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Properties of some non-random distributions

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PROPERTIES OF SOME NON-RANDOM DISTRIBUTIONS

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C. S.

TABLE OF CONTENTS

INTRODUCTION

Ecologists have found that the distribution of organisms of most species in a field is either more clumped (contagious) or more regular than would be expected if the organisms were placed at random in the field. It should therefore be beneficial to study the properties of certain contagious and regular distributions. In this paper, the summed Poisson and compound Poisson distributions will be introduced.

These are generalizations of the Poisson distribution, the distribution which arises when considering the number of random events occurring in a certain amount of time or space. A contagious distribution is then developed which is both a summed Poisson and a compound Poisson distribution. Lastly, the sampling variance of a very regular distribution will be derived.

CHAPTER 1

CONTAGIOUS DISTRIBUTIONS

I. SUMMED POISSON DISTRIBUTIONS

In the Poisson distribution, the probability of exactly n events is given by

$$(1) \qquad \pi(n, \quad) = \frac{e^{-\lambda} \lambda^n}{n!}$$

where $\lambda > 0$ is the expected number of events. If different individuals of a population have different values of λ, and if λ is distributed according to the cumulative distribution function $U(\lambda)$, the probability of n events in the total population is given by the Reimann-Stieltjes integral.

$$(2) \qquad \pi_n = \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} \, dU(\quad).$$

The distribution defined by (2) will be called the summed Poisson distribution.

The mean value of the summed Poisson distribution is $\sum_{n=0}^{\infty} n \, \pi_n = \sum_{n=0}^{\infty} n \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} \, dU(\lambda)$. Since $\sum_{n=0}^{\infty} \frac{n \, e^{-\lambda} \lambda^n}{n!}$ converges uniformly to λ on any internal $[0, a]$, $a \geq 0$, the order of integration and summation may be interchanged to obtain

$$\int_0^\infty \sum_{n=0}^{\infty} \frac{n\, e^{-\lambda} \lambda^n}{n!} \, dU(\lambda) = \int_0^\infty \lambda \, dU(\lambda) = m.$$

Therefore the mean value of the summed Poisson distribution equals the mean value of the cumulative distribution function $U(\lambda)$.

It is natural to compare distributions related to the Poisson with the Poisson distribution with the same mean. Theorem 1 adopted from Feller [1] is such a comparison.

THEOREM 1.

Let $U(\lambda)$ be any c.d.f. with mean m then

(3) $$\pi_o \geq (0,\ m)$$

and

(4) $$\frac{\pi_1}{\pi_o} \leq m.$$

For the proof of (3) we have $\pi_o = \int_0^\infty e^{-\lambda} \, dU(\lambda)$ $= e^{-m} \int_0^\infty e^{m-\lambda} \, dU(\lambda)$. Replacing $e^{m-\lambda}$ by a Taylor series expansion with a remainder we obtain

$$\pi_o = e^{-m} \int_0^\infty 1 + (m-\lambda) + \frac{(m-\lambda)^2}{2!} e^{\theta(m-\lambda)} \, dU(\lambda)$$

where $0 < \theta < 1.$

Since $\int_0^\infty dU(\lambda) = 1$, $\int_0^\infty \lambda \, dU(\lambda) = m$ and $\int_0^\infty (m-\lambda)^2 e^{\theta(m-\lambda)} \geq 0$, we get

$$\pi_o \geq e^{-m} = \pi(0,\ m).$$

The proof of (4) is similar. Writing $m\pi_o - \pi_1$ as

$$e^{-m} \int_0^\infty (m-\lambda) \, e^{m-\lambda} \, dU(\lambda)$$

and replacing $e^{m-\lambda}$ with a Taylor series expansion, we obtain

$$m\pi_o - \pi_1 = e^{-m} \int_o^\infty m-\lambda \quad + (m-\lambda)^2 \; e^{\theta(m-\lambda)} \; dU(\lambda)$$

where $0 < \theta < 1$. Again, since $\int_o^\infty dU(\lambda) = 1$, $\int_o^\infty \lambda \, dU(\lambda) = m$, and $\int_o^\infty (m-\lambda)^2 \; e^{\theta(m-\lambda)} \; dU(\lambda) \geq 0$, we have

$$m \; \pi_o - \pi_1 \geq 0$$

or

$$\frac{\pi_1}{\pi_o} \leq m.$$

The first part of Theorem 1 states that when sampling a summed Poisson distribution, the proportion of samples in which no events occur is greater than or equal to the proportion of samples in which no events occur with no entries for the Poisson distribution with the same mean.

