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Separation axioms in topology

Elaine D. Nelson
The University of Montana

THE SEPARATION AXIOMS IN TOPOLOGY

By

Elaine D. Nelson

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Approved by:

James Druemmel

Chairman, Board of Examiners

Dean, Graduate School

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E.D.N.

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CHAPTER I

INTRODUCTION

The concept of a topological space is very general. It is often desirable to be more specific. One way in which this is done is to use the separation axioms to define topological spaces with more restricted properties. For example, it is not true in general that a sequence in a topological space has at most one limit. But with the use of the separation axioms a type of space may be defined in which the limit, if it exists, is unique.

In this chapter some of the basic concepts used in the paper will be discussed briefly. The separation axioms are stated and the resulting types of topological spaces are defined in Chapter II. The relationships among these spaces are also considered. In Chapter III a few interesting properties of the types of spaces defined are also discussed.

Two terms which are used repeatedly in this paper are "topology" and "topological space." If S is a nonempty set, a collection $\mathcal{T}$ of subsets of S is called a topology on S if it satisfies the following properties:

(i) The union of any subcollection of $\mathcal{T}$ belongs to $\mathcal{T}$.

(ii) The intersection of any nonempty finite subcollection of $\mathcal{T}$ belongs to $\mathcal{T}$.

(iii) $S \in \mathcal{T}$.

The topological space consisting of the set S and the topology $\mathcal{T}$ will be denoted by $(S, \mathcal{T})$. A collection of sets $\mathcal{A}$ is a subbase for a topology $\mathcal{T}$ if and only if each member of $\mathcal{T}$ is the union of finite intersections of sets in $\mathcal{A}$ and $\mathcal{A} \subset \mathcal{T}$. The subbase $\mathcal{A}$ is said to generate the topology $\mathcal{T}$.

To avoid possible confusion it is necessary to define some of the terms which will be used throughout this paper. A subset of S is open if and only if it is in $\mathcal{T}$. A subset of S is closed if and only if its complement is open. The complement of a set X with respect to S will be denoted by $\mathcal{C}X$ or S-X.

As an example of a topological space, consider the set S of positive integers. Let $\mathcal{T}$ be the family consisting of $\emptyset$, S, and all subsets of S of the form $\{1, 2, \ldots, n_1\}$. Let $\{1, 2, \ldots, n_1\}$, $\{1, 2, \ldots, n_2\}$, ... be any sets in $\mathcal{T}$. $\{1, 2, \ldots, n_1\} \cup \{1, 2, \ldots, n_2\} \cup \ldots = \{1, 2, \ldots, n_k\}$, where $n_k = \sup \{n_1, n_2, \ldots\}$, if this set has a supremum. If the set does not have a supremum, the union of the sets is S. In either case the union of the sets is in $\mathcal{T}$. Let $\{1, 2, \ldots, n_1\}$, $\{1, 2, \ldots, n_2\}$, ..., $\{1, 2, \ldots, n_m\}$ be in $\mathcal{T}$. $\{1, 2, \ldots, n_1\} \cap \{1, 2, \ldots, n_2\} \cap \ldots \cap \{1, 2, \ldots, n_m\} = \{1, 2, \ldots, n_k\}$, where $n_k = \inf \{n_1, n_2, \ldots, n_m\}$. $\{1, 2, \ldots, n_k\} \in \mathscr{L}$. Also, $S \in \mathscr{L}$. Therefore $\mathscr{L}$ is a topology on S.

The closure of a set X is the intersection of all closed sets containing X. This is denoted by $\overline{X}$. $\overline{X}$ may also be described as the smallest closed set containing X. A point x is a cluster point of a set X if and only if $x \in \overline{X - \{x\}}$.

A subset N of S is a neighborhood of a point $x \in S$ if there exists an open set U such that $x \in U \subset N$. N is a neighborhood of a subset A of S if there exists an open set V such that $A \subset V \subset N$. Note that a neighborhood need not be open.

A sequence in a topological space $(S, \mathscr{L})$ is a function from the set of positive integers to S. It is denoted by (x_n). A sequence (x_n) converges to x or has x as a limit if for every neighborhood N of x there exists a positive integer M such that $x_n \in N$ if $n \geq M$. This definition does not require that the limit of a sequence be unique.

One special kind of a topological space is a space in which a distance is defined. A metric or distance function on a set S is a function d from SXS to R, the set of real numbers, satisfying the following properties:

(i) If $x, y \in S$, $d(x,y) \geq 0$; $d(x,y) = 0$ if and only if $x = y$.

(ii) If x, $y \in S$, $d(x,y) = d(y,x)$.

(iii) If x, y, $z \in S$, $d(x,y) \leq d(x,z) + d(z,y)$.

The metric d is used to define a topology on S. If $\alpha > 0$ and $x \in S$, the set $C(x,\alpha) = \{y \in S\colon d(x,y) < \alpha\}$ is called the open ball with center x and radius α. The topology generated by the collection of all such open balls is the natural topology on S. The topological space thus formed is denoted by (S,d) and is called a metric space.

A well known example of a metric space is n-dimensional Euclidean space. The metric in this space can be defined in various ways but the most common is $d(x,y) = (\sum_{k=1}^{n}(x_k - y_k)^2)^{\frac{1}{2}}$, where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. In Euclidean space $d(x,y)$ is often written $|x-y|$.

