Weierstrass theorem and some generalizations

Rick Billstein

The University of Montana

Follow this and additional works at: https://scholarworks.umt.edu/etd
Let us know how access to this document benefits you.

Recommended Citation
https://scholarworks.umt.edu/etd/8066
THE WEIERSTRASS THEOREM AND SOME GENERALIZATIONS

By

Richard W. Billstein

B.A., Eastern Montana College, 1966

Presented in partial fulfillment of the requirements for the degree of

Master of Arts

UNIVERSITY OF MONTANA

1969

Approved by:

[Signatures and dates]

[Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.]
ACKNOWLEDGEMENTS

I wish to express my sincerest gratitude to Professor William M. Myers for his encouragement, guidance, and above all his patience without which this paper could not have been written. In addition, I wish to thank Professors Elizabeth Papousek and Robert P. Banaugh for their reading of the manuscript.

R. W. B.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>SOME PROOFS OF THE WEIERSTRASS THEOREM</td>
</tr>
<tr>
<td>II</td>
<td>APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS</td>
</tr>
<tr>
<td>III</td>
<td>GENERALIZING THE WEIERSTRASS THEOREM</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>105</td>
</tr>
</tbody>
</table>
INTRODUCTION

The classical Weierstrass Theorem states that any continuous real function defined on a bounded closed interval of real numbers can be approximated uniformly by polynomials. This theorem and its generalizations will be the basis of the following paper. The problem that Weierstrass was faced with in 1885 was that if \( f(x) \) is a real valued and continuous function on \([a, b]\), and if \( \varepsilon > 0 \), is it possible to find a polynomial function \( P(x) \) such that \( |P(x) - f(x)| < \varepsilon \) for all \( x \in [a, b] \)? He answered this question in the affirmative in the classical theorem which bears his name.

In chapter one, several proofs of the Weierstrass Theorem are given along with one form of Stone's generalization of the Weierstrass Theorem. The first proof of the Weierstrass Theorem makes use of Bernstein polynomials. This method of proof gives a constructive method of finding a sequence of polynomials which converge uniformly on the interval to the given continuous function. Also, the rapidity of the convergence can be estimated.

The second proof given here of the Weierstrass Theorem is credited to Lebesque in 1898. This method of proof is to show that \( f(x) \) can be approximated by a piecewise linear function which in turn can be approximated by a polynomial. This is a constructive proof, but it does...
not lead to a practical method of approximation. A third proof of the Weierstrass Theorem making use of integrals is also presented.

To Weierstrass is due also the corresponding theorem on approximation by trigonometric sums:

If \( f(x) \) is a given periodic function of period \( 2\pi \), continuous for all real values of \( x \), and \( \varepsilon \) is a given positive quantity, it is always possible to define a trigonometric sum \( T(x) \) such that \( |f(x) - T(x)| < \varepsilon \) for all real values of \( x \).

This theorem will be the main subject in chapter two. The similarity between these two theorems of Weierstrass will also be examined.

Since polynomials are smooth functions, we might expect that for a function to be well approximated by a polynomial, it should be smooth. It is with this in mind that we examine some properties of derivatives in this chapter to see if there is any relationship between the smoothness of \( f(x) \) and the degree of approximation of polynomials to \( f(x) \).

Finally in chapter three, we attempt to generalize the Weierstrass Theorem. The generalization seeks to lighten the restrictions on the domain in which the given functions are defined. In brief, chapter three examines the question: what functions can be built from the functions of a presecribed family by application of certain
algebraic operations and uniform passage to the limit?
CHAPTER I

Some Proofs of the Weierstrass Approximation Theorem

1.1 Among the multitude of proofs of the Weierstrass Theorem, the proof using Bernstein polynomials is one of the most popular. Before establishing this proof, several facts are needed.

Definition 1.1: Let \( f(x) \) be a real valued function defined on the closed interval \([0, 1]\). The polynomial

\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is called the Bernstein polynomial of degree \( n \) for the function \( f(x) \).

Lemma 1.1: For every \( x \), \( \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1 \).

Proof: The binomial theorem states that

\[
(s+t)^n = \sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k}
\]

If \( s = x \) and \( t = 1-x \) in the binomial theorem, then \( 1 = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \).

Lemma 1.2: For all real \( x \), \( \sum_{k=0}^{n} \binom{n}{k} (k-nx)^2 x^k (1-x)^{n-k} \leq \frac{n}{4} \).

Proof: From the binomial formula, we can observe that \( \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n \). Differentiating this with respect to \( z \), and multiplying the result by \( z \), we obtain

\[
\sum_{k=0}^{n} \binom{n}{k} k z^k = nz(1+z)^{n-1}
\]

Again differentiating this quantity and multiplying the result by \( z \), we obtain

\[
\sum_{k=0}^{n} k^2 \binom{n}{k} z^k = nz(1+nz)(1+z)^{n-2}
\]

Now let \( z = \frac{x}{1-x} \) in the above equations and multiply each equation by \( (1-x)^n \), then

\[
\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1
\]
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k (1-x)^{n-k} = nx, \quad (2) \]
\[ \sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x + nx). \quad (3) \]

Now multiply (1) by \( n^2 x^2 \), (2) by \(-2nx\), (3) by 1, then (1), (2), and (3) become
\[ \sum_{k=0}^{n} \binom{n}{k} n^2 x^{k+2} (1-x)^{n-k} = n^2 x^2, \quad (4) \]
\[ \sum_{k=0}^{n} \binom{n}{k} (-2nk)x^k+1 (1-x)^{n-k} = -2n^2 x^2, \quad (5) \]
\[ \sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k} = nx(1-x + nx). \quad (6) \]

Now add together (4), (5), and (6) and group terms, we obtain
\[ \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k (k-nx)^2 = nx(1-x). \]

The conclusion of the lemma follows from the fact that \( x(1-x) \leq \frac{1}{4} \) for all real \( x \). Hence
\[ \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k (k-nx)^2 = nx(1-x) \leq \frac{n}{4}. \]

1.2 With the help of these lemmas, it is now possible to prove the Bernstein Theorem.

**THEOREM 1.1** (Bernstein): If the function \( f(x) \) is continuous on the segment \([0, 1]\), then \( B_n(x) \rightarrow f(x) \) uniformly with respect to \( x \) as \( n \rightarrow \infty \).

**Proof:** Let \( M = \max_{x \in [0, 1]} |f(x)| \), and suppose \( \varepsilon > 0. \)

Since \( f \) is continuous on the closed and bounded interval \([0, 1]\), there exists \( \delta_\varepsilon > 0 \) such that if \( |y-z| < \delta_\varepsilon, \)
\( y \in [0, 1], \ z \in [0, 1], \) then \( |f(y) - f(z)| < \varepsilon. \) Suppose \( x \in [0, 1]; \) then from Lemma 1.1 it follows that
\[ f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}. \]
Hence
\[ |B_n(x) - f(x)| \leq \frac{n}{2} |\sum_{k=0}^{n-1} (\frac{k}{n})x^k(1-x)^{n-k}| \leq \frac{n}{2} |f(\frac{k}{n}) - f(x)||x^k(1-x)^{n-k}|. \]

To finish the theorem, it is necessary to divide all the numbers \( k = 0, 1, 2, \ldots, n \) into two categories. Let \( A = \{ k \mid \frac{k}{n} - x < \delta \varepsilon \} \), and \( B = \{ k \mid \frac{k}{n} - x \geq \delta \varepsilon \} \). Consider the problem in two cases:

**Case 1:** Suppose \( k \in A \), then since \( \frac{k}{n} - x < \delta \varepsilon \), this implies \( |f(\frac{k}{n}) - f(x)| < \varepsilon \), and from Lemma 1.1, \( \Sigma |f(\frac{k}{n}) - f(x)| \leq \varepsilon \). From this fact along with Lemmas 1.1 and 1.2, we see that \( \Sigma |f(\frac{k}{n}) - f(x)||x^k(1-x)^{n-k}| \leq \varepsilon \sum_{k=0}^{n-1} x^k(1-x)^{n-k} = \varepsilon \).

**Case 2:** Suppose \( k \in A \), then \( |\frac{k}{n} - x| \geq \delta \varepsilon \). This implies that \( \frac{k}{n} - nx \geq \delta \varepsilon \), and finally that \( \frac{(k-nx)^2}{n^2 \delta^2 \varepsilon} \geq 1 \).

From this fact along with Lemmas 1.1 and 1.2, we see that \( \Sigma |f(\frac{k}{n}) - f(x)||x^k(1-x)^{n-k}| \leq \frac{2M}{n^2 \delta^2 \varepsilon} \sum_{k \in B} (k-nx)^2 x^k(1-x)^{n-k} \leq \frac{2M}{n^2 \delta^2 \varepsilon} n = \frac{M}{2n \delta^2 \varepsilon} \). Combining this information, we see that for all \( x \in [0, 1] \),

\[ |B_n(x) - f(x)| < \varepsilon + \frac{M}{2n \delta^2 \varepsilon}. \]

Hence, if \( n > \frac{M}{2\varepsilon \delta^2} \), then

\[ |B_n(x) - f(x)| < 2\varepsilon \] which is the desired condition.

Note that the techniques used in the above proof can be modified to show that if \( f \) is bounded on \([0, 1]\), and \( f \) is continuous at \( x_o \in [0, 1] \), then the sequence \( B_n(x_o) \to f(x_o) \) on \([0, 1]\).

We are now ready to establish the following:
THEOREM 1.2 (Weierstrass): Let \( f(x) \) be a continuous function defined on the closed interval \([a, b]\). For every \( \varepsilon > 0 \), there exists a polynomial \( P(x) \) such that \( |f(x) - P(x)| < \varepsilon \) for all \( x \in [a, b] \).

Proof: If \([a, b] = [0, 1]\), then the theorem follows directly from the Bernstein Theorem. So now suppose that \([a, b] \neq [0, 1]\). From the hypothesis, \( f \) is a continuous function on \([a, b]\), that is, \( f: [a, b] \to \mathbb{R} \) where \( \mathbb{R} \) is the set of all real numbers. Now define a new function \( \phi: [0, 1] \to [a, b] \) by \( \phi(y) = a + y(b-a) \). Also define a function \( g: [0, 1] \to \mathbb{R} \) by \( g(y) = f(\phi(y)) = f \circ \phi \). Since \( \phi \) is continuous and \( f \) is continuous, we can conclude that \( g = f \circ \phi \) is continuous. Since \( g \) is continuous on \([0, 1]\), by the Bernstein Theorem, for \( \varepsilon > 0 \), there exists a polynomial, \( H(x) \), such that \( |g(x) - H(x)| < \varepsilon \) for all \( x \in [0, 1] \). Now define another function \( \psi: [a, b] \to [0, 1] \) by \( \psi(x) = \frac{x - a}{b - a} \). \( \psi(x) \) is a polynomial. Let \( P: [a, b] \to \mathbb{R} \) be defined by \( P(x) = H(\psi(x)) \) for \( x \in [a, b] \), i.e., \( P = H \circ \psi \), then \( P(x) = H(\frac{x - a}{b - a}) \) and \( P \) is a polynomial in \( x \). Since \( \psi \circ \phi \) is the identity mapping on \([0, 1]\), \( \psi \circ \phi \) is the identity mapping on \([a, b]\), and \( g = f \circ \phi \), we see that \( \psi \circ g = f \circ \phi \circ \psi = f \). Let \( x \in [a, b] \); then since \( \psi(x) \in [0, 1] \) we have \( |P(x) - f(x)| = |H(\psi(x)) - g(\psi(x))| < \varepsilon \).

1.3 The method of using Bernstein polynomials to prove the Weierstrass Theorem gives us a constructive method of finding a sequence of polynomials which converge.
uniformly on \([a, b]\) to the given continuous function. Another method of proof was introduced by Lebesgue in 1898. This method of proof was to show that \(f(x)\) can be approximated closely by a broken line which in turn can be approximated arbitrarily closely by a polynomial. The first step may be accomplished by establishing the following theorem:

**Theorem 1.3:** Let \(f\) be a continuous function on a closed interval \([a, b]\). Then for any \(\varepsilon > 0\), there exists a continuous piece-wise linear function \(F\) on \([a, b]\) which approximates \(f\) uniformly within \(\varepsilon\) on \([a, b]\).

Proof: Since \(f\) is continuous on a closed and bounded interval \([a, b]\), it is uniformly continuous there. Hence if we have any \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(x, y \in [a, b]\), \(|x - y| < \delta\), then \(|f(x) - f(y)| < \frac{\varepsilon}{2}\). Now divide \([a, b]\) into \(N\) equal subintervals at points \(a = x_0 < x_1 < x_2 < \ldots < x_N = b\), and choose \(N\) large enough that \(\frac{b - a}{N} < \delta\).

Let \(P_k = (x_k, y_k)\), where \(y_k = f(x_k)\). \(P_k\) is on the graph of \(f\). Define \(F\) by \(F(x) = \frac{(x_{k+1} - x)y_k + (x - x_k)y_{k+1}}{x_{k+1} - x_k}\) for all \(x_k \leq x \leq x_{k+1}\).

Then if \(x = x_k\), we have \(F(x_k) = y_k = f(x_k)\), \(F(x_{k+1}) = y_{k+1} = f(x_{k+1})\). Thus this portion of the graph of \(F\) goes from \(P_k\) to \(P_{k+1}\) and is a straight line. Thus if \(x_k \leq x \leq x_{k+1}\), we have
\[ |F(x) - f(x_k)| = \left| \frac{(x_{k+1} - x)f(x_k) + (x - x_k)f(x_{k+1}) - f(x_k)(x_{k+1} - x)}{x_{k+1} - x_k} \right| \]

\[ = \left| \frac{(x - x_k)(f(x_{k+1}) - f(x_k))}{x_{k+1} - x_k} \right| \leq |f(x_{k+1}) - f(x_k)| < \frac{\varepsilon}{2}. \]

Thus if \( x_k \leq x \leq x_{k+1} \), then \( |F(x) - f(x)| \leq |F(x) - f(x_k)| + |f(x_k) - f(x)| < \varepsilon \). Hence we have a continuous piece-wise linear function which approximates \( f \) uniformly on \([a, b]\).

**THEOREM 1.4:** Let \( f(x) \) be a continuous function on the closed interval \([0, 1]\). Then, given \( \varepsilon > 0 \), there exists a polynomial \( P(x) \) such that \( |P(x) - f(x)| < \varepsilon \) for all \( x \in [0, 1] \).

Proof: Since \( f(x) \) is continuous on the closed and bounded interval \([0, 1]\), it is uniformly continuous there. Hence given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if \( |x - y| < \delta \) and if \( x \in [0, 1], y \in [0, 1] \), then \( |f(x) - f(y)| < \frac{\varepsilon}{2} \). Choose \( m \) such that \( \frac{1}{m} < \delta \) and let \( x_i = \frac{i}{m} \) for \( i = 0, 1, \ldots, m \), and define the continuous piece-wise linear function \( b(x) \) by \( b(x_k) = f(x_k) \) \( k = 0, 1, \ldots, m \), and the condition that \( b(x) \) is linear on \([x_{i-1}, x_i], i = 1 \ldots m\).

Since \( b(x_k) = f(x_k) \), then \( |f(x) - b(x)| < \frac{\varepsilon}{2} \) for all \( x \in [0, 1] \) from the last theorem. The continuous piece-wise linear function \( b(x) \) can be written in the form

\[ b(x) = c_0 + \sum_{k=0}^{m-1} b_k (x - x_k + |x - x_k|) \] where \( c_0 = f(x_0) \) and \( b_k = \frac{a_k + 1 - a_k}{2} \) for \( k = 0, 1, \ldots m - 1 \) and \( a_0 = 0 \),

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
We shall approximate each term \( b_k(x - x_k + |x - x_k|) \) uniformly on \([-1, 1]\). Now, if \( |x| \leq 1 \), \( |x| = \sqrt{1 - (1-x^2)} \) or, with \( u = 1 - x^2 \), \( |x| = \sqrt{1 - u} = (1-u)^{1/2} \). The Taylor's series expansion for \((1-u)^{1/2}\) for \( u = 0 \) is 
\[
1 - \frac{u}{2} - \frac{u^2}{2^2 2!} - \frac{3u^3}{2^3 3!} - \ldots .
\]

Applying the ratio test + 0 this binomial series shows that it is uniformly convergent on \([-u, u]\), if \( 0 \leq u < 1 \). By applying the Bernstein Theorem, [see The Elements of Real Analysis, Bartle, p. 414], we find that the Taylor series for \((1-u)^{1/2}\) at \( u = 0 \) converges to \((1-u)^{1/2}\) on \([0, 1]\).