The inequalities (3) and (4) can be made sharper by using two more terms of the Taylor series of $e^{m-\lambda}$. If this is done we obtain

(3') $$\pi_o \geq e^{-m} \quad 1 + \frac{\sigma^2}{2} - \frac{1}{6} M$$

and

(4') $$m\pi_o - \pi_1 \geq e^{-m} \quad \sigma^2 - \frac{1}{2} M$$

where σ^2 and M are respectively the second and third moments about the mean of $U(\lambda)$.

Thus,

$$\sigma^2 = \int_0^\infty (\lambda - m)^2 \, dU(\lambda)$$

and

$$M = \int_0^\infty (\lambda - m)^3 \, dU(\lambda).$$

Feller states that

$$\pi_1 < \pi\,(1, m)$$

whenever $\quad 2\sigma^2 - M > 0.$

This statement is untrue as shown by the following counterexample. Let the distribution of be given by

$$p(\lambda = 8) = 1/2$$

$$p(\lambda = 10) = 1/2.$$

Then

$$m = \tfrac{1}{2}\cdot 8 + \tfrac{1}{2}\cdot 10 = 9,$$

$$\sigma^2 = \tfrac{1}{2}(8-9)^2 + \tfrac{1}{2}(10-9)^2 = 1$$

and $\quad M = \tfrac{1}{2}(8-9)^3 + \tfrac{1}{2}(10-9)^3 = 0.$

Therefore $\quad M = 0 < 2 = 2\cdot\sigma^2,$

but $\quad \pi_1 > \pi(1, m)$, since

$$\pi_1 = \tfrac{1}{2}\, 8\, e^{-8} + \tfrac{1}{2}\, 10\, e^{-10} = .0015690$$

$$\pi(1, m) = 9\, e^{-9} = .0011106$$

It is true, however, that (4) becomes a strict inequality if $2\sigma^2 - M > 0$. This is easily seen from (4') and might be what Feller intended to say.

A property of the summed Poisson distribution which is not contained in Feller's paper is that samples in which many events occur have a greater probability than would be expected if the distribution were random. This is stated formally in Theorem 2.

THEOREM 2. For any distribution $U(\lambda)$ such that $0 < \sigma^2 = \int_0^\infty (\lambda - m)^2 \, dU(\lambda)$, there exists an N such that $\pi(n, m) < \pi_n$ for all $n > N$.

Proof: Since $\sigma^2 > 0$, $\exists$ $a > 0$ and $b \geq m + a$ such that $\int_{m+a}^{b} dU(\lambda) = p > 0$. Since $\frac{m+a}{a} > 1$, $(\frac{m+a}{m})^n$ is unbounded and increasing, there exists N such that $n > N$ implies

$$\frac{e^{-m}}{p\ e^{-b}} < \left(\frac{m+a}{m}\right)^n$$

or $\quad m^n e^{-m} < p(m+a)^n e^{-b}$.

Since for λ in the interval $[m+a, b]$ $\frac{\lambda^n e^{-\lambda}}{n!} \geq \frac{(m+a)^n e^{-b}}{n!}$, we have

$$\pi_n = \int_0^\infty \frac{\lambda^n e^{-\lambda}}{n!} \, dU(\lambda) \geq \int_{m+a}^{b} \frac{\lambda^n e^{-\lambda}}{n!} \, dU(\lambda)$$

$$\geq \int_{m+a}^{b} \frac{(m+a)^n}{n!} e^{-b} \, dU(\quad)$$

$$= \frac{(m+a)^n e^{-b}}{n!} p > \frac{m^n e^{-n}}{n!} = \pi(n; m)$$

We note that the variance of the summed Poisson distribution is

$$\sum_{n=0}^{\infty} n^2 \pi_n - m^2 = \sum_{n=0}^{\infty} n^2 \int_0^\infty \frac{e^{-\lambda} \lambda^n}{n!} \, dU(\lambda) - m^2$$

$$= \int_0^\infty \Sigma \frac{n^2 e^{-\lambda} \lambda^n}{n!} dU(\lambda) - m^2 =$$

$$\int_0^\infty (\lambda + \lambda^2) dU(\lambda) - m^2 =$$

$$\int_0^\infty \lambda dU(\lambda) + \int_0^\infty \lambda^2 dU(\lambda) - m^2 = m + \sigma^2$$

