \sum_{n=0}^{\infty} f^n - \sum_{n=1}^{\infty} f^n = 1,$$

which proves (a).

If $|\lambda| > \|f\|$ then $\left\|\frac{f}{\lambda}\right\| < 1$. By (a), $1 = \left(1 - \frac{f}{\lambda}\right) \sum_{n=0}^{\infty} \left(\frac{f}{\lambda}\right)^n$ which yields,

$$1 = (\lambda - f) \sum_{n=0}^{\infty} \frac{f^n}{\lambda^{n+1}},$$

proving (b).

If $\|f-g\| < \frac{1}{\|f^{-1}\|}$ then $\|1-f^{-1}g\| = \|f^{-1}(f-g)\| \leq \|f^{-1}\|\|f-g\| < 1$. Thus by (a),

$$\begin{aligned}1 &= \left(1-\left(1-f^{-1}g\right)\right)\sum_{n=0}^{\infty}\left(1-f^{-1}g\right)^n = f^{-1}g\sum_{n=0}^{\infty} f^{-n}\left(f-g\right)^n \\ &= g\sum_{n=0}^{\infty} f^{-n-1}\left(f-g\right)^n,\end{aligned}$$

which completes the proof. □

Corollary 2.3.2. *If B is a commutative Banach algebra, then B^{-1} is an open subset of B.*

Proof. By Lemma 2.3.1(c), for each point $f \in B^{-1}$, the open ball centered at f of radius $\frac{1}{\|f^{-1}\|}$ consists entirely of invertible elements. This proves B^{-1} is open in B. □

Lemma 2.3.3. *Let B be a unital, commutative Banach algebra. For every $f \in B$, the set $\sigma(f)$ is a non-empty compact subset of* $\overline{\mathbb{D}}_{\|f\|} := \{\lambda \in \mathbb{C} : |\lambda| \leq \|f\|\}$.

Proof. Fix $f \in B$. For any $|\lambda| > \|f\|$, Lemma 2.3.1(a) gives, $\lambda - f$ is invertible. Thus $f - \lambda$, is invertible, i.e., $\lambda \in \mathbb{C}\backslash\sigma(f)$. The contrapositive of this result gives $\sigma(f) \subset \overline{\mathbb{D}}_{\|f\|} = \{\lambda \in \mathbb{C} : |\lambda| \leq \|f\|\}$.

Let $z_0 \in \mathbb{C}\backslash\sigma(f)$, and take $z \in \mathbb{C}$ such that $|z_0 - z| < \dfrac{1}{\|(f-z_0)^{-1}\|}$. If we define $h = z_0 - f$ and $k = z - f$, then h is invertible and $\|h - k\| = |z_0 - z| < \dfrac{1}{\|h^{-1}\|}$. From Lemma 2.3.1($c$), we have two conclusions. First $k = z - f$ is invertible, as is $f - z$, i.e., $z \in \mathbb{C}\backslash\sigma(f)$. This demonstrates that for every point z_0 in $\mathbb{C}\backslash\sigma(f)$, the open ball centered at z_0 of radius $\dfrac{1}{\left\|(z_0 - f)^{-1}\right\|}$ is entirely contained in $\mathbb{C}\backslash\sigma(f)$. This proves $\mathbb{C}\backslash\sigma(f)$ is open, hence $\sigma(f)$ is closed, and thus compact since we have already established it is bounded. The second conclusion is a formula for $(z - f)^{-1}$ that is valid whenever $|z - z_0| < \dfrac{1}{\left\|(z_0 - f)^{-1}\right\|}$. Specifically

$$\begin{aligned}(z-f)^{-1} = k^{-1} &= \sum_{n=0}^{\infty} h^{-n-1}(h-k)^n \\ &= \sum_{n=0}^{\infty} (z_0 - f)^{-n-1}(z_0 - z)^n \\ &= \sum_{n=0}^{\infty} -(f - z_0)^{-n-1}(z - z_0)^n \end{aligned} \tag{2.3.1}$$

This will be helpful in proving $\sigma(f)$ is non-empty.

Let $r : \mathbb{C}\backslash\sigma(f) \to B$ be defined by $r(z) = (z - f)^{-1}$. By (2.3.1), $r(z)$ has a local power series expansion centered at each $z_0 \in \mathbb{C}\backslash\sigma(f)$ of the form

$$r(z) = \sum_{n=0}^{\infty} -(f - z_0)^{-n-1}(z - z_0)^n \text{ for all } z \text{ with } |z - z_0| < \frac{1}{\left\|(z_0 - f)^{-1}\right\|}.$$

Let $\varphi \in B^*$ be a continuous linear functional and define $r_\varphi : \mathbb{C}\backslash\sigma(f) \to \mathbb{C}$ by $r_\varphi = \varphi \circ r$. The continuity and linearity of φ gives

$$r_\varphi(z) = \sum_{n=0}^{\infty} -\varphi\left((f - z_0)^{-n-1}\right)(z - z_0)^n \text{ for all } z \text{ with } |z - z_0| < \frac{1}{\left\|(z_0 - f)^{-1}\right\|}.$$

Thus r_φ has a local power series expansion for each point $z_0 \in \mathbb{C}\backslash\sigma(f)$ with positive radius of convergence. Therefore r_φ is analytic on its domain. If we were to assume $\sigma(f) = \emptyset$, then r_φ would be entire.

Assume (for contradiction) that $\sigma(f) = \emptyset$, i.e., r_φ is entire. If $z \in \mathbb{C}$ such that $|z| \geq \|f\|$ then by 2.3.1(b) we have,

$$r(z) = (z - f)^{-1} = \sum_{n=0}^{\infty} \frac{f^n}{z^{n+1}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{f}{z}\right)^n.$$

Thus $|r_\varphi(z)| \leq \|\varphi\|\|r(z)\| \leq \|\varphi\|\dfrac{1}{|z|}\dfrac{1}{1 - \|f/z\|} = \dfrac{\|\varphi\|}{|z| - \|f\|} \to 0$ as $|z| \to \infty$. Thus r_φ is bounded and by Liouville's Theorem, constant. By these limits this constant can only be zero. Finally we have $\varphi\left((z - f)^{-1}\right) = 0$ for all $\varphi \in B^*$. A standard result from functional analysis implies that the only element which every linear functional takes to zero is zero [2, III.6.7]. Thus, $(z - f)^{-1} = 0$, but this is a contradiction since 0 is not invertible. Therefore, $\sigma(f)$ is not empty. □

2.4 Maximal Ideal Space

In the theory of Banach spaces, (bounded) linear functionals and codimension-1 subspaces play a key role. In the theory of Banach *algebras,* that role is taken by multiplicative linear functionals and maximal ideals. Let B be a unital commutative Banach algebra. We denote the set of non-trivial, multiplicative linear functionals on B by $\mathcal{M}_B$.

To establish the connection between multiplicative linear functionals and maximal ideals, we first recall a theorem of Gelfand and Mazur.

Theorem 2.4.1 (Gelfand-Mazur Theorem)**.** *Any Banach algebra that is a field is isometrically isomorphic to* $\mathbb{C}$.

Proof. Suppose $B\backslash\{0\} = B^{-1}$, and let $f \in B^{-1}$. By Theorem 2.3.3, there exists $\lambda \in \sigma(f)$, i.e., $f - \lambda$ is not invertible. Since we have assumed B is a field, the only non-invertible element is zero and so $f - \lambda = 0$, i.e, $f = \lambda$. In other words, the standard embedding $j : \mathbb{C} \to B$ given by $j(\lambda) = \lambda \cdot 1_B$ is surjective. Thus j is an isometric algebra isomorphism between $\mathbb{C}$ and B. □

Lemma 2.4.2. *Let B be a commutative Banach algebra. If $\varphi : B \to \mathbb{C}$ is a non-trivial multiplicative linear functional, then $\varphi(f) \in \sigma(f)$ for all $f \in B$, $\|\varphi\| = 1$ (i.e, φ is an element of the unit sphere of the dual of B), and* $\ker\varphi$ *is a maximal ideal.*

Proof. Let φ be a multiplicative linear functional on B. Then $\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)^2$ implies $\varphi(1) = 0$ or 1. If $\varphi(1) = 0$ then $\varphi(f \cdot 1) = \varphi(f)\varphi(1) = 0$ for all $f \in B$, i.e., $\varphi = 0$. If φ is non-trivial, then $\varphi(1) = 1$, and, by linearity, $\varphi(\lambda) = \lambda$ for all $\lambda \in \mathbb{C}$.

If $f \in B^{-1}$, then $1 = \varphi(f)\varphi(f^{-1})$. Thus $\varphi(f) \neq 0$, i.e., φ does not take any invertible element to zero. Let $f \in B$ and $\lambda \notin \sigma(f)$. Then $f - \lambda$ is invertible, and $0 \neq \varphi(f - \lambda) = \varphi(f) - \lambda$ implies $\varphi(f) \neq \lambda$. The contrapositive of this gives us that $\varphi(f) \in \sigma(f)$. By Lemma 2.3.3, $\varphi(f) \in \sigma(f) \subset \overline{\mathbb{D}}_{\|f\|}$ so $|\varphi(f)| \leq \|f\|$ and $\|\varphi\| \leq 1$. Since $\varphi(1) = 1 = \|1\|$, we obtain $\|\varphi\| = 1$.

Since φ a ring homomorphism, $\ker\varphi$ is an ideal and since $\varphi \in B^*$, the ideal has codimension 1. Thus the only subspace of B containing $\ker\varphi$ is $\ker\varphi$ itself of B. Suppose

M is an ideal containing $\ker \varphi$. Since $M$is a subspace of B either $M = \ker \varphi$ or $M = B$ and this proves $\ker \varphi$ is a maximal ideal. $\square$

Lemma 2.4.3. *Let B be a unital commutative Banach algebra. If M is a maximal ideal, there exists a unique multiplicative linear functional on B such that* $\ker \varphi = M$.

Proof. As a proper ideal, M cannot contain invertible elements, i.e., $M \subset B \backslash B^{-1}$ and so its closure, which is clearly an ideal, must be proper since $B \backslash B^{-1}$ is closed. By maximality, M is equal to its closure, i.e., M is closed. Since M is closed, the quotient B/M is a Banach space, and since M is a maximal ideal, B/M is a field. By the Gelfand-Mazur Theorem, B/M is isometrically isomorphic to $\mathbb{C}$. Thus the standard embedding of $\mathbb{C}$ into B/M, $j : \mathbb{C} \to B/M$, is surjective. Let $q : B \to B/M$ be the quotient map and $\varphi = j^{-1} \circ q : B \to \mathbb{C}$. Then φ is a multiplicative linear functional, since it is the composition of multiplicative linear maps, and $\ker \varphi = \ker q = M$. For uniqueness, let ψ be a multiplicative linear functional with kernel M. As the kernel of a linear functional, M has codimension 1. Thus $B = M \oplus \mathbb{C}$ (as vector spaces). Every $f \in B$ can be uniquely written as $f = m + \lambda$ with $m \in M$ and $\lambda \in \mathbb{C}$ and

$$\psi(f) = \psi(m + \lambda) = \psi(m) + \lambda = \lambda = \varphi(m) + \lambda = \varphi(m + \lambda) = \varphi(f),$$

i.e., $\psi = \varphi$. $\square$

These two lemmas show that multiplicative linear functionals and maximal ideals are in bijective correspondence. So, even though $\mathcal{M}_B$ is defined to be the set of multiplicative linear functionals, it is customary to refer to $\mathcal{M}_B$ as the *maximal ideal space.*

Lemma 2.4.4. *Let B be a commutative, unital, Banach algebra. Then*

$$\sigma(f) = \{\varphi(f) : \varphi \in \mathcal{M}_B\}$$

for all $f \in B$.

Proof. In Lemma 2.4.2 it was shown that $\{\varphi(f) : \varphi \in \mathcal{M}_B\} \subset \sigma(f)$. If $\lambda \in \sigma(f)$ then $f - \lambda \notin B^{-1}$. Let $M = B(f - \lambda) = \{g(f - \lambda) : g \in B\}$. Clearly M is an ideal, and it is proper since $1 \in M$ implies there exists a $g \in B$ such that $g(f - \lambda) = 1$ which contradicts $f - \lambda \notin A^{-1}$. By commutative, unital ring theory, M is contained in a maximal ideal. Let φ be the corresponding multiplicative linear functional whose kernel contains $f - \lambda \in M$. So $\varphi(f - \lambda) = 0$ and $\varphi(f) = \lambda$. □

By Lemma 2.4.2, $\mathcal{M}_B$ is a subset of the unit sphere of the dual space, B^*. By the Banach-Alaoglu Theorem [2, V 3.1], the unit sphere of B^* is weak-* compact, so we topologize $\mathcal{M}_B$ by giving it the inherited weak-* topology of the unit sphere in B^*. This is called the *Gelfand topology* on $\mathcal{M}_B$.

Lemma 2.4.5. *The maximal ideal space,* $\mathcal{M}_B$*, is compact in the Gelfand topology.*

Proof. It suffices to show that $\mathcal{M}_B$ is closed in the unit sphere of B^* with the weak-* topology. In this topology a net $\{\varphi_\alpha\}_{\alpha \in \mathcal{I}}$ converges to φ, $\varphi_\alpha \to \varphi$, if and only if $\varphi_\alpha(f) \to \varphi(f)$ (in $\mathbb{C}$) for all $f \in B$. Let φ_α be a net in $\mathcal{M}_B$ with limit φ in the unit sphere of B^*. Then φ is a non-trivial linear functional, and we just need to show it is multiplicative. Thus for every $f, g \in B$,

$$\varphi(f)\varphi(g) = \left(\lim_\alpha \varphi_\alpha(f)\right)\left(\lim_\alpha \varphi_\alpha(g)\right) = \lim_\alpha \varphi_\alpha(f)\varphi_\alpha(g) = \lim_\alpha \varphi_\alpha(fg) = \varphi(fg),$$

due to the continuity of multiplication in $\mathbb{C}$. Thus in the weak-* topology, $\mathcal{M}_B$ is closed subset of the unit ball of B^*. □

Example 2.4.6. Let X be a compact Hausdorff topological space and consider the commutative, unital, Banach algebra $C(X)$. For every $x \in X$ let $\varphi_x : C(X) \to \mathbb{C}$ be given by $\varphi_x(f) = f(x)$. Clearly this is a multiplicative linear functional so $\varphi_x \in \mathcal{M}_B$.

In fact, as we show below, every multiplicative functional on $C(X)$ is of this type. First, however, we need the following lemma.

Lemma 2.4.7. *Let X be a compact Hausdorff space. If I is a proper ideal of $C(X)$, then the elements of I have a common zero.*

Proof. We will prove the contrapositive. Let I be an ideal of $C(X)$ with no common zero. Then $\{X \backslash f^{-1}(0) : f \in I\}$ is an open cover of X. Since X is compact, it has a finite subcover, i.e., there exists $f_1, \ldots, f_n \in I$ with no common zero. Since I is an ideal and $\overline{f_1}, \ldots, \overline{f_n} \in C(X)$ then $g = \sum_{i=1}^{n} f_i \overline{f_i} = \sum_{i=1}^{n} |f_i|^2 \in I$. Since the f_i's have no common zero, g never takes the value zero and is thus invertible, and $I = C(X)$. □

Lemma 2.4.8. *Let X be a compact Hausdorff space. The assignment $x \mapsto \varphi_x$ is a homeomorphism from X onto $\mathcal{M}_{C(X)}$.*

Proof. For any two distinct points $x, y \in X$ there exists a continuous (real-valued) function $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$ by the well-known Urysohn's Lemma ([4, Thm 5.3]). Thus we have $\varphi_x \neq \varphi_y$ for $x \neq y$, and the assignment is injective. To show the assignment is onto, let φ be an arbitrary element of $\mathcal{M}_{C(X)}$, and so $M = \ker \varphi$ is a maximal ideal of $C(X)$. By the previous lemma, the functions in M have at least one common zero, say at x. Thus $M \subset \ker \varphi_x$ so by maximality $\ker \varphi = M = \ker \varphi_x$. The uniqueness in Lemma 2.4.3 implies $\varphi = \varphi_x$.

For continuity we need to show that the assignment preserves convergence of nets, i.e., $x_\alpha \to x$ in X implies $\varphi_{x_\alpha} \to \varphi_x$ in $\mathcal{M}_{C(X)}$. Let $x_\alpha \to x$ be a convergent net in X. For every $f \in C(X)$, $\varphi_{x_\alpha}(f) = f(x_\alpha)$ and f is continuous so $f(x_\alpha) \to f(x) = \varphi(x)$. Thus

$\varphi_{x_\alpha}(f) \to \varphi_x(f)$ for every $f \in C(X)$ which gives $\varphi_{x_\alpha} \to \varphi_x$ in $\mathcal{M}_{C(X)}$. Finally, we have a continuous bijective map from a compact space to a Hausdorff space and thus the map is a homeomorphism. $\square$

2.5 The Gelfand Transform

The connection between $C(X)$ and its maximal ideal space can be applied in general to get a much better understanding of the class of unital, commutative, Banach algebras.

Definition 2.5.1. Let f be an element of a unital, commutative, Banach algebra B. Define the *Gelfand transform* of f, $\hat{f} : \mathcal{M}_B \to \mathbb{C}$, by $\hat{f}(\varphi) = \varphi(f)$. Let $\hat{B} = \{\hat{f} : f \in B\}$ and $\Lambda : B \to \hat{B} \subset C(\mathcal{M}_B)$ be the mapping $f \mapsto \hat{f}$.

Note the Gelfand topology (i.e., relative weak-* topology) given to $\mathcal{M}_B$ insures that each $\hat{f}$ is continuous.