By Raabe's Test and Abel's Theorem, the series is uniformly convergent in the interval \(-1 \leq u \leq 1\). Hence, there are polynomials in \( u \) which approximate \((1-u)^{1/2}\) uniformly on \([0, 1]\). Thus for \( \varepsilon > 0 \), there exists \( n \) large enough that
\[
\left| (1-u)^{1/2} - (1 - \frac{u^2}{2} - \frac{u^2}{2^2 2!} - \frac{3u^3}{2^3 3!} - \ldots ) \right| < \varepsilon .
\]

Suppose \( x \in [0, 1] \), then
\[
|x - x_k| \leq 1 \text{ and } 0 \leq 1 - (x-x_k)^2 \leq 1 .
\]

If we replace \( u \) by \( 0 \leq 1 - (x-x_k)^2 \leq 1 \), then
\[
\left| |x - x_k| - (1 - \frac{1}{2}(1 - (x-x_k)^2) - \frac{(1 - (x-x_k)^2)^2}{2^2 2!} - \ldots \right|
\]
\[
\leq 1 - \frac{3 \ldots (2n-3)u^n}{2^n n!} \left[ 1 - (x - x_k^2) \right] \right) < \varepsilon .
\]

Thus \( |x - x_k| \) can be approximated uniformly to within \( \varepsilon \) by a polynomial in \( x \).
Also \( b_k(x - x_k + |x - x_k|) \) can be approximated uniformly by polynomials on \([0, 1]\). Thus there exist polynomials \( P_k(x) \), \( k = 0 \ldots m-1 \), such that

\[
|P_k(x) - b_k(x - x_k + |x - x_k|)| < \frac{\varepsilon}{2m} \quad \text{for all } x \in [0, 1],
\]

and \( k = 0, 1, \ldots m-1 \). Therefore

\[
\sum_{k=0}^{m-1} |P_k(x) - b_k(x - y_k + |x - y_k|)| < \frac{\varepsilon}{2m} \quad \text{for all } x \in [0, 1].
\]

This implies that \( |b(x) - (c_0 + \sum_{k=0}^{m-1} P_k(x))| < \frac{\varepsilon}{2} \) for all \( x \in [0, 1] \). Hence

\[
|f(x) - (c_0 + \sum_{k=0}^{m-1} P_k(x))| = |f(x) - b(x) + b(x) - (c_0 + \sum_{k=0}^{m-1} P_k(x))| \leq |f(x) - b(x)| + |b(x) - (c_0 + \sum_{k=0}^{m-1} P_k(x))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Since \( c_0 + \sum_{k=0}^{m-1} P_k(x) \) is the sum of polynomials, it is also a polynomial. If we let \( P_k(x) = c_0 + \sum_{k=0}^{m-1} P_k(x) \), then \( |P(x) - f(x)| < \varepsilon \) for all \( x \in [0, 1] \).

1.4 The preceding theorem does not lead to any more refined results on the nature of the convergence of the approximating polynomials. It is in the study of the nature of the convergence that other proofs of the Weierstrass Theorem arise. One such proof is the following:

**Theorem 1.5:** Any function which is continuous on \([a, b]\) may be uniformly approximated by polynomials in this interval.

**Proof:** Assume that \([a, b]\) lies entirely in the interval \( 0 < x < 1 \) and a function \( f \) is continuous on \([a, b]\).
Then there exists $\delta_1$ such that $0 < \delta_1 < 1$ and $0 < a - \delta_1$ and $b + \delta_1 < 1$. Since $f$ is continuous on $[a, b]$, then it is uniformly continuous, hence for $\varepsilon > 0$, there exists a $\delta_2 > 0$, $\delta_2 \in (0, 1)$, such that if $|v| < \delta_2$ and $a \leq x \leq b$, $a \leq v + x \leq b$, then $|f(v + x) - f(x)| < \varepsilon$. Let

$\delta = \min(\delta_1, \delta_2)$ and let $\alpha = a - \delta$ and $\beta = b + \delta$. Hence we have $0 < \alpha < a < b < \beta < 1$. Without loss of generality, assume that the function $f(x)$ which is continuous on $[a, b]$ has been extended continuously to $[\alpha, \beta]$. Consider the integral $J_n = \int_0^1 (1 - v^2)^n dv$. By the Lebesgue Bounded Convergence Theorem, $\{J_n\}$ converges to zero. Also consider the integral $J_n^* = \int_0^1 (1 - v^2)^n dv$. If $n \geq 1$,

$J_n = \int_0^1 (1 - v^2)^n dv > \int_0^1 (1 - v)^n = \frac{1}{n + 1},$

$J_n^* = \int_0^1 (1 - v^2)^n dv < (1 - \delta^2)^n (1 - \delta) < (1 - \delta^2)^n$, hence

$0 < \frac{J_n^*}{J_n} < (n + 1)(1 - \delta^2)^n$. It follows readily, (by L'Hospital's rule, for example), that $\lim_{n \to \infty} \frac{J_n^*}{J_n} = 0$. Now assume that $a \leq x \leq b$ and form the expression $P_n(x) = \frac{\int_\alpha^\beta f(u)[1 - (u-x)^2]^n du}{\int_1^1 (1-u^2)^n du}$, $n = 1, 2 \ldots$, which are polynomials in $x$ of degree $2n$. If $I$ equals the numerator in $P_n(x)$ and $u = v + x$, then

$I = \int_\alpha^\beta f(u)[1 - (u-x)^2]^n du = \int_\alpha^\beta f(v+x)[1 - v^2]^n dv = \int_\alpha-\delta f(v+x)[1 - v^2]^n dv + \int_\delta^\beta f(v+x)[1 - v^2]^n dv + \int_\delta^\beta f(v+x)[1 - v^2]^n dv.$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Let $I_1 = \int_{-v}^{v} f(v+x)(1-v^2)^n \, dv$, $I_2 = \int_{-v}^{v} f(v+x)(1-v^2)^n \, dv$, and $I_3 = \int_{-v}^{v} f(v+x)(1-v^2)^n \, dv$; then $I = I_1 + I_2 + I_3$. Now $I_2 = \int_{-v}^{v} f(v+x)(1-v^2)^n \, dv = f(x)\int_{-v}^{v} (1-v^2)^n \, dv$ 
$+ \int_{-v}^{v} [f(v+x) - f(x)](1-v^2)^n \, dv = 2f(x)(J_n - J_n^*)$ 
$+ \int_{-v}^{v} [f(v+x) - f(x)](1-v^2)^n \, dv$. If $I_4 = \int_{-v}^{v} [f(v+x) - f(x)](1-v^2)^n \, dv$, then $|I_4|^n = |\int_{-v}^{v} [f(v+x) - f(x)](1-v^2)^n \, dv| 
\leq \varepsilon \int_{-v}^{v} (1-v^2)^n \, dv \leq \varepsilon \int_{-v}^{v} (1-v^2)^n \, dv = 2 \varepsilon J_n$. Further, if $M$ is the maximum of $|f(x)|$ for $\alpha \leq x \leq \beta$, then $|I_1| \leq M \int_{-v}^{v} (1-v^2)^n \, dv = MJ_n^*$, $|I_3| \leq M \int_{-v}^{v} (1-v^2)^n \, dv = MJ_n^*$. Also, $\int_{-v}^{v} (1-u^2)^n \, dv = 2J_n$. Further, we have $P_n(x) = \frac{I_1 + I_3 + I_4 + 2f(x)(J_n - J_n^*)}{2J_n}$.

Therefore $|P_n(x) - f(x)| \leq \frac{|I_1|}{2J_n} + \frac{|I_3|}{2J_n} + \frac{|I_4|}{2J_n}$
$+ |f(x)| \leq \frac{M}{2J_n} + \frac{M}{2J_n} + 2\varepsilon \frac{J_n}{J_n} + \frac{M}{J_n} = 2M \frac{J_n}{J_n} + \varepsilon$.

Since the limit $\frac{J_n}{J_n} \to 0$ and since $M \frac{J_n}{J_n}$ does not depend upon $x$, it follows that there exists an $N$ such that $|P_N(x) - f(x)| < 2 \varepsilon$ for all $x$ in $[a, b]$ provided $[a, b] \subset (0, 1)$. If $[a, b] \neq (0, 1)$, then a change of variable yields the desired result.

Using this same type of reasoning, we can prove the following theorem:

**THEOREM 1.6:** Any function in $m$ variables $x_1, x_2, \ldots, x_m$
which is continuous for $a_i \leq x_i \leq b_i$, $i = 1, \ldots, m$ may be approximated uniformly by polynomials in $x_1, x_2, \ldots, x_n$.

We will not prove this theorem since it will be a consequence of a stronger theorem proven later.

1.5 One of the most famous generalizations of the Weierstrass Theorem was made by Marshall H. Stone in 1937. To prove the Stone-Weierstrass Theorem, we do not need the full strength of the Weierstrass Theorem but only the following special case which can be stated as a Corollary.

**Corollary 1.1:** For every interval $[-a, a]$, there is a sequence of real polynomials $P_n(x)$ such that $P_n(0) = 0$ and such that $\lim_{n \to \infty} P_n(x) = |x|$ uniformly on $[-a, a]$.

**Proof:** By the Weierstrass Theorem, there exists a sequence $\{P_n^*\}$ of real polynomials which converge to $|x|$ uniformly on $[-a, a]$. In particular $P_n^*(0) \to 0$ as $n$ approaches infinity. The polynomials $P_n(x) = P_n^*(x) - P_n^*(0) (n = 1, 2, 3, \ldots)$ have the desired properties.

Before proceeding with the proof, we need to isolate some properties of polynomials which make the following generalization possible.

**Definition 1.2:** A family $\mathcal{Q}$ of real-valued functions defined on a set $E$ is said to be an algebra if

(a) $f + g \in \mathcal{Q}$ for all $f \in \mathcal{Q}, g \in \mathcal{Q}$
(b) $f \cdot g \in \mathcal{Q}$ for all $f \in \mathcal{Q}, g \in \mathcal{Q}$
(c) $c \cdot f \in \mathcal{Q}$ for all $f \in \mathcal{Q}, c \in \mathbb{R}$
that is, if \( \mathcal{A} \) is closed under addition, multiplication, and scalar multiplication.

**Definition 1.3:** A set \( \mathcal{A} \) of real-valued functions on a set \( E \) is said to be uniformly closed if \( f_n \in \mathcal{A} \) (\( n = 1, 2, 3 \ldots \)) and \( f_n \rightarrow f \) uniformly implies that \( f \in \mathcal{A} \).

**Definition 1.4:** Let \( \mathcal{A} \) be the set of all real-valued functions which are limits of uniformly convergent sequences of members of \( \mathcal{A} \), a set of real-valued functions on a set \( E \). Then \( \mathcal{A} \) is called the uniform closure of \( \mathcal{A} \).

**Theorem 1.7:** Let \( \mathcal{A} \) be the uniform closure of an algebra \( \mathcal{A} \) of bounded real-valued functions on a set \( E \). Then \( \mathcal{A} \) is a uniformly closed algebra.

**Proof:** If \( f \in \mathcal{A} \), \( g \in \mathcal{A} \), then there exist uniformly convergent sequences \( \{f_n\}, \{g_n\} \), such that \( f_n \rightarrow f \), \( g_n \rightarrow g \) and \( f_n \in \mathcal{A} \), \( g_n \in \mathcal{A} \). Since we are dealing with bounded functions, it can be shown that \( f_n + g_n \rightarrow f + g \), \( f_n g_n \rightarrow fg \), \( cf_n \rightarrow cf \), where \( c \) is any constant, the convergence being uniform in each case. So now we have \( f + g \in \mathcal{A} \), \( f \cdot g \in \mathcal{A} \), and \( cf \in \mathcal{A} \), so \( \mathcal{A} \) is an algebra. Now we show that \( \mathcal{A} \) is uniformly closed. Let \( \{f_n\} \) be a uniformly convergent sequence of members of \( \mathcal{A} \). Since \( f_n \in \mathcal{A} \), there exists a function \( g_n \in \mathcal{A} \) such that \( |f_n(x) - g_n(x)| < \frac{1}{n} \) for all \( x \in E \). If \( f_n \rightarrow f \) uniformly, then \( g_n \rightarrow f \) uniformly. Hence we have \( f \in \mathcal{A} \), and \( \mathcal{A} \) is uniformly closed.

**Definition 1.4:** Let \( \mathcal{A} \) be a family of real-valued functions on a set \( E \). Then \( \mathcal{A} \) is said to separate points.
on $E$ if to every pair of distinct points $x, y \in E$, there corresponds a function $f \in \mathcal{P}$ such that $f(x) \neq f(y)$.

**Definition 1.5:** Let $\mathcal{P}$ be a family of real-valued functions on a set $E$. If to each $x \in E$, there corresponds a function $g \in \mathcal{P}$ such that $g(x) \neq 0$, we say that $\mathcal{P}$ vanishes at no point of $E$.

**Theorem 1.8:** Suppose $\mathcal{P}$ is an algebra of real-valued functions on a set $E$, $\mathcal{P}$ separates points on $E$, and $\mathcal{P}$ vanishes at no point of $E$. Suppose $x_1, x_2$ are distinct points of $E$, and $c_1, c_2$ are constants. Then $\mathcal{P}$ contains a function $f$ such that $f(x_1) = c_1$, $f(x_2) = c_2$.

**Proof:** By hypothesis, $\mathcal{P}$ contains functions $g$ and $h$ such that $g(x_1) \neq g(x_2)$ and $h(x_1) \neq 0$. Let $u = g + \lambda h$ where $\lambda$ is a constant chosen in the following manner: If $g(x_1) \neq 0$, then $\lambda = 0$; if $g(x_1) = 0$, then $g(x_2) \neq 0$, and there is $\lambda \neq 0$ such that $\lambda[h(x_1) - h(x_2)] \neq g(x_2)$. Then $u \in \mathcal{P}$ and from the way $\lambda$ was chosen $u(x_1) \neq u(x_2)$ and $u(x_1) \neq 0$. Let $\alpha = u^2(x_1) - u(x_1)u(x_2)$; then $\alpha \neq 0$. Let $f_1 = \alpha^{-1}[u^2 - u(x_2)u]$, then $f_1(x_1) = \alpha^{-1}[u^2(x_1) - u(x_1)u(x_2)] = \alpha^{-1}\alpha = 1$, $f_1(x_2) = \alpha^{-1}[u^2(x_2) - u(x_2)u(x_2)] = \alpha^{-1}0 = 0$, and $f_1 \in \mathcal{P}$. Similarly, there exists an $f_2 \in \mathcal{P}$ with $f_2(x_1) = 0$, $f_2(x_2) = 1$. Let the function $f$ be defined by $f = c_1f_1 + c_2f_2$, then $f(x_1) = c_1f_1(x_1) + c_2f_2(x_1) = c_1$, $f(x_2) = c_1f_1(x_2) + c_2f(x_2) = c_2$, and also $f \in \mathcal{P}$.

1.6 We now have all the material needed for the proof of the Stone-Weierstrass Theorem. The following
proof will be divided into four steps for simplicity.

**THEOREM 1.9** (Stone-Weierstrass): Let \( \mathcal{A} \) be an algebra of real-valued continuous functions on a closed interval \([a, b]\). If \( \mathcal{A} \) separates points on \([a, b]\) and if \( \mathcal{A} \) vanishes at no point of \([a, b]\), then the uniform closure \( \mathcal{A} \) of \( \mathcal{A} \) consists of all real continuous functions on \([a, b]\).

**Step 1:** If \( f \in \mathcal{A} \), then \( |f| \in \mathcal{A} \).

Proof: Let \( w = \text{lub} |f(x)| \) for \( x \in [a, b] \) and let \( \varepsilon > 0 \) be given. By Corollary 1.1, there exist real numbers \( c_1, \ldots, c_n \) such that \( \sum_{i=1}^{n} c_i y^i - |y| < \varepsilon \), for \( y \in [-w, w] \). Let \( g = \sum_{i=1}^{n} c_i f^i \); then \( g \in \mathcal{A} \) since \( \mathcal{A} \) is closed under addition, multiplication, and scalar multiplication. Since \( f(x) \in [-w, w] \), by Corollary 1.1 and the case above, we see that
\[
|g(x) - |f(x)|| = \left| \left( \sum_{i=1}^{n} c_i f^i \right)(x) - |f|(x) \right| < \varepsilon
\]
for all \( x \in [a, b] \). Since \( \mathcal{A} \) is uniformly closed and \( g \in \mathcal{A} \), it follows that \( |f| \in \mathcal{A} \).

**Step 2:** If \( f, g \in \mathcal{A} \), then \( \max(f, g), \min(f, g) \in \mathcal{A} \).

Proof: The \( \max(f, g) \) and \( \min(f, g) \) are defined by \( (\max(f, g))(x) = \max(f(x), g(x)) \) and \( (\min(f, g))(x) = \min(f(x), g(x)) \). The following identities are needed to complete this step: \( \max(f, g) = \frac{f + g + |f - g|}{2}, \min(f, g) = \frac{f + g - |f - g|}{2} \). These identities follow from the corresponding identities involving real numbers.
By closure \( f + \epsilon \in \mathcal{G} \), and by step 1, \( \frac{|f - g|}{2} \in \mathcal{G} \). Hence \( \max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2} \in \mathcal{G} \), and \( \min(f, g) = \frac{f + g}{2} - \frac{|f - g|}{2} \in \mathcal{G} \). By induction, the result can be extended to any finite set of functions. Hence if \( f_1, \ldots, f_n \in \mathcal{G} \), then \( \max(f_1, \ldots, f_n) \in \mathcal{G} \) and \( \min(f_1, \ldots, f_n) \in \mathcal{G} \).