Ecologists have used a variance - mean ratio greater than one an indicator of contagion. The variance - mean ratio of the summed Poisson distribution is $\frac{\sigma^2 + m}{m} \geq 1$ and therefore represents a contagious distribution in this sense.

II. GENERATING FUNCTIONS AND THE COMPOUND POISSON DISTRIBUTION

Before considering another generalization of the Poisson distribution, the concept and properties of generating functions will be introduced. For the proof of these properties see Feller [2].

Definition. Let X be a random variable assuming the values 0, 1, 2, ... with distribution given by

$$p(X = n) = P_n$$

The series

$$f(s) = \sum_{n=0}^{\infty} P_n s^n$$

is called the generating function for the distribution of X if $\sum_{n=0}^{\infty} P_n s^n$ converges in some interval $-s_0 < s < s_0$.

Examples

(a) If X has the distribution $p(X = 0) = q$ $p(X = 1) = p$, where $p + q = 1$; then

(5) $f(s) = q + ps$

(b) If X has the Poisson distribution, then

(6) $f(s) = \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!} s^n = e^{-\lambda + \lambda s}$

The mean value of any random variable X with generating function f(s) is

(7) $E(X) = m = f'(1)$.

If $\lim_{s \to 1} f'(s) = \infty$

then the mean of X does not exist.

The variance of X is

(8) $\text{var}(X) = \sigma^2 = f''(1) + f'(1) - [f'(1)]^2$

THEOREM 3. If X and Y are independent random variables with generating functions f(s) and g(s) respectively then

(9) $h(s) = f(s) \cdot g(s)$

is the generating function of X + Y.

THEOREM 4: Let X_k be a sequence of mutually independent random variables with the common distribution $p(X_k = n) = p_n$ with generating function f(s).

Let the distribution of the random variable N have the generating function g(s). The generating function, h(s), of the sum $S_N = X_1 + \ldots + X_N$ is the compound function

(10) $h(s) = g(f(s))$

Two examples of Theorem 4 are of special interest

(a) If the X_i have distribution

$P(X_i = 1) = p$ and $P(X_i = 0) = q$

then by (5)

(11) $$h(s) = g(q + ps)$$

(b) If N has a Poisson distribution with mean λ,

then by (6)

(12) $$h(s) = e^{-\lambda + \lambda f(s)}.$$

Any distribution which has a generating function with the form of (12) will be called a compound Poisson distribution. The properties of the compound Poisson distribution are treated in detail by Feller and will not be considered here.

III. NEYMAN'S TYPE -A DISTRIBUTION

As an illustration of the importance of compound Poisson and summed Poisson distributions in the study of contagious distributions, Neyman's Type A contagious distribution [3] will be considered. Neyman was interested in the distribution of larvae in a field, which is divided into plots of equal areas. He wished to determine the probability c_K that exactly K larvae are found in a certain plot.

In his paper, Neyman kept his discussion general and only at a late stage did he make the simplifying assumptions. Therefore he had to use very complicated differential arguments. Here we will start by making the simplifying assumptions, which will allow us to use the powerful tool of generating functions. Neyman assumed (a) The larvae may come from various litters. (b) The probability that exactly V litters are represented on the plot is given by the Poisson distribution (1). (c) The probability f_n that there are exactly n survivors is the same for all litters; the generating function for this distribution will be represented by f(s). (d) The probability that a given individual of a particular litter is found on the plot is p. The generating function for the distribution of the probability of the given individual being found on the plot is therefore given by (5) where q = 1-p. (e) The parameter u generally depends on the litter of which the individual is a member (and varies, in particular, with the position of the litter relative to the plot being considered). However, u here will be taken as a constant, say $u = u_o$.