CHAPTER II

THE SEPARATION AXIOMS AND RELATED TOPOLOGICAL SPACES

The separation axioms which will be used to define the types of topological spaces in this chapter may be stated as follows:

(O) If x and y are distinct points of a topological space, then there exists an open set U which contains one of the points but not the other.

(I) If x and y are distinct points of a topological space, then there exists an open set U which contains x but not y and an open set V which contains y but not x.

(II) If x and y are distinct points of a topological space, then there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

(III) If X is a closed set in a topological space $(S, \mathscr{L})$ and y is a point in S but not in X, then there exist disjoint open sets U and V such that $X \subset U$ and $y \in V$.

(IV) If X and Y are disjoint closed sets in a topological space, then there exist disjoint open sets U and V such that $X \subset U$ and $Y \subset V$.

Topological spaces may be defined which satisfy one or more of these axioms. In this chapter seven types of spaces will be defined and the relationships among them will be discussed.

Definition: A topological space which satisfies Axiom (0) is a T_0-space.

A necessary and sufficient condition for a space to be a T_0-space may be given in terms of closures of singletons in the space.

Theorem 1: A topological space is a T_0-space if and only if the closures of any two distinct singletons are distinct.

Proof: Suppose $(S, \mathscr{T})$ is a T_0-space. Let $x, y \in S$, $x \neq y$. Then there exists an open set U containing one of the points but not the other. Suppose $x \in U$. $\mathscr{C} U$ is a closed set containing y. Since $\overline{\{y\}}$ is the smallest closed set containing y, $\{y\} \subset \overline{\{y\}} \subset \mathscr{C} U$. Hence $x \notin \overline{\{y\}}$. Since $x \in \overline{\{x\}}$, $\overline{\{x\}} \neq \overline{\{y\}}$.

Conversely, suppose the closures of two distinct singletons $\{x\}$ and $\{y\}$ in S are distinct. Then there exists some point $z \in S$ which is contained in one of the closures but not the other. Suppose $z \in \overline{\{x\}}$, but $z \notin \overline{\{y\}}$. If $x \in \overline{\{y\}}$, then $\overline{\{x\}} \subset \overline{\{y\}}$, so $z \in \overline{\{y\}}$. This is a contradiction. Hence $x \notin \overline{\{y\}}$. Thus $\mathscr{C} \overline{\{y\}}$ is an open set containing x but not y. Therefore $(S, \mathscr{T})$

is a T_0-space.

The next type of space to be considered is a T_1-space.

Definition: A topological space which satisfies Axiom (I) is a T_1-space.

The following theorem is often useful when working with T_1-spaces.

Theorem 2: A topological space $(S, \mathcal{T})$ is a T_1-space if and only if every singleton in $(S, \mathcal{T})$ is closed.

Proof: Suppose $(S, \mathcal{T})$ is a T_1-space. Let $x \in S$. For every $y \in \mathcal{C}\{x\}$ there exists an open set U satisfying $y \in U \subset \mathcal{C}\{x\}$. Hence $\mathcal{C}\{x\}$ is a neighborhood of each of its points. Therefore, $\mathcal{C}\{x\}$ is open and $\{x\}$ is closed.

Conversely, suppose every singleton in $(S, \mathcal{T})$ is closed. Let $x, y \in S$, $x \neq y$. $\mathcal{C}\{y\}$ is open and contains x, but not y. Also, $\mathcal{C}\{x\}$ is open and contains y, but not x. Therefore $(S, \mathcal{T})$ is a T_1-space.

It is obvious from the definitions that a T_1-space is a T_0-space. But a T_0-space is not necessarily a T_1-space. This is shown by the following example.

Example 1: Let S be the set of positive integers. Let $\mathcal{T}$ be the family consisting of $\emptyset$, S, and all subsets of the form $\{1, 2, \ldots, n\}$. It was proved in the introduction that $\mathcal{T}$ is a topology on S. Let x,

$y \varepsilon S$, $x \neq y$. Suppose $x < y$. Then $x \varepsilon \{1, 2, \ldots, x\}$, but $y \notin \{1, 2, \ldots, x\}$. Hence there exists an open set which contains x but not y. Therefore $(S, \mathscr{T})$ is a T_0-space. Let $\{1, 2, \ldots, n\}$ be any set containing y. Since $x < y$, $x \varepsilon \{1, 2, \ldots, n\}$. Thus there are no open sets containing y but not x. Therefore $(S, \mathscr{T})$ is not a T_1-space.

In Axiom (I) which was used to define a T_1-space, the open sets U and V were not necessarily disjoint. If the condition $U \cap V = \emptyset$ is added another type of space, the T_2-space may be defined. A few interesting properties which this space has but T_0 and T_1-spaces do not have will be discussed in Chapter III.

Definition: A topological space which satisfies Axiom (II) is a T_2-space or a Hausdorff space.

The following theorem gives another way of determining if a space is a T_2-space.

Theorem 3: A topological space is a T_2-space if and only if the singleton consisting of any point x is the intersection of all closed neighborhoods of x.