**Step 3:** Given a real-valued function \( f \), continuous on \([a, b]\), a point \( x \in [a, b] \), and \( \epsilon > 0 \), there exists a function \( g_x \in \mathcal{G} \) such that \( g_x(x) = f(x) \) and \( g_x(t) > f(t) - \epsilon \) for all \( t \in [a, b] \).

**Proof:** Suppose \( f \) is a real-valued continuous function on \([a, b]\), \( x \in [a, b] \), and \( \epsilon > 0 \). Since \( \mathcal{G} \subseteq \mathcal{G} \) and \( \mathcal{G} \) satisfies the hypothesis of the previous theorem, so does \( \mathcal{G} \). Hence for every \( y \in [a, b] \), there exists a function \( h_y \in \mathcal{G} \) such that \( h_y(x) = f(x) \) and \( h_y(y) = f(y) \). Since \( h_y \) is continuous, there exists a \( \delta_1 > 0 \) such that if \( t \in [a, b] \) and \( |t - y| < \delta_1 \), then \( |h_y(t) - h_y(y)| < \frac{\epsilon}{2} \). Also, since \( f \) is continuous, there exists \( \delta_2 > 0 \) such that if \( t \in [a, b] \) and \( |t - y| < \delta_2 \), then \( |f(t) - f(y)| < \frac{\epsilon}{2} \). Let \( \delta_y = \min(\delta_1, \delta_2) \); if \( t \in [a, b] \) and \( |t - y| < \delta_y \), then, since \( h_y(y) = f(y) \), it follows that \( |h_y(t) - f(t)| \leq |h_y(t) - h_y(y)| + |f(y) - f(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \). Hence \( h_y(t) > f(t) - \epsilon \) if \( t \in [a, b] \) and \( |t - y| < \delta_y \). Let \( J_y = \{t \mid t \in \mathbb{R}, |t - y| < \delta_y \} \); then \( [a, b] = \bigcup_{y \in [a, b]} J_y \), and each \( J_y \) is open. Since \( [a, b] \) is compact, there is a finite set of points \( y_1, \ldots, y_n \) such that
[a, b] = \bigcup_{i=1}^{n} J_{y_i} \subseteq \bigcup_{i=1}^{n} J_{y_n}. \text{ Let } g_x = \max(h_{y_1}, \ldots, h_{y_n}), \text{ then by step 2, } g_x \in \mathcal{M}. \text{ From above } h_{y_j}(x) = f(x), \text{ so } g_x(x) = f(x). \text{ Further, if } t \in [a, b], \text{ then there is an } i \text{ such that } t \in J_{y_i} \text{ and hence } g_x(t) \geq h_{y_i}(t) > f(t) - \epsilon.

**Step 4:** Given a real-valued function $f$ continuous on $[a, b]$, and $\epsilon > 0$, there exists a function $h \in \mathcal{M}$ such that $|h(x) - f(x)| < \epsilon$ for all $x \in [a, b]$.

Since $\mathcal{M}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

**Proof:** Consider the functions $g_x$, for each $x \in [a, b]$, constructed in step 3. Since $g_x$ is continuous, there exists $\delta_1 > 0$ such that $t \in [a, b], \ |t - x| < \delta_1$ implies $|g_x(t) - g_x(x)| < \frac{\epsilon}{2}$. Since $f$ is continuous, there exists $\delta_2 > 0$ such that $t \in [a, b], \ |t - x| < \delta_2$ implies $|f(t) - f(x)| < \frac{\epsilon}{2}$. Let $\delta_x = \min(\delta_1, \delta_2)$. Then $\delta_x > 0$.

Now since $g_x(x) = f(x)$, it follows that $|g_x(t) - f(t)| \leq |g_x(t) - g_x(x)| + |f(x) - f(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, for $t \in [a, b], \ |t - x| < \delta_x$. Let $V_x = \{t \in \mathbb{R}, \ |t - x| < \delta_x\}$. Then $V_x$ is an open set, and $[a, b] = \bigcup_{x \in [a, b]} V_x$. Since $[a, b]$ is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that $[a, b] = \bigcup_{i=1}^{m} V_{x_i}$. Let $h = \min(g_{x_1}, \ldots, g_{x_m})$; by step 2, $h \in \mathcal{M}$ and since $g_x(t) > f(t) - \epsilon$ for all $t \in [a, b]$, this implies $h(t) > f(t) - \epsilon$ for all $t \in [a, b]$. Also if $t \in [a, b]$,
then there is an $i$ such that $t \in X_i$ and so $h(t) \leq g(x_i(t)) < f(t) + \varepsilon$. Hence we have the desired property 
$|h(t) - f(t)| < \varepsilon$ for all $t \in [a, b]$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER II

Approximation By Trigonometric Polynomials

Not only did Weierstrass first enunciate the theorem than an arbitrary continuous function can be approximated by a polynomial with any assigned degree of accuracy, but to Weierstrass is also due the following theorem which will be the main subject of chapter two:

**THEOREM 2.1:** If \( f(x) \) is a given function of period \( 2\pi \), continuous for all real values of \( x \), and if \( \varepsilon > 0 \), it is always possible to define a trigonometric sum \( T(x) \) such that \( |f(x) - T(x)| < \varepsilon \) for all real values of \( x \).

By a polynomial is meant an expression of the form \( a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \). This expression will be said to represent a polynomial of the \( n \)th degree, not only when \( a_n \) is different from zero, but, also when \( a_n = 0 \). That is to say, the words "polynomial of the \( n \)th degree" will be used in place of the longer expression "polynomial of the \( n \)th degree at most".

**Definition 2.1:** A trigonometric sum of the \( n \)th order in \( x \) is an expression of the form \( a_0 + a_1\cos x + a_2\cos 2x + \ldots + a_n\cos nx + b_1\sin x + b_2\sin 2x + \ldots + b_n\sin nx \), where \( a_i \) and \( b_i \) are real numbers for \( i = 1 \ldots n \).

This definition is inclusive once more; the simultaneous vanishing of \( a_n \) and \( b_n \) is not ruled out. Before
establishing the previous theorem, it will be necessary to
prove several preliminary lemmas and theorems.

Lemma 2.1: If $m$ is a positive integer, the expres-
sion $\frac{\sin^4\left(\frac{mx}{2}\right)}{\sin^4\left(\frac{x}{2}\right)}$ is a trigonometric sum in $x$, of order $2m - 2$.

Proof: Because of the identities $\cos px \cos qx = \frac{1}{2}[\cos(p+q)x + \cos(p-q)x]$, $\sin px \sin qx = \frac{1}{2}[\cos(p-q)x - \cos(p+q)x]$, and $\sin px \cos qx = \frac{1}{2}[\sin(p+q)x - \sin(p-q)x]$, it can be seen that the product of two trigonometric sums of order $p$ and $q$ respectively is a trigonometric sum of order $p + q$. We have another iden-
tity, $\cos \theta = 1 - 2 \sin^2\frac{\theta}{2}$; this implies $1 - \cos mx = 2 \sin^2\left(\frac{mx}{2}\right)$ and $1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right)$. Putting these
together, we obtain $\frac{1 - \cos mx}{1 - \cos x} = \frac{\sin^2\left(\frac{mx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}$. We now claim
that this is a trigonometric sum of order $m - 1$. This
claim stems from the identity $1 - \cos mx$
$= \sum_{p=0}^{m-1} [\cos px - \cos(p+1)x]$. Examining this summation, we
see $\cos px - \cos(p+1)x = (1-\cos x)$
- $\sum_{q=1}^{p} [\cos(q-1)x - 2 \cos qx + \cos(q+1)x]$. Breaking this
down, $\cos(q-1)x - 2 \cos qx + \cos(q+1)x$
$= [\cos(q-1)x + \cos(q+1)x] - 2 \cos qx = 2 \cos qx \cos x$
- $2 \cos qx = -2 \cos qx(1-\cos x)$. Putting this breakdown
together, $1 - \cos mx = \sum_{p=0}^{m-1} [(1 - \cos x) + \sum_{q=1}^{p} 2 \cos qx(1-\cos x)]$
\[
\frac{m-1}{p} \sum_{p=0}^{m-1} (1 - \cos x)(1 + \sum_{q=1}^{p} 2 \cos qx)
\]

= (1 - \cos x) \frac{m-1}{p} \sum_{p=0}^{m-1} (1 + \sum_{q=1}^{p} 2 \cos qx). \text{ Hence}

\[
\frac{1 - \cos mx}{1 - \cos x} = m + \frac{m-1}{p} \sum_{p=0}^{m-1} (1 + \sum_{q=1}^{p} 2 \cos qx) = m + \frac{m-1}{p} \sum_{q=1}^{p} 2(m-q)\cos qx.
\]

So now \(\frac{\sin^2(mx)}{\sin^2(x)}\) is a trigonometric sum in \(x\) of order \(m - 1\); its square will then be a sum of order \(2m - 2\).

2.2 We now establish the following:

**THEOREM 2.2:** If \(f(x)\) is a function of period \(2\pi\), and if there is a real number \(\lambda\) such that \(|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|\) for all real values of \(x_1\) and \(x_2\), then there will exist for every positive integer \(n\) a trigonometric sum \(T_n(x)\), of the \(n\)th order, such that, for all real values of \(x\), \(|f(x) - T_n(x)| \leq \frac{K\lambda}{n}\), where \(K\) is an absolute constant, depending neither on \(x\), nor on \(n\), nor on \(\lambda\), \(n\) or on any further specification with regard to the function \(f(x)\).

**Proof:** Let \(F_0(u) = \left[\frac{\sin mu}{m \sin u}\right]^4\) for \(0 < u \leq \frac{\pi}{2}\), and let \(F_0(u) = 1\) for \(u = 0\). Also, if \(x\) is real, let \(I_m(x) = h_m \int_{-\pi/2}^{\pi/2} f(x+2u)F_m(u)du\), where \(m\) is any positive integer, and \(h_m\) is defined by \(\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u)du\). If \(v = x + 2u\), then \(I_m(x) = h_m \int_{-\pi}^{\pi} f(v)F_m\left[\frac{v-x}{2}\right]dv\). Both factors in this last integral have period \(2\pi\) with regard to \(v\), so that the value of the integral is unchanged if
the interval of integration is replaced by any other interval of length $2\pi$. So now $I_m(x) = \frac{h}{2} \int_0^\pi f(v) F_m \left( \frac{v - x}{2} \right) dv$.

The expression $F_m \left( \frac{v - x}{2} \right)$, by Lemma 2.1, is a trigonometric sum of order $2m - 2$ in $(v - x)$. Thus $F_m \left( \frac{v - x}{2} \right)$ may be written as $a_0 + a_1 \cos (v - x) + b_1 \sin (v - x) + \ldots$

+ $a_{2m-2} \cos (2m-2)(v - x) + b_{2m-2} \sin (2m-2)(v - x)$. A typical term looks like $b_k \sin k(v - x) + a_k \cos k(v - x)$

= $b_k (\sin kv \cos kx - \cos kv \sin kx)$

+ $a_k (\cos kv \cos kx + \sin kv \sin kx)$

= $(b_k \sin kv + a_k \cos kv) \cos kx$

+ $(-b_k \cos kv + a_k \sin kv) \sin kx$. Thus $F_m \left( \frac{v - x}{2} \right)$ may be regarded as a trigonometric sum of the same order, i.e., $2m - 2$, in $x$ with coefficients which are trigonometric functions of $v$. The whole integrand is a trigonometric sum of order $2m - 2$ in $x$ with coefficients which are continuous functions of $v$, and $I_m(x)$ is therefore a trigonometric sum of order $2m - 2$ in $x$, with constant coefficients.

Since $\frac{1}{h} \int_{-\pi/2}^{\pi/2} F_m(u) du$, if we multiply both sides by $f(x) \cdot h_m$, we obtain $f(x) = h_m \int_{-\pi/2}^{\pi/2} f(x) F_m(u) du$. Hence

$I_m(x) - f(x) = h_m \int_{-\pi/2}^{\pi/2} f(x + 2u) F_m(u) du - h_m \int_{-\pi/2}^{\pi/2} f(x) F_m(u) du$

= $h_m \int_{-\pi/2}^{\pi/2} [f(x + 2u) - f(x)] F_m(u) du$. By hypothesis,

$|f(x + 2u) - f(x)| \leq \lambda |x + 2u - x| = \lambda |2u| = 2\lambda |u|$. Hence $|I_m(x) - f(x)| \leq 2\lambda h_m \int_{-\pi/2}^{\pi/2} |u| |F_m(u)| du \leq 4\lambda h_m \int_{0}^{\pi/2} u F_m(u) du$
\[
\frac{2\lambda}{\ell} \int_0^{\pi/2} u \frac{F_m(u)}{u} \, du = \int_0^{\pi/2} \frac{\sin^4 t}{t^4} \, dt.
\]

Now let \( c_1 = \int_0^{\pi/2} \frac{\sin^4 t}{t^4} \, dt \), where the integrand is defined to have value 1 when \( t = 0 \). Let \( c_2 = \int_0^{\infty} \frac{\sin^4 t}{t^3} \, dt \), where the integrand is defined to have the value 0 when \( t = 0 \). We observe that
\[
\left| \int_0^{(n+1)\pi} \frac{\sin^4 t}{t^3} \, dt \right| 
\leq \int_0^{(n+1)\pi} \left| \frac{\sin^4 t}{t^3} \right| \, dt 
\leq \int_0^{(n+1)\pi} \frac{1}{t^3} \, dt 
\leq \frac{1}{(n\pi)^3} \cdot \pi = \frac{1}{n^2} \cdot \frac{1}{n^3},
\]
for \( n \geq 1 \), hence
\[
\int_0^{n\pi} \left| \frac{\sin^4 t}{t^3} \right| \, dt \leq \pi + \frac{1}{n^2} \cdot \frac{1}{1} + \frac{1}{n^2} \cdot \frac{1}{2} + \ldots
\]

\[
+ \frac{1}{n^2} \cdot \frac{1}{(n-1)^2} \leq \pi + \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m^2} \leq \pi + \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{1}{m^2}
\]
and
\[
\int_0^{\infty} \left| \frac{\sin^4 t}{t^3} \right| \, dt \text{ converges since it is a p-series. Thus } \int_0^{\infty} \frac{\sin^4 t}{t^3} \, dt \text{ is convergent, and } |c_2| = \int_0^{\infty} \frac{\sin^4 t}{t^3} \, dt \leq \int_0^{\infty} \left| \frac{\sin^4 t}{t^3} \right| \, dt \leq \pi + \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{1}{m^2}.
\]

Since \( 0 < \sin u < u \) for \( 0 < u \leq \frac{\pi}{2} \), then \( \frac{1}{\sin u} > \frac{1}{u} \). If \( mu = t \), then
\[
\int_0^{\pi/2} \frac{\sin^4 m u}{m^4 u^4} \, du = \int_0^{\pi/2} \frac{\sin^4 t}{t^4} \, dt \geq \frac{1}{m} \int_0^{\pi/2} \frac{\sin^4 t}{t^4} \, dt = \frac{c_1}{m}.
\]
It can be shown that \( \frac{\sin u}{u} \) decreases monotonically as \( u \) goes from 0 to \( \frac{\pi}{2} \), so \( \frac{\sin u}{u} > \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}, \frac{1}{\sin u} < \frac{\pi}{2u} \). Therefore
\[
\int_0^{\pi/2} u F_m(u) \, du = \int_0^{\pi/2} \frac{u \sin^4 m u}{m^4 u^4} \, du
\]
\[ \int_{-\pi/2}^{\pi/2} \frac{u \sin \frac{4}{m} \mu u (\pi/2)^4}{u} \, du = \frac{1}{m} (\pi/2)^4 \int_{-\pi/2}^{\pi/2} \frac{t \sin \frac{4}{m} t \, dt}{t} \]

\[ \leq \frac{1}{m^2} (\pi/2)^4 \int_{0}^{\infty} \frac{\sin t}{t^2} \, dt = \left( \frac{\pi}{2} \right)^4 \frac{c_2}{m^2}. \] Thus \(|I_m(x) - f(x)|\)

\[ \frac{2 \lambda \int_{-\pi/2}^{\pi/2} u F_m(u) \, du}{\int_{-\pi/2}^{\pi/2} F_m(u) \, du} \leq \frac{2 \lambda (\pi/2)^4 \left( \frac{c_2}{m^2} \right)}{\frac{c_1}{m}} = \frac{\pi^4 c_2 \lambda}{8 c_1 m}. \] Now let \(n\) be an arbitrary integer and choose \(m = \frac{n}{2} + 1\) if \(n\) is even and \(m = \frac{n + 1}{2}\) if \(n\) is odd. In either case \(2m - 2 \leq n < 2m\).