Now, by Theorem 4, the generating function for the distribution of survivors of a particular litter in the plot under observation is

(13) $$f(q + ps)$$

Since the number of litters in the plot has a Poisson distribution, the generating function of the distribution of the number of larvae in the plot is

(14) $$C(s) = e^{-\lambda} + \lambda f(q+ps)$$

by (12) and (13).

Now assume that the distribution of the number of survivors from a particular litter is Poisson with mean M, thus

$$f(s) = e^{-M + Ms}$$

and

$$f(q + ps) = e^{-M + M(q + ps)}$$

Under this added assumption C(s) becomes

$$C(s) = e^{-\lambda + \lambda e^{-M + M(q + ps)}}$$

$$= e^{-\lambda + \lambda e^{-Mp + Mps}}$$

or

(15) $$C(s) = e^{-\lambda + \lambda e^{-a + as}}$$

where $a = Mp$.

But

$$e^{-\lambda + \lambda e^{-a + as}} = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n (e^{-a} e^{as})^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n e^{-an}}{n!} \sum_{k=0}^{\infty} \frac{(asn)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{e^{-\lambda} a^k s^k}{k!} \sum_{n=0}^{\infty} \frac{\lambda^n e^{-an} n^k}{n!}$$

$$= \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda} a^k}{k!} \sum_{n=0}^{\infty} \frac{n^k (\lambda e^{-a})^n}{n!}$$

Therefore

(16) $$c_k = \frac{e^{-\lambda} a^k}{k!} \sum_{n=0}^{\infty} \frac{n^k (\lambda e^{-a})^n}{n!}$$

This agrees with the result announced by Neyman [3] as equation (43).

Neyman's distribution is a compound Poisson distribution by (15). It is also a summed distribution since, if λ takes on values na only, $a > 0$, $n = 0, 1, \ldots,$ and if λ has the Poisson distribution $p(\lambda = na) = \frac{e^{-\lambda}\lambda^n}{n!}$ then (2) becomes

$$\pi_k = \sum_{n=0}^{\infty} \frac{e^{-na}(na)^k}{k!} \quad \frac{e^{-\lambda}\lambda^n}{n!} = \frac{e^{-\lambda} a^k}{k!} \sum_{n=0}^{\infty} \frac{n^k(e^{-a}\lambda)^n}{n!}$$

which is the same as (16).

The mean and the variance of Neyman's distribution can be calculated from (15) by using (7) and (8).

$$C'(s) = a\lambda \quad e^{-a + as} \; e^{-\lambda} \quad + \lambda \; e^{-a + as}$$

and $$c''(s) = (a\lambda \quad e^{-a + as})^2 \; e^{-\lambda} \quad + \lambda \; e^{-a + as}$$

$$+ a^2\lambda \quad e^{-a + as} \; e^{-\lambda} \quad + \lambda \; e^{-a + as} \quad .$$

Therefore

$$m = C'(1) = a\lambda = Mp\lambda$$

and

$$\sigma^2 = C''(1) + C'(1) - [C'(1)]^2$$

$$= (a\lambda)^2 + a^2\lambda + a\lambda - [a\lambda]^2$$

$$= a^2\lambda + a\lambda =$$

$$M^2p^2\lambda + Mp\lambda \; .$$

CHAPTER 2

REGULAR DISTRIBUTIONS

Botanists have noticed that the variance of the distribution of the number of plants found in a quadrat (square or rectangle) depends on the area of the quadrat. They have used this fact to investigate the contagion and regularity in the distribution of plants. Instead of sampling randomly with different quadrat sizes, a grid of contiguous quadrats is laid out, and increasing quadrat sizes are then built up by blocking adjacent quadrats in pairs, fours, eights, etc. For a survey of the literature using this method see P. Greig-Smith [4].

I. SAMPLING THE SQUARE LATTICE

The effect of quadrat size on the variance when randomly sampling a regular distribution in the form of a square lattice (all points in a plane with integral coordinates) will be obtained below.