Theorem 2.5.2. *Let B be a commutative, unital, Banach algebra. Then the Gelfand transform $\Lambda : B \to C(\mathcal{M}_B)$ is an algebraic homomorphism which does not increase the norm (i.e., $\|\Lambda\| \leq 1$). Moreover, $\hat{B}$ separates the points in $\mathcal{M}_B$ and contains the constant functions.*

Proof. Let $f, g \in B$, then for all $\varphi \in \mathcal{M}_B$, $\widehat{fg}(\varphi) = \varphi(fg) = \varphi(f)\varphi(g) = \hat{f}(\varphi)\hat{g}(\varphi)$ which shows Λ is multiplicative. Linearity is similar. In the notation of the Gelfand transform, Theorem 2.4.4 becomes $\hat{f}(\mathcal{M}_B) = \sigma(f) \subset \overline{\mathbb{D}}_{\|f\|}$. Thus $\|\hat{f}\| = \max\{|f(\varphi)| : \varphi \in \mathcal{M}_B\} \leq \|f\|$. If $\varphi_1 \neq \varphi_2$ then there is some $f \in B$ where they differ so $\hat{f}(\varphi_1) = \varphi_1(f) \neq \varphi_2(f) = \hat{f}(\varphi_2)$. Finally, $\hat{1}(\varphi) = \varphi(1) = 1$ for all $\varphi \in \mathcal{M}_B$, and, by linearity, Λ takes constants to the corresponding constant functions. $\square$

The Gelfand transform maps any unital, commutative, Banach algebra to a subalgebra of continuous functions on a compact Hausdorff space. Thus, we can gain a lot of understanding of the more general class by examining the, more concrete, class of subalgebras of $C(X)$ where X is a compact Hausdorff space. Uniform algebra theory is equivalent to the study of unital, commutative Banach algebras for which the Gelfand transform is an isometry. In general, Λ need not even be injective. In the most extreme case, it is possible that $\hat{B} = \mathbb{C}$ as the following example shows.

Example 2.5.3. Consider the unital, commutative Banach algebra $\mathbb{C}^n$, equipped with coordinate-wise operations and the sup norm. Let $s : \mathbb{C}^n \to \mathbb{C}^n$ be the *right shift operator* given by

$$s(z_1, z_2, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$

This is clearly a linear operator with operator norm 1. Thus $s \in \mathcal{B}(\mathbb{C}^n)$. Let A be the algebra generated by $s^0 = id_{\mathbb{C}^n}$ and s in $\mathcal{B}(\mathbb{C}^n)$ which has the basis, $s^0, \ldots, s^{n-1}$. Similarly, let A_0 be the algebra generated by s which has the basis $s^1, \ldots, s^{n-1}$. Any $a \in A_0$ has the form $a = \displaystyle\sum_{m=1}^{n-1} \alpha_m s^m$ for some choice of α_m's in $\mathbb{C}$. Since $s^m = 0$ for all $m \geq n$, then

$$a^n = \left(\sum_{m=1}^{n-1} \alpha_m s^m \right)^n = \sum_{m=n}^{n^2-n} \beta_m s^m = 0,$$

for some β_m's in $\mathbb{C}$. In other words, each element of A_0 is nilpotent. Let φ be any multiplicative linear functional on A, then $\varphi(a)^n = \varphi(a^n) = \varphi(0) = 0$ implies $\varphi(a) = 0$ for all $a \in A_0$. Thus every multiplicative linear functional is zero on all of A_0. Since a multiplicative linear functional must preserve the identity, the only multiplicative linear functional on A is the one given by $\displaystyle\sum_{m=0}^{n-1} \alpha_m s^m \overset{\varphi}{\mapsto} \alpha_0$. Thus the kernel of the Gelfand transform of A is A_0,[2] which is a codimension 1 subspace, and $\hat{A} = \mathbb{C}$.

[2]The algebra A_0 is an example of a *radical* algebra.

Of course, if Λ is injective,[3] then $\hat{B}$ is algebraically isomorphic to B, though possibly not isometrically. In particular, $\hat{B}$ need not be closed in the uniform norm. In that case, the Gelfand transform is still quite useful. We consider two norms on $\hat{B}$: the norm from B induced by Λ ($\|\hat{b}\|_1 := \|b\|$) and the uniform norm that $\hat{B}$ has as a subalgebra of $C(\mathcal{M}_B)$. We can then use the uniform norm (which, by the theorem, is dominated by the inherited one) and we may consider $\hat{B}$ as a dense subalgebra of the uniform algebra obtained by closing $\hat{B}$ in the uniform norm.

In the case that the Gelfand transform is not only injective but also isometric, then $\hat{B}$ is a uniform algebra. On the other hand, if B is a uniform algebra on X, then each point evaluation is a multiplicative linear functional and thus in $\mathcal{M}_B$. In fact, X embeds topologically into $\mathcal{M}_B$. The interpretation for this is that $\mathcal{M}_B$ is the largest topological space containing X such that all the functions in B can be continuously extended from X to $\mathcal{M}_B$. The Gelfand transform of each function in B is that extension. This is remarkable since usually one asks if extensions of functions exist in a specifically given topological space, Y containing X. With the Gelfand transform, one need not guess the space Y on which to seek extentions.

One can interpret the Gelfand transform as dividing the study of commutative, unital Banach algebras into two classes: algebras for which the Gelfand transform is trivial ($\hat{B} = \mathbb{C}$) and algebras for which the Gelfand transform is injective. A refinement of the latter case, is the class for which the Gelfand transform is not only injective but also isometric. This is the class (up to isomorphism) of uniform algebras which is the class we will consider in the sequel.

[3] If the Gelfand transform is injective the algebra is *semisimple*.

Chapter 3

Boundaries of Uniform Algebras

3.1 Boundaries and the Peripheral Spectrum

It is well known that analytic functions on the unit disk $\mathbb{D}$ take their maximum modulus on $\mathbb{T}$, which is the topological boundary of $\mathbb{D}$. This phenomenon can be extended for commutative Banach algebras in general.

Definition 3.1.1. Let B be a unital, commutative Banach algebra. A subset E of $\mathcal{M}_B$ is a called a *boundary* if $\max\{|\hat{f}(\varphi)| : \varphi \in E\} = \|\hat{f}\|$ for every $f \in B$.

Clearly $\mathcal{M}_B$ itself is a boundary for B. If A is a uniform algebra on X, then X is a boundary, identified with its embedding in $\mathcal{M}_A$. In this way we will also consider subsets E of X satisfying $\max\{|f(x)| : x \in E\} = \|f\|$ to be boundaries. If we wish to consider properties that require the Gelfand transform of the algebra on the entire maximal ideal space, we simply take the hypothesis that "A is a uniform algebra on $\mathcal{M}_A$" as in the following lemma. This can always be done formally via the Gelfand transform which is an isometric algebra isomorphism.

Lemma 3.1.2. *Let A be a uniform algebra on $\mathcal{M}_A$ and $E \subset \mathcal{M}_A$ be a boundary for A. Then the restriction map $r : A \to A|_E = \{f|_E : f \in A\} \subset C(E)$ given by $r(f) = f|_E$ is an isometric algebra isomorphism.*

This result follows simply by making a few observations. The map is clearly linear and multiplicative. By the definition of boundary, it is norm-preserving and thus an isometric embedding, i.e., injective. It is surjective by definition, and that is all that is required to show. However, we also know the restrictions of constants are constant and E separates points. So, although E need not be closed, if it is, then $A|_E$ is a uniform algebra on E.

A key consequence of this result is that if $f|_E = g|_E$ then $f = g$. This follows from the injectivity in the lemma above but can also be seen directly from the definition of a boundary. If $f|_E = g|_E$ then $(f - g)\,|_E = 0$ thus the maximum modulus of $f - g$ on E is zero, so the maximum modulus on all of X (i.e., the norm) is zero. Thus $\|f - g\| = 0$ implies $f = g$.

Let A be a uniform algebra on X, and define $\mathrm{Ran}(f) = f(X)$. Recall that the point evaluations are multiplicative linear functionals, so $\mathrm{Ran}(f) = f(X) \subset \hat{f}(\mathcal{M}_A) = \sigma(f)$ by Lemma 2.4.4. However this containment may be strict. Consider $A(\mathbb{D})$, which is a uniform algebra on $\overline{\mathbb{D}}$. By the previous lemma, $A(\mathbb{D})|_{\mathbb{T}} \cong A(\mathbb{D})$ is a uniform algebra on $\mathbb{T}$. If $f \in A(\mathbb{D})|_{\mathbb{T}}$ is the function given by $f(z) = z$ then $\mathrm{Ran}(f) = \mathbb{T}$ but $\sigma(f) = \overline{\mathbb{D}}$. It is convenient to have $\mathrm{Ran}(f) = \sigma(f)$, that way all values of the spectrum may be realized by evaluating the function, which can simplify proofs. (This is another case where we add the assumption that A is a uniform algebra on $\mathcal{M}_A$.) We can get an analogous property by considering a portion of the spectrum.

Definition 3.1.3 ([11, 13])**.** Let A be a uniform algebra on X. For every $f \in A$ let

$\sigma_\pi(f) = \{\lambda \in \sigma(f) : |\lambda| = \|f\|\}$. This set is called the *peripheral spectrum [5, 13]* of f. We introduce the notation, $M(f) = \{x \in X : |f(x)| = \|f\|\} = f^{-1}(\sigma_\pi(f))$ which we call the *maximizing set* of f. Let $\mathrm{Ran}_\pi(f) = \{f(x) : x \in M(f)\} = f(M(f))$ denote the *peripheral range.*

Note the spectrum and the peripheral spectrum are invariant under an isometric algebraic isomorphisms. However, $M(f)$ is tied to a specific representation of the algebra. For example if $A(\mathbb{D})$ is the disk algebra, which is isometrically isomorphic to $A(\mathbb{D})|_{\mathbb{T}}$, then $M(1_{A(\mathbb{D})}) = \overline{\mathbb{D}}$ and yet $M(1_{A(\mathbb{D})|_{\mathbb{T}}}) = \mathbb{T}$. It appears the same issue could occur with the peripheral range. However, we will provide an alternative characterization of the peripheral spectrum, which will show this is not the case. To assist with this we give the following lemma.

Lemma 3.1.4. *Let $g \in C(X)$ for X compact Hausdorff and let $\epsilon > 0$. Then for all $x \in X$ we have the following dichotomy,*

$$|g(x) + \epsilon| = \|g\| + \epsilon \textit{ if and only if } g(x) = \|g\|$$
$$|g(x) + \epsilon| < \|g\| + \epsilon \textit{ if and only if } g(x) \neq \|g\|.$$

Proof. Suppose $|g(x) + \epsilon| = \|g\| + \epsilon$, then

$$\|g\| + \epsilon = |g(x) + \epsilon| \leq |g(x)| + \epsilon \leq \|g\| + \epsilon,$$

and there is equality throughout. In particular, $\|g\| = |g(x)|$. Using this we have,

$$\begin{aligned}
|g(x)|^2 + 2|g(x)| + \epsilon^2 &= |g(x) + \epsilon|^2 = (\|g\| + \epsilon)^2 \\
&= \|g(x)\|^2 + 2\|g(x)\| + \epsilon^2 \\
&= |g(x) + \epsilon|^2 = (g(x) + \epsilon)\left(\overline{g(x)} + \epsilon\right) \\
&= |g(x)|^2 + g(x) + \overline{g(x)} + \epsilon^2 \\
&= |g(x)|^2 + 2\operatorname{Re} g(x) + \epsilon^2
\end{aligned}$$

which implies $\|g\| = |g(x)| = \operatorname{Re} g(x) = g(x)$. The converse is clear. Thus we have established the first assertion and its contrapositive which is $|g(x) + \epsilon| \neq \|g\| + \epsilon$ if and only if $g(x) \neq \|g\|$. However for all x we have, $|g(x) + \epsilon| \leq \|g + \epsilon\| \leq \|g\| + \epsilon$, so $|g(x) + \epsilon| \neq \|g\| + \epsilon$ if and only if $|g(x) + \epsilon| < \|g\| + \epsilon$. □

Lemma 3.1.5. *Let A be a uniform algebra on $\mathcal{M}_A$ and let $E \subset \mathcal{M}_A$ be a boundary of A. For any $f \in A$ the following are equivalent:*

(a) $\alpha \in \sigma_\pi(f)$,

(b) $\|\overline{\alpha} f + 1\| = \|f\|^2 + 1$ *and* $|\alpha| = \|f\|$,

(c) $\|\overline{\alpha} f + 1\| \geq \|f\|^2 + 1$ *and* $|\alpha| = \|f\|$,

(d) there exists $x \in E$ such that $x \in M(f)$ and $f(x) = \alpha$.

Proof. Let $\alpha \in \sigma_\pi(f)$. Since the functions of A are defined on their maximal ideal space Lemma 2.4.4 applies to give an $x \in \mathcal{M}_A$ such that $f(x) = \alpha$ and $|\alpha| = \|f\|$. Lemma 3.1.4 gives $\|\overline{\alpha} f + 1\| = \|\overline{\alpha} f\| + 1 = \|f\|^2 + 1$ justifying (b).

The fact that (b) implies (c) is trivial. We now show show (c) implies (d). If $\alpha = 0$ the justification is trivial so assume $\alpha \neq 0$. By definition of boundary there exists $x \in E$

such that $|\overline{\alpha}f(x)+1| = \|\overline{\alpha}f+1\|$. We have,

$$\|\overline{\alpha}f\|+1 = \|f\|^2+1 \leq \|\overline{\alpha}f(x)+1\| = |\overline{\alpha}f(x)+1| \leq |\overline{\alpha}f(x)|+1 \leq \|\overline{\alpha}f(x)\|+1$$

and we have equality throughout. By Lemma 3.1.4, we have $\overline{\alpha}f(x) = \|\overline{\alpha}f\| = |\overline{\alpha}|\,|\alpha| = \overline{\alpha}\alpha$, which implies $f(x)=\alpha$ since $\alpha \neq 0$. Clearly $|f(x)| = |\alpha| = \|f\|$, i.e., $x \in M(f)$.

Finally we show (d) implies (a). If $f(x)=\alpha$ and $x \in M(f)$, then $\alpha \in \sigma(f)$ and $|\alpha| = \|f\|$. These are the necessary conditions for $\alpha \in \sigma_\pi(f)$. □

Corollary 3.1.6. *If A is a uniform algebra on X and $E \subset X$ is a boundary then for all $f \in A$ we have* $\mathrm{Ran}_\pi(f) = \sigma_\pi(f) = f(M(f) \cap E)$.

This result is an immediate consequence of $(a) \implies (b)$ with both E and X considered as boundaries of $\mathcal{M}_A$. In other words, this says that every point in the peripheral spectrum can be assumed to be a maximum modulus value taken by the function on X or any other boundary.

3.2 Peaking Functions and the Choquet Boundary

In this section we are seeking a "small" boundary for a given uniform algebra. If X is metrizable, it can be shown that A does have a smallest boundary, which is automatically closed. However, for general compact Hausdorff spaces there is no smallest boundary, although there is a smallest *closed* boundary, called the *Shilov* boundary. One of the purposes of this chapter is to give an alternative proof of this well-known result. We first develop another well-known boundary, called the *Choquet* boundary, that is not in general closed, but is contained in every closed boundary.

For the remainder of the section let A be a fixed uniform algebra on $X = \mathcal{M}_A$.

Definition 3.2.1. A non-empty set $E \subset X$ is called an m-set if it is the (arbitrary) intersection of maximizing sets, i.e., $E = \bigcap_{f \in A_E} M(f) \neq \emptyset$ for some subset $A_E \subset A$. For an m-set E, let $\mathcal{E}_E = \{E' \subset E : E' \text{ is an } m\text{-set}\}$ be the the family of all m-sets contained in E. Note that inclusion in a partial order on $\mathcal{E}_E$ and the set of all m-sets is simply $\mathcal{E}_X$.

Lemma 3.2.2. *For each m-set E, the family $\mathcal{E}_E$ contains minimal elements.*

Proof. We will apply Zorn's lemma. Let $\mathcal{C}$ is a chain of $\mathcal{E}_E$. Clearly $\bigcap_{E' \in \mathcal{C}} E'$ is a lower-bound for the chain and an m-set in $\mathcal{E}_E$, provided it is non-empty. Since X is compact and m-sets are closed, the finite intersection property applies. Thus it suffices to show that a finite chain has non-empty intersection. If $E_1 \subset E_2 \subset \ldots \subset E_n$ is a finite chain, then clearly $\emptyset \neq E_1 = \bigcap_{i=1}^{n} E_i$. Therefore, by Zorn's lemma, $\mathcal{E}_E$ has minimal elements. $\square$

Lemma 3.2.3. *Let $\delta A = \bigcup \{E : E \text{ is minimal in } \mathcal{E}_X\}$. The set δA is a boundary for A and each m-set meets δA.*

Proof. First we show that each m-set meets δA. Let $E \subset X$ be an m-set and E_0 be a minimal element of $\mathcal{E}_E$. Suppose $F \in \mathcal{E}_X$ such that $F \subset E_0$, then $F \in \mathcal{E}_E$ and, by minimality of E_0 in $\mathcal{E}_E$, $F = E_0$. Thus E_0 is also minimal in $\mathcal{E}_X$ and $E_0 \subset \delta A$. Thus $\emptyset \neq E_0 \subset \delta A \cap E$. Finally, for any $f \in A$, $M(f)$ is an m-set so $M(f) \cap \delta A \neq \emptyset$, which shows δA is a boundary for A. $\square$

We call the minimal elements in $\mathcal{E}_X$ *minimal m-sets*. When dealing with m-sets, it is useful to note that if a finite number of non-zero functions share a common maximizer, then the maximizing set of their product is the intersection of their maximizing sets. This is made precise by the following lemma.

Lemma 3.2.4. *Let* $\{f_i\}_{i=1}^n \subset A\setminus\{0\}$*. The following are equivalent:*

(a) the functions have a common maximizer, i.e.,

$$\bigcap_{i=1}^n M(f_i) \neq \emptyset,$$

(b) the norm of the product is the product of the norms, i.e.,

$$\left\|\prod_{i=1}^n f_i\right\| = \prod_{i=1}^n \|f_i\|\,, \text{ and}$$

(c) the maximizers of the product are the common maximizers,

$$M\left(\prod_{i=1}^n f_i\right) = \bigcap_{i=1}^n M(f_i).$$

Proof. By induction it suffices to show this for $n = 2$.