Proof: Suppose $(S, \mathscr{T})$ is a T_2-space. Let $x \varepsilon S$. For each $y \varepsilon S$, $y \neq x$, there exist disjoint open sets U_y and V_y such that $x \varepsilon U_y$, $y \varepsilon V_y$, and $U_y \cap V_y = \emptyset$. $\mathscr{C} V_y$ is closed and $U_y \subset \mathscr{C} V_y$. This is true for all $y \varepsilon \mathscr{C}\{x\}$. Let $\mathscr{V}$ be the collection of all V_y. Then

$\bigcup_{V_y \varepsilon \mathcal{V}} V_y = \mathcal{C}\{x\}$ and $\bigcap_{V_y \varepsilon \mathcal{V}} \mathcal{C}V_y = \mathcal{C}(\bigcup_{V_y \varepsilon \mathcal{V}} V_y) =$ $\mathcal{C}(\mathcal{C}\{x\}) = \{x\}$. The intersection of all closed neighborhoods of x is a subset of $\bigcap_{V_y \varepsilon \mathcal{V}} \mathcal{C}V_y = \{x\}$. But x is in all these neighborhoods. Therefore the intersection of all closed neighborhoods of x is $\{x\}$.

Conversely, let $x, y \varepsilon S$, $x \neq y$. Since the intersection of all closed neighborhoods of x is $\{x\}$, there exists a closed neighborhood N of x such that $y \not\varepsilon N$. Since N is a neighborhood of x, there exists an open set U such that $x \varepsilon U \subset N$. $\mathcal{C}N$ is an open set containing y. Thus U and $\mathcal{C}N$ are disjoint open sets containing x and y, respectively. Therefore $(S, \mathcal{T})$ is a T_2-space.

Every Hausdorff space is obviously a T_1-space, and hence, a T_0-space, but the converse is not true. The following is an example of a space which is T_1 but not Hausdorff.

<u>Example 2</u>: Let S be an infinite set and $\mathcal{T}$ the cofinite topology on S, i.e., the collection consisting of the empty set and those subsets of S whose complements are finite. Let x be any point of S. $\{x\}$ is closed, since $\mathcal{C}\{x\}$ is open. Therefore $(S, \mathcal{T})$ is a T_1-space. Let $x, y \varepsilon S$. Let U be an open set containing x but not y. $\mathcal{C}U$ is finite. Thus there are no nonempty open sets disjoint from U. Therefore $(S, \mathcal{T})$ is not Hausdorff.

The next types of spaces to be considered are regular spaces and T_3-spaces.

Definition: A topological space which satisfies Axiom (III) is a regular space.

Definition: A T_3-space is a regular T_1-space.

Theorem 4: A topological space is a T_3-space if and only if it is a regular T_2-space.

Proof: Suppose $(S, \mathscr{T})$ is a T_3-space. Then it is a regular T_1-space. Let $x, y \in S$. Since $(S, \mathscr{T})$ is a T_1-space, $\{x\}$ and $\{y\}$ are closed. Because $(S, \mathscr{T})$ is regular, there exist open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Therefore $(S, \mathscr{T})$ is a regular T_2-space.

Conversely, suppose $(S, \mathscr{T})$ is a regular T_2-space. Then it is a T_1-space. Therefore $(S, \mathscr{T})$ is a T_3-space.

There are several conditions which are equivalent to a space being regular.

Theorem 5: The following statements are equivalent:

(i) A topological space $(S, \mathscr{T})$ is regular.

(ii) For every nonempty open set U in $(S, \mathscr{T})$ and for every $x \in U$, there exists an open set Q_x such that $x \in Q_x \subset \overline{Q_x} \subset U$.

(iii) For every closed set A in $(S, \mathscr{T})$ the intersection of all closed neighborhoods of A is A.

Proof: This theorem will be proved if the implications

(i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) are proved.

(i) $\Rightarrow$ (ii): Let U be an open set in $(S, \mathscr{T})$ and $x \in U$. Then $\mathscr{C}\, U$ is closed and $x \notin \mathscr{C}\, U$. Since $(S, \mathscr{T})$ is regular, there exists open sets Q_x and Q such that $x \in Q_x$, $\mathscr{C}\, U \subset Q$, and $Q_x \cap Q = \emptyset$. Hence $Q_x \subset \mathscr{C}\, Q \subset U$. Since $\mathscr{C}\, Q$ is closed, $\overline{Q_x} \subset \mathscr{C}\, Q \subset U$. Therefore, $x \in Q_x \subset \overline{Q_x} \subset U$.

(ii) $\Rightarrow$ (iii): Let A be a closed set in $(S, \mathscr{T})$ and $x \notin A$. Then $\mathscr{C}\, A$ is open and $x \in \mathscr{C}\, A$. Hence there exists an open set Q_x such that $x \in Q_x \subset \overline{Q_x} \subset \mathscr{C}\, A$. $\mathscr{C}\, Q_x$ is closed and $A \subset \mathscr{C}\, \overline{Q_x} \subset \mathscr{C}\, Q_x$. Hence $\mathscr{C}\, Q_x$ is a closed neighborhood of A which does not contain x. Therefore, the intersection of all closed neighborhoods of A is a subset of A. Since A is in every closed neighborhood of A, A is in their intersection. Therefore the intersection of every closed neighborhood of A is A.

(iii) $\Rightarrow$ (i): Let A be a closed set in $(S, \mathscr{T})$ and $x \notin A$. Since the intersection of all closed neighborhoods of A is A, there exists a closed neighborhood B of A such that $x \notin B$. Since B is a neighborhood of A there exists an open set U such that $A \subset U \subset B$. $\mathscr{C}\, B$ is an open set containing x. Thus U and $\mathscr{C}\, B$ are disjoint open sets containing A and x, respectively. Therefore $(S, \mathscr{T})$ is regular.