Let the corresponding expression \(I_m(x)\) be denoted by \(T_n(x)\).

Then since \(2m - 2 \leq n\) and \(T_n(x)\) has order \(2m - 2\), \(T_n(x)\) has order \(n\), and since \(\frac{1}{m} < \frac{2}{n}\), \(|I_n(x) - f(x)| \leq \frac{\pi^4 c_2 \lambda}{8 c_1 m}\)

\[ \leq \frac{\pi^4 c_2 \lambda}{4 c_1 n} = \frac{k \lambda}{n} \] for all \(x\) if \(k\) is taken to be \(\frac{\pi^4 c_2}{4 c_1}\).

**Definition 2.2:** If \(E \subset \mathbb{R}\), and if \(E\) is a closed and bounded set and if \(f(x)\) is a real-valued and uniformly continuous function on \(E\) and if \(w(\delta)\) is defined for \(\delta > 0\) by \(w(\delta) = \max|f(x_2) - f(x_1)|\) when \(|x_2 - x_1| \leq \delta, x_1 \in E, x_2 \in E\), then \(w\) is called the modulus of continuity of \(f\).

**Theorem 2.3:** If \(f(x)\) is a real-valued continuous function of period \(2\pi\), with modulus of continuity \(w(\delta)\), there exists for every positive integer \(n\) a trigonometric sum \(T_n(x)\), of the \(n\)th order, such that, for all real values of \(x\), \(|f(x) - T_n(x)| \leq k'w(\frac{2\pi}{n})\), where \(k'\) is an absolute constant.
Proof: Let \( f(x) \) be an arbitrary continuous function of period \( 2\pi \), and let \( \omega(\delta) \) be the modulus of continuity of \( f(x) \). Let \( \phi(x) \) be the continuous piece-wise linear function of period \( 2\pi \) which takes on the same values as \( f(x) \) at the points \(-\pi, -\pi + \frac{2\pi}{n}, -\pi + \frac{4\pi}{n}, \ldots, \pi - \frac{2\pi}{n}, \pi \) and is linear from each point of this set to the next. The graph of \( \phi \) is a broken line, no segment of which has slope greater than \( \frac{\omega(\frac{2\pi}{n})}{\frac{2\pi}{n}} \). Then \( \phi(x) \) satisfies the hypothesis of Theorem 2.2 with \( \lambda = \frac{\omega(\frac{2\pi}{n})}{\frac{2\pi}{n}} \). Hence for every positive integer \( n \), there is a trigonometric sum \( T_n(x) \) of the nth order such that \( |\phi(x) - T_n(x)| \leq \frac{k}{2\pi} \omega(\frac{2\pi}{n}) \). If \(-\pi \leq x \leq \pi\), then \( x \) differs by less than \( \frac{2\pi}{n} \) from one of the numbers \(-\pi, -\pi + \frac{2\pi}{n}, -\pi + \frac{4\pi}{n}, \ldots, \pi \) for which \( \phi \) is defined to be equal to \( f \). Let \( t \) be such a number, then \( |f(x) - \phi(x)| \leq |f(x) - f(t)| + |f(t) - \phi(t)| + |\phi(t) - \phi(x)| \leq 2\omega(\frac{2\pi}{n}) \) for all \( x \). So now \( |T_n(x) - f(x)| \leq |T_n(x) - \phi(x)| + |\phi(x) - f(x)| \leq \omega(\frac{2\pi}{n})(\frac{k}{2\pi} + 2) \). If we let \( k' = \frac{k}{2\pi} + 2 \), then \( |T_n(x) - f(x)| \leq k' \omega(\frac{2\pi}{n}) \) which is the desired property.

Since the limit \( \omega(\frac{2\pi}{n}) = 0 \), the Weierstrass Theorem for approximation by trigonometric polynomials is now established.

2.3 Recall in the proof of Theorem 2.2, that to
an arbitrary positive integer \( n \), a second positive integer \( m \) was assigned, in terms of which a function \( F_m(u) \) was constructed. Also, a trigonometric sum \( T_n(x) \), approximating \( f(x) \) was defined equal to an expression which could be reduced to the form \[ \frac{1}{2} h_m \int_{-\pi}^{\pi} f(v)F_m\left(\frac{v-x}{2}\right)dv, \] \( h_m \) being independent of \( x \). Lemma 2.1 stated that \( F_m\left(\frac{u}{2}\right) \) is a trigonometric sum in \( u \) of order \( 2m - 2 < n \), therefore we can write \[ F_m\left(\frac{v-x}{2}\right) = \frac{1}{2} A_0 + \sum_{k=1}^{n} [A_k \cos k(v-x) + B_k \sin k(v-x)]dv. \]

Hence \( T_n(x) = \frac{1}{2} h_m \int_{-\pi}^{\pi} f(v)F_m\left(\frac{v-x}{2}\right)dv = \frac{1}{4} h_m A_0 \int_{-\pi}^{\pi} f(v)dv + \int_{-\pi}^{\pi} \frac{1}{2} h_m \sum_{k=1}^{n} [A_k \cos k(v-x) + B_k \sin k(v-x)]dv. \) Thus \[ \int_{-\pi}^{\pi} f(v)dv \] is the constant term. It can be shown that \( h_m \) and \( A_0 \) are positive real numbers. Hence the constant term in \( T_n(x) \) is equal to zero if and only if \( \int_{-\pi}^{\pi} f(v)dv = 0. \)

**Lemma 2.2:** If \( f \) is a continuous function of period \( 2\pi \), then \( f \) is the derivative of a function of period \( 2\pi \) if and only if \( \int_{-\pi}^{\pi} f(x)dx = 0. \)

**Proof:** Suppose \( f \) is the derivative of a function of period \( 2\pi \), and let \( F'(x) = f(x) \). Then \[ \int_{-\pi}^{\pi} f(x)dx = F(\pi) - F(-\pi) = 0. \] Conversely let \( G(x) = \int_{-\pi}^{x} f(t)dt \). Then \( G'(x) = f(x) \). Suppose \( \int_{-\pi}^{\pi} f(t)dt = 0 \), then

\[
G(x + 2\pi) - G(x) = \int_{x}^{x+2\pi} f(t)dt - \int_{-\pi}^{x} f(t)dt = \int_{x}^{x+2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt = 0,
\]
and so \( G \) is periodic.
Lemma 2.3: Suppose \( f(x) \) is a function of period \( 2\pi \), which has an everywhere continuous derivative \( f'(x) \). For a particular value of \( n \), let \( t_n'(x) \) be a trigonometric sum of the \( n \)th order, without constant term:
\[
t_n'(x) = k \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx),
\]
and let \( \varepsilon_n \) be a constant such that \(|f'(x) - t_n'(x)| \leq \varepsilon_n\) for all \( x \); then there exists a trigonometric sum of the \( n \)th order such that
\[
|f(x) - T_n(x)| \leq \frac{k\varepsilon_n}{n}
\]
for all \( x \) where \( k \) is a constant; moreover \( T_n(x) \) may be chosen such that \( T_n(x) \) has constant term zero whenever \( f(x) \) is a derivative of a function of period \( 2\pi \), i.e., if \( \int_0^{2\pi} f(x)dx = 0 \).

Proof: Let \( t_n(x) \) be a trigonometric sum, without a constant term, which has \( t_n'(x) \) for its derivative, that is, \( t_n(x) = \sum_{k=1}^{n} \alpha_k \sin kx - \beta_k \cos kx \). Let \( r_n(x) = f(x) - t_n(x) \). Then since \( f(x) \) and \( t_n(x) \) have period \( 2\pi \), \( r_n(x) \) has period \( 2\pi \). Suppose \( x_1 \) and \( x_2 \) are real numbers, by the mean value theorem, there exists an \( x_0 \in (x_1, x_2) \) such that
\[
|r_n(x_2) - r_n(x_1)| = |r_n'(x_0)| |x_1 - x_2|.
\]
Since
\[
|r_n'(x_0)| = |f'(x_0) - t_n'(x_0)| \leq \varepsilon_n,
\]
then
\[
|r_n(x_2) - r_n(x_1)| \leq \varepsilon_n |x_1 - x_2|.
\]
Therefore the conditions of Theorem 2.2 are satisfied with \( \lambda = \varepsilon_n \). Hence there exists a trigonometric sum of the \( n \)th order, \( T_{n,1}(x) \), such that
\[
|r_n(x) - T_{n,1}(x)| \leq \frac{k\varepsilon_n}{n}
\]
for all \( x \). Let \( T_n(x) = t_n(x) + T_{n,1}(x) \). Then
\[
f(x) - T_n(x) = f(x) - t_n(x) - T_{n,1}(x)
\]
= r_n(x) - T_n,1(x) and so |f(x) - T_n(x)| ≤ \frac{k\varepsilon_n}{n}. Let 
F: [-\pi, \pi] \rightarrow R be defined by F(x) = \int_{-\pi}^{\pi} f(t)dt.

Then F'(x) = f(x) for all x ∈ [-\pi, \pi]. Suppose 
F(-\pi) = F(\pi), i.e., suppose \int_{-\pi}^{\pi} f(t)dt = 0. Then 
\int_{-\pi}^{\pi} r_n(x)dx = \int_{-\pi}^{\pi} f(x)dx - \int_{-\pi}^{\pi} t_n(x)dx = 0. Hence from the 
previous observation, T_n,1(x) can be chosen to have constant term zero. Also, t_n(x) has constant term zero, so 
T_n(x) has constant term zero.

**THEOREM 2.4:** If f(x) is a function of period 2\pi, 
having a pth derivative f^{(p)}(x) and if there is a real 
number \lambda such that |f^{(p)}(x_2) - f^{(p)}(x_1)| ≤ \lambda |x_2 - x_1|
for all real values of x_1 and x_2, there will exist for 
every integer n a trigonometric sum \(T_n(x), of the nth 
order, such that, for all real values of x, |f(x) - T_n(x)|
≤ \frac{k^{p+1}\lambda}{n^{p+1}}, where k is the constant found in Theorem 2.2.

Also if \int_{-\pi}^{\pi} f(x)dx = 0, then \(T_n(x)\) may be chosen to have 
constant term zero.

**Proof:** Since f(x) has period 2\pi, f(x) = f(x + 2\pi).
Also f'(x_0 + 2\pi) = \lim_{x \to x_0 + 2\pi} \frac{f(x) - f(x_0 + 2\pi)}{x - (x_0 + 2\pi)}
= \lim_{x \to x_0} \frac{f(x - 2\pi) - f(x_0)}{(x - 2\pi) - x_0} = f'(x_0). So f'(x) has period 
2\pi. Continuing in this fashion, we find that f(x), f'(x), 
... f^{(p)}(x) are periodic of period 2\pi. We now show by
induction that the desired condition holds. If \( p = 1 \), and 
\[ |f'(x_2) - f'(x_1)| \leq \lambda |x_2 - x_1|, \]
then by Theorem 2.2 there is a trigonometric sum \( T_n(x) \) such that 
\[ |f'(x) - T_n(x)| < \frac{k\lambda}{n} \]
for all \( x \). Since \( f' \) is the derivative of a periodic function of period \( 2\pi \), \( T_n(x) \) may be chosen to have constant term zero. By Lemma 2.3 there exists a trigonometric sum \( T_{n,1}(x) \) such that 
\[ |f(x) - T_{n,1}(x)| < \frac{k}{n} \cdot \frac{k\lambda}{n} = \frac{k^2\lambda}{n^2} \]
for all \( x \), moreover \( T_{n,1}(x) \) may be chosen to have constant term zero if \( \int_0^{2\pi} f(x) \, dx = 0 \). Now assume that if \( g(x) \) is a function of period \( 2\pi \) having a continuous \((p-1)\)th derivative on \( R \) and if \( \lambda \) is a real number such that 
\[ |g^{(p-1)}(x_2) - g^{(p-1)}(x_1)| \leq \lambda |x_2 - x_1|, \]
then there exists a trigonometric sum of order \( n \) such that 
\[ |g(x) - T_n(x)| \leq \frac{k^p\lambda}{n^p} \]
for all \( x \). Also, assume that if \( g(x) \) is the derivative of a periodic function of period \( 2\pi \), then \( T_n(x) \) may be chosen to have constant term zero. Now suppose that the periodic function \( f(x) \) of period \( 2\pi \) has a \( p \)th derivative on \( R \), and that there is a real number \( \lambda \) such that, if \( x_1 \) and \( x_2 \) are real numbers, then 
\[ |f^{(p)}(x_2) - f^{(p)}(x_1)| \leq \lambda |x_2 - x_1|. \]
Then \( f'(x) \) has a \((p-1)\)th derivative on \( R \), and 
\[ |(f')^{(p-1)}(x_2) - (f')^{(p-1)}(x_1)| \leq \lambda |x_2 - x_1| \]
where \( f' \) is periodic of period \( 2\pi \). Therefore there exists a trigonometric sum \( T_n(x) \) such that 
\[ |f'(x) - T_n(x)| < \frac{k^p\lambda}{n^p} \]
for all \( x \). Also, \( f' \) is the derivative of a periodic
function of period $2\pi$, thus $T_n(x)$ can be chosen to have constant term zero. By Lemma 2.3, there is a trigonometric sum $T_{n,1}(x)$ such that $|f(x) - T_{n,1}(x)| \leq \frac{k.k^p.n^p}{n^{p+1}} = \frac{k^p.n^p}{n^{p+1}}$.

If $f$ is the derivative of a periodic function of period $2\pi$, then by Lemma 2.3, $T_{n,1}(x)$ may be chosen to have constant term zero. Hence the induction is complete and the theorem is proved.

**Theorem 2.5:** If $f(x)$ is a continuous function of period $2\pi$, with modulus of continuity $\omega(\delta)$, and if $f$ is the derivative of a continuous function of period $2\pi$, then there exists a trigonometric sum $T_n(x)$, of the $n$th order without constant term, such that $|f(x) - T_n(x)| < k''\omega(\frac{2\pi}{n})$ for real $x$, where $k$ is the absolute constant of Theorem 2.2 and $k'' = \frac{k}{2\pi} + 4$.

**Proof:** Let $\Phi(x)$ be the piece-wise linear continuous function of period $2\pi$ which takes on the same values as $f(x)$ at the points $-\pi, -\pi + \frac{2\pi}{n}, -\pi + \frac{4\pi}{n}, \ldots, \pi - \frac{2\pi}{n}, \pi$ and is linear from each point to the next. The graph of $\Phi$ is a broken line, no segment of which has a slope greater than $\frac{\omega(\frac{2\pi}{n})}{\frac{2\pi}{n}}$. If $c = \int_{-\pi}^{\pi} \Phi(x)dx$, let $\Phi_1(x) = \Phi(x) - \frac{c}{2\pi}$.

Then $\int_{-\pi}^{\pi} \Phi_1(x)dx = \int_{-\pi}^{\pi} (\Phi(x) - \frac{c}{2\pi})dx = \int_{-\pi}^{\pi} \Phi(x)dx - \int_{-\pi}^{\pi} \frac{c}{2\pi}dx = c - c = 0$. Also $|\Phi_1(x_2) - \Phi_1(x_1)| = |\Phi(x_1) - \frac{c}{2\pi} - \Phi(x_2)|$
\[ \frac{c}{2\pi} = |\phi(x_1) - \phi(x_2)| < \lambda |x_2 - x_1| \text{ where } \lambda = \frac{w(\frac{2\pi}{n})}{\frac{2\pi}{n}}. \]

Hence by Theorem 2.2, there exists a trigonometric sum \( T_{n,1}(x) \) without constant term such that \( |\phi_1(x) - T_{n,1}(x)| \leq \frac{k\lambda}{n} = \frac{k}{2\pi} \cdot \frac{w(\frac{2\pi}{n})}{\frac{2\pi}{n}} \) for all \( x \). From Lemma 2.2, \( \int_{-\pi}^{\pi} f(x)dx = 0 \), so \( |c| = |\int_{-\pi}^{\pi} \phi(x)dx| = |\int_{-\pi}^{\pi} \phi(x)dx - \int_{-\pi}^{\pi} f(x)dx| = |\int_{-\pi}^{\pi} [\phi(x) - f(x)]dx|. \]

If \( x \in [-\pi, \pi] \), then there is a number within \( \frac{2\pi}{n} \) of one of the numbers \( -\pi, -\pi + \frac{2\pi}{n}, \ldots, \pi \). Let \( t \) be such a number. Then \( \phi(t) = f(t) \), so \( |f(x) - \phi(x)| \leq |f(x) - f(t)| + |f(t) - \phi(t)| + |\phi(t) - \phi(x)| \leq 2w(\frac{2\pi}{n}). \)

So \( |c| \leq 2\pi \cdot 2w(\frac{2\pi}{n}) \). Also \( |\phi(x) - \phi_{1}(x)| = |\phi(x) - \phi(x) + \frac{c}{2\pi}| \leq 2w(\frac{2\pi}{n}) \), and \( |f(x) - \phi_{1}(x)| \leq |f(x) - \phi(x)| + |\phi(x) - \phi_{1}(x)| \leq 4w(\frac{2\pi}{n}) \) for all \( x \). Hence \( |f(x) - T_{n,1}(x)| \leq |f(x) - \phi_{1}(x)| + |\phi_{1}(x) - T_{n,1}(x)| \leq w(\frac{2\pi}{n})[\frac{k}{2\pi} + 4] \) for all \( x \). If \( k'' = \frac{k}{2\pi} + 4 \), then we have the desired form of the theorem.