To prove (a) implies (b) assume $f, g \in A$ such that $M(f) \cap M(g) \neq \emptyset$. Then there exists $x \in M(f) \cap M(g)$ and $\|f\|\,\|g\| = |f(x)|\,|g(x)| \leq \|fg\| \leq \|f\|\,\|g\|$ and so $\|fg\| = \|f\|\,\|g\|$.

To show (b) implies (c), suppose $\|fg\| = \|f\|\,\|g\|$ and let $x \in M(fg)$. Thus $|f(x)g(x)| = \|fg\| = \|f\|\,\|g\|$. Since $f \neq 0$ we have $\dfrac{|f(x)|}{\|f\|} \leq 1$ so $\|g\| = \dfrac{\|f\|\,\|g\|}{\|f\|} = \dfrac{|f(x)g(x)|}{\|f\|} \leq |g(x)|$ which implies $x \in M(g)$ and $x \in M(f)$ similarly. Thus $\emptyset \neq M(fg) \subset M(f) \cap M(g)$. If $x \in M(f) \cap M(g)$ then $|f(x)g(x)| = \|f\|\,\|g\| = \|fg\|$ and $x \in M(fg)$ which proves (b). Since clearly (c) implies (a) the proof is complete. $\square$

Definition 3.2.5. Let A be a uniform algebra. A function $h \in A$ is called a *peaking function* if $\sigma_\pi(h) = \{1\}$. Let $\mathcal{F}(A)$ be the family of all peaking functions of A, namely, $\mathcal{F}(A) = \{h \in A\colon \sigma_\pi(h) = \{1\}\}$. For the case special case of $h \in \mathcal{F}(A)$, the maximizing set $M(h)$ is called the *peak set* of h, and it is customary to denote it instead by $P(f)$. For any $E \subset X$ define $\mathcal{F}_E(A) = \{h \in \mathcal{F}(A) : E \subset P(h)\}$, and for any $x \in X$ define

$\mathcal{F}_x(A) = \mathcal{F}_{\{x\}}(A)$. The non-empty intersection of peak sets is called a p-set which is clearly an m-set.

This next lemma directly establishes the existence and, indeed, the prevalence of peaking functions.

Lemma 3.2.6. *Let A be a uniform algebra.*

(a) *For every $f \in A \backslash \{0\}$, $\lambda \in \sigma_\pi(f)$, and $\epsilon > 0$ the function $h = \dfrac{f/\lambda + \epsilon}{1+\epsilon}$ is a peaking function of A with peak set $f^{-1}(\lambda)$ and $\sigma_\pi(fh) = \{\lambda\}$.*

(b) *$A = \mathbb{C} \cdot \mathcal{F}(A) + \mathbb{C}$, i.e., the linear span of $\mathcal{F}(A)$ is A.*

(c) *For each fixed $r \in (0,1)$ then $\mathbb{C}\left(\mathcal{F}(A) - r\right) = A$.*

Proof. Clearly if $f(x) = \lambda$ then $h(x) = 1$. Note

$$\begin{aligned}|h(x)| &= \left|\frac{f(x)/\lambda}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\right| \leq \left\|\frac{f(x)/\lambda}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\right\| \leq \frac{\|f/\lambda\|}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} \\ &= \frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon} = 1.\end{aligned}$$

By Lemma 3.1.4, equality holds if only if $\dfrac{f(x)/\lambda}{1+\epsilon} = \left\|\dfrac{f(x)/\lambda}{1+\epsilon}\right\| = \dfrac{1}{1+\epsilon}$, i.e., $f(x) = \lambda$ in which case, not only is $|h(x)| = 1$ but $h(x) = 1$. Thus $h \in \mathcal{F}(A)$ and $P(h) = f^{-1}(\lambda)$. The assertion in (b) comes by solving, $f = (\lambda + \lambda\epsilon)\, h - \lambda\epsilon$, then (c) follows by $f = (\lambda + \lambda\epsilon)\left(h - \dfrac{\epsilon}{1+\epsilon}\right)$ for $\epsilon = \dfrac{r}{1-r}$. Since $M(h) = P(h) = f^{-1}(\lambda) \subset M(f)$, the functions h and f share a common maximizer. Thus $\sigma_\pi(fh) = \big\{f(x)h(x) : x \in P(h) \cap M(f) = P(h) = f$ $\{\lambda \cdot 1\}$. $\square$

Parts (b) and (c) of the lemma show that the set of peaking functions in a uniform algebra is a very large class.

Lemma 3.2.7. *Every minimal m-set is a p-set.*

Proof. Let E be a minimal m-set and $x \in E$. By definition $E = \bigcap_{f \in A_E} M(f)$ for some family $S \subset A$. Without loss of generality we may assume $f \neq 0$ for all $f \in S$. Since $f(x) \in \sigma_\pi(f)$, Lemma 3.2.6(a) implies that there exists some $h_f \in \mathcal{F}(A)$ such that $P(h_f) = f^{-1}(f(x)) \subset M(f)$. Thus $x \in P(h_f)$ and

$$x \in \bigcap_{f \in S} P(h_f) \subset \bigcap_{f \in S} M(f) = E$$

which implies equality since E is minimal. □

Lemma 3.2.8. *Let E be a p-set. For every open set U containing E there exists $h \in \mathcal{F}_E(A)$ such that $P(h) \subset U$.*

Proof. By definition of p-set, there is some set $S \subset \mathcal{F}_E(A)$ such that $E = \bigcap_{f \in S} P(f)$. Thus $E \subset \bigcap_{f \in \mathcal{F}_E(A)} P(f) \subset \bigcap_{f \in S} P(f) = E$ which gives

$$\bigcup_{f \in \mathcal{F}_E(A)} X \backslash P(f) = X \backslash E.$$

Since $E \subset U$, we have $\{X \backslash P(f) : f \in \mathcal{F}_E(A)\} \cup \{U\}$ is an open cover of X (compact). Thus there exists, $f_1, \ldots, f_n \in \mathcal{F}_E(A)$ such that $\bigcup_{i=1}^{n} X \backslash P(f_i) \cup U = X$, i.e., $\bigcap_{i=1}^{n} P(f_i) \cap (X \backslash U) = \emptyset$. Let $h = \prod_{i=1}^{n} f_i$. By Lemma 3.2.4, $M(h) = \bigcap_{i=1}^{n} P(f_i)$, so clearly $h \in \mathcal{F}_E(A)$. Thus $P(h) \cap (X \backslash U) = \emptyset$, i.e., $P(h) \subset U$. □

The following is a generalization of a important result called Bishop's Lemma.

Lemma 3.2.9 (Bishop's Lemma for p-sets [11])**.** *Let A be a uniform algebra on X and*

E be a p-set of A. If $f \in C(X)$ is such that $f|_E \not\equiv 0$, then there is a peaking function $h \in \mathcal{F}_E(A)$ such that fh takes its maximum modulus on E.

Proof. Without loss of generality, assume that the maximum modulus of f on E is 1. F or each $n \in \mathbb{N}$ define the open set

$$U_n = \left\{ x \in X \colon |f(x)| < 1 + \frac{1}{2^{n+1}} \right\},$$

and observe that $U_1 \supset U_2 \supset \ldots \supset E$. For each fixed $n \in \mathbb{N}$, Lemma 3.2.8 gives that there exists $k \in \mathcal{F}_E(A)$ such that $P(k) \subset U_n$. For each $x \in X \backslash U_n$ then $|k(x)| < 1$ and since $X \backslash U_n$ is compact the maximum modulus on $X \backslash U_n$ is strictly less than 1. For a large enough power $m \in \mathbb{N}$, $|k^m| < \dfrac{1}{2\|f\|}$ on $X \backslash U_n$. Define $h_n = k^m$.

Now define $h = \displaystyle\sum_{n=1}^{\infty} \frac{h_n}{2^n}$. It is clear that the series is absolutely convergent (since $\|h_n\| = 1$) and $\|h\| \leq 1$. In fact $|h(x)| = 1$ if and only if $|h_n(x)| = 1 = h_n(x)$ for all n, which implies $x \in E$, thus $h(x) = 1$. Since $|h(x)| = 1$ implies $h(x) = 1$, it follows that $h \in \mathcal{F}_E(A)$. Clearly $\|fh\| \geq 1$, since $\max_{x \in E} |f(x)| = 1$ and $h \equiv 1$ on E. We claim that $\|fg\| \leq 1$. Allow $U_0 = X$ and fix $x \in X$. There are two cases to consider.

Case 1: $x \in U_{N-1} \backslash U_N$ for some $N \geq 1$. Then $x \in U_1, U_2, \ldots, U_{N-1}$ and $x \in X \backslash U_n$ for all $n \geq N$. Thus $|f(x)| < 1 + \dfrac{1}{2^N}$, $|h_n(x)| < \dfrac{1}{2\|f\|} < \dfrac{1}{2}$ for all $n \geq N$, and

$$\begin{aligned}|h(x)| &\leq \sum_{n=1}^{N-1} \frac{|h_n(x)|}{2^n} + \sum_{n=N}^{\infty} \frac{|h_n(x)|}{2^n} \leq \sum_{n=1}^{N-1} \frac{1}{2^n} + \sum_{n=N}^{\infty} \frac{1/2}{2^n} \\ &= \left(1 - \frac{1}{2^{N-1}}\right) + \frac{1}{2} \cdot \frac{1}{2^{N-1}} = 1 - \frac{1}{2^{N-1}} \left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2^N}.\end{aligned}$$

Consequently, $|f(x)h(x)| < \left(1 + \dfrac{1}{2^N}\right)\left(1 - \dfrac{1}{2^N}\right) = 1 - \dfrac{1}{4^N} < 1.$

Case 2: $x \in \bigcap_{n=1}^{\infty} U_n$. Then $x \in U_n$ for all n. Hence $|f(x)| < 1 + \frac{1}{2^{n+1}}$ for all n, so $|f(x)| \leq 1$ and therefore, $|f(x)h(x)| \leq 1$, since, as a peaking function, $|h(x)| \leq 1$. □

The original Bishop's Lemma can be obtained as a corollary of Lemma 3.2.9.

Lemma 3.2.10 (*Bishop's Lemma [1]*). *Let E be a peak set of a uniform algebra A and let $f \in A$ be such that $f|_E \not\equiv 0$. Then there is a peaking function $h \in \mathcal{F}_E(A)$ such that fh takes its maximum modulus only within $E = P(h)$.*

Proof. According to Bishop's Lemma for p-sets, there exists $k \in \mathcal{F}_E(A)$ such fk takes its maximum modulus on E, i.e., $M(fk) \cap E \neq \emptyset$. By definition of peak sets, there exists some $k' \in \mathcal{F}(A)$ such that $P(k) = E$. Thus fk and k' have a common maximizer and $M(fkk') = M(fk) \cap P(k') \subset E$. The lemma is satisfied by $h = kk'$. □

Lemma 3.2.11. *For all $f \in A$ and all $x \in \delta A$ such that $f(x) \neq 0$ there exists $h \in \mathcal{F}_x(A)$ such that $\sigma_\pi(fh) = \{f(x)\}$.*

Proof. Let E be the minimal p-set containing x. Bishop's Lemma for p-sets asserts that there exists $k \in \mathcal{F}_E(A)$ such that fk takes its maximum modulus on E, i.e., $M(fk) \cap E \neq \emptyset$. By minimality $E \subset M(fk)$ and $f(x) \in \sigma_\pi(fk)$. By Lemma 3.2.6(a) there exists $k' \in \mathcal{F}(A)$ with $P(k') = (fk)^{-1}(f(x))$ such that $\sigma_\pi(fkk') = \{f(x)\}$. Thus $k' \in \mathcal{F}_x(A)$, since $f(x) = f(x)k(x)$ and $h = kk'$ is as desired. □

Lemma 3.2.12. *Minimal m-sets are singletons. Equivalently, $x \in \delta A$ if and only if $\{x\}$ is a p-set.*

Proof. Let E be a minimal m-set and $x, y \in E$. If we show $f(x) = f(y)$ for all $f \in A$, then, by the "separation of points" condition required in the definition of a uniform algebra, $x = y$ and E is a singleton. Note by Lemma 3.2.7 E is a p-set.

Let $f \in A$. If both $f(x)$ and $f(y)$ are zero then there is nothing to show. Without loss of generality, assume $f(x) \neq 0$. By 3.2.11 there exists $h \in \mathcal{F}_x(A)$ such that $\sigma_\pi(fh) = \{f(x)\}$. So $P(h) \cap E \neq \emptyset$ and, by minimality, $E \subset P(h)$. Similarly, $E \subset M(fh)$. Thus $y \in M(fh) \cap P(h)$, which gives, $f(y) = f(y)h(y) \in \sigma_\pi(fh) = \{f(x)\}$ which completes the proof. $\square$

The following corollary is a restatement of Lemma 3.2.8 combined this result.

Corollary 3.2.13. *For every $x \in \delta A$ and every open neighborhood U of x there exists $h \in \mathcal{F}_x(A)$ such that $P(h) \subset U$.*

Definition 3.2.14. A point $x \in \mathcal{M}_A$ is a *p-point* [3] if for every open neighborhood U of x there exists $h \in \mathcal{F}_x(A)$ such that $P(h) \subset U$. The *Choquet boundary* is the set of all p-points.

Corollary 3.2.13 and the previous results of this section provide an alternative proof of the existence of the Choquet boundary expressed in the following theorem.

Theorem 3.2.15 (Choquet Boundary Theorem for Uniform Algebras)**.** *If A is a uniform algebra then the set of all p-points is a boundary of A.*

Lemma 3.2.16. *The boundary δA is contained in any closed boundary $E \subset \mathcal{M}_A$ of A.*

Proof. Let $E \subset \mathcal{M}_A$ be a closed boundary of A. Suppose that $\delta A \backslash E \neq \emptyset$ and let $x \in \delta A \backslash E \subset X \backslash E$. Then $X \backslash E$ is an open neighborhood of x in X and, by Corollary 3.2.13, there exists $h \in \mathcal{F}_x(A)$ such that $P(h) \subset X \backslash E$. Thus $|h(x)| < 1 = \|h\|$ on E, which contradicts the assumption that E is a boundary. Consequently, $\delta A \backslash E = \emptyset$, thus $\delta A \subset E$. Therefore, δA is contained in every closed boundary. $\square$

Theorem 3.2.17 (Shilov's Theorem)**.** *The intersection of all closed boundaries of a unital commutative Banach is a closed boundary.*

Proof. Let ∂A be the intersection of all closed boundaries. By the previous lemma, δA is contained in every closed boundary, so $\delta A \subset \partial A \subset \overline{\delta A}$. Since ∂A contains a boundary it is itself a boundary and, as the intersection of closed sets, it is closed. Thus $\partial A = \overline{\delta A}$ is a boundary. □

Theorem 3.2.17 implies there exists a smallest closed boundary $\partial A = \overline{\delta A}$. This result is well known and ∂A is the famous Shilov boundary. The standard proof found of its existence in [17, Thm 1.5.2] is much shorter but uses the Gelfand topology which is not elementary. Our proof of Theorem 3.2.17 is longer, but more constructive. Also we simultaneously prove the existence of the Choquet boundary, which is useful when we cannot use the Shilov boundary, e.g. [11].

3.3 Multiplicatively Isolating Families in a Uniform Algebra

Recall a subset of X is a *p-set* if it is an (arbitrary, non-empty) intersection of peak sets and x is a *p-point* if $\{x\}$ is a p-set. The Choquet boundary, δA, the is set of all p-points.

We define a general class of functions that have properties similar to the set of peaking functions, $\mathcal{F}(A)$.

Definition 3.3.1. Let $\mathcal{A}$ be a subset of a uniform algebra A and for each $x \in \delta A$ let $\mathcal{A}_x = \{f \in \mathcal{A}\backslash\{0\} : x \in M(f)\}$. We say $\mathcal{A}$ is a *multiplicatively isolating family* (m.i.f) of A if

(i) each $\mathcal{A}_x$ is multiplicatively closed and

(ii) for every open neighborhood U of x there is $f \in \mathcal{A}_x$ such that $M(f) \subset U$.

The set of peaking functions, $\mathcal{F}(A)$, is itself a multiplicatively isolating family. Clearly if two functions peak on $x \in \delta A$ then so does their product, which is then a peaking function, so (i) is satisfied. Condition (ii) is satisfied by Corollary 3.2.13. The entire algebra, A, is multiplicatively isolating since it is closed under multiplication and $\mathcal{F}(A) \subset A$ provides the functions whose existence is required by (ii).

Lemma 3.3.2. *If $\mathcal{A}$ is a m.i.f. of A then $\mathcal{A}_{x_1} \subset \mathcal{A}_{x_2}$ implies $x_1 = x_2$.*

Proof. We prove the contrapositive. If $x_1 \neq x_2$ then there exists an open neighborhood U of x_1 that excludes x_2. By condition (ii) of the definition there exists $f \in \mathcal{A}_{x_1}$ such that $M(f) \subset U$. Thus $f \notin \mathcal{A}_{x_2}$, i.e., $\mathcal{A}_{x_1} \not\subset \mathcal{A}_{x_2}$. □

The importance of multiplicatively isolating families is illustrated by the following lemma.

Lemma 3.3.3. *Let $\mathcal{A}$ be a multiplicatively isolating family of A. Then for every $f \in A$ and $x \in \delta A$,*

$$\inf_{h \in \mathcal{A}_x} \frac{\|fh\|}{\|h\|} = |f(x)|. \tag{3.3.1}$$

Proof. For all $h \in \mathcal{A}_x \backslash \{0\}, |h(x)| = \|h\|$ so $\dfrac{\|fh\|}{\|h\|} \geq \dfrac{|f(x)||h(x)|}{\|h\|} = |f(x)|$ and this gives $\inf\limits_{h \in \mathcal{A}_x} \dfrac{\|fh\|}{\|h\|} \geq |f(x)|$. To show the opposite inequality, for each $\epsilon > 0$ we will produce $h \in \mathcal{A}_x$ such that $\dfrac{\|fh\|}{\|h\|} < |f(x)| + \epsilon$. Note if $f = 0$ there is nothing to show. Let X be the carrier space of the uniform algebra, and define $U = \{y \in X : |f(y)| < |f(x)| + \epsilon\}$.