Consider the one-dimensional lattice consisting of the points $0, \pm 1, \pm 2, \ldots$. This is to be randomly sampled by an interval of length

$$\ell = n + \alpha$$

where n is an integer and $0 \leq \alpha < 1$. Let the number of lattice points in the interval of length ℓ be the random variable X. The distribution of X is given by

$p(X = n) = 1-\alpha$ and $p(X = n + 1) = \alpha$

Then

(17) $$m = E(X) = n(1-\alpha) + (n+1)\alpha = n - n\alpha + n\alpha + \alpha = n + \alpha = \ell$$

and

(18) $$\mathrm{var}(X) = \sigma^2 = (n-\ell)^2(1-\alpha) + (n+1-\ell)^2\alpha = (n-(n+\alpha))^2(1-\alpha) + (n+1-(n+\alpha))^2\alpha = \alpha^2(1-\alpha) + (1-\alpha)^2\alpha = \alpha(1-\alpha)$$

For a very elegant derivation of (17) and (18) see Kendall and Moran [4].

From (17) and (18) it is easy to derive the mean and variance of the number of points in a rectangle placed randomly on a square lattice with sides parallel to the x and y axis. The number of lattice points in a rectangle may be regarded as the product of two independent random variables X and Y which are respectively the numbers of lattice points on a linear lattice in the intervals corresponding to the sides of the rectangle in the x and y directions. Let the length of the sides of the rectangle be $r = n + \alpha_x$ in the x direction and $s = N + \alpha_y$ in the Y direction. By (17) and (18)

$$E(X) = r,\ \sigma_x^2 = \mathrm{Var}\ (X) = \alpha_x(1-\alpha_x)$$

$$E(Y) = s \text{ and } \sigma_y^2 = \mathrm{Var}\ (Y) = \alpha_y(1-\alpha_y)$$

Therefore

(19) $$m = E(X,Y) = E(X)E(Y) = rs$$

and

$$\sigma^2 = E(X^2Y^2) - E(XY)^2 =$$

$$E(X^2)E(Y^2) - E(X^2)E(Y)^2 +$$

$$E(X^2)E(Y)^2 - E(X)^2E(Y)^2 =$$

$$E(X^2)\,[E(Y^2) - E(Y)^2]$$

$$+ E(Y)^2[E(X^2) - E(X)^2] =$$

$$E(X^2)\,\sigma_y^2 - E(X)^2\,\sigma_y^2 + E(X)^2\,\sigma_y^2$$

$$+ E(Y)^2\,\sigma_x^2 =$$

$$\sigma_y^2\,\sigma_x^2 + E(X)^2\,\sigma_y^2 + E(Y)^2\,\sigma_x^2$$

(20) $$= \alpha_x(1-\alpha_x)\,\alpha_y\,(1-\alpha_y) + r^2\,\alpha_y(1-\alpha_y) + s^2\,\alpha_x(1-\alpha_x).$$

In particular, if the rectangle is a square, then $\ell_o = r = s$ and $\alpha_o = \alpha_x = \alpha_y$ therefore by (19) and (20).

(21) $$m = \ell_o^2$$

(22) $$\sigma^2 = \alpha_o^2\,(1-\alpha_o)^2 + 2\,\ell_o^2\,\alpha_o(1-\alpha_o)$$

As another example, consider the case where $r = 2s$. In particular let

$$r = \sqrt{2}\,\ell \quad \text{and} \quad s = \frac{\ell}{\sqrt{2}}$$

where ℓ is the length of the side of the square with the same area as the rectangle determined by r and s.

Then (20) becomes

(23) $$\sigma^2 = \alpha_x(1-\alpha_x)\,\alpha_y\,(1-\alpha_y) + 2\,\ell^{\,2}\,\alpha_y\,(1-\alpha_y) + \frac{\ell^{\,2}}{2}\,\alpha_x(1-\alpha_x)$$

where $r = \sqrt{2}\,\ell = n + \alpha_x$ and

$$s = \frac{\ell}{\sqrt{2}} = N + \alpha_y.$$

Equations (22) and (23) have been graphed (Fig. 1 and Fig. 2). The equations and graphs show that variance as function of area depends on the slope of the quadrat.

II. SAMPLING THE DIAGONAL LATTICE

The orientation of the quadrat in relation to the x and y axis also affects the variance as a function of area. This is shown below by sampling the square lattice with a square which is oriented at a 45° angle with respect to the x and y axis.