**Theorem 2.6:** If \( f(x) \) is a function of period \( 2\pi \) which has everywhere a continuous \( p \)th derivative with modulus of continuity \( w(\delta) \), there exists for every integer \( n \) a trigonometric sum \( T_{n}(x) \), of the \( n \)th order, such that for all real values of \( x \), \( |f(x) - T_{n}(x)| \leq \frac{k'' \cdot k^{p}}{n^{p}} \cdot w(\frac{2\pi}{n}) \) where \( k \) is the absolute constant given in Theorem 2.2 and
\[ k'' = \frac{k}{2\pi} + 4. \] Further if \( f \) is the derivative of a continuous function of period \( 2\pi \), then \( T_n(x) \) may be chosen to have constant term zero.

Proof: The proof of the theorem will be by induction. If \( p = 1 \), then by Theorem 2.5 there exists a trigonometric sum \( T_{n,1}(x) \) of order \( n \) without constant term such that
\[ |f(x)' - T_{n,1}(x)| < k'' \omega\left(\frac{2\pi}{n}\right) \]
for all \( x \). By Lemma 2.3, there exists a trigonometric sum \( T_n(x) \) of the \( n \)th order such that
\[ |f(x) - T_n(x)| < \frac{k}{n} \cdot k'' \omega\left(\frac{2\pi}{n}\right) \]
for all \( x \). Also, by Lemma 2.3, if \( f \) is the derivative of a continuous function of period \( 2\pi \), then \( T_n(x) \) may be chosen to have constant term zero.

Now assume that if \( g(x) \) is a function of period \( 2\pi \) having an everywhere continuous \((p-1)\)th derivative with modulus of continuity \( \omega(\delta) \), then there exists a trigonometric sum \( T_n(x) \) of the \( n \)th order such that for all real \( x \),
\[ |g(x) - T_n(x)| < \frac{k'' k^{p-1}}{n^{p-1}} \omega\left(\frac{2\pi}{n}\right). \]
Further, assume that if \( g \) is the derivative of a continuous function, then \( T_n(x) \) may be chosen to have constant term zero. Now suppose \( f(x) \) is a function of period \( 2\pi \) with a continuous \( p \)th derivative with modulus of continuity \( \omega(\delta) \). From the hypothesis applied to \( f'(x) \), it follows that there exists a trigonometric sum \( T_n(x) \) such that
\[ |f'(x) - T_n(x)| < \varepsilon \]
for all real \( x \), where
\[ \varepsilon_n = \frac{k'' k^{p-1}}{n^{p-1}} \omega\left(\frac{2\pi}{n}\right). \] Since \( \int_0^\pi f'(x)dx = 0 \), \( T_n(x) \) can be chosen to have constant term zero. By Lemma 2.3, there is
a trigonometric sum $T_n^*(x)$ of the nth order such that

$$|f(x) - T_n^*(x)| < \frac{k\varepsilon_n}{n}$$

for all real $x$. Therefore

$$|f(x) - T_n^*(x)| < \frac{kk''k^{p-1}}{n^p} \omega(\frac{2\pi}{n}) = \frac{k''k^p}{n^p} \omega(\frac{2\pi}{n}).$$

Finally, if $f(x)$ is the derivative of a function of period $2\pi$, then, by Lemma 2.3, it follows that $T_n^*(x)$ may be chosen to have constant term zero.

**Corollary 2.1:** If $f(x)$ is a function of period $2\pi$ which has everywhere a continuous $p$th derivative, there exists for every positive integer $n$ a trigonometric sum $T_n(x)$, of the nth order, such that $\lim_{n \to \infty} \varepsilon_n = 0$, where

$$\varepsilon_n = \max_{x \in R} |f(x) - T_n(x)|.$$

2.4 Thus we have developed the Weierstrass Theorem on trigonometric approximation and several results concerning trigonometric approximations of functions having several derivatives. We now turn to polynomial approximation and see how polynomial approximation and trigonometric approximation are related. To aid in this transition, however, one more lemma is needed.

**Lemma 2.4:** If $f(x)$ is an even function of period $2\pi$, and if there is a trigonometric sum $T_n(x)$ of the nth order such that $|f(x) - T_n(x)| < \varepsilon$ for all $x$, then there exists a cosine sum (that is, a trigonometric sum without sine terms) of the nth order such that for all real $x$, $|f(x) - C_n(x)| \leq \varepsilon$.

**Proof:** Let $C_n(x) = \frac{1}{2}[T_n(x) + T_n(-x)]$. Then $C_n(x)$
is a cosine sum. Since \( f(x) \) is an even function, then
\[
f(x) = \frac{1}{2}[f(x) + f(-x)].
\]
Hence \( |f(x) - C_n(x)| \)
\[
= \left| \frac{1}{2}[f(x) - T_n(x)] + \frac{1}{2}[f(-x) - T_n(-x)] \right| < \varepsilon.
\]

**THEOREM 2.7:** If \( f(x) \) satisfies the condition
\[
|f(x^2) - f(x_1)| < \lambda |x^2 - x_1| \quad \text{throughout } [a, b] \quad \text{of length } t,
\]
there exists for every positive integer \( n \) a polynomial
\( P_n(x) \) of the nth degree such that \( |f(x) - P_n(x)| \leq \frac{L \lambda}{n} \) for
\( a \leq x \leq b \) with \( L = \frac{k}{2} \), where \( k \) is the constant of Theorem 2.2.

**Proof:** Suppose \( f : [a, b] \rightarrow \mathbb{R} \) satisfies the condi­
tions of the theorem. Then define \( f_1[-1, 1] \rightarrow \mathbb{R} \) by
\[
f_1(y) = f\left(\frac{y(b-a)}{2} + \frac{(a+b)}{2}\right) \quad \text{for } y \in [-1, 1].
\]
Hence
\[
|f_1(y_2) - f_1(y_1)| < \frac{\lambda}{2} |y_2 - y_1| \quad \text{for } y_2 \in [-1, 1], y_1 \in [-1, 1].
\]
Let \( y = \cos \Theta \) and \( \Theta(\Theta) = f_1 \cos \Theta; \) then \( \Theta(\Theta) \) is an even
function defined for all real values of \( \Theta \). Also
\[
|\Theta(\Theta_2) - \Theta(\Theta_1)| = |f_1(\cos \Theta_2) - f_1(\cos \Theta_1)|
\]
\[
< \frac{kL}{2} |\cos \Theta_2 - \cos \Theta_1| < \frac{kL}{2} |\Theta_2 - \Theta_1|.
\]
Hence by Theorem 2.2, there exists a trigonometric sum of the nth order, \( T_n(\Theta) \),
such that \( |\Theta(\Theta) - T_n(\Theta)| \leq \frac{kL \lambda}{2n} = \frac{\lambda \lambda L}{n} \) where \( L = \frac{k}{2} \), for all
real \( \Theta \). By Lemma 2.4, since \( \Theta(\Theta) \) is an even function, there
is a cosine sum \( C_n(\Theta) \) such that \( |\Theta(\Theta) - C_n(\Theta)| \leq \frac{\lambda \lambda L}{n} \) for
\( 0 \leq \Theta \leq \pi \). We now show by induction that a cosine sum of
the nth order in \( \Theta \) is a polynomial of the nth degree in \( y \).

In \( n = 1 \), then \( \cos \Theta = \cos \Theta \); if \( n = 2 \), then \( \cos 2\Theta = 2 \cos^2 \Theta - 1 \). So now assume \( \cos j\Theta \) is a polynomial in \( \cos \Theta \) for \( j = 1 \)
\( \ldots \), \( k \), then \( \cos (k+1)\Theta = 2 \cos k\Theta \cos \Theta - \cos (k-1)\Theta \) which

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
is a polynomial of degree \( k + 1 \) in \( \cos \theta \). Hence \( C_n(\theta) \) can be considered as a polynomial of the \( n \)th degree in \( \cos \theta \).

Let \( Q_n: [-1, 1] \rightarrow \mathbb{R} \) be defined by \( Q_n(y) = C_n(\cos^{-1}y) \) for \( y \in [-1, 1] \). Then \( Q_n(y) \) is a polynomial of degree \( n \) in \( y \) and \( |f_1(y) - Q_n(y)| \leq \frac{kL}{2n} = \frac{\lambda L}{n} \) for \(-1 \leq y \leq 1\). Now let \( P_n(x) = Q_n(\frac{2x - a - b}{b - a}) = Q_n(y) \) for \( x \in [a, b] \). Then \( P_n(x) \) is a polynomial of degree \( n \) in \( x \). Therefore \( |P_n(x) - f(x)| = |Q_n(y) - f_1(y)| \leq \frac{\lambda L}{n} \) for \( x \in [a, b] \) where \( L = \frac{k}{2} \).

**Theorem 2.8:** If \( f(x) \) is a continuous function with modulus of continuity \( \omega(\delta) \) in the closed interval \([a, b]\) of length \( l \), then there exists for every positive integer \( n \), a polynomial \( P_n(x) \) of the \( n \)th degree such that for \( a \leq x \leq b \), \( |f(x) - P_n(x)| \leq L'\omega\left(\frac{\delta}{n}\right) \) where \( L' \) is an absolute constant.

**Proof:** Suppose \( f(x) \) is an arbitrary continuous function for \( a \leq x \leq b \) and let \( \omega(\delta) \) be its modulus of continuity in this interval. Let \( \phi(x) \) be the continuous function which takes on the same values as \( f(x) \) at the points \( a, \; a + \frac{k}{n}, \ldots, \; b - \frac{k}{n}, \; b \) and is linear from each point of this set to the next. Then \( \phi(x) \) is a continuous piece-wise linear function on \([a, b]\), no segment of which has slope greater than \( \frac{\omega(\frac{k}{n})}{\frac{k}{n}} \). Hence if \( \lambda = \frac{\omega(\frac{k}{n})}{\frac{k}{n}} \), then \( \phi(x) \) satisfies Theorem 2.7 since \( |\phi(x_2) - \phi(x_1)| \leq \lambda|x_2 - x_1| \). Thus there exists a polynomial \( P_n(x) \) of the \( n \)th degree such that
\[ |\phi(x) - P_n(x)| \leq \frac{L\epsilon}{n} = Lw\left(\frac{x}{n}\right). \]

If \( x \in [a, b] \), then \( x \) differs from one of the numbers \( a, a + \frac{k}{n}, \ldots, b - \frac{k}{n}, b \) by less than \( \frac{k}{n} \). Let \( t \) be such a number, then \( f(t) = \phi(t) \), and

\[ |f(x) - \phi(x)| \leq |f(x) - f(t)| + |f(t) - \phi(t)| + |\phi(t) - \phi(x)| \]

\[ \leq 2w\left(\frac{x}{n}\right). \]

Thus \( |f(x) - P_n(x)| \leq |f(x) - \phi(x)| + |\phi(x) - P_n(x)| \)

\[ \leq 2w\left(\frac{x}{n}\right) + Lw\left(\frac{x}{n}\right) = w\left(\frac{x}{n}\right)(L+2). \]

Hence if \( L' = L + 2 \), then

\[ |f(x) - P_n(x)| \leq L'w\left(\frac{x}{n}\right). \]

2.5 Since limit \( w\left(\frac{x}{n}\right) \rightarrow 0 \), we have Weierstrass's Theorem for polynomial approximation. Thus we have shown that the Weierstrass Theorem for trigonometric approximation implies his theorem for polynomial approximation. We now show the corresponding theorems concerning derivatives, and finally that the polynomial approximation theorem implies the trigonometric approximation theorem.

**Lemma 2.5:** Suppose \( f(x) \) has a continuous derivative \( f'(x) \) for \( a \leq x \leq b \), and that there is a polynomial \( p_n'(x) \), of degree \( n - 1 \), such that \( |f'(x) - p_n'(x)| < \epsilon_n \). Then there exists a polynomial \( P_n(x) \) of the \( n \)th degree such that

\[ |f(x) - P_n(x)| \leq \frac{L\epsilon}{n}. \]

Proof: Let \( p_n(x) = \int_a^x p_n'(x)dx \) and let \( r_n(x) = f(x) - p_n(x) \). Then \( |r_n'(x)| = |f'(x) - p_n'(x)| < \epsilon_n \). By the Mean Value Theorem, there exists \( x_o \in (x_1, x_2) \) such that

\[ |r_n(x_2) - r_n(x_1)| \leq |r_n'(x_o)| |x_2 - x_1|. \]

Thus

\[ |r_n(x_2) - r_n(x_1)| \leq \epsilon_n |x_2 - x_1|. \]

Thus by Theorem 2.7,
there exists a polynomial $\pi_n(x)$ of the nth degree such that

$$|r_n(x) - \pi_n(x)| < \frac{L \varepsilon}{n}.$$  

If $P_n(x) = p_n(x) + \pi_n(x)$, then $P_n(x)$ is a polynomial of the nth degree and $|f(x) - P_n(x)|$

$$= |r_n(x) - \pi_n(x)| \leq \frac{L \varepsilon}{n}.$$

**Theorem 2.9:** If $f(x)$ has a pth derivative $f^{(p)}(x)$ satisfying the condition that $|f^{(p)}(x_2) - f^{(p)}(x_1)|$

$\leq \lambda |x_2 - x_1|$ throughout $[a, b]$ of length $\ell$, there exists for every integral value of $n > p$ a polynomial $P_n(x)$ of the nth degree such that for $a < x < b$, $|f(x) - P_n(x)|$

$$\leq \frac{L_p \ell^{p+1} \lambda}{n^{p+1}}$$

where $L_p = \frac{(p+1)! \ell^{p+1}}{p!}$ and $L$ is the constant of Theorem 2.7.

**Proof:** The proof of this theorem will be by induction. If $p = 1$, then $f(x)$ has a derivative $f'(x)$ such that $|f'(x_2) - f'(x_1)| \leq \lambda |x_2 - x_1|$ for all $x \in [a, b]$. By Theorem 2.7, for $n > 1$, there is polynomial of degree $n - 1$, $P_{n-1}(x)$, such that $|f'(x) - P_{n-1}(x)| \leq \frac{L \lambda}{n-1}$ for all $x \in [a, b]$. By Lemma 2.5, there is a polynomial of degree $n$, $P_n(x)$, such that $|f(x) - P_n(x)| \leq \frac{L \ell \lambda}{(n-1)}$. Let $L = \frac{L \ell \lambda}{n(n-1)}$ for all $x \in [a, b]$. Now proceed by induction and assume that if $g(x)$ has a $(p-1)$th derivative $g^{(p-1)}(x)$ satisfying $|g^{(p-1)}(x_2) - g^{(p-1)}(x_1)| \leq \lambda |x_2 - x_1|$ and if $k > p - 1$, then assume that there is a polynomial $P_k(x)$ of the kth degree such that for $a \leq x \leq b$, $|g(x) - P_k(x)| \leq \frac{L^p \ell^p \lambda}{(k-p+1)\ldots(k-1)k}$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Now suppose \( f(x) \) has a pth derivative on \([a, b]\) satisfying 
\[
|f^{(p)}(x_2) - f^{(p)}(x_1)| \leq \lambda |x_2 - x_1| \text{ for } x_1 \in [a, b],
\]
\( x_2 \in [a, b] \), then \( f'(x) \) has a continuous \((p-1)\)th derivative 
\( f^{(p)}(x) \) for \( a \leq x \leq b \). If \( n > p \), then \( n - 1 > p - 1 \), and so 
by hypothesis there is a polynomial \( P_{n-1}(x) \) of degree \( n - 1 \) 
such that 
\[
|f'(x) - P_{n-1}(x)| \leq \frac{L_p}{(n-p)(n-p+1) \cdots (n-1)}.
\]
Then 
by Lemma 2.5, there is a polynomial, \( P_n(x) \), of the nth 
degree such that 
\[
|f(x) - P_n(x)| \leq \frac{L_p L^{p+1} \lambda}{n (n-p)(n-p+1) \cdots (n-1)}
\]
for all \( x \in [a, b] \). The conclusion of 
the proof is dependent on the observation that 
\[
\frac{1}{(n-p)(n-p+1) \cdots n} = \frac{n}{n-p} \frac{n}{n-p+1} \cdots \frac{1}{n+1} \leq \frac{(p+1)^{p+1}}{p^{p+1}} \frac{1}{n^{p+1}}
\]
for \( n \geq p + 1 \). Hence 
\[
|f(x) - P_n(x)| \leq \frac{L_{p+1} \lambda \lambda^p}{p^n n^{p+1}} \text{ for all } x \in [a, b].
\]
So let 
\[
L_p = \frac{(p+1)^p L_p^{p+1}}{p^{p+1}}, \text{ then } |f(x) - P_n(x)| \leq \frac{L_p L^{p+1} \lambda}{n^{p+1}} \text{ for all }
\]
for all \( x \in [a, b] \).