Clearly U is an open neighborhood of $x \in \delta A$, and there exists $k \in \mathcal{A}_x$ such that $M(k) \subset U$. If $U = X$, there is nothing to show, otherwise let $\delta = \max_{y \in X \setminus U} |k(y)|$. Since $X \setminus U$ is compact, the maximum is justified. Since $X \setminus U$ is disjoint from $M(k)$, $\delta < \|k\|$, i.e., $\dfrac{\delta}{\|k\|} < 1$. Thus, for a sufficiently high power n, $\dfrac{\delta^n}{\|k\|^n} < \dfrac{|f(x)| + \epsilon}{\|f\|}$. By taking $h = k^n \in \mathcal{A}_x$ this inequality becomes

$$\max_{y \in X \setminus U} \frac{|h(y)|}{\|h\|} < \frac{|f(x)| + \epsilon}{\|f\|},$$

and so for all $y \in X \setminus U$ we have $\dfrac{|f(y)||h(y)|}{\|h\|} < \dfrac{|f(y)|}{\|f\|}(|f(x)| + \epsilon) \leq |f(x)| + \epsilon$. For $y \in U$, we have $\dfrac{|f(y)||h(y)|}{\|h\|} \leq |f(y)| < |f(x)| + \epsilon$, which completes the proof. $\square$

For each $f \in A$, consider the series $e^f = \sum_{i=0}^{\infty} \dfrac{f^n}{n!}$. Note $\sum_{i=0}^{\infty} \dfrac{\|f\|^n}{n!} = e^{\|f\|} < \infty$ which shows the series is convergent in A. The set $e^A = \{e^f : f \in A\}$ is called the *exponent* of the algebra. Since $e^{f(x)} e^{-f(x)} = 1$ for all $x \in X$, we see e^A consists of invertible elements.

Lemma 3.3.4. *The set $\mathcal{A} = e^A \cap \mathcal{F}(A)$ is a multiplicatively isolating family.*

Proof. Let $x \in \delta A$ and $f, g \in \mathcal{A}_x$, i.e., $f, g \in e^A \cap \mathcal{F}(A)$ and $x \in P(f) \cap P(g)$. Since both f and g are in $\mathcal{F}(A)$ with a common maximizer, x, their product is in $\mathcal{F}(A)$ with maximizer x, i.e., $fg \in \mathcal{F}_x(A)$. There exists $f', g' \in A$ such that $f = e^{f'}$ and $g = e^{g'}$. Thus $fg = e^{f'+g'} \in e^A$ and condition (i) of Definition 3.3.1 is satisfied.

Let $x \in \delta A$ and U be an open neighborhood of x. Then there exists $k \in \mathcal{F}(A)$ such that $P(k) \subset U$. Let $h = e^{k-1}$. If $x \in P(k)$ then $h(x) = e^{k(x)-1} = e^0 = 1$. If $x \notin P(k)$, then $|k(x)| < 1$ and $\operatorname{Re} k(x) < 1$ which gives $\operatorname{Re} k(x) - 1 < 0$. Thus $e^{\operatorname{Re} k(x) - 1} < 1$ and $|h(x)| = \left| e^{k(x)-1} \right| = e^{\operatorname{Re} k(x) - 1} < 1$. Thus $h \in e^A \cap \mathcal{F}(A)$ and $P(h) = P(k) \subset U$ so condition (ii) of Definition 3.3.1 is satisfied. $\square$

Example 3.3.5. Let A be a uniform algebra then the following sets are multiplicatively isolating,

(a) $\mathcal{F}(A)$

(b) $\mathcal{F}(A) \cap e^A$

(c) $\mathcal{F}(A) \cap A^{-1}$

(d) A^{-1}

(e) A

Let A be any of the above sets. In all of the cases, $\mathcal{A}_x$ is closed under products and thus condition (i) of the definition is satisfied. Condition (ii) requires for each $x \in \delta A$ and each open neighborhood there exists $h \in \mathcal{A}_x$ such that $M(f) \subset U$. Since $\mathcal{F}(A) \cap e^A \subset A$ in all the above cases, the existence required is met by an $f \in \mathcal{F}(A) \cap e^A$ by Lemma 3.3.4.

The following lemma is a stronger version of Bishop's Lemma.

Lemma 3.3.6. *If $E \subset X$ is a peak set, and $f \in A$ is such that $f|_E \neq 0$, then there exists $h \in \mathcal{F}(A) \cap e^A$ such that fh attains its maximum modulus exclusively on E. In particular, h is invertible.*

Proof. We will use the following inequality, easily verified with Rolle's Theorem.

$$e^{n(x-1)} < x, \quad \forall\, x \in \left[2^{-1}, 1\right), \quad \forall\, n \geq 2 \tag{3.3.2}$$

Let E be a peak set and $f \in A$ such that $f|_E \neq 0$. By Lemma 3.2.10 there exists $h \in \mathcal{F}(A)$ such that $P(h) = E$ and fh takes its maximum modulus only on E. Choose

$n \geq 2$ such that $\left(e^{-\frac{1}{2}}\right)^n < \dfrac{\|fh\|}{\|f\|}$ and define $k = e^{n(h-1)}$. Firstly, for all $x \in X, |k(x)| = e^{n(\operatorname{Re} h(x)-1)} \leq e^{\operatorname{Re} h(x)-1} \leq 1$ since $\operatorname{Re} h(x) - 1 \leq 0$. Also

$$1 = |k(x)| \iff n(\operatorname{Re} h(x) - 1) = 0 \iff \operatorname{Re} h(x) = 1 \iff h(x) = 1,$$

and thus $1 = |k(x)|$ implies $k(x) = 1$. Therefore $k \in \mathcal{F}(A)$ and $P(k) = P(h) = E$. Also for any $x \in X$ with $\|fh\| = |f(x)h(x)|$, we have $x \in P(h) = P(k)$, which implies $\|fh\| = |f(x)h(x)| = |f(x)k(x)| \leq \|fk\|$.

Finally we show that fk attains its maximum modulus exclusively on E. Let $x \in X$ such that $x \notin P(k)$. Then $x \notin P(h)$ and $-1 \leq \operatorname{Re} h(x) < 1$.

Case 1: $\dfrac{1}{2} \leq \operatorname{Re} h(x) < 1$. Then $|k(x)| = e^{n(\operatorname{Re} h(x)-1)} < \operatorname{Re} h(x) \leq |h(x)|$, by (3.3.2), so $|f(x)k(x)| < |f(x)h(x)| \leq \|fh\| \leq \|fk\|$.

Case 2: $\operatorname{Re} h(x) < \dfrac{1}{2}$. Then $\operatorname{Re} h(x) - 1 < -\dfrac{1}{2}$ and $|k(x)| = \left(e^{\operatorname{Re} h(x)-1}\right)^n < \left(e^{-\frac{1}{2}}\right)^n < \dfrac{\|fh\|}{\|f\|}$. Thus $|f(x)k(x)| < \|f\|\dfrac{\|fh\|}{\|f\|} = \|fh\| \leq \|fk\|$.

Since $|f(x)k(x)| < \|fk\|$ for all $x \notin P(k) = E$, fk attains its maximum modulus exclusively on E. $\square$

Chapter 4

Weakly Peripherally-Multiplicative Mappings Between Uniform Algebras

4.1 Norm Multiplicative Mappings

In this chapter we show a given mapping between uniform algebras is an isometric algebra isomorphism if it satisfies rather general conditions. Suppose A is a uniform algebra on the compact Hausdorff space X, and there exists a homeomorphism $\psi : Y \to X$. Let $T : A \to C(Y)$ be given by $Tf = f \circ \psi$. Since f and ψ are continuous, so is Tf. It is easy to show this map is linear, multiplicative, injective and continuous. For example, $T(f + g) = (f + g) \circ \psi = f \circ \psi + g \circ \psi = T(f) + T(g)$, shows T is additive and mainly consists of applying definitions. Maps defined in this way are called *composition operators*. So, if T is a composition operator, then T is an isometric algebraic isomorphism. However this condition is not necessary. Thus we have the following lemma.

Lemma 4.1.1. *Let $\psi : Y \to X$ be a homeomorphism between topological spaces. Let A be a Banach algebra of continuous functions on X with pointwise operations and uniform norm then $T : A \to C(Y)$ given by $T(f) = f \circ \psi$ is an isometric algebra isomorphism to its image.*

Consider a uniform algebra A; let $\hat{A}$ be its Gelfand transform and $\hat{A}|_E$ be the restriction of the Gelfand transforms to a boundary E. Clearly the restriction map is an algebraic homomorphism, and, since E is a boundary, it is isometric. This implies the restriction is injective and thus an isometric algebraic isomorphism between Banach algebras. If E is closed, for example $E = \partial A$, then the restriction map is an isometric algebraic isomorphism between uniform algebras. However in general ∂A will not be homeomorphic to $\mathcal{M}_A$. In particular, $\partial A(\mathbb{D}) = \mathbb{T}$ and $\mathcal{M}_{A(\mathbb{D})} = \overline{\mathbb{D}}$ which are clearly not homeomorphic. This demonstrates that isomorphic uniform algebras can exist on non-homeomorphic carrier spaces. They will, however, have homeomorphic maximal ideal spaces. If A and B are uniform algebras on their maximal ideal spaces $X = \mathcal{M}_A$ and $Y = \mathcal{M}_B$ correspondingly, then an isometric algebraic isomorphism $T\colon A \to B$ induces a homeomorphism $\psi\colon \mathcal{M}_B \to \mathcal{M}_A$ such that $(Tf)(y) = f(\psi(y))$ for all $f \in A$ and $y \in Y$. Additionally the Shilov and Choquet boundaries will be homeomorphic by restricting ψ. We will seek to find conditions under which a given map $T : A \to B$ is a composition operator of the form $T(f) = f \circ \psi$ for some homeomorphism $\psi : \delta B \to \delta A$. Then the linearity and multiplicativity come immediately. The first step in this task is to show that $|T(f)| = |f \circ \psi|$ as in the following theorem.

Theorem 4.1.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be multiplicatively isolating subsets of uniform algebras A and B respectively. If $T : \mathcal{A} \to \mathcal{B}$ is a surjective mapping such that*

$$\|TfTg\| = \|fg\| \tag{4.1.1}$$

for all $f, g \in \mathcal{A}$ then there exists a homeomorphism $\psi : \delta B \to \delta A$ such that

$$|Tf| = |f \circ \psi| \text{ on } \delta B.$$

In the following lemmas we assume that hypotheses of Theorem 4.1.2 hold.

Lemma 4.1.3. *For all $f \in \mathcal{A}$, $\|Tf\| = \|f\|$.*

Proof. By (4.1.1), $\|Tf\|^2 = \left\|Tf^2\right\| = \|TfTf\| = \|ff\| = \left\|f^2\right\| = \|f\|^2$. □

Lemma 4.1.4. *The inequality $\dfrac{|f|}{\|f\|} \leq \dfrac{|g|}{\|g\|}$ holds on δA if and only if the inequality $\dfrac{|Tf|}{\|Tf\|} \leq \dfrac{|Tg|}{\|Tg\|}$ holds on δB.*

Proof. For the forward direction assume $\dfrac{|f|}{\|f\|} \leq \dfrac{|g|}{\|g\|}$ on δA. Let $y \in \delta B$, $k \in \mathcal{B}_y$ and $h \in \mathcal{A}$ with $T(h) = k$. Since the maximum modulus must be taken on a boundary, $\dfrac{|fh|}{\|f\|} \leq \dfrac{|gh|}{\|g\|}$ on δA implies $\dfrac{\|fh\|}{\|f\|} \leq \dfrac{\|gh\|}{\|g\|}$ and $\dfrac{\|Tf \cdot k\|}{\|k\| \, \|Tf\|} \leq \dfrac{\|Tg \cdot k\|}{\|k\| \, \|Tg\|}$ using condition (4.1.1) and norm equality from Lemma 4.1.3. Taking the infimum over all $k \in \mathcal{B}_y$ and applying Lemma 3.3.3, we have $\dfrac{|Tf(y)|}{\|Tf\|} \leq \dfrac{|Tg(y)|}{\|Tg\|}$.

For the converse, assume $\dfrac{|Tf|}{\|Tf\|} \leq \dfrac{|Tg|}{\|Tg\|}$ on δB. Let $x \in \delta A$, $h \in \mathcal{A}_x$. Then $\dfrac{|TfTh|}{\|Tf\|} \leq \dfrac{|TgTh|}{\|Tg\|}$ on δB implies $\dfrac{\|TfTh\|}{\|Tf\|} \leq \dfrac{\|TgTh\|}{\|Tg\|}$ and $\dfrac{\|fh\|}{\|h\| \, \|f\|} \leq \dfrac{\|gh\|}{\|h\| \, \|g\|}$ using condition (4.1.1) and norm equality from Lemma 4.1.3. Taking the infimum over all $h \in \mathcal{A}_x$, we have $\dfrac{|f(x)|}{\|f\|} \leq \dfrac{|g(x)|}{\|g\|}$. □

Lemma 4.1.5. *There exists a unique bijection $\psi : \delta B \to \delta A$ such that*

$$\mathcal{A}_{\psi(y)} = T^{-1}(\mathcal{B}_y)$$

for all $y \in \delta B$.

Proof. Fix $y \in \delta B$ and define,

$$F_y = \bigcap_{f \in T^{-1}(\mathcal{B}_y)} M(f).$$

First we show that F_y is nonempty. By the finite intersection property, it suffices to show $\bigcap_{k=1}^{n} M(f_k) \neq \emptyset$ for $f_k \in T^{-1}(\mathcal{B}_y)$, since X is compact. Let $f \in T^{-1}\left(\prod_{k=1}^{n} Tf_k\right)$, i.e., $Tf = \prod_{k=1}^{n} Tf_k$. Note

$$|Tf(y)| \leq \|Tf\| = \left\| \prod_{k=1}^{n} Tf_k \right\| \leq \prod_{k=1}^{n} \|Tf_k\| \leq \prod_{k=1}^{n} |Tf_k(y)| = |Tf(y)|$$

so $\|Tf\| = \prod_{k=1}^{n} \|Tf_k\|$ and hence

$$\frac{|Tf|}{\|Tf\|} = \frac{\prod_{k=1}^{n} |Tf_k(y)|}{\prod_{k=1}^{n} \|Tf_k\|} = \prod_{k=1}^{n} \frac{|Tf_k|}{\|Tf_k\|}.$$

Since each factor is clearly less than or equal to one, $\frac{|Tf|}{\|Tf\|} \leq \frac{|Tf_k|}{\|Tf_k\|}$ for all $k = 1 \ldots n$. Thus $\frac{|f|}{\|f\|} \leq \frac{|f_k|}{\|f_k\|} \leq 1$ on δA, by Lemma 4.1.4. By Lemma 3.2.3, $M(f) \cap \delta A \neq \emptyset$ and for $x \in M(f) \cap \delta A$ we have $1 = \frac{|f(x)|}{\|f\|} \leq \frac{|f_k(x)|}{\|f_k\|} \leq 1$. Consequently, $x \in M\left(\frac{f_k}{\|f_k\|}\right) = M(f_k)$ for any k, implying $x \in \bigcap_{k=1}^{n} M(f_k)$. By Lemma 3.2.3 there exists $x \in \delta A \cap F_y$. Thus $x \in \bigcap_{f \in T^{-1}(B'_y)} M(f)$, i.e., all the functions in $T^{-1}(\mathcal{B}_y)$ maximize on x. Thus we have shown

$$\forall y \in \delta B, \exists x \in \delta A \text{ such that } T^{-1}(\mathcal{B}_y) \subset \mathcal{A}_x. \tag{4.1.2}$$

We now repeat the procedure above to produce a $y \in \delta B$ associated with each given

$x \in \delta A$. Fix $x \in \delta A$ and define

$$E_x = \bigcap_{g \in T(\mathcal{A}_x)} M(g).$$

Similarly we show that E_x is nonempty by establishing $\bigcap_{k=1}^{n} M(T(f_k)) \neq \emptyset$ for $f_k \in \mathcal{A}_x$. Let $f = \prod_{k=1}^{n} f_k$. Note $|f(x)| \leq \|f\| \leq \prod_{k=1}^{n} \|f_k\| = \prod_{k=1}^{n} |f_k(x)| = |f(x)|$ so $\|f\| = \prod_{k=1}^{n} \|f_k\|$ and hence

$$\frac{|f|}{\|f\|} = \prod_{k=1}^{n} \frac{|f_k|}{\|Tf\|}.$$

Since each factor is clearly less than or equal to one, $\dfrac{|f|}{\|f\|} \leq \dfrac{|f_k|}{\|f\|}$ for all $k = 1 \ldots n$. Thus $\dfrac{|Tf|}{\|Tf\|} \leq \dfrac{|Tf_k|}{\|Tf_k\|} \leq 1$ on δB, by Lemma 4.1.4. By Lemma 3.2.3, there exists $y \in M(Tf) \cap \delta B$. For such y we have $1 = \dfrac{|Tf(y)|}{\|Tf\|} \leq \dfrac{|Tf_k(y)|}{\|Tf_k\|} \leq 1$ implies $x \in M\left(\dfrac{Tf_k}{\|Tf_k\|}\right) = M(Tf_k)$ for any k, so $y \in \bigcap_{k=1}^{n} M(Tf_k)$ as desired. By Lemma 3.2.3, there exists $y \in \delta B \cap E_x$. Thus $y \in \bigcap_{g \in T(A'_x)} M(g)$, i.e., all the functions in $T(\mathcal{A}_x)$ maximize on y. Thus we have shown

$$\forall x \in \delta A, \exists y \in \delta B \text{ such that } T(\mathcal{A}_x) \subset B'_y. \tag{4.1.3}$$