The definitions and theorems given thus far show that a T_3-space is regular, T_2, T_1, and T_0. Now consider an example of a space which is T_2 but not regular and hence not T_3.

<u>Example 3</u>: Let S be the subset of the Cartesian plane R^2 consisting of all pairs of real numbers (x,y) such that $y \geq 0$, and let L be the subset of S consisting of all (x,y) such that y = 0. For each point p of S and each real number r > 0, define a set $U_r(p)$ as follows: $U_r(p) = S_r(p) \cap S$ if $p \in S - L$, $U_r(p) = [S_r(p) \cap (S - L)] \cup \{p\}$ if $p \in L$, where $S_r(p) = \{q \in R^2: |p - q| < r\}$. Let $\mathscr{X}$ be the topology generated by the collection of subsets of S consisting of all sets of the form $U_r(p)$, where $p \in S$ and $r > 0$. If p and q are distinct points of S, let $|p - q| = 2r$. Then $U_r(p)$ and $U_r(q)$ are disjoint open subsets of S containing p and q, respectively. Hence, $(S, \mathscr{X})$ is a T_2-space. Consider the point p = (0,0) and the open set $U_r(p)$. $p \in U_r(p)$. Open subsets of $U_r(p)$ which contain p contain sets of the form $U_s(p)$, where $0 < s < r$. Let $q \in \{q \in L: |p - q| \leq s\}$. Open sets containing q contain sets of the form $U_t(q)$, $t > 0$. Since $|p - q| \leq s$, $U_t(q) \cap U_s(p) \neq \emptyset$ and $q \in \overline{U_s(p)}$. But $q \notin U_r(p)$. Thus $\overline{U_s(p)}$ is not a subset of $U_r(p)$. Therefore $(S, \mathscr{X})$ is not regular.

The next question to be considered is whether or not a regular space is also T_0, T_1, T_2, and T_3. To show that a regular space need not be T_0, T_1, T_2, and T_3 it will suffice to give an example of a space which is regular but not T_0.

<u>Example 4</u>: Consider the set $S = \{a, b, c,\}$. Let $\mathcal{T} = \{\emptyset, \{a\}, \{b,c\}, S\}$. The closed sets in $(S,\mathcal{T})$ are $\emptyset$, $\{a\}$, $\{b,c\}$, and S. Consider the possible pairs of closed sets and points not in the sets.

$\{a\}$, b: $\{a\} \subset \{a\}$ and $b \varepsilon \{b,c\}$.

$\{a\}$, c: $\{a\} \subset \{a\}$ and $c \varepsilon \{b,c\}$.

$\{b,c\}$, a: $\{b,c\} \subset \{b,c\}$ and $a \varepsilon \{a\}$.

Since $\{a\} \cap \{b,c\} = \emptyset$, a closed set and a point not contained in the set are contained in disjoint open sets. Therefore $(S,\mathcal{T})$ is regular. There is no open set containing b but not c and no open set containing c but not b. Therefore $(S,\mathcal{T})$ is not a T_0-space.

The other two types of spaces to be considered in this paper are normal spaces and T_4-spaces.

<u>Definition</u>: A topological space which satisfies Axiom (IV) is a normal space.

<u>Definition</u>: A T_4-space is normal T_1-space.

An equivalent definition of a T_4-space could have been given in terms of normal and T_2-spaces.

<u>Theorem 6</u>: A topological space is a T_4-space if and only if it is a normal T_2-space.

<u>Proof</u>: The proof is similar to the proof of Theorem 4 and will not be given here.

There are several conditions which are equivalent to a topological space being normal.

Theorem 7: The following statements are equivalent:

(i) A topological space $(S, \mathcal{T})$ is normal.

(ii) Given any closed set A in $(S, \mathcal{T})$ and any open set U containing A, there exists an open set V such that $A \subset V \subset \overline{V} \subset U$.

(iii) Given any two disjoint closed subsets A and B of S, there exists a continuous mapping $f: S \longrightarrow [0,1]$, such that $f(x) = 0$ if $x \in A$, and $f(x) = 1$ if $x \in B$.

Proof: This can be proved by proving that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let $(S, \mathcal{T})$ be a normal space. Let A be a closed subset of S and let U be an open set containing A. $\mathcal{C}\, U$ is a closed set which is disjoint from A. Since $(S, \mathcal{T})$ is normal, there exist disjoint open sets V and W such that $A \subset V$ and $\mathcal{C}\, U \subset W$. Then $V \subset \mathcal{C}\, W$. Hence $\overline{V} \subset \overline{\mathcal{C}\, W} = \mathcal{C}\, W \subset \mathcal{C}\,(\mathcal{C}\, U) = U$. Therefore $A \subset V \subset \overline{V} \subset U$.