Theorem 2.10: If \( f(x) \) has a continuous pth derivative 
with modulus of continuity \( \omega(\delta) \) throughout the closed 
interval \([a, b]\) of length \( \ell \), there exists for every integer 
\( n > P \) a polynomial \( P_n(x) \) of the nth degree such that for 
\( a \leq x \leq b \), 
\[
|f(x) - P_n(x)| \leq \frac{L_p L^{p+1} \lambda}{n^{p+1}} \omega \left( \frac{1}{n-p} \right) \text{ where }
\]
\[ L_p = \frac{(p+1)p^{-1}P(L+2)}{p!} \] and \( L \) is the constant of Theorem 2.7.

Proof: This proof will be shown by induction. If \( p = 1 \), then \( f(x) \) has a continuous derivative \( f'(x) \) with modulus of continuity \( \omega(\delta) \) throughout \([a, b]\) of length \( \ell \). By Theorem 2.8, if \( n > 1 \), there is a polynomial \( P_{n-1}(x) \) of degree \( n - 1 \) such that
\[
|f'(x) - P_{n-1}(x)| \leq L'\omega\left(\frac{\ell}{n-1}\right)
\]
for all \( x \in [a, b] \). By Lemma 2.5, there is a polynomial of degree \( n \) such that
\[
|f(x) - P_n(x)| \leq \frac{L\ell}{n} \cdot L'\omega\left(\frac{\ell}{n-1}\right).
\]

Now proceed by induction and assume that if \( g(x) \) has a continuous \((p-1)\)th derivative with modulus of continuity \( \omega(\delta) \) throughout \([a, b]\), and if \( k > p - 1 \), then there exists a polynomial \( P_k(x) \) of the \( k \)th degree such that
\[
|g(x) - P_k(x)| \leq \frac{LP_{p-1}p^{-1}L_p}{(p-2)\ldots(k-1)k} \omega\left(\frac{\ell}{k-1}\right)
\]
for all \( x \in [a, b] \). Now suppose \( f(x) \) has a continuous \( p \)th derivative, \( f^{(p)}(x) \), with modulus of continuity throughout \([a, b]\), then \( f'(x) \) has a continuous \((p-1)\)th derivative, \( f^{(p-1)}(x) \), for \( a \leq x \leq b \) with modulus of continuity throughout \([a, b]\). If \( n > p \) then \( n - 1 > p - 1 \), and so by hypothesis there is a polynomial \( P_{n-1}(x) \), of degree \( n - 1 \) such that
\[
|f'(x) - P_{n-1}(x)| \leq \frac{LP_{p-1}p^{-1}L_p}{(n-p+1)\ldots(n-1)} \omega\left(\frac{\ell}{n-p}\right)
\]
for all \( x \in [a, b] \). By Lemma 2.5, there exists a polynomial \( P_n(x) \) of the \( n \)th degree such that
\[
|f(x) - P_n(x)| \leq \frac{L\ell}{n} \cdot \frac{LP_{p-1}p^{-1}L_p}{(n-p+1)\ldots(n-1)} \omega\left(\frac{\ell}{n-p}\right)
\]
The conclusion of the proof again depends on the observation made in Theorem 2.9. Thus $|f(x) - P_n(x)| \leq \frac{L_p'}{(n-p+1)\cdots(n-1)n} \omega\left(\frac{x}{n-p}\right)$.

Thus \[ |f(x) - P_n(x)| \leq \frac{L_p'}{(n-p+1)\cdots(n-1)n} \omega\left(\frac{x}{n-p}\right). \]

Let \[ L_p' = \frac{(p+1)^{p-1}L_p'}{p}. \]

then \[ |f(x) - P_n(x)| \leq \frac{L_p'}{n^p} \omega\left(\frac{x}{n-p}\right). \]

**Corollary 2.2:** If $f(x)$ has a continuous $p$th derivative for $a \leq x \leq b$, there exists for every positive integral value of $n$ a polynomial $P_n(x)$ of the $n$th degree such that \[ \lim_{n \to \infty} n^p \epsilon_n = 0 \] where $\epsilon_n = \max |f(x) - P_n(x)|$ in $[a, b]$.

2.6 The preceding theorems can be made to serve as a basis for a discussion of the Fourier series. The Fourier series for a given integrable function $f(x)$ on $[-\pi, \pi]$ is a series of the form \[ \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \]
in which the coefficients have the values\[ a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt, \quad k = 0, 1, \ldots. \]
The expression $s_n(x)$\[ = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \]
is called the $n$th partial sum of the series. Previously we have shown that if $f(x)$ is a given function continuous on $[a, b]$, then there exists a sequence of trigonometric sums, $T_n(x)$, where $T_n(x)$ is a sum of order $n$ which converges uniformly to $f(x)$. It is natural to ask if the sequence $s_n(x)$ converges to $f(x)$,
and also if it converges uniformly to $f(x)$. The answer is yes if additional hypotheses are added. A common method of inducing convergence for a divergent sequence is to form a sequence $\{C_n\}$ from $a_n$ by $C_n = \frac{a_1 + a_2 + \ldots + a_n}{n}$. It can be shown that if a sequence $\{a_n\}$ of real numbers converges, then the sequence $\{C_n\}$ converges, and to the same limit.

There are also cases where $\{a_n\}$ diverges, but the sequence $\{C_n\}$ converges. Since the sequence $\{s_n(x)\}$ may not always converge for continuous functions, it is reasonable to consider the sequence of arithmetic means of $\{s_n(x)\}$ in an approach due to Fejer which we present here without proof. For the proof, see [6], pp. 129-131.

**THEOREM 2.11** (Fejer) Let $f(x)$ be a continuous function of period $2\pi$, let $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt$, $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt$, $k = 0, 1, \ldots$. Also let $s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$, and define $\sigma_n(x) = \frac{1}{n} + \sum_{k=0}^{n} s_n(x)$, then $\sigma_n(x) \xrightarrow{U} x$ in $\mathbb{x}$.

2.7 We now proceed to show that the polynomial approximation theorem implies the trigonometric approximation theorem. To prove this it is necessary to establish several facts which shall be presented here in the form of lemmas.

**Lemma 2.6:** The function $\cos^k(x)$ can be written as a cosine sum of order $k$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof: The method of proof used here is induction. If \( k = 1 \), then we get \( \cos x \) which is a cosine sum of order 1. So now assume \( \cos^{k-1} x \) can be written as a cosine sum of order \( k-1 \), i.e., \( \cos^{k-1} x = \frac{a_0}{2} + \sum_{n=1}^{k-1} a_n \cos nx \). Then

\[
\cos^k x = \frac{a_0 \cos x}{2} + \sum_{n=1}^{k-1} a_n \cos nx \cos x = \frac{a_0 \cos x}{2} + \sum_{n=1}^{k-1} a_n \cos nx \cos x
\]

\[
+ \frac{1}{2} \sum_{n=1}^{k-1} a_n [\cos((n+1)x) + \cos((n-1)x)] = \frac{a_0 \cos x}{2} + \frac{1}{2} \sum_{n=1}^{k-1} a_n \cos(n+1)x
\]

\[
+ \frac{1}{2} \sum_{n=1}^{k-1} a_n \cos(n-1)x = \frac{a_0 \cos x}{2} + \frac{1}{2} \sum_{n=2}^{k} a_n \cos nx
\]

\[
+ \frac{1}{2} \sum_{n=0}^{k-1} a_{n+1} \cos nx = \frac{1}{2} \sum_{n=1}^{k} a_{n-1} \cos nx + \frac{1}{2} \sum_{n=1}^{k-1} a_{n+1} \cos nx
\]

\[
+ \frac{1}{2} = \frac{a_1}{2} + \sum_{n=1}^{k-1} \left( \frac{a_{n+1} + a_{n-1}}{2} \right) \cos nx + \frac{1}{2} a_{k-2} \cos(k-1)x
\]

\[
+ \frac{1}{2} a_{k-1} \cos kx. \quad \text{Thus } \cos^k(x) \text{ is a cosine sum of order } k.
\]

Lemma 2.7: If \( T_n(x) \) is a trigonometric sum of order \( n \), \( T_n(x) \sin x \) is a trigonometric sum of order \( n + 1 \).

Proof: Let \( T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \), then \( T_n(x) \sin x = \frac{a_1 \sin x}{2} + \sum_{k=1}^{n} (a_k \cos kx \sin x + b_k \sin kx \sin x) \)

\[
= \frac{a_0 \sin x}{2} + \sum_{k=1}^{n} a_k \left[ \sin(kx+x) - \sin(kx-x) \right] + \frac{b_k}{2} \left[ \cos(kx-x) - \cos(kx+x) \right]
\]

which is in the form of a trigonometric sum.

Lemma 2.8: If \( T_n(x) \) is a trigonometric sum of order \( n \), then \( T_n(x+a) \) is also a trigonometric sum of order \( n \).

Lemma 2.9: If the function \( f(x) \) is defined and continuous on \([0, \pi]\), then for \( \epsilon > 0 \), there exists a cosine sum, \( C_n(x) \), such that \( |f(x) - C_n(x)| < \epsilon \) for all \( x \in [0, \pi] \).
Proof: Consider the function $\varnothing(y) = f(\arccos y)$ where $y \in [-1, 1]$; this function is defined and continuous on $[-1, 1]$. By Weierstrass's Theorem, for $\varepsilon > 0$, there is a polynomial $\sum_{k=0}^{n} a_k y^k$ such that for all $y \in [-1, 1]$, $|\varnothing(y) - \sum_{k=0}^{n} a_k y^k| = |f(\arccos y) - \sum_{k=0}^{n} a_k y^k| < \varepsilon$. Let $g(y) = \arccos y$ and $h(x) = \cos x$, then $g:[-1, 1] \to [0, \pi]$ and $h:[0, \pi] \to [-1, 1]$ and $\varnothing = f \circ g$. We see that $g \circ h$ is the identity mapping on $[0, \pi]$, and $h \circ g$ is the identity mapping on $[-1, 1]$; $g$ and $h$ are both one-to-one, onto, and continuous, thus $\varnothing \circ h = (f \circ g) \circ h = f \circ (g \circ h) = f$ on $[0, \pi]$. If $x \in [0, \pi]$, then $|f(x) - \sum_{k=0}^{n} a_k \cos^k x| = |\varnothing(h(x)) - \sum_{k=0}^{n} a_k \cos^k x| = |\varnothing(\cos x) - \sum_{k=0}^{n} a_k \cos^k x| < \varepsilon$ since $\cos x \in [-1, 1]$. By Lemma 2.6, $\cos^k x$ can be written as a cosine sum of order $k$, so $\sum_{k=0}^{n} a_k \cos^k$ is a cosine sum $C_n(x)$ of order $n$, thus $|f(x) - C_n(x)| < \varepsilon$ for all $x \in [0, \pi]$.

Lemma 2.10: If the even function $f(x)$ is defined for all $x$, has period $2\pi$, and is continuous everywhere, then for all $\varepsilon > 0$, there exists a trigonometric sum, in fact a cosine sum, $C_n(x)$, such that $|f(x) - C_n(x)| < \varepsilon$ for all real numbers $x$.

Proof: By Lemma 2.9, there is a cosine sum $C_n(x)$ such that $|f(x) - C_n(x)| < \varepsilon$ for all $x \in [0, \pi]$. Since $f(x)$ is even, $|f(x) - C_n(x)| = |f(-x) - C_n(-x)| < \varepsilon$ for all $x \in [-\pi, \pi]$. Since $C_n(x)$ and $f(x)$ are periodic functions...
with period $2\pi$, it follows that $|f(x) - C_n(x)| < \varepsilon$ for all real numbers $x$.

**Theorem 2.12**: (Weierstrass) Let $f(x)$ be a continuous periodic function of period $2\pi$; then for all $\varepsilon > 0$, there exists a trigonometric sum $T_n(x)$ such that $|f(x) - T_n(x)| < \varepsilon$ for all real numbers $x$.

Proof: By Lemma 2.5, for the even continuous functions of period $2\pi$, $f(x) + f(-x)$ and $[f(x) - f(-x)] \sin x$, there exist trigonometric sums $T_1(x)$ and $T_2(x)$ such that $|f(x) + f(-x) - T_1(x)| < \frac{\varepsilon}{2}$ and $|f(x) - f(-x))\sin x - T_2(x)| < \frac{\varepsilon}{2}$ for all real $x$. This implies that $f(x) + f(-x) = T_1(x) + \alpha_1(x)$ where $|\alpha_1(x)| < \frac{\varepsilon}{2}$ for all real $x$ and $(f(x) - f(-x))\sin x = T_2(x) + \alpha_2(x)$ where $|\alpha_2(x)| < \frac{\varepsilon}{2}$ for all real $x$. Multiplying the first of these equalities by $\sin^2 x$ and the second by $\sin x$, then adding them and dividing by two, we obtain $f(x)\sin^2 x$

$$
\frac{\sin^2(x)T_1(x) + \sin(x)T_2(x)}{2} + \frac{\sin^2 x \alpha_1(x) + \sin x \alpha_2(x)}{2},
$$

Let $\frac{\sin^2(x)T_1(x) + \sin x T_2(x)}{2} = T_3(x)$; then by the use of Lemma 2.7, $T_3(x)$ is a trigonometric sum. Let

$$
\alpha_3(x) = \frac{\sin^2(x) \alpha_1(x) + \sin(x) \alpha_2(x)}{2},
$$

then $|\alpha_3(x)| < \frac{\varepsilon}{2}$ for all real $x$. Hence $f(x)\sin^2 x = T_3(x) + \alpha_3(x)$ where $|\alpha_3(x)| < \frac{\varepsilon}{2}$ for all real $x$. Since $f(x)$ is an arbitrary continuous periodic function, a similar equality is true.
for the continuous periodic function of period 2π, f(x - \frac{\pi}{2}),
so there is a trigonometric sum T_4(x) such that f(x - \frac{\pi}{2})\sin^2 x
= T_4(x) + \alpha_4(x) where |\alpha_4(x)| < \frac{\epsilon}{2} for all real x. Suppose u
is real and let x = u + \frac{\pi}{2}. Then u = x - \frac{\pi}{2}, and f(u)\cos^2 u
= f(u)\sin^2 (u + \frac{\pi}{2}) = f(x - \frac{\pi}{2})\sin^2 x = T_4(u + \frac{\pi}{2}) + \alpha_4(u + \frac{\pi}{2}).
Now T_4(u + \frac{\pi}{2}) is a trigonometric sum in u + \frac{\pi}{2}, so T_4(u + \frac{\pi}{2})
is a trigonometric sum in u. Let T_5(u) = T_4(u + \frac{\pi}{2}), and
let \alpha_5(u) = \alpha_4(u + \frac{\pi}{2}). Then |\alpha_5(u)| < \frac{\epsilon}{2} for all u, and
f(u)\cos^2 u = T_5(u) + \alpha_5(u) for all u. Let T(x) = T_3(x) +
T_5(x), and let \alpha(x) = \alpha_3(x) + \alpha_5(x). Then f(x) = f(x)\sin^2 x
+ f(x)\cos^2 x = T_3(x) + \alpha_3(x) + T_5(x) + \alpha_5(x) = T(x) + \alpha(x),
where |\alpha(x)| \leq |\alpha_3(x)| + |\alpha_5(x)| < \epsilon for all real x. There­
fore |f(x) - T(x)| < \epsilon for all real x, and T(x) is a trigo­
nometric sum in x.
CHAPTER III

Generalizations of the Weierstrass Theorem

3.1 The generalization in Chapter Three deals with the problem of relaxing the restrictions imposed on the domain over which the given functions are defined. The main problem lies in constructing functions from a prescribed family by the application of certain algebraic operations, (i.e., addition, multiplication, scalar multiplication, formation of absolute value functions, formation of maximum and minimum functions, and uniform passage to the limit). In the classical case just discussed, the prescribed family consists of just two functions, $f_1$ and $f_2$, where $f_1(x) = 1$ and $f_2(x) = x$ for all $x$ in the basic interval, and the particular algebraic operations of addition, scalar multiplication, and multiplication, together with the operation of uniform passage to the limit.

Before beginning the discussion, there are several definitions which will be needed throughout the chapter. We present them here as a refresher to the reader.

Definition 3.1: Let $F$ be a collection of subsets of $X$ such that:

1.) $\emptyset \in F$, $X \in F$
2.) $A \in F$, $B \in F$ implies $A \cap B \in F$
3.) $F_0 \subseteq F$ implies $\bigcup_{F_0} \in F$
Then $F$ is called a topology for $X$ and $X$ is called a topological space with topology $F$. The sets of the collection $F$ are called open sets.