Equation (4.1.2) implies there exists $\psi : \delta B \to \delta A$ such that such that $T^{-1}(\mathcal{B}_y) \subset \mathcal{A}_{\psi(y)}$. We now show this function is unique and the containment is actually equality. Fix $y \in \delta B$. Suppose $x_1, x_2 \in \delta A$ such that

$$T^{-1}(\mathcal{B}_y) \subset \mathcal{A}_{x_1} \text{ and } T^{-1}(\mathcal{B}_y) \subset \mathcal{A}_{x_2} \tag{4.1.4}$$

as in (4.1.2). Since T is surjective, this implies $\mathcal{B}_y \subset T(\mathcal{A}_{x_1})$ and $\mathcal{B}_y \subset T(\mathcal{A}_{x_2})$. By (4.1.3) there exists $y_1, y_2 \in \delta B$ such that $T(\mathcal{A}_{x_i}) \subset \mathcal{B}_{y_i}$ for $i = 1, 2$. Thus $\mathcal{B}_y \subset \mathcal{B}_{y_i}$ and by Lemma 3.3.2, $y = y_1 = y_2$. Thus we have $\mathcal{A}_{x_1} \subset T^{-1}T(\mathcal{A}_{x_1}) \subset T^{-1}(\mathcal{B}_{y_1}) = T^{-1}(\mathcal{B}_y) \subset \mathcal{A}_{x_2}$ by (4.1.4). Thus $x_1 = x_2$ by Lemma 3.3.2 and $T^{-1}(\mathcal{B}_y) = \mathcal{A}_{x_1}$. This shows $\psi : \delta B \to \delta A$ is the unique map such that $T^{-1}(\mathcal{B}_y) = \mathcal{A}_{\psi(y)}$. Lastly we show ψ is onto. By (4.1.3) for each $x \in \delta A$ there exists y such that $T(\mathcal{A}_x) \subset \mathcal{B}_y$. This gives $\mathcal{A}_x \subset T^{-1}T(\mathcal{A}_x) \subset T^{-1}(\mathcal{B}_y)$. We also have $T^{-1}(\mathcal{B}_y) = \mathcal{A}_{\psi(y)}$ and so $\mathcal{A}_x \subset \mathcal{A}_{\psi(y)}$ which gives $x = \psi(y)$ by Lemma 3.3.2. □

Lemma 4.1.6. *For all $f \in \mathcal{A}$ and $y \in \delta B$,*

$$|T(f)(y)| = |f(\psi(y))|.$$

Proof. By Lemmas 3.3.3 and 4.1.5 we get

$$\begin{aligned}
|Tf(y)| &= \inf_{k \in \mathcal{B}_y} \|Tf \cdot k\| = \inf_{h \in T^{-1}(\mathcal{B}_y)} \|TfTh\| \\
&= \inf_{h \in \mathcal{A}_{\psi(y)}} \|TfTh\| = \inf_{h \in \mathcal{A}_{\psi(y)}} \|fh\| \\
&= |f(\psi(y))|.
\end{aligned}$$

□

To establish that ψ is a homeomorphism, we will consider a particular topological basis for the Choquet boundary.

Lemma 4.1.7. *The family of sets $\mathfrak{B} = \{|f|^{-1}((\delta, \infty)) : \delta \geq 0, f \in \mathcal{A} \backslash \{0\}\}$ is a basis for the topology of δA.*

Proof. Clearly every element in $\mathfrak{B}$ is open in X. Let $x \in \delta A$ and U be an open neighborhood of x in X (and thus $U \cap \delta A$ is an arbitrary neighborhood of x in δA.) We now show that every such neighborhood contains a neighborhood of x in $\mathfrak{B}$. Since $\mathcal{A}$ is a multiplicatively isolating family of A, there exists an $h \in \mathcal{A}_x \backslash \{0\}$ such that $M(h) \subset U$. If $U = X$, then $|h|^{-1}((0, \infty)) \subset U$. Otherwise $X \backslash U$ is a non-empty compact set and so is $|h|(X \backslash U)$, which is disjoint from the compact set $|h|(M(h))$. Thus there exists a δ such that $\max |h|(X \backslash U) < \delta < \|h\|$. If $|h(y)| > \delta$ then $y \notin X \backslash U$, i.e., $y \in U$. Thus $|h|^{-1}((\delta, \infty) \subset U$ and $\mathfrak{B}$ is a basis for the topology of δA. □

Lemma 4.1.8. *The bijection ψ from Lemma 4.1.5 is a homeomorphism.*

Proof. First we show continuity of ψ. Let $\mathfrak{B}_A$ be the basis as in Lemma 4.1.7 for δA. We need only show $\psi^{-1}(U)$ is open for $U \in \mathfrak{B}_A$. So for some $\delta > 0$, $U = |f|^{-1}((\delta, \infty))$ and $y \in \psi^{-1}(|f|^{-1}((\delta, \infty))$ if and only if $\delta < |f(\psi(y))|$ if and only if $\delta < |Tf(y)|$. Thus $\psi^{-1}(U) = |Tf|^{-1}((\delta, \infty))$.

For the continuity of ψ^{-1}, let $\mathfrak{B}_B$ be the basis as in Lemma 4.1.7 for δB. We need only show $\psi(U)$ is open for $U \in \mathfrak{B}_B$. So for some $\delta > 0$, $U = |Tf|^{-1}((\delta, \infty))$ and $x \in \psi(|Tf|^{-1}((\delta, \infty))$ if and only if $\delta < |Tf(\psi^{-1}(x))|$ if and only if $\delta < |f(x)|$, which yields $\psi(U) = |Tf|^{-1}((\delta, \infty))$. □

We have now proven Theorem 4.1.2, stated at the beginning of the section.

Theorem (**4.1.2**). *Let $\mathcal{A}$ and $\mathcal{B}$ be multiplicatively isolating subsets of uniform algebras A and B, respectively. If $T : \mathcal{A} \to \mathcal{B}$ is a surjective mapping such that*

$$\|TfTg\| = \|fg\|$$

for all $f, g \in \mathcal{A}$ then there exists a homeomorphism $\psi : \delta B \to \delta A$ such that

$$|Tf| = |f \circ \psi| \text{ on } \delta B.$$

The following theorem follows immediately from Theorem 4.1.2.

Theorem 4.1.9. *Let $T : A \to B$ be a mapping between uniform algebras. If there exist a multiplicatively isolating set $\mathcal{A}$ such that $T(\mathcal{B})$ is a multiplicatively isolating set and*

$$\|TfTg\| = \|fg\|$$

for all $f \in A$ and $g \in \mathcal{A}$ then there exists a homeomorphism $\psi : \delta B \to \delta A$ such that

$$|Tf| = |f \circ \psi| \text{ on } \delta B.$$

Proof. Let $\mathcal{B} = T(\mathcal{A})$. The restriction $T|_{\mathcal{A}}$ satisfies Theorem 4.1.2 so we get ψ and $|Tf| = |f \circ \psi|$ for all $f \in \mathcal{A}$. However Lemmas 4.1.5 and 3.3.3 give for all $f \in A$,

$$\begin{aligned} |Tf(y)| &= \inf_{k \in \mathcal{B}_y} \|Tf \cdot k\| = \inf_{h \in T^{-1}(\mathcal{B}_y)} \|TfTh\| \\ &= \inf_{h \in \mathcal{A}_{\psi(y)}} \|TfTh\| = \inf_{h \in \mathcal{A}_{\psi(y)}} \|fh\| \\ &= |f(\psi(y))|. \end{aligned}$$

□

The result obtained in [11, Theorem 1] is a particular case of Theorem 4.1.2 with $\mathcal{A} = \mathcal{F}(A)$. From the examples of multiplicatively isolating families given in Example 3.3.5 we have the following corollaries.

Corollary 4.1.10. *Let $T : A \to B$ be a mapping between uniform algebras satisfying $T(A^{-1}) = B^{-1}$ and*

$$\|TfTg\| = \|fg\|$$

for all $f \in A$ *and* $g \in A^{-1}$. *Then there exists a homeomorphism* $\psi : \delta B \to \delta A$ *such that*

$$|Tf| = |f \circ \psi| \text{ on } \delta B.$$

Corollary 4.1.11. *Let* $T : A \to B$ *be a surjective mapping between uniform algebras such that*

$$\|TfTg\| = \|fg\|$$

for all $f, g \in A$. *Then there exists a homeomorphism* $\psi : \delta B \to \delta A$ *such that*

$$|Tf| = |f \circ \psi| \text{ on } \delta B.$$

In [9] O. Hatori asks if the preservation of peaking functions hypothesis of Theorem 4.1.9 (as was published in [11]) may be replaced by surjectivity. Corollary 4.1.11 answers this in the affirmative.

4.2 Unital Weakly Peripherally-Multiplicative Mappings

Theorem 4.1.2 shows that under its assumptions T resembles an algebra isomorphism. The following examples show, however, that the conditions of Theorem 4.1.2 are necessary but not sufficient for T to be an algebra isomorphism.

Example 4.2.1.

(a) Let A be a uniform algebra, $B = \overline{A}$ and let $T\colon A \to B$ be the conjugation mapping $Tf = \overline{f}$.

(*b*) If $A = B = C(\{x_0\})$, then any $f \in A$ is of the form $re^{i\theta}$. Let $T\colon A \to A$ be defined as $T(re^{i\theta}) = re^{2i\theta}$.

The set of peaking functions of A is a multiplicatively isolating family of A, and, in both cases, $T(\mathcal{F}(A)) = \mathcal{F}(B)$ and T is unital. Also, T is norm-multiplicative since, in case (a), $\|\overline{fg}\| = \|fg\|$ and in case (b), $\left|r_1 e^{2i\theta_1} r_2 e^{2i\theta_2}\right| = r_1 r_2 = \left|r_1 e^{i\theta_1} r_2 e^{i\theta_2}\right|$. In both cases T is multiplicative. However in the first case $T(\lambda f) = T(\lambda)T(\overline{f}) = \overline{\lambda}T(f)$, which is not equal to $\lambda T(f)$ in general. In the second case $T(\lambda f) = T(\lambda)T(f) = \lambda^2 T(f)$, which is not equal $\lambda T(f)$ in general. Thus T is not homogeneous and so T is not an isomorphism, though it satisfies the conditions of Theorem 4.1.2. These examples demonstrate that even unital mappings that preserve the peaking functions and are norm-multiplicative need not be algebra isomorphisms. We now consider strengthening the hypothesis of Theorem 4.1.2 so that we may conclude T is an isometric algebra isomorphism. In [13] it is shown that it suffices to assume T is surjective and the peripheral spectra of elements fg and $TfTg$ are equal, i.e., $\sigma_\pi(TfTg) = \sigma_\pi(fg)$ for all $f, g \in A$. Such mappings are called *peripherally multiplicative*. Strengthening the result of [13], we consider mappings $T\colon A \to B$ between uniform algebras that satisfy the condition

$$\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \emptyset \tag{WPM}$$

for all $f, g \in A$. Such mappings we call *weakly peripherally-multiplicative* mappings.

Lemma 4.2.2 ([11]). *Let $T\colon A \to B$ be a weakly peripherally-multiplicative mapping between two uniform algebras. Then*

(*a*) *T is norm-multiplicative*

(*b*) *$\sigma_\pi(T1Tf) \cap \sigma_\pi(f) \neq \emptyset$ for every $f \in A$*

(c) $1 \in \sigma_\pi\left((Th)^2\right)$ *for every* $h \in \mathcal{F}(A)$*; in particular,* $1 \in \sigma_\pi\left((T1)^2\right)$

(d) $1 \in \sigma_\pi(T1Th)$ *for every* $h \in \mathcal{F}(A)$*.*

If, in addition, T *is unital, then*

(e) $\sigma_\pi(Tf) \cap \sigma_\pi(f) \neq \emptyset$ *for every* $f \in A$

(f) $1 \in \sigma_\pi(Th)$ *for every* $h \in \mathcal{F}(A)$*.*

(g) $1 \in \sigma_\pi(h)$ *for every* $h \in T^{-1}(\mathcal{F}(B))$*.*

(h) *If* $Tf \in \mathcal{F}(B)$*, then* $1 \in \sigma_\pi(f)$*.*

Proof. (a) follows immediately from the weakly peripherally-multiplicative property (WPM), since $|\lambda| = \|f\|$ for every $\lambda \in \sigma_\pi(f)$; (b), (c) and (d) follow by substituting $h = 1$, $f = g$ and $f = 1$ in (WPM) correspondingly. The remaining statements are straightforward. $\square$

However the following example shows that the weakly peripherally-multiplicative condition is not strong for T to be an isometric algebra isomorphism.

Example 4.2.3. Let X be a compact Hausdorff space. Consider two disjoint copies, X_1, X_2 of X. Let $A = C(X)$, $B = C(X_1 \sqcup X_2)$, and define $T\colon A \to B$ by $T(0) = 0, (Tf)|_{X_1} = f$, $(Tf)|_{X_2} = f^2/\|f\|$, $f \neq 0$.

Here T is unital, $\sigma_\pi(f) \subset \sigma_\pi(Tf)$, and $\sigma_\pi(fg) \subset \sigma_\pi(TfTg)$. Therefore, T is weakly peripherally-multiplicative, without being peripherally multiplicative, i.e., the sets $\sigma_\pi(TfTg)$ and $\sigma_\pi(fg)$ do not necessarily coincide. For example these sets are different for $f = 1$ and $g = -1$. Thus the weakly peripherally-multiplicative condition alone does not suffice. Note that T in this example fails to preserve a multiplicatively isolating set of elements. In particular, T fails to perserve the (multiplicatively isolating) family of peaking functions. Thus we will consider mappings that *preserve the peaking functions, i.e.,*

$$\mathcal{F}(B) = T\left(\mathcal{F}(A)\right). \tag{PPF}$$

Lemma 4.2.4. *If $T : A \to B$ is norm multiplicative mapping and preserves the peaking functions then it is unital.*

Proof. By Theorem 4.1.2, $|T1| = |1 \circ \psi| = 1$ on δB, but $T1$ is a peaking function, which implies $T1 \equiv 1$ on δA and so $T1 = 1$. □

Condition (PPF) implies, in particular, that $\sigma_\pi\left(Th\right) = \sigma_\pi\left(h\right)$ for all $h \in \mathcal{F}(A)$, i.e., that T preserves the peripheral spectrum of peaking functions. Note that all unital mappings considered in [5, 6, 7, 13, 14, 15] automatically preserve the peaking functions and are weakly peripherally-multiplicative. The next theorem shows, any such mapping is an algebra isomorphism.

Theorem 4.2.5 ([11])**.** *Let $T\colon A \to B$ be a mapping between uniform algebras. If T is weakly peripherally-multiplicative and preserves the peaking functions,*[1] *then T is an isometric algebra isomorphism.*

Proof. Since weakly peripherally-multiplicative mappings are norm-multiplicative and the set peaking functions of A (resp. B) are a multiplicatively isolating family of A (resp. B), Theorem 4.1.2 implies that the mapping $\psi\colon \delta B \to \delta A$ is a homeomorphism, and

$$|Tf(y)| = |f(\psi(y))| \tag{4.2.1}$$

for all $y \in \delta B$ and $f \in A$. We now show that $(Tf)(y) = f(\psi(y))$.

[1] Note that we do not assume T is surjective.

Take $f \in A$ and $y \in \delta B$. Equation (4.2.1) implies $f(\psi(y)) = 0$ if and only if $(Tf)(y) = 0$, so we may assume that $f(\psi(y)) \neq 0$. Let U be an open neighborhood of y. Bishop's Lemma 3.2.10 implies that there exists a $Th \in \mathcal{F}(B)$ such that $TfTh$ assumes its maximum modulus only within $P(Th) \subset U$. By Lemma 3.2.6 (a), if $\lambda \in \sigma_\pi(TfTh) \cap \sigma_\pi(fh)$, then $(fh)^{-1}(\{\lambda\})$ and $(TfTh)^{-1}(\{\lambda\})$ are peak sets of A and B correspondingly. Hence, by Lemma 3.2.3 these sets meet the Choquet boundaries δA and δB correspondingly. Therefore we can choose elements $y_1, y_2 \in \delta B$ such that

$$\lambda = Tf(y_1) \cdot Th(y_1) = f(\psi(y_2)) \cdot h(\psi(y_2)). \tag{4.2.2}$$

Applying (4.2.1) we obtain

$$\|TfTh\| = |\lambda| = |Tf(y_1) \cdot Th(y_1)| = |f(\psi(y_2)) \cdot h(\psi(y_2))| = |Tf(y_2) \cdot Th(y_2)|.$$

The assumption on Th implies $TfTh$ takes is maximum modulus only on $P(Th)$. Thus both points y_1 and y_2 belong to $P(Th) \subset U$, and therefore, $\psi(y_1), \psi(y_2) \in P(h)$, by (4.2.1). Hence $Th(y_1) = 1 = h(\psi(y_1))$ which applied to (4.2.2), yields $Tf(y_1) = f(\psi(y_2))$. Since for any neighborhood of y there exist two such points, the continuity of Tf and $f \circ \psi$ implies $Tf(y) = f(\psi(y))$, as intended.