(ii) $\Rightarrow$ (iii): Let A and B be closed subsets of S such that $A \cap B = \emptyset$. First define a collection $\{U_r : r \text{ rational}\}$ of open sets such that $\overline{U_r} \subset U_s$ whenever $r < s$ in the following way. For all $r < 0$ let $U_r = \emptyset$, and for all $r > 1$ let $U_r = S$. Next define $U_1 = \mathcal{C}\, B$, which is an open set containing A with the desired property. By (ii) U_1 contains an open set U_0 containing A whose closure is also contained in U_1. Now let (r_n) be a listing of all the rationals in $[0,1]$ with $r_1 = 0$ and $r_2 = 1$. For each $n \geq 3$ inductively define the open set U_{r_n} by taking the largest r_i and the smallest r_j such that $i, j < n$ and $r_i < r_n < r_j$. Then use (ii) to obtain the open set U_{r_n} with the property that $\overline{U_{r_i}} \subset U_{r_n} \subset \overline{U_{r_n}} \subset U_{r_j}$. Now define the mapping f by setting $f(x) = \inf \{r: x \in U_r\}$. Since every $x \in S$ belongs to some U_r and 0 is a lower bound for the r's, the infimum exists and f(x) is a well-defined real number in $[0,1]$. $f(x) < q$ if and only if $x \in U_r$ for some $r < q$, so $\{x: f(x) < q\} = \cup \{U_r: r < q\}$, which is open, since each U_r is open. Let $f(x) > p$. Then $\inf \{r: x \in U_r\} > p$ and there exists a rational number q such that $\inf \{r: x \in U_r\} > q > p$. Hence $r > q$ for all $r \in \{r: x \in U_r\}$. Since $q < \inf \{r: x \in U_r\}$, $x \notin U_q$. Also, there exists a rational number q_1 such that $q > q_1 > p$. By the construction of the open sets, $U_{q_1} \subset \overline{U_{q_1}} \subset U_q \subset \overline{U_q} \subset U_r$. Since $x \notin U_q$, $x \notin \overline{U_{q_1}}$. Therefore q_1 is a rational number greater than p such that $x \in \complement \overline{U_{q_1}}$. Let $x \in \complement \overline{U_r}$ for some $r > p$. Since $U_r \subset \overline{U_r}$ and $x \notin \overline{U_r}$, $x \notin U_r$. Hence $\inf \{r: x \in U_r\} \geq r > p$ and $f(x) > p$. Therefore $f(x) > p$ if and only if $x \in \complement \overline{U_r}$ for some $r > p$. Hence $\{x: f(x) > p\} = \cup \{\complement \overline{U_r}: r > p\}$, which is an open set. Also,

$\{x : p < f(x) < q\} = (\cup\{\mathscr{C}\,\overline{U_r} : r > p\}) \cap (\cup\{U_r : r < q\})$, which is open since it is the intersection of two open sets. Thus $f^{-1}((p,q))$ is always an open set and hence, f is continuous. Since $A \subset U_0$, $f(x) = 0$ if $x \in A$. Since $\mathscr{C}B = U_1$, $f(x) = 1$ if $x \in B$.

(iii) $\Rightarrow$ (i): Let A and B be disjoint closed subsets of S and f a continuous mapping from S to [0,1] such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$. Consider the subspace [0,1] of R, the set of real numbers. Let J_1 and J_2 be open subsets of [0,1] consisting of the intervals $0 \leq x < \frac{1}{4}$ and $\frac{3}{4} < x \leq 1$, respectively. Define $U = f^{-1}(J_1)$, $V = f^{-1}(J_2)$. U and V are disjoint open subsets of S containing A and B, respectively. Therefore $(S, \mathscr{L})$ is normal.

It has already been shown that a T_4-space is T_1, T_2, and normal. Since it is T_1, it is also T_0. It remains to show that a T_4-space is regular and T_3. Of course, if the space is regular, it will be T_3, since it is already T_1.

<u>Theorem 7</u>: Every T_4-space is regular.

<u>Proof</u>: Since a T_4-space is a T_1-space, singletons are closed. Let A be a closed set in a T_4-space $(S, \mathscr{L})$. Let $x \in S$, $x \notin A$. $\{x\}$ is closed. Hence there exist open sets U and V such that $x \in U$, $A \subset V$, and $U \cap V = \emptyset$. Therefore $(S, \mathscr{L})$ is regular.

The converse of this theorem is not true. In fact, it can be shown that a T_3-space is not necessarily normal. The Nested Interval Theorem will be used in the next example. The proof of this theorem will not be given here but can be found in many advanced calculus texts.

Nested Interval Theorem: Let (I_k) be a sequence of nonempty closed intervals which is nested in the sense that $I_1 \supset I_2 \supset \ldots \supset I_k \supset \ldots$. Then there exists a point which belongs to all of the intervals.

Example 5: Let S be the subset of the Cartesian plane R^2 consisting of all pairs of real numbers (x,y) such that $y \geq 0$. Let L be the subset of S consisting of all (x,y) such that $y = 0$. Let $S_r(p) = \{q \in R^2: |p - q| < r\}$. For each $p \in S$ and $r > 0$, define a set $U_r(p)$ as follows:

$U_r(p) = S_r(p) \cap S$ if $p \in S - L$.

$U_r(p) = S_r(q) \cup \{p\}$, where q is the point (x,r) if p is the point $(x,0)$.