Consider first the lattice that consists of all points (x, y) where x and y are integers and x + y is even. This will be called the diagonal lattice. The diagonal lattice will be randomly sampled with a square with sides parallel to the x and y axis. Let u be the number of rows and v be the number of columns that occur in the square. If either u or v is even, then the number of lattice points in the square is given by $\frac{uv}{2}$. If both u and v are odd, the number of lattice points is given by either $\frac{(u-1)v}{2} + \frac{v+1}{2} = \frac{uv+1}{2}$

or $$\frac{(u-1)v}{2} + \frac{v-1}{2} = \frac{uv-1}{2}$$

depending on whether the first row in the square contains $\frac{v+1}{2}$ or $\frac{v-1}{2}$ lattice points. If the random variables U and V are respectively the numbers of rows and columns then U and V are independent. If ℓ is the length of the side of the square and $2n \leq \ell < 2n + 1$, $\ell = 2n + \beta$, then U and V have the distribution

$$p(U = 2n) = p(V = 2n) = 1 - \beta$$

$$p(U = 2n+1) = p(V = 2n+1) = \beta$$

Let the random variable X be the number of lattice points in the square. The distribution of X for $2n \leq \ell < 2n + 1$ is therefore given by

$$P(X = 2n^2) = (1-\beta)^2$$

$$P(X = 2n^2 + n) = 2\beta(1-\beta)$$

$$P(X = 2n^2 + 2n + 1) = \frac{1}{2}\beta^2$$

$$P(X = 2n^2 + 2n) = \frac{1}{2}\beta^2$$

since the two cases when U and V are both odd have the same probability of occurring.

Thus the mean value of X for $2n \leq \ell < 2n + 1$ is

$m = E(X) =$

$$2n^2(1-\beta)^2 + (2n^2+n)2\beta(1-\beta) + (2n^2 + 2n + 1)\frac{1}{2}\;{}^2 +$$

$$(2n^2 + 2n)\frac{1}{2}\beta^2 = 2n^2[(1-\beta)^2 + 2\beta(1-\beta) + \beta^2] +$$

$$2n[\beta(1-\beta) + \beta^2] + \frac{1}{2}\beta^2 = 2n^2 + 2n\beta + \frac{1}{2}\beta^2 = \frac{4n^2 + 4n\beta + \beta^2}{2}$$

$$= \frac{(2n+\beta)^2}{2} = \frac{\ell^2}{2}$$

and

$$\sigma^2 = E(X^2) - m^2 = 4n^4(1-\beta)^2 + (2n^2+n)^2\, 2\beta(1-\beta)$$

$$+ (2n^2 + 2n+1)^2 \frac{1}{2}\beta^2 + (2n^2+2n)^2 \frac{1}{2}\beta^2 - \frac{\ell^4}{4}$$

$$= 4n^4[(1-\beta)^2 + 2\beta(1-\beta) + \beta^2] + 8n^3[\beta(1-\beta) + \beta^2]$$

$$+ n^2[2\beta(1-\beta) + 6\beta^2]\, 2n\,\beta^2 + \frac{1}{2}\beta^2 - (4n^4 + 8n^3\beta$$

$$+ 6n^2\beta^2 + 2n\beta^3 + \frac{\beta^4}{4})$$

$$= n^2\, 2\beta(1-\beta) + 2n\beta^2(1-\beta) + \frac{\beta^2}{2} - \frac{\beta^4}{4}$$

$$= 2n\,\beta(1-\beta)(n+\beta) + \frac{\beta^2}{2} - \frac{\beta^4}{4}$$

$$= (2n + \beta - \beta)\,\beta\,(1-\beta)\frac{(2n + \beta + \beta)}{2} + \frac{\beta^2}{2} - \frac{\beta^4}{4}$$

$$= (\ell - \beta)\,\beta\,(1-\beta)\left(\frac{\ell + \beta}{2}\right) + \frac{\beta^2}{2} - \frac{\beta^4}{4}$$

$$= \left(\frac{\ell^2 - \beta^2)\,\beta\,(1-\beta)}{2}\right. + \frac{\beta^2}{2} - \frac{\beta^4}{4}$$

$$= \frac{\ell^2\,\beta\,(1-\beta)}{2} + \frac{\beta^2}{2} - \frac{\beta^3}{2} + \frac{\beta^4}{4}$$

$$= \frac{\ell^2\,\beta\,(1-\beta)}{2} + \frac{\beta^2}{4} + \frac{\beta^2(1-\beta)^2}{4}$$