Finally note that since δA and δB are boundaries for A and B resp., the restrictions $r_A \colon A \to A|_{\delta A}$ and $r_B \colon B \to B|_{\delta B}$ are isometric algebra isomorphisms by Lemma 3.1.2. Define a mapping $\widetilde{T} \colon A|_{\delta A} \to B|_{\delta B}$ by $\widetilde{T}(f|_{\delta A}) = (Tf)|_{\delta B}$. Clearly, $\widetilde{T} = r_B \circ T \circ r_A^{-1}$ so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "r_A"'] & B \arrow[d, "r_B"] \\ A|_{\delta A} \arrow[r, "\widetilde{T}"] & B|_{\delta B} \end{tikzcd}$$

We have just shown that $\widetilde{T}f = f \circ \psi$. By Lemma 4.1.1, $\widetilde{T}$ is an injective isometric algebra homomorphism. Therefore, $T = r_B^{-1} \circ \widetilde{T} \circ r_A$ is an injective isometric algebra homomorphism. Recall $\mathcal{F}(B) \subset T(A)$ and, by Lemma 3.2.6, the linear span of $\mathcal{F}(B)$ is B. Thus T is also surjective and now we may conclude it is an isometric algebra isomorphism from A onto B. $\square$

Actually one can see from the proof that Theorem 4.2.5 is true if the condition $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \emptyset$ holds for all $f \in A$ and for peaking functions $g \in \mathcal{F}(A)$ only.

Clearly if T is surjective and preserves the peripheral spectra of algebra elements, i.e., $\sigma_\pi(Tf) = \sigma_\pi(f)$, then it preserves the families of peaking functions. In particular, we have the following corollary.

Corollary 4.2.6. *Let A and B be uniform algebras. If $T \colon A \to B$ is a surjective mapping such that*

(i) $\sigma_\pi(Tf) = \sigma_\pi(f)$ for every $f \in A$ and

(ii) $\sigma_\pi(TfTg) \subset \sigma_\pi(fg)$, or $\sigma_\pi(TfTg) \supset \sigma_\pi(fg)$ for all $f, g \in A$,

then T is an isometric algebra isomorphism.

In [13], maps $T : A \to B$ such that $\sigma_\pi(TfTg) = \sigma_\pi(fg)$ are called *peripherally-multiplicative*. If T is unital then $\sigma_\pi(Tf) = \sigma_\pi(T1Tf) = \sigma_\pi(1 \cdot f) = \sigma_\pi(f)$, i.e., T

preserves the peripheral spectra of algebra elements. Hence we get the main result of [13] as a corollary.

Corollary 4.2.7 ([13, Theorem 1]). *Let A and B be uniform algebras. If $T\colon A \to B$ is a surjective mapping such that*

(i) T is unital and

(ii) T is peripherally multiplicative, i.e., $\sigma_\pi(TfTg) = \sigma_\pi(fg)$, $f, g \in A$,

then T is an isometric algebra isomorphism.

The preservation of peaking functions property

$$\mathcal{F}(B) = T(\mathcal{F}(A)) \tag{PPF}$$

is clearly equivalent to the inclusions $\mathcal{F}(B) \subset T(\mathcal{F}(A))$ and $T(\mathcal{F}(A)) \subset \mathcal{F}(B)$. In fact as the next lemma shows, in the case when $\mathcal{F}(B) \subset T(A)$ either of these conditions is sufficient for a weakly peripherally-multiplicative mapping to preserve peaking functions, which leads to a stronger version of Theorem 4.2.5.

Lemma 4.2.8 ([11]). *Let $T\colon A \to B$ be a weakly peripherally-multiplicative mapping between two uniform algebras. If*

(a) $\mathcal{F}(B) \subset T(\mathcal{F}(A))$, or,

(b) $T(\mathcal{F}(A)) \subset \mathcal{F}(B) \subset T(A)$,

then $\mathcal{F}(B) = T(\mathcal{F}(A))$, i.e., T preserves the peaking functions.

Proof. (a) Let $f \in \mathcal{F}(A)$ and $\lambda \in \sigma_\pi(Tf)$. Thus $(Tf)^{-1}(\lambda)$ is a peak set by Lemma 3.2.6 (a). Bishop's Lemma implies that there exists $k \in \mathcal{F}_{(Tf)^{-1}(\lambda)}(B)$ such that $(Tf) \cdot k$ takes its maximum modulus only on $(Tf)^{-1}(\lambda)$, i.e., $\sigma_\pi((Tf) \cdot k) = \{\lambda\}$. Since T preserves the norms, $\|(Tf) \cdot k\| = |\lambda| = \|Tf\| = \|f\| = 1$. By the hypothesis there exists $h \in \mathcal{F}(A)$ such that $Th = k$. Norm-multiplicativity then implies $\|fh\| = \|TfTh\| = 1$. Since the product of peaking functions is a peaking function if and only if its norm is 1, we conclude that fh is a peaking function of A. Thus $\emptyset \neq \sigma_\pi(TfTh) \cap \sigma_\pi(fh) = \{\lambda\} \cap \{1\}$ implies $\lambda = 1$, and so $\sigma_\pi(Tf) = \{1\}$. Therefore Tf is a peaking function and consequently, $T(\mathcal{F}(A)) \subset \mathcal{F}(B)$. □

Proof (b). If $k \in \mathcal{F}(B)$, then by the hypotheses there is $f \in A$ such that $Tf = k \in \mathcal{F}(B)$. Let $\lambda \in \sigma_\pi(f)$. Thus $f^{-1}(\lambda)$ is a peak set by Lemma 3.2.6 (a). Bishop's Lemma implies that there exists $h \in \mathcal{F}_{f^{-1}(\lambda)}(A)$ such that fh takes its maximum modulus only on $f^{-1}(\lambda)$, i.e., $\sigma_\pi(fh) = \{\lambda\}$. Similarly $\|TfTh\| = \|fh\| = \|f\| = \|Tf\| = 1$ implies that $TfTh$ is a peaking function since, by hypothesis, $Th \in \mathcal{F}(B)$, i.e., $\sigma_\pi(TfTh) = \{1\}$. So $\emptyset \neq \sigma_\pi(TfTh) \cap \sigma_\pi(fh) = \{1\} \cap \{\lambda\}$ gives $\lambda = 1$ and so $\sigma_\pi(f) = \{\lambda\}$. Therefore f is peaking function and consequently, $\mathcal{F}(B) \subset T(\mathcal{F}(A))$. □

Theorem 4.2.5 implies the following

Corollary 4.2.9 ([11]). *If $T\colon A \to B$ is a weakly peripherally multiplicative mapping, not necessarily linear nor continuous, between uniform algebras such that*

(a) $\mathcal{F}(B) \subset T(\mathcal{F}(A))$, *or,*

(b) $T(\mathcal{F}(A)) \subset \mathcal{F}(B) \subset T(A)$,

then T is an isometric algebra isomorphism. Thus T is automatically continuous, linear and multiplicative.

4.3 Non-Unital Weakly Peripherally-Multiplicative Mappings

In general, neither weakly peripherally-multiplicative, nor norm-multiplicative mappings need to be unital.

Example 4.3.1. Let A be a uniform algebra and $T\colon A \to A$.

(a) If T is the identity, then $\mathrm{Ran}(T1) = \{1\}$.

(b) If T is the negative of the identity, then $\mathrm{Ran}(T1) = \{-1\}$.

(c) If $A = C(\{x_1, x_2\})$ and $T(z_1, z_2) = (z_1, -z_2)$ then $\mathrm{Ran}(T1) = \{\pm 1\}$.

In all cases T is weakly peripherally-multiplicative; in (a) and (b) it is surjective, while in (b) and (c) it is not unital. Observe that in (b) and (c) T is not an algebra isomorphism.

As mentioned before, weakly peripherally-multiplicative mappings that do not preserve the peaking functions might fail to be algebra isomorphisms. As established in this section, though, a large class of such mappings are closely related in certain sense to algebra isomorphisms. Recall that the peripheral spectrum of any peaking function $h \in \mathcal{F}(A)$ is a singleton, namely $\sigma_\pi(h) = \{1\}$. If $\mathbb{T}$ denotes the unit circle in the complex plane $\mathbb{C}$, then $\mathbb{T} \cdot \mathcal{F}(A)$ is the set of all functions $f \in A$ with $\|f\| = 1$ and singleton spectra.

Proposition 4.3.2 ([11])**.** *Let $T\colon A \to B$ be a weakly peripherally-multiplicative mapping between uniform algebras. If*

(a) $\mathcal{F}(B) \subset T(\mathbb{T} \cdot \mathcal{F}(A))$, *or,*

(b) $T(\mathcal{F}(A)) \subset \mathbb{T} \cdot \mathcal{F}(B) \subset T(A)$, *then either T or its negative is an isometric algebra isomorphism, and is, therefore, a bounded linear operator.*

Proof (a). According to Lemma 4.2.2 (a), T is norm-multiplicative, and by Lemma 4.1.3 it is norm preserving. By (a), for any $k \in \mathcal{F}(B)$ there is $f \in A$ with $Tf = k$ and $\sigma_\pi(f)$ a singleton. In particular, there is $e \in A$ such that $\sigma_\pi(e)$ is a singleton and $Te = 1$. Since $\sigma_\pi\left(e^2\right) \cap \sigma_\pi\left((Te)^2\right) = \sigma_\pi\left(e^2\right) \cap \{1\} \neq \emptyset$, we see that $\sigma_\pi\left(e^2\right) = \{1\}$. There are two possibilities for $\sigma_\pi(e)$: either $\sigma_\pi(e) = \{1\}$ or $\sigma_\pi(e) = \{-1\}$.

Suppose $\sigma_\pi(e) = \{1\}$. The set $\mathcal{A} = \mathbb{T} \cdot \mathcal{F}(A)$ is a multiplicatively isolating family and, by Lemma 3.3.3,

$$|e(x)| = \inf_{h \in \mathbb{T}\cdot\mathcal{F}(A)} \frac{\|eh\|}{\|h\|} = \inf_{h \in \mathbb{T}\cdot\mathcal{F}(A)} \frac{\|eh\|}{\|h\|} = \inf_{h \in \mathbb{T}\cdot\mathcal{F}(A)} \frac{\|1 \cdot Th\|}{\|Th\|} = 1,$$

for all $x \in \delta A$. Since $\sigma_\pi(e) = \{1\}$, $e = 1$ and T is unital. By Lemma 4.2.2 (g), if $f \in \mathcal{F}(A)$, then $1 \in \sigma_\pi(f)$. Consequently, $\sigma_\pi(f) = \{1\}$, since $\sigma_\pi(f)$ is a singleton. Hence, $f \in \mathcal{F}(A)$, and therefore $\mathcal{F}(B) \subset T(\mathcal{F}(A))$. By Corollary 4.2.9, T is an isomorphism.

If $\sigma_\pi(e) = \{-1\}$, then the above argument applies to the mapping $T'f = T(-f)$, which is weakly-peripherally multiplicative, and so T' is an isomorphism. In particular, T' is linear so $T' = -T$ and T is the negative of an isomorphism.

(b) As in part (a), T is norm-multiplicative and norm preserving. According to (b), $\sigma_\pi(Th)$ is a singleton for all $h \in \mathcal{F}(A)$. So $\sigma_\pi\left(1^2\right) \cap \sigma_\pi\left((T1)^2\right) \neq \emptyset$ implies $\sigma_\pi\left((T1)^2\right) = \{1\}$. There are two possibilities for $\sigma_\pi(T1)$: either $\sigma_\pi(T1) = \{1\}$ or $\sigma_\pi(T1) = \{-1\}$.

Suppose $\sigma_\pi(T1) = \{1\}$. The set $\mathcal{B} = \mathbb{T} \cdot \mathcal{F}(B)$ is a multiplicatively isolating family and, by Lemma 3.3.3,

$$|T1(y)| = \inf_{h \in T^{-1}(\mathbb{T}\cdot\mathcal{F}(B))} \frac{\|T1Th\|}{\|Th\|} = \inf_{h \in T^{-1}(\mathbb{T}\cdot\mathcal{F}(B))} \frac{\|1 \cdot h\|}{\|h\|} = 1,$$

for all $y \in \delta B$. Since $\sigma_\pi(T1) = \{1\}$, $T1 = 1$ and T is unital. Lemma 4.2.2 (f) implies that $1 \in \sigma_\pi(Th)$ for any $h \in \mathcal{F}(A)$. Consequently, $\sigma_\pi(Th) = \{1\}$, since $\sigma_\pi(Th)$ is a singleton. Hence, $Th \in \mathcal{F}(B)$, and therefore $T(\mathcal{F}(A)) \subset \mathcal{F}(B)$. By Corollary 4.2.9, T is an isomorphism.

If $-T1 \in \mathcal{F}(B)$, then the above argument applies to $-T$, so it is an isomorphism. □

If T is a weakly peripherally-multiplicative mapping, then $1 \in \sigma_\pi(T1Th)$ for any $h \in \mathcal{F}(A)$, by Lemma 4.2.2 (d). If, moreover, $\sigma_\pi(T1Th) = \{1\}$, then $T1Th \in \mathcal{F}(B)$ and consequently,

$$(T1) \cdot T(\mathcal{F}(A)) \subset \mathcal{F}(B). \tag{4.3.1}$$

Example 4.2.3 shows that the above condition is not sufficient to characterize weakly peripherally-multiplicative mappings. However, as the next theorem shows, the stronger condition

$$(T1) \cdot T(\mathcal{F}(A)) = \mathcal{F}(B) \tag{4.3.2}$$

is sufficient for this. Note that the mappings considered in [6, 13, 15, 17] automatically satisfy this condition.

Theorem 4.3.3 ([11])**.** *If a mapping $T\colon A \to B$, not necessarily linear, between two uniform algebras satisfies the conditions*

(i) $\mathcal{F}(B) = (T1) \cdot T(\mathcal{F}(A))$ *and*

(ii) T *is weakly peripherally-multiplicative,*

then T is the product of a function in B with values in the set $\{\pm 1\}$ and an algebra isomorphism from A onto B. More precisely, there exists an isometric algebra isomorphism $\widetilde{T}\colon A \to B$, such that for any $f \in A$

$$Tf = (T1) \cdot \widetilde{T}f, \tag{4.3.3}$$

and therefore T is automatically a bounded linear operator.

Proof. The mapping $\widetilde{T} = (T1)\cdot T$ satisfies the hypothesis of Theorem 4.2.5 and therefore is an isometric algebra isomorphism. Thus $1 = \widetilde{T}(1) = T1 \cdot T1 = (T1)^2$, $(T1)^{-1} = T1$, and the values of $T1$ are in the set $\{\pm 1\}$. This completes the proof since $T = (T1)\cdot\widetilde{T}$. $\square$

Actually Theorem 4.3.3 holds if the weak peripheral multiplicativity of T is replaced by the condition $\sigma_\pi(TfTh) \cap \sigma_\pi(fh) \neq \emptyset$ for all $f \in A$ and for peaking functions $h \in \mathcal{F}(A)$ only. Observe that if a unital mapping satisfies condition (4.3.2), then it preserves the peaking functions.

In the context of Theorem 4.3.3, we obtain the following:

Corollary 4.3.4 ([11]). *Let A and B be uniform algebras on their maximal ideal spaces X and Y correspondingly. If a mapping $T\colon A \to B$ satisfies the assumptions of Theorem 4.3.3, then there exists a homeomorphism $\psi\colon Y \to X$ such that*

$$(Tf)(y) = \kappa(y) \cdot f(\psi(y)) \tag{4.3.4}$$

for all $f \in A$, where $\kappa = T1 \in B$ and $\mathrm{Ran}(\kappa) \subset \{\pm 1\}$.

Corollary 4.3.4 implies that T is a weighted composition operator on A. Clearly, isomorphisms are weighted composition operators with trivial weight.

Example 4.3.5. Let $X = X_1 \sqcup X_2, X_1 \cong X_2$ be the compact Hausdorff space from Example 4.2.3, $A = C(X)$, and $B = C(X_1 \sqcup X_2)$. The mapping $T\colon A \to B$ defined by $(Tf)|_{X_1} = f$, $(Tf)|_{X_2} = -f$, is linear, but not of the type (4.3.3). Clearly, $\sigma_\pi(T1) = \{\pm 1\}$, $\sigma_\pi(f) \subset \sigma_\pi(Tf)$, and therefore, T is weakly peripherally-multiplicative. Here T violates condition (i) of Theorem 4.3.3. In particular $\mathcal{F}(B) \not\subset T(A)$.

As in the unital case, condition (i) in Theorem 4.3.3 can be relaxed. Indeed the proof follows the same line except that it is based on Corollary 4.2.9 instead of Theorem 4.2.5.

Corollary 4.3.6 ([11])**.** *Let $T\colon A \to B$ be a weakly peripherally-multiplicative mapping between uniform algebras such that (a) $\mathcal{F}(B) \subset (T1) \cdot T(\mathcal{F}(A))$, or, (b) $(T1) \cdot T(\mathcal{F}(A)) \subset \mathcal{F}(B) \subset (T1) \cdot T(A)$. Then T is as in Theorem 4.3.3.*

Peripherally-multiplicative mappings, mentioned earlier, are introduced in [13]. By definition, these mappings require

$$\sigma_\pi(fg) = \sigma_\pi(TfTg)$$

to hold for all $f, g \in A$. Peripherally-multiplicative mappings are automatically weakly peripherally-multiplicative. It is easy to show that surjective and peripherally-multiplicative mappings meet the assumptions of Corollary 4.3.6. Indeed, the second inclusion in (b) is satisfied by the surjectivity of T, so it is enough to check only the first inclusion in (b). If $h \in \mathcal{F}(A)$, then $T1Th \in \mathcal{F}(B)$ since $\sigma_\pi(T1Th) = \sigma_\pi(1 \cdot h) = \{1\}$, by the peripheral multiplicativity. Hence $(T1) \cdot T(\mathcal{F}(A)) \subset \mathcal{F}(B)$. Corollary 4.3.6 now implies the following main result of Luttman and Tonev in [13]:

Corollary 4.3.7. *Let A and B be uniform algebras. If the mapping $T\colon A \to B$, not necessarily linear, is surjective and peripherally-multiplicative, then T is as in Theorem 4.3.3.*

Chapter 5

Norm Conditions for Isomorphism - The Hatori Conjecture

5.1 Introduction

In 2005 O. Hatori proposed (in private communication) the following conjecture: Consider a surjective map $T : A \to B$ between uniform algebras satisfying

$$\|Tf\,Tg + 1\| = \|fg + 1\|\,, \text{ for all } f, g \in A. \tag{H}$$

Then T is an isometric algebra isomorphism. The conjecture is false as shown by the examples in the following two lemmas.