**Definition 3.2:** Suppose $X$ is a topological space and suppose $p \in X$. A function $f: X \rightarrow \mathbb{R}$ is said to be continuous at $p$ if $\varepsilon > 0$ implies that there exists $G \in F$ such that $p \in G$ and such that $q \in G$ implies $|f(p) - f(q)| < \varepsilon$.

**Definition 3.3:** A topological space $X$ is said to be compact if $\bigcap_{F' \subseteq F} X = \bigcup_{V \subseteq F'}$ implies that there exists $V \subseteq F'$ such that $V$ is a finite collection and $X = \bigcup_{V}$, that is, every open covering of $X$ has a finite subcovering.

We recall that if $X$ is a compact topological space and if $f: X \rightarrow \mathbb{R}$ continuous on $X$, then $f$ is bounded on $X$.

Let us also agree on the following terminology throughout the remainder of the chapter: let $X$ be an arbitrary topological space with at least two points; $\mathcal{X}$ the family of all continuous real functions on $X$; and $\mathcal{X}_0$ a prescribed subfamily of $\mathcal{X}$.

**Definition 3.4:** Suppose $f \in \mathcal{X}$, $g \in \mathcal{X}$. Define $f \cap g: X \rightarrow \mathbb{R}$ and $f \cup g: X \rightarrow \mathbb{R}$ by $(f \cap g)(x) = \min f(x), g(x)$, and $(f \cup g)(x) = \max f(x), g(x)$ for $x \in X$. Then $f \cap g$ and $f \cup g$ are called respectively the minimum of $f$ and $g$, and the maximum of $f$ and $g$.

Note that if $f \in \mathcal{X}$, $g \in \mathcal{X}$, then from the identity $|(f \cap g)(x) - (f \cap g)(y)| \leq \max (|f(x) - f(y)|, |g(x) - g(y)|), \]
we see that \( f \cap g \in \mathcal{X} \). Similarly if \( f, g \in \mathcal{X} \), then \( f \cup g \in \mathcal{X} \). Also, \( \cap \) and \( \cup \) are commutative and associative operations in \( \mathcal{X} \), \( \cap \) is distributive with respect to \( \cup \), and \( \cup \) is distributive with respect to \( \cap \). If \( f_i \in \mathcal{X}, \ i = 1 \ldots n \), we write \( \bigcap_{i=1}^{n} f_i \) for \( f_1 \cap f_2 \cap \ldots \cap f_n \) and \( \bigcup_{i=1}^{n} f_i \) for \( f_1 \cup f_2 \cup \ldots \cup f_n \). Note that \( (\bigcap_{i=1}^{n} f_i)(x) = \min_{i=1\ldots n} f_i(x) \) and \( (\bigcup_{i=1}^{n} f_i)(x) = \max_{i=1\ldots n} f_i(x) \). The operations \( \cap \) and \( \cup \) are called the lattice operations in \( \mathcal{X} \). The operations \( \cap \) and \( \cup \) are called, respectively, the minimum and maximum operations in \( \mathcal{X} \).

**Definition 3.5:** Suppose \( \mathcal{X}_0 \subseteq \mathcal{X} \). \( \mathcal{X}_0 \) is said to be closed under the maximum operation if \( f, g \in \mathcal{X}_0 \) implies \( f \cup g \in \mathcal{X}_0 \). \( \mathcal{X} \) is said to be closed under the minimum operation if \( f, g \in \mathcal{X}_0 \) implies \( f \cap g \in \mathcal{X}_0 \). Note that if \( \mathcal{X} \) is closed under the maximum operation, then \( f_i \in \mathcal{X}_0 \) for \( i = 1 \ldots n \) implies \( \bigcup_{i=1}^{n} f_i \in \mathcal{X}_0 \). Similarly for the minimum operation.

**Definition 3.6:** Suppose \( \mathcal{X}_0 \subseteq \mathcal{X} \). Let \( \mathcal{A}(\mathcal{X}_0) \) be the smallest collection which contains \( \mathcal{X}_0 \) and which is closed under the maximum operation. Note that \( \mathcal{A}(\mathcal{X}_0) \subseteq \mathcal{X} \).

If \( C \) is the family of all collections containing \( \mathcal{X}_0 \) and closed under the maximum operation, then \( \mathcal{C} \) is a collection which contains \( \mathcal{X}_0 \) and which is closed under the maximum operation and which is contained in every
collection that contains $X_0$ and is closed under the maximum operation. A similar statement may also be made about families of all collections containing $X_0$ and which are closed under different operations.

**Definition 3.7:** Suppose $X_0 \subseteq X$. Let $B(X_0)$ be the smallest collection which contains $X_0$ and which is closed under the minimum operation. Note that $B(X_0) \subseteq X$.

**Definition 3.8:** Let $U_1(X_0)$ be the smallest collection which contains $X_0$ and which is closed under the maximum operation and the minimum operation. Note that $U_1(X_0) \subseteq X$.

**Theorem 3.1:** Suppose $X_0 \subseteq X$, then $A(X_0) = \{f \mid f \in X; \text{there exists } f_i \in X_0 \text{ for } i = 1 \ldots n \text{ such that } f = \bigcup_{i=1}^{n} f_i\}$, $B(X_0) = \{f \mid f \in X; \text{there exists } f_i \in X_0 \text{ for } i = 1 \ldots n \text{ such that } f = \bigcap_{i=1}^{n} f_i\}$.

**Proof:** Let $Y = \{f \mid f \in X; \text{there exists } f_i \in X_0 \text{ for } i = 1 \ldots n \text{ such that } f = \bigcup_{i=1}^{n} f_i\}$. Suppose $f \in Y$, then $f \in X$ and there exists $f_i \in X_0$ for $i = 1 \ldots n$ such that $f = \bigcup_{i=1}^{n} f_i$. Since $X_0 \subseteq A(X_0)$, and since $A(X_0)$ is closed under the max operation, then $f \in A(X_0)$ and $Y \subseteq A(X_0)$.

Now suppose $g \in X_0$. Let $g_1 = g$, $g_2 = g$. Then $g = g_1 \cup g_2$, $g_1 \in X_0$, $g_2 \in X_0$, so $g \in Y$ and hence $X_0 \subseteq Y$. Now suppose $f \in Y$, $g \in Y$; then there exists $f_i \in X_0$, $i = 1 \ldots n$, such that $f = \bigcup_{i=1}^{n} f_i$, and there exists $g_i \in X_0$, $i = 1 \ldots n$, such that $g = \bigcup_{i=1}^{m} g_i$. Thus $g \cup h = (\bigcup_{i=1}^{n} g_i) \cup (\bigcup_{i=1}^{m} h_i) \in Y$ and $Y$ is closed under the maximum operation. Since $X_0 \subseteq Y$ and $Y$
is closed under the maximum operation, \( A(\mathbb{X}_o) \subseteq Y \). Thus we have \( A(\mathbb{X}_o) = Y \). The proof for \( B(\mathbb{X}_o) = [f | f \in \mathbb{X}; \text{there exists } f_i \in \mathbb{X}_o, i = 1 \ldots n \text{ such that } f = \bigcap_{i=1}^{n} f_i \] is similar.

**Theorem 3.2:** Suppose \( \mathbb{X}_o \subseteq \mathbb{X} \). Then \( A(B(\mathbb{X}_o)) = U_1(\mathbb{X}_o) \), and \( B(A(\mathbb{X}_o)) = U_1(\mathbb{X}_o) \). Hence \( U_1(\mathbb{X}_o) = [f | f \in \mathbb{X}; \text{there exists } f_i, j \in \mathbb{X}_o, i = 1 \ldots n; j = 1 \ldots k_i, \text{such that } f = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k_i} f_i, j] \), and \( U_1(\mathbb{X}_o) = [f | f \in \mathbb{X}; \text{there exists } f_i, j \in \mathbb{X}_o, i = 1 \ldots n; j = 1 \ldots k_i, \text{such that } f = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k_i} f_i, j] \).

**Proof:** Let \( Y = [f | f \in \mathbb{X}; \text{there exists } f_i, j \in \mathbb{X}_o, i = 1 \ldots n; j = 1 \ldots k_i, \text{such that } f = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k_i} f_i, j] \). Suppose \( f \in Y \), then \( f \in \mathbb{X} \) and there exists \( f_i, j \in \mathbb{X}_o, i = 1 \ldots n; j = 1 \ldots k_i, \text{such that } f = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k_i} f_i, j \).

Since \( \mathbb{X}_o \subseteq U_1(\mathbb{X}_o) \) and since \( U_1(\mathbb{X}_o) \) is closed under the max and min operations, then \( f \in U_1(\mathbb{X}_o) \) and \( Y \subseteq U_1(\mathbb{X}_o) \).

Now suppose \( g \in \mathbb{X}_o \). Let \( g_{i, j} = g \) for \( i = 1, 2; j = 1, 2 \). Then \( g = \bigcup_{i=1}^{2} \bigcup_{j=1}^{2} g_{i, j} \) with \( g_{i, j} \in \mathbb{X}_o \), so \( g \in Y \), and hence \( \mathbb{X}_o \subseteq Y \).

Now suppose \( f \in Y, g \in Y \), then there exists \( f_{i, j} \in \mathbb{X}_o, i = 1 \ldots n; j = 1 \ldots k_i \text{ such that } f = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k_i} f_{i, j} \) and there exists \( g_k, \ell \in \mathbb{X}_o, i = 1 \ldots r; \ell = 1 \ldots m_k \text{ such that } g = \bigcup_{k=1}^{r} \bigcup_{\ell=1}^{m_k} g_k, \ell \). Thus \( f = (f_{1, 1} \cup f_{1, 2} \cup \ldots \cup f_{1, k_1}) \).
\[ n \left( f_{2,1} \cup f_{2,2} \cup \ldots \cup f_{2,k_2} \right) \cap \ldots \cap n \left( f_{n,1} \cup f_{n,2} \cup \ldots \cup f_{n,k_n} \right) \]

and \( g \) could be written out likewise. Hence \( f \cap g \in Y \). Now

if \( g_i = (g_{i,1} \cup g_{i,2} \cup \ldots \cup g_{i,r_i}) \), \( i = 1 \ldots r \), and

\[ f_i = (f_{i,1} \cup f_{i,2} \cup \ldots \cup f_{i,k_i}) \], \( i = 1 \ldots n \), then

\( f = f_1 \cap f_2 \cap \ldots \cap f_n \), and \( g = g_1 \cap g_2 \cap \ldots \cap g_m \). Using

the identity for max and min operations, we see that

\[ f \cup g = (f_1 \cap f_2 \cap \ldots \cap f_n) \cup (g_1 \cap g_2 \cap \ldots \cap g_m) \]

\[ = (f_1 \cup g_1) \cap (f_1 \cup g_2) \cap \ldots \cap (f_1 \cup g_m) \cap (f_2 \cup g_1) \]

\[ \cap (f_2 \cup g_2) \cap \ldots \cap (f_2 \cup g_m) \cap \ldots \cap (f_n \cup g_1) \cap (f_n \cup g_2) \]

\[ \cap \ldots \cap (f_n \cup g_m) \in Y. \] Thus \( f \cup g \in Y, f \cap g \in Y, \) so \( Y \)

is closed under the maximum and minimum operations, and

\( x_0 \subseteq Y, \) and \( U_1(x_0) \subseteq Y. \) Now since \( U_1(x_0) \subseteq Y \) and \( Y \subseteq U_1(x_0), \)

then \( U_1(x_0) = Y. \) A similar argument can be used to show

that \( U_1(x_0) = \{ f \mid f \in x; \text{there exists } f_{i,j} \in x_0, i = 1 \ldots n; \)

\( j = 1 \ldots k_i \text{ such that } f = \sum_{i=1}^{n} \prod_{j=1}^{k_i} f_{i,j} \}. \)

**Definition 3.9:** Suppose \( x_0 \subseteq x. \) Let \( V_1(x_0) \) be the

smallest collection which contains \( x_0 \) and which is closed

under addition and scalar multiplication. Note that

\( V_1(x_0) \subseteq x. \)

**Theorem 3.3:** Suppose \( x_0 \subseteq x, \) then \( V_1(x_0) = \{ f \mid f \in x; \)

there exists \( c_i \in R, f_i \in x_0, i = 1 \ldots n, \) such that

\( f = \sum_{i=1}^{n} c_i f_i \}. \)

**Proof:** The proof follows in the same fashion as
Theorem 3.1.

Definition 3.10: Suppose $X \subset X$. Let $V(X_0)$ be the smallest collection which contains $X_0$ and which is closed under addition, scalar multiplication, and the lattice operations. Note that $V(X_0) \subset X$.

Theorem 3.4: Suppose $X_0 \subset X$. Then $V(X_0) = U_1(V_1(X_0))$, and $V(X_0) = \{f | f \in X; \exists f_i, j \in V_1(X_0), i = 1 \ldots n; j = 1 \ldots k_i \text{ such that } f = \sum_{i=1}^{n} \sum_{j=1}^{k_i} f_{i,j} \}$.

Proof: We know that $X_0 \subset V(X_0)$ and $V(X_0)$ is closed under addition and scalar multiplication, so $V_1(X_0) \subset V(X_0)$. Also, $V(X_0)$ is closed under the lattice operations, so $U_1(V_1(X_0)) \subset V(X_0)$, and since $U_1(V_1(X_0))$ is closed under the lattice operations, we must show that $U_1(V_1(X_0))$ is closed with respect to scalar multiplication and addition. Suppose $f \in U_1(V_1(X_0))$ and $c \in R$. Then by Theorem 3.2,

$$f = \sum_{i=1}^{n} \sum_{j=1}^{k_i} f_{i,j}, \text{ where } f_{i,j} \in V_1(X_0), i = 1 \ldots n; j = 1 \ldots k_i.$$ 

If $c \geq 0$, then $cf = (cf_{1,1} \cup cf_{1,2} \cup \ldots \cup cf_{1,k_1}) \cap (cf_{2,1} \cup cf_{2,2} \cup \ldots \cup cf_{2,k_2}) \cap \ldots \cap (cf_{n,1} \cup cf_{n,2} \cup \ldots \cup cf_{n,k_n}).$ Therefore $cf \in U_1(V_1(X_0))$ if $c \geq 0$. If $c < 0$, and if $f_{i} \in X, i = 1 \ldots n$, then $\sum_{i=1}^{n} cf_{i} = c \sum_{i=1}^{n} f_{i} = c i=1 f_{i}$. Hence $cf = (cf_{1,1} \cup cf_{1,2} \cup \ldots \cup cf_{1,k_1}) \cup (cf_{2,1} \cup cf_{2,2} \cup \ldots \cup cf_{2,k_2}) \cup \ldots \cup (cf_{n,1} \cup \ldots \cup cf_{n,k_n})$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Therefore cf ε $U_1(V_1(\mathcal{X}_o))$ if $c < 0$. Hence $U_1(V_1(\mathcal{X}_o))$ is closed with respect to scalar multiplication. Now suppose $f \in U_1(V_1(\mathcal{X}_o))$ and $g \in U_1(V_1(\mathcal{X}_o))$. Then $f = \sum_{i=1}^{n} \sum_{j=1}^{k_i} f_{i,j}$, where $f_{i,j} \in V_1(\mathcal{X}_o), i = 1 \ldots n; j = 1 \ldots k_i$ and

$g = \sum_{i=1}^{m} \sum_{j=1}^{l_i} g_{i,j}$ where $g_{i,j} \in V_1(\mathcal{X}_o), i = 1 \ldots m; j = 1 \ldots l_i$.

Let $f_i = (f_{i,1} \cup f_{i,2} \cup \ldots \cup f_{i,k_i}), i = 1 \ldots n$, and $g_i = (g_{i,1} \cup g_{i,2} \cup \ldots \cup g_{i,l_i}), i = 1 \ldots m$. Now $f + g$ can be written as $f + g = (f_1 \cap f_2 \cap \ldots \cap f_n) + (g_1 \cap g_2 \cap \ldots \cap g_m)$. Making use of the identities;

$$(f_1 \cap f_2 \cap \ldots \cap f_n) + (g_1 \cap g_2 \cap \ldots \cap g_m) = (f_1 + g_1) \cap (f_1 + g_2) \cap \ldots \cap (f_1 + g_m) \cap \ldots \cap (f_n + g_1) \cap (f_n + g_2) \cap \ldots \cap (f_n + g_m),$$

and so $U_1(V_1(\mathcal{X}_o))$ is closed under addition. Since $U_1(V_1(\mathcal{X}_o))$ contains $\mathcal{X}_o$ and is closed under addition, scalar multiplication, and the lattice operations, then $V(\mathcal{X}_o) \subseteq U_1(V_1(\mathcal{X}_o))$.

Also since $U_1(V_1(\mathcal{X}_o)) \subseteq V(\mathcal{X}_o)$, we have the desired conclusion that $V(\mathcal{X}_o) = U_1(V_1(\mathcal{X}_o))$. The second part of the theorem follows from Theorem 3.2 and the fact that $V(\mathcal{X}_o) = U_1(V_1(\mathcal{X}_o))$.