Similarly, if $2n - 1 \leq \ell < 2n$ then the distribution of X where $= 2n - \gamma$ is

$$P(X = 2n^2) = (1 - \gamma)^2$$

$$P(X = 2n^2 - n) = 2\gamma(1 - \gamma)$$

$$P(X = 2n^2 - 2n + 1) = \frac{1}{2}\gamma^2$$

$P(X = 2n^2 - 2n) = \frac{1}{2} \gamma^2.$

Therefore the mean value of X for $2n - 1 \le \ell < 2n$ is $m = E(X) = 2n^2(1- \gamma)^2 + (2n^2 - n)\, 2\gamma(1 - \gamma)$

$$+ (2n^2 - 2n + 1)\frac{1}{2}\gamma^2 + (2n^2 - 2n)\frac{1}{2}\gamma^2$$

$$= \frac{4n^2 - 4n \quad + \gamma^2}{2} = \frac{\ell^2}{2}$$

and $\sigma^2 = E(X^2) - m^2 = 4n^4(1 - \gamma)^2 + (2n^2 + n)^2\, 2\gamma(1- \gamma)$

$$+ (2n^2 - 2n + 1)^2 \frac{1}{2}\gamma^2 + (2n^2 - 2n)^2 \frac{1}{2}\gamma^2 - \frac{\ell^4}{4}$$

$$= 2n\gamma(1- \gamma)(n- \gamma) + \frac{1}{2}\gamma^2 + \frac{\gamma^4}{4}$$

$$= (\ell + \gamma)(1- \gamma)\left(\frac{\ell - \gamma}{2}\right) + \frac{1}{2}\gamma^2 + \frac{\gamma^4}{4}$$

$$= \frac{\ell^2\gamma(1- \gamma)}{2} + \frac{\gamma^2}{4} + \frac{\gamma^2(1- \gamma)^2}{4}$$

By letting

$= 2n \pm \alpha \qquad 0 \le |\alpha| \le 1$

the two previous cases can be combined to give

$$m = E(X) = \frac{\ell^2}{2}$$

and $\sigma^2 = E(X-m)^2 =$

$$\frac{\ell^2|\alpha|(1-|\alpha|)}{2} + \frac{|\alpha|^2}{4} + \frac{|\alpha|^2(1-|\alpha|)^2}{4}$$

Sampling the square lattice with a square which has sides of length ℓ_o and oriented at a 45^o angle with respect to the x and y axis is the same as sampling the diagonal lattice with a square which has sides of length $\ell = \sqrt{2}\, \ell_o$ and oriented with sides parallel to the x and y axes. Therefore the mean and variance of the number of points in a square with sides of length ℓ_o placed at random in a swuare lattice and oriented at a 45^o angle to the x and y axes is given by

$$m = \frac{(\sqrt{2}\,\ell_o)^2}{2} = \ell_o^2$$

and σ^2 =

$$\frac{(\sqrt{2}\,\ell_o)^2\,|\alpha|\,(1-|\alpha|)}{2} + \frac{|\alpha|^2}{4} + \frac{|\alpha|^2\,(1-|\alpha|)^2}{4}$$

where $\sqrt{2}\,\ell_o = 2n \pm \alpha$

$0 \leq |\alpha| \leq 1$

The variance (24) as a function of ℓ_o is graphed in Fig. 3.