Lemma 5.1.1. *Let $\kappa \in C(Y)$ such that $\kappa^2 = 1$. Define a map $\Psi : C(Y) \to C(Y)$ by $\Psi f = \kappa f \in C(Y)$. Then Ψ is an isometric $\mathbb{C}$-linear bijection with $\Psi^{-1} = \Psi$ and satisfies $\|\Psi f \Psi g + 1\| = \|fg + 1\|$ for all $f, g \in C(Y)$.*

Proof. Note $\kappa^2 = 1$. So $\Psi^2 = id_{C(Y)}$ which implies Ψ is a bijection with $\Psi^{-1} = \Psi$. For all $f, g \in C(Y)$ and $\lambda \in \mathbb{C}$ we find $\Psi(f + \lambda g) = \kappa(f + \lambda g) = \kappa f + \lambda \kappa g = \Psi f + \lambda \Psi g$ and Ψ is $\mathbb{C}$ linear. Since $\kappa^2 = 1, |\kappa| = 1$ and $|\Psi f| = |\kappa f| = |f|$. So Ψ preserves the modulus of functions and thus the norm proving Ψ is an isometry. Also $\Psi f \Psi g = (\kappa f)(\kappa g) = fg$ so Ψ satisfies (H). $\square$

The map Ψ is multiplicative if and only if $\kappa = 1$ (i.e., Ψ is the identity map) since $\Psi 1 \Psi 1 = \kappa^2 = 1$ and $\Psi 1 = \kappa$ and these are equal only if $\kappa = 1$. If Y is connected then κ must be constant in which case Ψ is either the identity map or the negation map. If Y is disconnected more complicated examples. Suppose Y is the disjoint union of two compact subspaces, say Y_1 and Y_2. Then κ defined by $\kappa|_{Y_1} = 1$ and $\kappa|_{Y_2} = -1$ is continuous and $\kappa^2 = 1$. In this example Ψ is neither the identity map nor the negation map.

Lemma 5.1.2. *Let $e \in C(Y)$ be an idempotent, i.e., $e^2 = e$ and let $e' = 1 - e$. Define a map $\Phi : C(Y) \to C(Y)$ by $\Phi f = ef + e'\overline{f} \in C(Y)$. Then Φ is an multiplicative, isometric, $\mathbb{R}$-linear bijection with $\Phi^{-1} = \Phi$ and satisfies $\|\Psi f \Psi g + 1\| = \|fg + 1\|$ for all $f, g \in C(Y)$.*

Proof. Note $ee' = e(1 - e) = e - e^2 = 0$ and $(e')^2 = e' - e'e = e'$. Thus e' is also idempotent, and $ee' = 0$. As idempotents, e and e' and only take the values 0 or 1. In particular, they are real-valued. Thus

$$\begin{aligned}\Phi \circ \Phi(f) &= e\left(ef + e'\overline{f}\right) + e'\overline{\left(ef + e'\overline{f}\right)} = e^2 f + ee'f + e'e\overline{f} + (e')^2\overline{\overline{f}} \\ &= ef + e'f = (e + e')\, f = f,\end{aligned}$$

which proves Φ is a bijection with $\Phi^{-1} = \Phi$. If $f, g \in C(Y)$ and $r \in \mathbb{R}$, then

$$\Phi(f + rg) = e\,(f + rg) + e'\overline{(f + rg)} = ef + reg + e'\overline{f} + re'\overline{g} = \Phi f + r\Phi g$$

proves $\mathbb{R}$-linearity. Note

$$\Phi f \Phi g = \left(ef + e'\overline{f}\right)(eg + e'\overline{g}) = e^2 fg + ee' f\overline{g} + e'e\overline{f}g + (e')^2\,\overline{fg} =$$

$$= efg + e'\overline{fg} = \Phi(fg) = \begin{cases} fg & \text{on } e^{-1}(1) \\ \overline{fg} & \text{on } e^{-1}(0) = (e')^{-1}(1) \end{cases}$$

for all $f, g \in C(Y)$. Thus Φ is multiplicative and $|\Phi f \Phi + 1| = |fg + 1|$ on $e^{-1}(1)$ and $|\Phi f \Phi + 1| = \left|\overline{fg} + 1\right| = |fg + 1|$ on $e^{-1}(0)$. Since the uniform norm is the maximum modulus, (H) holds. Similarly $|\Phi f| = |f|$ on $e^{-1}(1)$ and $|\Phi f| = \left|\overline{f}\right| = |f|$ on $e^{-1}(0)$. Thus Ψ preserves the modulus and thus the norm and is an isometry. □

Note that in general $\Phi(1) = 1$. The map Φ is $\mathbb{C}$-linear if and only if $e \equiv 1$ (i.e., Φ is the identity map) since $\Phi(i) = (2e - 1)i$ and $i\Phi(1) = i$ and these are equal if and only if $e \equiv 1$. If Y is connected then e is constant and thus Φ is either the identity map (if $e \equiv 1$) or the conjugation operator (if $e \equiv 0$). If Y is the disjoint union of two compact subspaces Y_1 and Y_2. Then e defined by $e|_{Y_1} = 1$ and $e|_{Y_2} = 0$ is continuous and e is idempotent and the corresponding map Φ is neither the identity map nor the conjugation map. In any case Φ fixes real-valued functions.

Another interesting fact is that maps satisfying (H) compose to a map satisfying (H).

Lemma 5.1.3. *If $T : A \to B$ and $S : B \to C$ are maps between uniform algebras satisfy* (H) *then so does $S \circ T : A \to C$.*

Proof. Let $f, g \in A$. Then $\|S(Tf)S(Tg)+1\| = \|TfTg+1\| = \|fg+1\|$. $\square$

In [11] it was shown that surjective maps satisfying condition (H) are isometric algebra isomorphisms provided that they are assumed to be homogeneous which, of course, Ψ and Φ are not in general. However in [12] we showed that surjective maps satisfying (H) have the form $\Psi \circ \Phi \circ \widetilde{T}$ where $\widetilde{T}$ is an isometric algebra isomorphism. Specifically we have the following theorem.

Theorem 5.1.4 ([12, Theorem 2.9])**.** *Let $T\colon A \to B$ be a surjective map that satisfies $\|TfTg+1\| = \|fg+1\|$ for all $f, g \in A$. Then there exist an idempotent $e \in B$ and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that*

$$Tf = (T1)\left(e\tilde{T}f + e'\overline{\tilde{T}f}\right)$$

for all $f \in A$, where $e' = 1 - e$ and $(T1)^2 = 1$.

The rest of this chapter is devoted to the proof of Theorem 5.1.4 also contained in [12].

5.2 Preliminary Results

The following result applies in any uniform algebra. However the hypothesis is quite relevant to condition (H).

Lemma 5.2.1. *Let A be a uniform algebra and $f \in A$. Then $\|f\| + 1 = \|f + 1\|$ if and only if $\|f\| \in \sigma_\pi(f)$.*

Proof. For any complex number $\lambda \in \mathbb{C}$, the following statements will be justified.

$$|\lambda + 1|^2 = |\lambda|^2 + 2\operatorname{Re}\lambda + 1 \tag{5.2.1}$$

$$\lambda = |\lambda| \iff |\lambda + 1| = |\lambda| + 1 \tag{5.2.2}$$

Equation (5.2.1) is verified by,

$$|\lambda + 1|^2 = (\lambda + 1)\left(\overline{\lambda} + 1\right) = |\lambda|^2 + \lambda + \overline{\lambda} + 1 = |\lambda|^2 + 2\operatorname{Re}\lambda + 1,$$

since $\dfrac{\lambda + \overline{\lambda}}{2} = \operatorname{Re}\lambda$.

Clearly $\lambda = |\lambda|$ if and only if λ is a non-negative real number, so $|\lambda + 1| = \lambda + 1 = |\lambda| + 1$. For the other direction we square both sides to get $|\lambda + 1|^2 = |\lambda|^2 + 2|\lambda| + 1$. Applying (5.2.1) we get $|\lambda|^2 + 2\operatorname{Re}\lambda + 1 = |\lambda|^2 + 2|\lambda| + 1$ which simplifies to $\operatorname{Re}\lambda = |\lambda|$. Clearly this gives $\lambda = |\lambda|$ and (5.2.2) is justified.

For the main part, if $\|f\| + 1 = \|f + 1\|$ then there exists $x \in \delta A$ such that $|f(x) + 1| = \|f + 1\|$. Thus

$$|f(x)| + 1 \leq \|f\| + 1 = \|f + 1\| = |f(x) + 1| \leq |f(x)| + 1,$$

and we have equality throughout. Thus $|f(x)| = \|f\|$, and $|f(x)| + 1 = |f(x) + 1|$. From (5.2.2) we have $f(x) = |f(x)|$, yielding $f(x) = \|f\|$. Thus $\|f\| = f(x) \in \sigma_\pi(f)$.

Conversely, if $\|f\| \in \sigma_\pi(f)$ then there exists some x such that $f(x) = \|f\|$. Thus $\|f\| + 1 = |\|f\| + 1| = |f(x) + 1| \leq \|f + 1\| \leq \|f\| + 1$ and we have equality throughout, justifying $\|f + 1\| = \|f\| + 1$. $\square$

For the remainder of this section will assume $T : A \to B$ is a surjective map between uniform algebras satisfying (H).

Lemma 5.2.2. *The map T is bijective and $T^{-1}: B \to A$ satisfies* (H).

Proof. Suppose $Tf = Tg$ for $f, g \in A$. Then $\|fk+1\| = \|TfTk+1\| = \|TgTk+1\| = \|gk+1\|$ for all $k \in A$. We now show that $f = g$ using,

$$\|fk+1\| = \|gk+1\| \text{ for all } k \in A$$

For all $k \in A$ and $n \in \mathbb{N}$ we have

$$\left\|fk + \frac{1}{n}\right\| = \frac{1}{n}\|f \cdot (nk) + 1\| = \frac{1}{n}\|g \cdot (nk) + 1\| = \left\|gk + \frac{1}{n}\right\|.$$

Taking the limit as $n \to \infty$, yields $\|fk\| = \|gk\|$.

For any $x \in \delta A$ such that $f(x) \neq 0$ then $g(x) \neq 0$ and we may apply Lemma 3.2.11 to get a peaking function $h \in \mathcal{F}_x(A)$ such that $\sigma_\pi(gh) = \{g(x)\}$. Let $k = f(x)^{-1}h$ so $\sigma_\pi(gk) = \left\{g(x)f(x)^{-1}\right\}$ and $\|gk\| = \left|g(x)f(x)^{-1}\right| = 1$ since $|f| = |g|$ on δA. Thus

$$\begin{aligned} 2 &= \left|f(x)f(x)^{-1}h(x) + 1\right| = |f(x)k(x) + 1| \\ &\leq \|fk+1\| \leq \|gk+1\| \leq \|gk\| + 1 = 2, \end{aligned}$$

and we have equality throughout. In particular, $\|gk+1\| = \|gk\| + 1$ and by Lemma 5.2.1, $1 = \|gk\| \in \sigma_\pi(gk) = \left\{g(x)f(x)^{-1}\right\}$, i.e., $f(x) = g(x)$. Thus $f = g$ on δA, so $f = g$ and T is injective.

Since T is surjective and injective, there exists a well-defined, bijective map $T^{-1}\colon B \to$

A. For all $f, g \in B$,

$$\|fg+1\| = \left\|T\left(T^{-1}(f)\right)T\left(T^{-1}(g)\right)+1\right\| = \|T^{-1}(f)T^{-1}(g)+1\|$$

and so T^{-1} satisfies (H). $\square$

Lemma 5.2.3. *The map T preserves invertibility, i.e., $f \in A^{-1}$ if and only if $Tf \in B^{-1}$. In particular*

$$(Tf)^{-1} = -T\left(-f^{-1}\right).$$

Proof. Let $f \in A^{-1}$. Then $0 = \left\|f\left(-f^{-1}\right)+1\right\| = \left\|TfT\left(-f^{-1}\right)+1\right\|$. Thus $TfT\left(-f^{-1}\right) = -1$, which implies that Tf is invertible and which proves $T(A^{-1}) \subset B^{-1}$. This result applies also to T^{-1} so $T^{-1}(B^{-1}) \subset A^{-1}$ and thus $T(A^{-1}) = B^{-1}$. $\square$

Recall, a mapping T is called *norm-multiplicative* if it satisfies $\|fg\| = \|TfTg\|$ for all $f, g \in A$. Following an argument similar to that of Honma [9, Lemma 3.3], we next show that T satisfies the norm-multiplicative property when at least one of factors is invertible.

Lemma 5.2.4. *For all $f \in A$ and $g \in A^{-1}$, $\|TfTg\| = \|fg\|$.*

Proof. For all $f, g \in A$,

$$\begin{aligned}\|fg\| = \|fg+1-1\| \le \|fg+1\|+1 &= \|TfTg+1\|+1\\ &\le \|TfTg\|+2.\end{aligned}$$

Since T^{-1} also satifies (H), we also have $\|TfTg\| \le \|fg\|+2$.

For any $n \in \mathbb{N}$, $f \in A$, and $g \in A^{-1}$ we have,

$$\begin{aligned} n\,\|fg\| = \|f\cdot(ng)\| &\le \|TfT(ng)\| + 2 \\ &= \left\|TfTg(Tg)^{-1}T(ng)\right\| + 2 \\ &\le \|TfTg\|\left\|(Tg)^{-1}T(ng)\right\| + 2 \\ &\le \|TfTg\|\left\|-T(-g^{-1})T(ng)\right\| + 2 \qquad (5.2.3) \\ &\le \|TfTg\|\left(\left\|-g^{-1}ng\right\| + 2\right) + 2 \\ &= \|TfTg\|\,(n+2) + 2, \end{aligned}$$

where line (5.2.3) is obtined by Lemma 5.2.3. Thus $\|fg\| \le \|TfTg\|\,\dfrac{n+2}{n} + \dfrac{2}{n}$ which shows that $\|fg\| \le \|TfTg\|$, by letting $n \to \infty$. Since T^{-1} also satisfies (H) we may apply this result to get, $\|TfTg\| \le \left\|T^{-1}(Tf)T^{-1}(Tg)\right\| = \|fg\|$ Thus $\|fg\| = \|TfTg\|$. ∎

With this result we now have the necessary conditions for the map $\psi : \delta B \to \delta A$ to exist as in Theorem 4.1.9. The multiplicatively isolating sets used to satisfy the hypothesis will be A^{-1} and $T(A^{-1}) = B^{-1}$ (Lemma 3.3.5).

Corollary 5.2.5. *There exists a homeomorphism $\psi : \delta B \to \delta A$ such that*

$$|Tf| = |f\circ\psi| \text{ on } \delta B$$

for all $f \in A$. Consequently $\|fg\| = \|TfTg\|\,, \|f\| = \|Tf\|$.

Lemma 5.2.6. *For all $\alpha, \beta \in \mathbb{C}$, $\operatorname{Re} T\alpha T\beta \le \operatorname{Re}\alpha\beta$*

Proof. First we justify,

$$|T\alpha| = |\alpha| \qquad (5.2.4)$$

for $\alpha \in \mathbb{C}$. By Corollary 5.2.5, T preserves the norm. Thus (5.2.4) follows for $\alpha = 0$, since $T0 = 0$. For the case $\alpha \in \mathbb{C}\setminus\{0\}$, then α is invertible and $|\alpha^{-1}| = |\alpha|^{-1}$, so using Lemma 5.2.3 we have,

$$\left|(T\alpha)^{-1}\right| \leq \left\|(T\alpha)^{-1}\right\| = \left\|-T(-\alpha^{-1})\right\| = \|-\alpha^{-1}\| = |\alpha|^{-1}.$$

Thus $|\alpha| \leq |T\alpha| \leq \|T\alpha\| = |\alpha|$, proving (5.2.4).

Now for all $\alpha, \beta \in \mathbb{C}$, $|T\alpha T\beta+1|^2 \leq \|T\alpha T\beta+1\|^2 = |\alpha\beta+1|^2$. Applying equation (5.2.1) to both sides we get $|T\alpha|^2|T\beta|^2 + 2\operatorname{Re} T\alpha T\beta + 1 \leq |\alpha|^2|\beta|^2 + 2\operatorname{Re}\alpha\beta + 1$. Simplifying this with (5.2.1) completes the proof. $\square$

Lemma 5.2.7. *The map T satisfies $(Ti)^2 = T(-i)^2 = T(1)T(-1) = -1$, $T(-i) = -Ti$, and $(T1)^2 = 1$.*

Proof. From (H) we have $0 = \|i \cdot i + 1\| = \|TiTi + 1\|$ so $(Ti)^2 = -1$. Similarly, $0 = \|-i \cdot -i + 1\| = \|T(-i)T(-i) + 1\|$ and $0 = \|1 \cdot -1 + 1\| = \|T(1)T(-1) + 1\|$ give $T(-i)^2 = -1 = T(1)T(-1)$.

Let $E = \dfrac{1 - T(i)T(-i)}{2}$. Note

$$\begin{aligned} \|-E+1\| &= \frac{\|T(i)T(-i)+1\|}{2} \\ &= \frac{\|i \cdot -i + 1\|}{2} \\ &= 1. \end{aligned}$$

Let $e = T^{-1}(T(i)E)$, i.e., $Te = T(i)E = T(-i)E$. Thus

$$\|ie+1\| = \|T(i)Te+1\| = \left\|T(i)^2E+1\right\| = \|-E+1\| = 1, \text{ and}$$
$$\|-ie+1\| = \|T(-i)Te+1\| = \left\|T(-i)^2E+1\right\| = \|-E+1\| = 1.$$

Applying equation (5.2.1) we have

$$|ie|^2 + \operatorname{Re} ie + 1 = |ie+1| \leq \|ie+1\| = 1, \text{ and}$$
$$|-ie|^2 - \operatorname{Re} ie + 1 = |-ie+1| \leq \|-ie+1\| = 1$$

which, when added and simplified, gives $2\,|e|^2 \leq 0$. Thus $0 = \|e\| = \|T(i)E\|$, i.e., $E = 0$ since $T(i)$ is invertible. Finally $0 = E = \dfrac{1 - T(i)T(-i)}{2}$ implies $T(i)T(-i) = 1$. So using formula for the inverse in Lemma 5.2.3, $T(-i) = (Ti)^{-1} = -T(-i^{-1}) = -Ti$.