Let $\mathscr{T}$ be the topology generated on S by the collection of sets of the form $U_r(p)$. Let $p, q \in S$ and let $0 < r \leq \frac{1}{2}|p - q|$. Then $p \notin U_r(q)$ and $q \notin U_r(p)$. Hence $(S,\mathscr{T})$ is a T_1-space. Let U be any nonvoid open set in $(S,\mathscr{T})$. Let $x \in U$. Since U is open, there exists $r > 0$ such that $x \in U_{\frac{r}{2}}(x)$. If $x \in S - L$, $\overline{U_{\frac{r}{2}}(x)} = \{q \in R^2: |x-q| \leq \frac{r}{2}\}$

$\cap S \subset U_r(x)$. If $x \varepsilon L$, $U_{\frac{r}{2}}(x) = \left\{ q \varepsilon R^2 \mid y - q| \leq \frac{r}{2} \right\}$, where $y = (x_1, \frac{r}{2})$ when $x = (x_1, 0)$. Hence $\overline{U_{\frac{r}{2}}(x)} \subset U_r(x)$

Therefore, $x \varepsilon U_{\frac{r}{2}}(x) \subset \overline{U_{\frac{r}{2}}(x)} \subset U_r(x)$ for all $x \varepsilon S$ and $(S, \mathscr{T})$ is regular. Now it must be shown that $(S, \mathscr{T})$ is not normal. Let $A = \left\{(x, 0): x = \text{rational}\right\}$ and $B = \left\{(x, 0): x = \text{irrational}\right\}$. Every subset of L is closed. Hence A and B are closed. Suppose U_A and U_B are open sets containing A and B, respectively. For each $p \varepsilon B$ there exists $r_p > 0$ such that $U_{r_p}(p) \subset U_B$. Let B_n denote the set of those irrational numbers x for which $r_p \geq \frac{1}{n}$. Now it will be shown that for some interval $I \subset L$, every point of I is arbitrarily close to B_n. Suppose that for each n and for each interval I there exists a subinterval J in I such that $B_n \cap J = \emptyset$. Let the rational numbers be ordered in a single sequence $(a_0, a_1, \ldots, a_n, \ldots)$. Then construct a sequence of closed intervals $I_n \subset I$ such that $I_{n+1} \subset I_n$, $a_n \notin I_n$, and $B_n \cap I_n = \emptyset$. By the Nested Interval Theorem there exists a real number z which belongs to all of these intervals. Since $a_n \notin I_n$, z is not rational. Hence, for sufficiently large n, $z \varepsilon B_n$. This a contradiction, since $z \varepsilon I_n$ and $B_n \cap I_n = \emptyset$. Hence, for some n, there exists an interval I with the property that every subinterval of I contains points of B_n. Consequently, there are points of B_n arbitrarily close to a. Since $a \in A$, there exists $r > 0$ such that $U_r((a, 0)) \subset U_A$. To each $x \in B_n$ there corresponds a set $U_r((x, 0)) \subset U_B$ with $r \geq \frac{1}{n}$. Hence, if x is sufficiently close to a, the sets $U_r((a, 0))$ and $U_r((x, 0))$ intersect. Thus U_A and U_B are not disjoint. Therefore $(S, \mathcal{T})$ is not normal and not T_4.

From this it can be concluded that any one of the spaces T_0, T_1, T_2, T_3, or regular does not imply T_4. It can also be shown that a space which is normal is not necessarily T_0, T_1, T_2, T_3, T_4, or regular. Two examples will be given to show this. The first one will be a normal space which is not T_0 and the second a normal space which is not regular.

Example 6: Let $S = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{b, c\}, S\}$. The only disjoint nonempty closed sets in $(S, \mathcal{T})$ are $\{a\}$ and $\{b, c\}$. Since these sets are also open, $(S, \mathcal{T})$ is normal. In Example 4 it was shown that $(S, \mathcal{T})$ is not a T_0-space. This also shows that $(S, \mathcal{T})$ is not T_1, T_2, T_3, or T_4.

Example 7: Let $S = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, S\}$. The closed sets in $(S, \mathcal{T})$ are S, $\{b, c\}$, $\{a, c\}$, $\{c\}$, and $\emptyset$. Since there are no nonempty disjoint closed sets, $(S, \mathcal{T})$ is normal. There is no open set containing the closed set $\{b, c\}$, but not containing a. Consequently, $(S, \mathcal{T})$ is not regular.

The results in this chapter are summarized in the diagram below.

$$
\begin{array}{l}
T_0 \\
\Uparrow \\
T_1 \\
\Uparrow \\
T_2 \\
\Uparrow \\
T_3 \Rightarrow \text{regular} \\
\Uparrow \\
T_4 \Rightarrow \text{normal}
\end{array}
$$

$T_3 \Leftrightarrow T_1$ and regular $\Leftrightarrow T_2$ and regular

$T_4 \Leftrightarrow T_1$ and normal $\Leftrightarrow T_2$ and normal

CHAPTER III

APPLICATIONS OF THE SEPARATION AXIOMS TO METRIC SPACES, COMPACT SPACES, AND SEQUENCES

An interesting and useful class of topological spaces are the metric spaces which were defined in Chapter I. The question may be asked: How is a metric space related to the spaces defined in Chapter II? This question is answered by the following theorem.

<u>Theorem 8</u>: A metric space is a T_4-space.