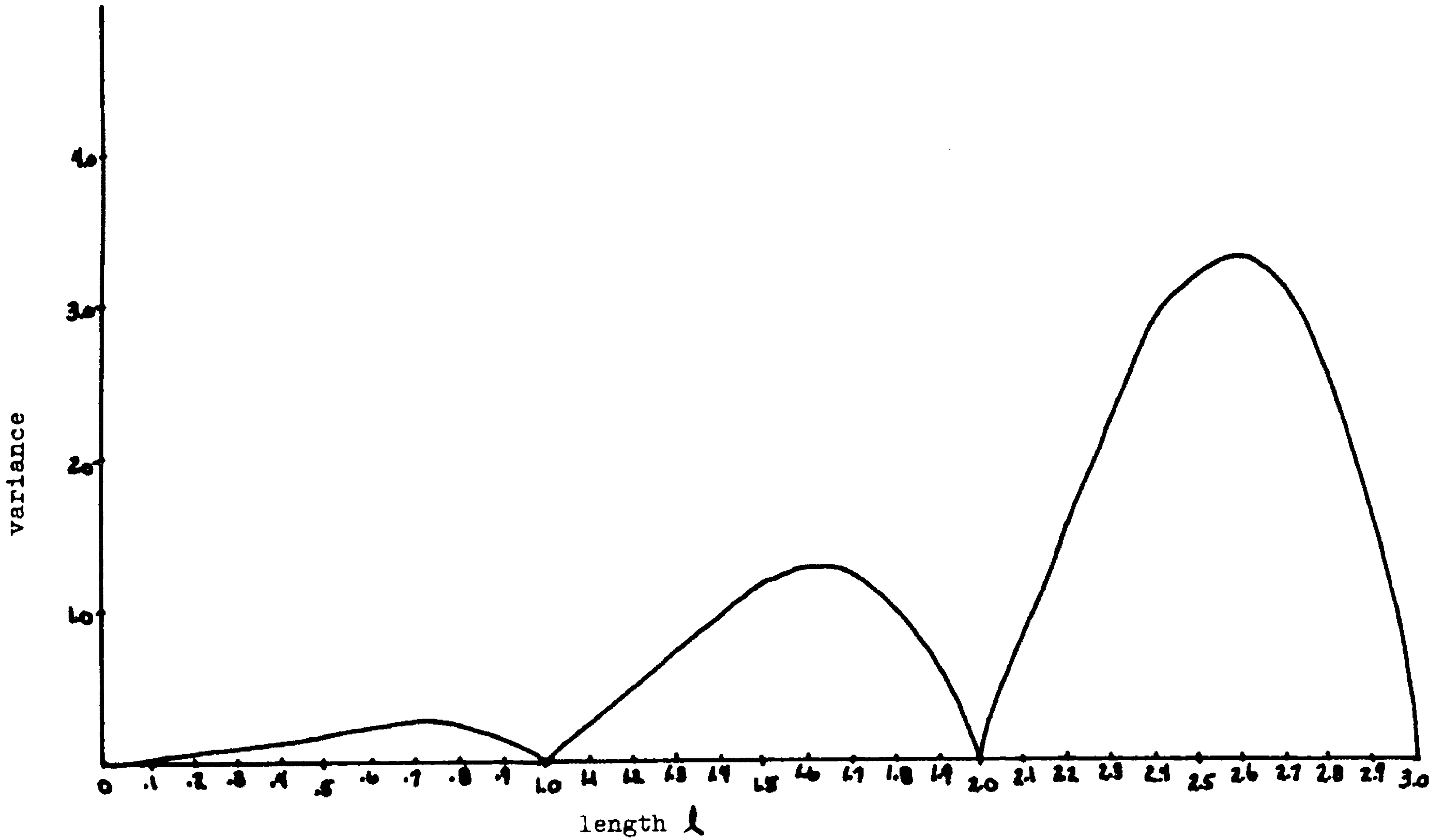
4.0
3.0
2.0
1.0
variance
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3.0
length l

Figure 1

REFERENCES CITED

(1) Feller, W., "On a General Class of 'Contagious' Distributions," Annals of Mathematical Statistics, Vol. 14 (1943), pp. 389-400

(2) Feller, W., An Introduction to Probability Theory and Its Applications, Volume I. John Wiley and Sons, 1957

(3) Neyman, J., "On a new class of 'contagious' distributions, applicable in entomology and bacteriology," Annals of Mathematical Statistics Vol. 10 (1939), pp. 35-57

(4) Greig-Smith, P., Quantitative Plant Ecology, Butterworths, 1964

(5) Kendall, M. G. and P. A. P. Moran, Geometrical Probability, Hafner Publishing Company, 1963