Finally we must justify $(T1)^2 = 1$. Since $|T1| = |1|$ by (5.2.4) it suffices to show $\operatorname{Im} T1 = 0$. By Lemma 5.2.6 $\operatorname{Re} TiT1 \leq \operatorname{Re} i \cdot 1 = 0$ and $-\operatorname{Re} TiT1 = \operatorname{Re} T(-i)T1 \leq \operatorname{Re} -i \cdot 1 = 0$, using also $T(-i) = -T(i)$. Thus $0 = \operatorname{Re} TiT1$. Since $(Ti)^2 = -1$, Ti takes purely imaginary values and so iTi is an invertible function taking purely real values. So $0 = iTi \operatorname{Re} TiT1 = \operatorname{Re} i(Ti)^2T1 = \operatorname{Re} -iT1 = \operatorname{Im} T1$. □

5.3 Special Case: T Preserves 1 and i

In this section, in addition to assuming $T : A \to B$ is surjective and satisfies (H), we will assume $T1 = 1$ and $Ti = i$.

Lemma 5.3.1. *The map T preserves all constants, i.e., $T\alpha = \alpha$ for all $\alpha \in \mathbb{C}$.*

Proof. In Lemma 5.2.7 we established $T(1)T(-1) = -1$ and $T(-i) = -Ti$. Thus the assumption that $T1 = 1$ and $Ti = i$ produces $T(-1) = -1$ and $T(-i) = -i$. Applying Lemma 5.2.6 successively with $\beta = 1$ and -1, yields $\operatorname{Re} T\alpha = \operatorname{Re} T\alpha T1 \leq \operatorname{Re}\alpha$ and $-\operatorname{Re} T\alpha = \operatorname{Re} T\alpha T(-1) \leq -\operatorname{Re}\alpha$. Thus $\operatorname{Re} T\alpha = \operatorname{Re}\alpha$. Now apply Lemma 5.2.6 with $\beta = i$ and $-i$ to get, $\operatorname{Re} iT\alpha = \operatorname{Re} TiT\alpha \leq \operatorname{Re} i\alpha$ and $-\operatorname{Re} iT\alpha = \operatorname{Re} T(-i)T\alpha \leq -\operatorname{Re} i\alpha$. Thus $\operatorname{Re} iT\alpha = \operatorname{Re} i\alpha$. Note that $\operatorname{Re} i\lambda = -\operatorname{Im}\lambda$ for any $\lambda \in \mathbb{C}$. So $\operatorname{Im} T\alpha = \operatorname{Im}\alpha$ and $T\alpha = \alpha$. □

Lemma 5.3.2. *The map T preserves the peripheral spectrum, i.e., $\sigma_\pi(f) = \sigma_\pi(Tf)$ for all $f \in A$.*

Proof. Since T preserves the norm (Corollary 5.2.5) $T0 = 0$ and the peripheral spectrum is preserved for $f = 0$. For the case that $f \neq 0$, let $\alpha \in \sigma_\pi(f)$ so $\left\|\alpha^{-1}f\right\| = 1 \in \sigma_\pi\left(\alpha^{-1}f\right)$. Lemma 5.2.1 asserts $\left\|\alpha^{-1}f\right\| \in \sigma_\pi\left(\alpha^{-1}f\right)$ if and only if

$$\left\|\alpha^{-1}f\right\| + 1 = \left\|\alpha^{-1}f + 1\right\|. \tag{5.3.1}$$

Note $1 = \left\|\alpha^{-1}f\right\| = \left\|T\left(\alpha^{-1}\right)Tf\right\|$ and from condition (H) we have,

$$\left\|\alpha^{-1}f + 1\right\| = \left\|T\left(\alpha^{-1}\right)Tf + 1\right\|.$$

Substituting these into (5.3.1) we have $\left\|T\left(\alpha^{-1}\right)Tf\right\| + 1 = \left\|T\left(\alpha^{-1}\right)Tf + 1\right\|$. Using Lemma 5.2.1 again we obtain, $1 = \left\|T\left(\alpha^{-1}\right)Tf\right\| \in \sigma_\pi\left(T\left(\alpha^{-1}\right)Tf\right) = \sigma_\pi\left(\alpha^{-1}Tf\right)$ since T preserves constants. Thus $\alpha \in \sigma_\pi(Tf)$, i.e., $\sigma_\pi(f) \subset \sigma_\pi(Tf)$ for all $f \in A$. Since this result also applies to T^{-1}, we have $\sigma_\pi(Tf) \subset \sigma_\pi\left(T^{-1}(Tf)\right) = \sigma_\pi(f)$, and thus $\sigma_\pi(f) = \sigma_\pi(Tf)$. □

These results lead to the following theorem, which is a special case of [7, Corollary 7.5], provided here with an alternate proof.

Theorem 5.3.3 ([12]). *If $T\colon A \to B$ is a surjective map between uniform algebras that satisfies* (H) *and preserves* 1 *and i, i.e $T1 = 1$ and $Ti = i$, then T is an isometric algebra isomorphism.*

Proof. By Corollary 5.2.5 for all $f \in A$ we have $|Tf| = |f \circ \psi|$ on δB. We will show that T is an isometric algebra isomorphism by proving it is a composition operator, i.e.,

$$Tf = f \circ \psi \tag{5.3.2}$$

on δB for all $f \in A$.

Fix $f \in A$ and $y \in \delta B$. If $Tf(y) = 0$ then $|f(\psi(y))| = 0$ and (5.3.2) is satisfied for that case. For $Tf(y) \neq 0$ then $f(\psi(y)) \neq 0$. By 3.2.11 there exist $k \in \mathcal{F}_y(B)$ such that $\sigma_\pi (Tf \cdot k) = \{Tf(y)\}$. Abbreviate $f(\psi(y))^{-1}$ by λ and observe $|\lambda| = |f(\psi(y))|^{-1} = |Tf(y)|^{-1}$. Thus $\sigma_\pi (Tf \cdot \lambda k) = \{\lambda Tf(y)\}$ and $\|Tf \cdot \lambda k\| = |\lambda| \, |Tf(y)| = 1$. Let $h = T^{-1}(\lambda k)$. By Lemma 5.3.2, T preserves the peripheral spectrum so $\sigma_\pi (Th) = \sigma_\pi (\lambda k) = \{\lambda\}$, since k is a peaking function. Also note $|h(\psi(y)| = |Th(y)| = |\lambda k(y)| = |\lambda \cdot 1|$, since k was chosen in $\mathcal{F}_y(B)$. Since $|h(\psi(y))| = |\lambda| = \|h\|$, then $h(\psi(y)) \in \sigma_\pi (h) = \{\lambda\}$, i.e., $h(\psi(y)) = \lambda = f(\psi(y))^{-1}$. Putting these facts together we have,

$$\begin{aligned} 2 &= |f(\psi(y))f(\psi(y)^{-1} + 1| = |f(\psi(y))h(\psi(y)) + 1| \\ &\leq \|fh + 1\| = \|TfTh + 1\| \leq \|TfTh\| + 1 = 2 \end{aligned}$$

so we have equality throughout. In particular, $\|TfTh + 1\| = \|TfTh\| + 1$ so by Lemma 5.2.1 $1 = \|TfTh\| \in \sigma_\pi (TfTh) = \left\{Tf(y)f(\psi(y))^{-1}\right\}$, i.e., $Tf(y) = f(\psi(y))$ and 5.3.2 is justified. As in the proof of Theorem 4.2.5 this proves that T is an injective isometric algebra homomorphism. Since T was assumed from the beginning to be surjective it actually an isomorphism. □

5.4 Proof of Theorem Theorem 5.1.4

We return to analyzing T without assuming it preserves 1 and i to get our main result. This is done by using the counterexamples Ψ and Φ from Lemmas 5.1.1 and 5.1.2 to "correct" T so that preserves both 1 and i.

Theorem (5.1.4). *Let $T\colon A \to B$ be a surjective map that satisfies $\|T(f)T(g)+1\| = \|fg+1\|$ for all $f,g \in A$. Then there exist an idempotent $e \in B$ and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that*

$$Tf = (T1)\left(e\tilde{T}f + e'\overline{\tilde{T}f}\right) \tag{5.4.1}$$

for all $f \in A$, where $e' = 1-e$.

Proof. Note that e' is also idempotent and $ee' = e(1-e) = e - e^2 = 0$. This property allows B to be written as the internal direct sum (as rings) of the ideals Be and Be'. Also $Be \oplus \overline{B}e' \subset C(Y)$ is clearly a uniform algebra on Y.

For e satisfying (5.4.1) we note $Ti = (T1)(ie - ie') = i(T1)(e-e') = (2e-1)$. Multiplying both sides by $-i(T1)$ and applying $(T1)^2 = 1$, we obtain $-i(T1)(Ti) = 2e-1$, i.e., $e = \dfrac{1-iT1Ti}{2}$.

Now let $e = \dfrac{1-iT1Ti}{2}$. Note

$$e^2 = \frac{1-2iT1Ti + i^2(T1)^2(Ti)^2}{4} = \frac{1-2iT1Ti - 1\cdot 1\cdot -1}{4} = \frac{2-2iT1Ti}{4} = e,$$

using $(T1)^2 = 1$ and $(Ti)^2 = -1$, from Lemma 5.2.7. Let $\Phi : B \to Be \oplus \overline{B}e'$ be given by $\Phi f = ef + e'\overline{f} \in Be \oplus \overline{B}e'$, as in Lemma 5.1.2. Note $\Phi e = e^2 + e'e = e$ and $\Phi e' = ee' + (e')^2 = e'$. To show Φ is surjective we take and arbitrary element $h \in Be \oplus \overline{B}e'$, which necessarily has the form $h = ef + e'\overline{g}$ for $f, g \in B$. Thus $ef + e'g \in B$, and, since Φ is additive and multiplicative,

$$\begin{aligned}\Phi(ef + e'g) &= \Phi e \Phi f + \Phi e' \Phi g = e\left(ef + e'\overline{f}\right) + e'\left(eg + e'\overline{g}\right)\\ &= ef + 0 \cdot \overline{f} + 0 \cdot g + e'\overline{g} = h.\end{aligned}$$

Let $\Psi : B \to B$ be given by $\Psi(f) = T1 \cdot f \in B$ as in 5.1.1 for $\kappa = T1$. Clearly Ψ is bijective since $\Psi = \Psi^{-1}$.

Let $\tilde{T} = \Phi \circ \Psi \circ T : A \to Be \oplus \overline{B}e'$. By Lemma 5.1.3, $\widetilde{T}$ satisfies (H) and is surjective, since it is the composition of surjective maps. By Lemma 5.2.7 and the fact that Φ fixes real-valued constants, $\widetilde{T}1 = \Phi(\Psi(T1)) = \Phi(T1 \cdot T1) = \Phi 1 = 1$. From $(Ti)^2 = -1$ and $(T1)^2 = 1$ we get $\overline{Ti} = -T(i)$ and $\overline{T1} = T1$. Thus

$$\begin{aligned}\widetilde{T}i = \Phi(\Psi(Ti)) = \Phi(T1Ti) &= eT1Ti + e'\overline{T1Ti} = eT1Ti - e'T1Ti\\ &= T1Ti(e - e') = T1Ti\,(e - (1 - e)) = T1Ti\,(2e - 1)\\ T1Ti(1 - iT1Ti - 1) &= -i(T1)^2(Ti)^2 = -i \cdot 1 \cdot -1 = i,\end{aligned}$$

recalling $e' = 1 - e$.

Thus, by Lemma 5.3.3, $\tilde{T}$ is an isometric algebra isomorphism onto $Be \oplus \overline{B}e'$. Recall $\Psi^{-1} = \Psi$ and $\Phi^{-1} = \Phi$, so $T = \Psi^{-1} \circ \Phi^{-1} \circ \widetilde{T} = \Psi \circ \Phi \circ \widetilde{T}$ which is explicitly given in (5.4.1). $\square$

5.5 A Generalization of Theorem 5.1.4

We can generalize the condition (H) to get a deeper result as given bellow.

Theorem 5.5.1. *Let $\lambda \in \mathbb{C}\setminus\{0\}$ be fixed. If $T : A \to B$ is a surjective map satisfying*

$$\|TfTg + \lambda\| = \|fg + \lambda\| \tag{H_λ}$$

for all $f, g \in A$, then there exist an idempotent $e \in B$, a function $\kappa \in B$ with $\kappa^2 = 1$, and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that

$$Tf = \kappa\left(e\tilde{T}f + \gamma e'\overline{\tilde{T}f}\right)$$

for all $f \in A$, where $e' = 1 - e$ and $\gamma = \dfrac{\lambda}{|\lambda|}$.

Proof. Choose α such that $\alpha^2 = \lambda$, and define $T'(f) = \alpha^{-1}T(\alpha f)$. Since α is invertible, T' is surjective, and

$$\|T'fT'g+1\| = \|\alpha^{-2}T(\alpha f)T(\alpha g)+1\| = \frac{1}{|\lambda|}\|T(\alpha f)T(\alpha g)+\lambda\| = \frac{1}{|\lambda|}\|\alpha^2 fg+\lambda\| = \|fg+1\|,$$

proves T' satisfies (H). By Theorem 5.1.4 there exist an idempotent $e \in B$ and an isometric algebra isomorphism $\tilde{T}\colon A \to Be \oplus \overline{B}e'$ such that $T'f = \kappa\left(e\tilde{T}f + e'\overline{\tilde{T}f}\right)$, where $\kappa = T'1 = \alpha^{-1}T\alpha$. Using the fact that $\widetilde{T}$ is an isomorphism we get

$$\begin{aligned} Tf = \alpha T'(\alpha^{-1}f) &= \alpha\kappa\left(e\tilde{T}(\alpha^{-1}f) + e'\overline{\tilde{T}(\alpha^{-1}f)}\right) \\ &= \kappa\left(e\tilde{T}f + \alpha\overline{\alpha}^{-1}e'\overline{\tilde{T}f}\right) = \kappa\left(e\tilde{T}f + \gamma e'\overline{\tilde{T}f}\right) \end{aligned}$$

since

$$\frac{\alpha}{\overline{\alpha}} = \frac{\alpha^2}{\alpha\overline{\alpha}} = \frac{\alpha^2}{|\alpha^2|} = \frac{\lambda}{|\lambda|} = \gamma. \tag{5.5.1}$$

□

Curiously this formula holds even though T' was defined by an arbitrary choice of one of two solutions to $\alpha^2 = \lambda$. The appearance of α is suppressed, but, in fact, $\kappa = \alpha^{-1}T\alpha$. The other option is $-\alpha$. From the formula given for T, it is clear that T is an $\mathbb{R}$-linear isometry. This addresses the mystery, since we may use this to show $(-\alpha)^{-1}T(-\alpha) = \alpha^{-1}T\alpha$. Also using this formula, we may give sufficient conditions for T to be an isomorphism or the conjugate of an isomorphism.

Corollary 5.5.2. *Let A and B be uniform algebras, $\lambda \in \mathbb{C} \setminus \{0\}$, and $T\colon A \to B$ a surjective map such that*

$$\|TfTg + \lambda\| = \|fg + \lambda\|$$

for all $f, g \in A$. Then T is an isometric algebra isomorphism if and only if $T1 = 1$ and $Ti = i$. Similarly, T is a conjugate-isomorphism if and only if $T1 = 1$ and $Ti = -i$. The equivalence if vacuous unless $\lambda \in \mathbb{R}$.

Proof. Theorem 5.5.1 implies $T(1) - iT(i) = \kappa(e + \gamma e') - i\kappa(ie - \gamma ie') = 2\kappa e$, in general. Thus, since $\kappa^2 = 1$,

$$\frac{T(1) - iT(i)}{2}\kappa = e \tag{5.5.2}$$

for any map T satisfying (H_λ).

If $T1 = 1$ and $Ti = i$, then (5.5.2) gives $\kappa = e$. Since κ can only take the values ± 1 and e can only take the values 0 and 1, $\kappa = e = 1$, which forces $e' = 0$ and $T = \widetilde{T}$. Conversely if T is an isometric isomorphism $T1 = 1$ and $Ti = 1$.

If $T1 = 1$ and $Ti = -i$ then (5.5.2) gives $0 = e$, forcing $e' = 1$ and $T = \kappa\gamma\overline{\widetilde{T}}$. Applying

$T1 = 1$ forces $\gamma\kappa = 1$, i.e., $\gamma = \kappa$. Under these conditions clearly T is a conjugate isomorphism.

For T to be a conjugate isomorphism then necessarily $T1 = 1$ and $Ti = i$, i.e., $0 = e$. This only occurs if $e' = 1$, since κ is invertible. Thus $e' = 1$ and $T(f) = \kappa\gamma\overline{\tilde{T}(f)}$, but $T(1) = 1$ additionally requires that $\kappa\gamma = 1$ so $T(f) = \overline{\tilde{T}(f)}$. Thus T is a conjugate isomorphism. However if T is a conjugate isomorphism then (H_λ) gives $2\,|\mathrm{Re}\,\lambda| = \left|\overline{\lambda} + \lambda\right| = \|T(\lambda)T1 + \lambda\| = \|\lambda \cdot 1 + \lambda\| = 2\,|\lambda|$ implies $\lambda \in \mathbb{R}$. So it is impossible for T to be a conjugate isomorphism unless the λ in the original condition happens to be a real number. $\square$

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