<u>Proof</u>: Let (S, d) be a metric space, $x, y \in S$, $x \neq y$. Then $d(x,y) = \alpha > 0$. $C(x, \frac{\alpha}{2}) = \left\{ z \in S: d(x,z) < \frac{\alpha}{2} \right\}$ and $C(y, \frac{\alpha}{2}) = \left\{ z \in S: d(x,z) < \frac{\alpha}{2} \right\}$ are disjoint open sets containing x and y, respectively. Therefore (S, d) is a T_2-space. Let A and B be disjoint closed sets in (S, d) and let $x \in A$. Since B is closed and $x \notin B$, there exists $\alpha_x > 0$ such that $C(x, \alpha_x) \cap B = \emptyset$, that is, $d(x, y) \geq \alpha_x$ for all $y \in B$. Similarly, for every $y \in B$, there exists $\beta_y > 0$ such that $d(x,y) \geq \beta_y$ for all $x \in A$. Then the sets $U = \cup \left\{ C(x, \frac{\alpha_x}{2}): x \in A \right\}$ and $V = \cup \left\{ C(y, \frac{\beta_y}{2}): y \in B \right\}$ are open sets containing A and B, respectively. Suppose $U \cap V \neq \emptyset$. Let $z \in U \cap V$. Then there exists $x \in A$ and $y \in B$ such that $z \in C(x, \frac{\alpha_x}{2}) \cap C(y, \frac{\beta_y}{2})$ and $d(x,y) \leq d(x,z) + d(y,z) < \frac{\alpha_x}{2} + \frac{\beta_y}{2} \leq \mu$, where

$\mu = \max(\alpha_x, \beta_y)$. But this contradicts either $d(x,y) \geq \alpha_x$ or $d(x,y) \geq \beta_y$. Hence $U \cap V = \emptyset$ and (S,d) is normal. Since (S, d) is also a T_2-space, it is a T_4-space.

This result, together with results from Chapter II, shows that a metric space is T_0, T_1, T_2, T_3, T_4, regular, and normal.

Hausdorff or T_2-spaces are important in the study of sequences. Not all sequences have unique limits, but in a Hausdorff space the limit of a sequence is unique, if it exists.

Theorem 9: In a Hausdorff space any sequence has at most one limit.

Proof: Let $(S, \mathcal{T})$ be a Hausdorff space. Suppose there is a sequence (x_n) in S which converges to x and y, $x \neq y$. There exist disjoint open sets U and V such that $x \in U$ and $y \in V$. Since U and V are neighborhoods of x and y, respectively, there exist M_1, M_2 such that $n \geq M_1$ implies $x_n \in U$ and $n \geq M_2$ implies $x_n \in V$. If $n \geq \sup \{M_1, M_2\}$, $x_n \in U$ and $x_n \in V$. This is impossible since U and V are disjoint. Therefore (x_n) has at most one limit.

The converse of this theorem is not true. There are spaces which are not Hausdorff in which every convergent sequence has a unique limit.

Example 8: Let S be an uncountable set and $\mathscr{T}$ be the collection of all subsets of S with countable complements and $\emptyset$. Let $x, y \in S$, $x \neq y$. Let U be an open set containing x, but not y. Such a set exists, since $\complement\{y\}$ is open and contains x. $\complement U$ is countable and contains no nonempty open sets. Therefore $(S, \mathscr{T})$ is not Hausdorff. Let (x_n) be a sequence which is constant after a certain point, say x_{n_0}. $x = x_{n_0}$ is a limit of (x_n). Consider $y \neq x$. $\complement\{x\}$ is a neighborhood of y, but there is no M such that $n \geq M$ implies $x_n \in \complement\{x\}$. Therefore y is not a limit of (x_n), so (x_n) has only one limit. Let (z_n) be a sequence which is not constant after a certain point. Let $z \in S$. $N = \{z\} \cup (S - \bigcup_{n=1}^{\infty} \{z_n\})$ is a neighborhood of z. But there is no M such that $z_n \in N$ if $n \geq M$. Hence (z_n) does not converge to z. Therefore every sequence in $(S, \mathscr{T})$ converges to at most one point.

Although a space in which sequences have unique limits is not necessarily T_2, it must be T_1.

Theorem 10: If $(S, \mathscr{T})$ is a topological space in which every sequence has at most one limit, then $(S, \mathscr{T})$ is a T_1-space.

Proof: Suppose $(S, \mathscr{T})$ is not a T_1-space. Then there exists $x, y \in S$, $x \neq y$ such that for all open sets

U containing x, $y \in U$. The sequence (y_n) with $y_1 = y_2 = \ldots = y$ converges to y. Since $y \in U$ for all U containing x, (y_n) also converges to x. This a contradiction, since $x \neq y$ and (y_n) has only one limit. Hence $y \notin U$. Therefore $(S, \mathcal{T})$ is a T_1-space.

It is possible to have T_1-spaces in which sequences do not have unique limits.

Example 9: Let S be the set of positive integers. Let $\mathcal{T}$ be the collection of all subsets of S with finite complements and $\emptyset$. Let $x, y \in S$, $x \neq y$. $\complement\{x\}$ is an open set containing y and $\complement\{y\}$ is an open set containing x. Hence $(S, \mathcal{T})$ is a T_1-space. Let (x_n) be the sequence with $x_n = n$ for all $n \in S$. Let x be any point in S. Let N be a neighborhood of x. N contains all but a finite number of points of S. Hence N contains all but a finite number of points of (x_n). Therefore (x_n) converges to x for all $x \in S$.

In the study of compact spaces, T_2-spaces are of special interest.

Definition: If $(S, \mathcal{T})$ is a topological space, $X \subset S$, and $\mathcal{U}$ is a collection of open subsets U of S such that $X \subset \bigcup_{U \in \mathcal{U}} U$, the collection $\mathcal{U}$ is an open cover of X.

Definition: A set X in a topological space $(S, \mathcal{T})$ is compact in case every open cover of X contains a finite cover of X. If S is compact, $(S,\mathscr{T})$ is said to be a compact topological space.

Theorem 11: Every compact T_2-space is a T_4-space.

Proof: Since a space is T_4 if it is a normal T_2-space, it is sufficient to show that every compact T_2-space is normal. Let $(S,\mathscr{T})$ be a compact T_2-space. Let A and B be disjoint closed subsets of S. Since $(S,\mathscr{T})$ is compact, A and B are compact. Let $x \varepsilon A$, $y \varepsilon B$. There exist disjoint open sets $U(x,y)$ and $V(x,y)$ such that $x \varepsilon U(x,y)$ and $y \varepsilon V(x,y)$. For a fixed y, the collection $\{U(x,y): x \varepsilon A\}$ is an open cover of A. Since A is compact, there exists a finite subset X_y of A such that $A \subset U(y) = \cup \{U(x,y): x \varepsilon X_y\}$. If such a set X_y is chosen for each $y \varepsilon B$, and $V(y)$ is defined as $\cap \{V(x,y): x \varepsilon X_y\}$, then the collection $\{V(y): y \varepsilon B\}$ is an open cover of B. Moreover, $U(y) \cap V(y) = \emptyset$ for all $y \varepsilon B$. Because B is compact, there is a finite subset Y of B such that $B \subset V = \cup \{V(y): y \varepsilon Y\}$. Now if U is taken to be the set $\cap \{U(y): y \varepsilon Y\}$, U is an open set containing A, because it is the intersection of a finite collection of open sets containing A. Similarly, V is an open set containing B. Since $U(y) \cap V(y) = \emptyset$ for all $y \varepsilon B$, $U \cap V = \emptyset$. Therefore $(S,\mathscr{T})$ is normal.

The Heine - Borel Theorem for Euclidean spaces states that a set is compact if and only if it is closed and bounded. The concept of boundedness is not defined in general topological spaces, so this theorem has no meaning there. Also, there are topological spaces in which not all compact sets are closed. But it can be proved that in a T_2-space every compact set is closed.

<u>Theorem 12</u>: If a topological space $(S, \mathscr{T})$ is a T_2-space, then every compact set in $(S, \mathscr{T})$ is closed.

<u>Proof</u>: Let A be a compact set in a T_2-space $(S, \mathscr{T})$, $x \notin A$. For each $y \in A$ there exist disjoint open sets U_x and U_y such that $x \in U_x$ and $y \in U_y$. The collection $\mathcal{U}_y = \{U_y : y \in A\}$ is an open cover for A. Since A is compact, there exists a finite subset $\mathcal{V}_y$ of $\mathcal{U}_y$ which covers A. Let $\mathcal{V}_x$ be the corresponding collection of sets containing x and let $U = \bigcap_{U_x \in \mathcal{V}_x} U_x$. U is an open set containing x, since it is the intersection of a finite number of open sets containing x. Also, $U = \bigcap_{U_x \in \mathcal{V}_x} U_x \subset \bigcap_{U_y \in \mathcal{V}_y} \mathscr{C}\, U_y = \mathscr{C} \bigcup_{U_y \in \mathcal{V}_y} U_y \subset \mathscr{C} A$. Thus each point in $\mathscr{C} A$ is contained in an open subset of $\mathscr{C} A$. Therefore, $\mathscr{C} A$ is open and A is closed.

The converse of this theorem is not true. A topological space may be found which is not T_2, but every compact set in the space is closed.

<u>Example 10</u>: Let S be an uncountable set and $\mathscr{T}$ the collection of all sets with countable complements

and $\emptyset$. In Example 8 it was shown that $(S,\mathscr{T})$ is T_1 but not T_2. Let A be an infinite subset of S. $A = B \cup \{a_1, a_2, \ldots\}$, where $\{a_1, a_2, \ldots\}$ is a countably infinite subset of A and $B = A - \{a_1, a_2, \ldots\}$. $U_i = S - \{a_{i+1}, a_{i+2}, \ldots\}$ is open. $\bigcup_{i=1}^{\infty} U_i$ is an open cover of A, but a finite number of U_i's do not cover A. Therefore A is not compact. Finite subsets of S are compact. They are also closed, since all countable subsets of S are closed. Therefore, every compact set in $(S,\mathscr{T})$ is closed.

<u>Theorem 13</u>: If every compact set in a topological space is closed, the space is T_1.

<u>Proof</u>: Let $(S,\mathscr{T})$ be a topological space in which every compact set is closed. Let $x \in S$. $\{x\}$ is compact, hence closed. Therefore $(S,\mathscr{T})$ is a T_1-space.

The following example shows that the converse of this theorem is false.

<u>Example 11</u>: Consider an infinite set S with the cofinite topology $\mathscr{T}$. $(S,\mathscr{T})$ is a T_1-space. Let A be a nonempty subset of S and let $\mathscr{G}$ be any open cover of A. Let $G \in \mathscr{G}$, $G \neq \emptyset$. Since G is open, it is infinite and $\mathscr{C}G$ is finite. Hence $\mathscr{C}G$ contains at most a finite number of points, say n, of A. The number of open sets from $\mathscr{G}$ needed to cover these n points does

not exceed n. Hence the maximum number of sets from $\mathcal{S}$ needed to cover A is n + 1. Therefore A is compact. Subsets of S with finite complements are not closed. Hence there are compact subsets of S which are not closed.

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