**Theorem 3.5:** Suppose $\mathcal{X}_o \subseteq \mathcal{X}$, and suppose $\mathcal{X}_o$ is closed under scalar multiplication and the maximum operation.
Then \( f \in X_o \) implies \(|f| \in X_o \), i.e., \( X_o \) is closed under the absolute value operation.

Proof: Suppose \( f \in X_o \). Since \( X_o \) is closed with respect to scalar multiplication, \((-1)(f) = -f \in X_o \). Since \( X_o \) is closed with respect to max operation, then \(|f| = f \cup (-f) \in X_o \) and \( X_o \) is closed under the absolute value operation.

**Theorem 3.6:** Suppose \( X_o \subseteq X \). Then \( V(X_o) \) is the smallest collection which contains \( X_o \) and which is closed under addition, scalar multiplication, the lattice operations, and the absolute value operations.

Proof: Let \( T \) be the smallest collection which contains \( X_o \) and which is closed under addition, scalar multiplication, the lattice operations, and the absolute value operation. Since \( X_o \subseteq T \) and \( T \) is closed under addition, scalar multiplication, and the lattice operations, then \( V(X_o) \subseteq T \). Also \( T \subseteq V(X_o) \) by Theorem 3.5, thus \( V(X_o) = T \).

**Definition 3.11:** Suppose \( X_o \subseteq X \). Suppose \( f_n \in X_o \), \( n = 1, 2, \ldots \) and \( f_n \xrightarrow{U} f \) on \( X \). Then \( X_o \) is said to be closed under uniform passage to the limit if \( f \in X_o \).

**Definition 3.12:** Suppose \( X_o \subseteq X \). Let \( U_2(X_o) \) be the smallest collection which contains \( X_o \) and which is closed under uniform passage to the limit.

**Theorem 3.7:** Suppose \( X_o \subseteq X \). Then \( U_2(X_o) = \{ f | f \in X; \text{there exists } f_n \in X_o \text{ such that } f_n \xrightarrow{U} f \text{ on } X \} \).
Proof: Let \( Y = \{ f \mid f \in X, \text{there exists } f_n \in X_0 \text{ such that } f_n \xrightarrow{U} f \text{ on } X \} \). Suppose \( f \in Y \), then \( f \in X \) and there exists \( f_n \in X_0 \) such that \( f_n \xrightarrow{U} f \) on \( X \). Since \( X_0 \subseteq U_2(X_0) \) and since \( U_2(X_0) \) is closed under uniform limits, then \( f \in U_2(X_0) \) and \( Y \subseteq U_2(X_0) \). Now suppose \( f \in X_0 \) and let \( f_n = f \) for all \( n \). Then \( f_n \xrightarrow{U} f \) on \( X \) and \( f_n \in X_0 \), so \( f \in Y \) and hence \( X_0 \subseteq Y \). Now we must show that \( Y \) is closed under the uniform limit operation. Suppose \( f_n \in Y \), \( f_n \xrightarrow{U} f \) on \( X \), then there exists \( g_n \in X_0 \) such that \( |f_n(x) - g_n(x)| < \frac{1}{n} \) for all \( x \in X \). Let \( \varepsilon > 0 \), then there exists \( N_1 \), such that \( n \geq N_1 \) implies \( |f_n(x) - f(x)| < \frac{\varepsilon}{2} \) for all \( x \in X \). Also there exists \( N_2 \) such that \( \frac{1}{N_2} < \frac{\varepsilon}{2} \). Let \( N = \max N_1, N_2 \). If \( n \geq N \), then \( \frac{1}{n} \leq \frac{1}{N} \leq \frac{1}{N_2} < \frac{\varepsilon}{2} \), and \( |g_n(x) - f(x)| \leq |g_n(x) - f_n(x)| + |f_n(x) - f(x)| < \frac{1}{n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \) for all \( x \in X \). Hence \( g_n \xrightarrow{U} f \) on \( X \). Since \( g_n \in X_0 \), it follows that \( f \in Y \). Thus \( U_2(X_0) \subseteq Y \) and so \( U_2(X_0) = Y \).

**Definition 3.13:** Suppose \( X_0 \subseteq X \). Then we define \( U(X_0) \) to be the smallest collection which contains \( X_0 \), which is closed under the lattice operations, and which is closed under uniform passage to the limit. Note that \( U(X_0) = X_0 \).

**Theorem 3.8:** Suppose \( X_0 \subseteq X \). Then \( U(X_0) = U_2(U_1(X_0)) \).

Proof: We know that \( X_0 \subseteq U(X_0) \) and \( U(X_0) \) is closed.
under the lattice operations, so \( U_1(\mathbb{X}_0) \subseteq U(\mathbb{X}_0) \). Also, \( U(\mathbb{X}_0) \) is closed under the uniform limit operation, so \( U_2(U_1(\mathbb{X}_0)) \subseteq U(\mathbb{X}_0) \). We now want to show that \( U(\mathbb{X}_0) \subseteq U_2(U_1(\mathbb{X}_0)) \). Since \( \mathbb{X}_0 \subseteq U_1(\mathbb{X}_0) \subseteq U_2(U_1(\mathbb{X}_0)) \), we must show that \( U_2(U_1(\mathbb{X}_0)) \) is closed under the lattice operations.

Suppose \( f \in U_2(U_1(\mathbb{X}_0)) \) and \( g \in U_2(U_1(\mathbb{X}_0)) \). Then by Theorem 3.7, \( f \in \mathbb{X} \) and there exists \( f_n \in U_1(\mathbb{X}_0) \) such that \( f_n \to f \) on \( \mathbb{X} \), and there exists \( g_n \in U_1(\mathbb{X}_0) \) such that \( g_n \to g \) on \( \mathbb{X} \). Therefore there exists \( N_1 \) such that for \( n \geq N_1 \), \( |f_n(x) - f(x)| < \frac{\varepsilon}{2} \), and there exists \( N_2 \) such that for \( n \geq N_2 \), \( |g_n(x) - g(x)| < \frac{\varepsilon}{2} \). Thus for \( n \geq \max N_1, N_2 \),

\[
|\max(f(x), g(x)) - \max(f_n(x), g_n(x))| \leq |f(x) - f_n(x)| + |g(x) - g_n(x)| < \varepsilon.
\]

Therefore \( f \cup g \) and \( f \cap g \) are the uniform limits of \( f_n \cup g_n \) and \( f_n \cap g_n \) respectively. Since \( f_n \cup g_n \in U_1(\mathbb{X}_0) \) and \( f_n \cap g_n \in U_1(\mathbb{X}_0) \), then \( f \cup g \in U_2(U_1(\mathbb{X}_0)) \) and \( f \cap g \in U_2(U_1(\mathbb{X}_0)) \). Thus \( U_2(U_1(\mathbb{X}_0)) \) is closed with respect to the lattice operations. Hence \( U(\mathbb{X}_0) \subseteq U_2(U_1(\mathbb{X}_0)) \).

Also since \( U_2(U_1(\mathbb{X}_0)) \subseteq U(\mathbb{X}_0) \), we have the desired conclusion that \( U(\mathbb{X}_0) = U_2(U_1(\mathbb{X}_0)) \).

**Definition 3.14:** Suppose \( \mathbb{X}_0 \subseteq \mathbb{X} \). Then we define \( W(\mathbb{X}_0) \) to be the smallest collection which contains \( \mathbb{X}_0 \), which is closed under addition, scalar multiplication, and the lattice operations, and which is closed under
uniform passage to the limit. Note that \( W(X_0) \subseteq X \).

**Theorem 3.9:** Suppose \( X_0 \subseteq X \). Then \( W(X_0) = U_2(V(X_0)) \) = \( U_2(U_1(V_1(X_0))) = U(V_1(X_0)) \).

**Proof:** By Theorem 3.4, \( U_2(V(X_0)) = U_2(U_1(V_1(X_0))) \) and by Theorem 3.8, \( U_2(U_1(V_1(X_0))) = U(V_1(X_0)) \). To finish the theorem, we must show that \( U_2(V(X_0)) = W(X_0) \). We know that \( X_0 \subseteq W(X_0) \) and \( W(X_0) \) is closed under addition, scalar multiplication, and lattice operations, so \( V(X_0) = W(X_0) \).

Since \( V(X_0) \subseteq W(X_0) \), and \( W(X_0) \) is closed under the uniform limit operation, we have \( U_2(V(X_0)) \subseteq W(X_0) \). Also, \( X_0 \subseteq W(X_0) = U_2(V(X_0)) \), so \( X_0 \subseteq U_2(V(X_0)) \). We must now show that \( U_2(V(X_0)) \) is closed under addition, scalar multiplication, and the lattice operations. Suppose \( f \in U_2(V(X_0)) \) and \( g \in U_2(V(X_0)) \) and \( c \in \mathbb{R} \). Then \( f \in X \) and there exists \( f_n \in V(X_0) \) such that \( f_n \xrightarrow{U} f \) on \( X \). Also \( g \in X \) and there exists \( g_n \in V(X_0) \) such that \( g_n \xrightarrow{U} g \) on \( X \). Then we can readily see that \( f_n + g_n \xrightarrow{U} f + g \), \( cf_n \xrightarrow{U} cf \), \( f_n \cup g_n \xrightarrow{U} f \cup g \), and \( f_n \cap g_n \xrightarrow{U} f \cap g \) on \( X \), \( f_n + g_n \in V(X_0) \), \( cf_n \in V(X_0) \), \( f_n \cup g_n \in V(X_0) \), and \( f_n \cap g_n \in V(X_0) \). Therefore \( f + g \in U_2(V(X_0)) \), \( cf \in U_2(V(X_0)) \), \( f \cup g \in U_2(V(X_0)) \) and \( f \cap g \in U_2(V(X_0)) \). Hence \( U_2(V(X_0)) \) is closed with respect to addition, scalar multiplication, and the lattice operations, and the uniform limit operation. Thus \( W(X_0) \subseteq U_2(V(X_0)) \) and \( W(X_0) = U_2(V(X_0)) \).

**Theorem 3.10:** Suppose \( X_0 \subseteq X \). Then \( W(X_0) \) is the
smallest collection which contains \( X_o \), which is closed under addition, scalar multiplication, the lattice operations, the absolute value operation, and which is closed under uniform passage to the limit.

Proof: The proof follows in the same fashion as Theorem 3.6.

**Definition 3.15**: Suppose \( X_o \subseteq X \). Then we define \( P(X_o) \) to be the smallest collection which contains \( X_o \) and which is closed under addition, scalar multiplication, and multiplication. Note that \( P(X_o) \subseteq X \).

**Theorem 3.11**: Suppose \( X_o \subseteq X \). Then \( P(X_o) = \{ f \mid f \in X; \text{there exists } c_i \in \mathbb{R}, i = 1 \ldots n, \text{there exists } f_{i,j} \in X, i = 1 \ldots n; j = 1 \ldots k_i \text{ such that} \}

\[
f = \sum_{i=1}^{n} (c_i \prod_{j=1}^{k_i} f_{i,j}).
\]

Proof: The proof is similar to the reasoning used in Theorem 3.1 and Theorem 3.3.

**Definition 3.16**: Suppose \( X_o \subseteq X \). Then we define \( L(X_o) \) to be the smallest collection which contains \( X_o \), and which is closed under addition, scalar multiplication, and multiplication, and which is closed under uniform passage to the limit.

**Theorem 3.12**: Suppose \( X \) is a compact topological space, and suppose \( X_o \subseteq X \). Then \( L(X_o) = \bigcup_{2} (P(X_o)) \).

Proof: We know \( X_o \subseteq L(X_o) \) and \( L(X_o) \) is closed
under addition, scalar multiplication, and multiplication, so \( P(X_o) = L(X_o) \). Since \( P(X_o) = L(X_o) \) and \( L(X_o) \) is closed under the uniform limit operation, we have that \( U_2(P(X_o)) = L(X_o) \). Also, \( X_o \subseteq P(X_o) \subseteq U_2(P(X_o)) \), so \( X_o \subseteq U_2(P(X_o)) \). We now must show that \( U_2(P(X_o)) \) is closed under addition, scalar multiplication, and multiplication. Suppose \( f \in U_2(P(X_o)) \), then \( f \in X \) and there exists \( f_n \in P(X_o) \) such that \( f_n \overset{U}{\rightarrow} f \) on \( X \). Suppose \( g \in U_2(P(X_o)) \), then \( g \in X \) and there exists \( g_n \in P(X_o) \) such that \( g_n \overset{U}{\rightarrow} g \) on \( X \). As before, it can be shown that \( f + g \) is the uniform limit of \( f_n + g_n \in P(X_o) \), and \( cf \) is the uniform limit of \( cf_n \in P(X_o) \) if \( c \in \mathbb{R} \). Thus \( U_2(P(X_o)) \) is closed under addition and scalar multiplication. To show the closure of multiplication, we make use of the fact that since \( X \) is compact, the functions \( f \) and \( g \) are bounded. Therefore there exist \( M_1 \) and \( M_2 \) such that \( |f(x)| \leq M_1 \) and \( |g(x)| \leq M_2 \) for all \( x \in X \). Since \( f_n \overset{U}{\rightarrow} f \), there exists \( N_1 \), such that for \( n \geq N_1 \),

\[
|f_n(x) - f(x)| \leq \frac{\varepsilon}{3M_2}, \quad \text{for all } x \in X, \quad \text{where } Q_1 = \min \frac{\varepsilon}{3M_2}.
\]

\[
\sqrt{\frac{\varepsilon}{\frac{1}{3}}} > 0. \quad \text{Since } g_n \overset{U}{\rightarrow} g, \text{there exists } N_2 \text{ such that for } n \geq N_2, \quad |g_n(x) - g(x)| \leq \frac{\varepsilon}{3} \quad \text{for all } x \in X \text{ where } Q_2 = \min \frac{\varepsilon}{3M_1}.
\]

\[
\frac{\varepsilon}{\frac{1}{3}} > 0. \quad \text{Thus } |(f_n g_n)(x) - (fg)(x)| = |f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x) - f(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|
\]

\[
+ |f(x)||g_n(x) - g(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \text{Therefore } fg \text{ is the uniform limit of } f_n g_n \in P(X_o) \text{ and so } U_2(P(X_o)) \text{ is closed.}
\]
under multiplication. We can now conclude that \( P(X_o) \subseteq U_2(P(X_o)) \) and finally that \( P(X_o) = U_2(P(X_o)) \).

Theorem 3.1: Suppose \( X \) is a compact topological space and suppose \( X_o \subseteq X \). Suppose \( X_o \) is closed under addition, scalar multiplication, and multiplication, and suppose \( X_o \) is closed under uniform passage to the limit. Then \( f \in X_o \) implies \( |f| \in X_o \). Further, \( X_o \) is closed under the lattice operations.

Proof: Since \( X \) is compact, if \( f \in X_o \) then \( f \) is bounded, that is, there exists \( M \) such that \( |f(x)| \leq M \) for \( x \in X \). By Corollary 1.1, there exists a polynomial, \( P(y) \), such that \( p(0) = 0 \) and \( |P(y) - |y|| < \epsilon \) for all \( y \in [-M, M] \). Since \( f(x) \in [-M, M] \), then \( |p(f(x)) - |f(x)|| < \epsilon \) for all \( x \in X \). Since \( p(f(x)) = \sum_{i=1}^{n} c_i (f(x))^i = (\sum_{i=1}^{n} c_i f^i)(x) \), we can write \( |(\sum_{i=1}^{n} c_i f^i)(x) - |f|(x)| < \epsilon \) for all \( x \in X \).

Since \( X_o \) is closed under addition, scalar multiplication, multiplication and uniform passage to the limit, we see that \( \sum_{i=1}^{n} c_i f^i \in X_o \). Therefore \( |f| \) can be uniformly approximated by functions in \( X_o \) and since \( X_o \) is closed under uniform passage to the limit, we conclude that \( |f| \in X_o \).

Further, suppose \( f \in X_o \), \( g \in X_o \), then \( f \cup g = \max(f, g) = \frac{1}{2}(f + g + |f - g|) \) and \( f \cap g = \min(f, g) = \frac{1}{2}(f + g - |f - g|) \), and since \( X_o \) is closed with respect to addition, multiplication, and scalar multiplication along with the absolute value operation, we see that \( f \cup g \in X_o \).
and \( f \cap g \in X_0 \).

**Theorem 3.14:** Suppose \( X \) is a compact topological space and suppose \( X_0 \subseteq X \). Then \( L(X_0) = U(P(X_0)) \).

**Proof:** By Theorem 3.12, \( L(X_0) = U_2(p(X_0)) \subseteq U(p(X_0)) \), so \( L(X_0) \subseteq J(p(X_0)) \). Since \( L(X_0) \) is closed under the lattice operations, by Theorem 3.13, we see that \( U(p(X_0)) \subseteq L(X_0) \). Thus \( L(X_0) = U(p(X_0)) \).

**Theorem 3.15:** Suppose \( X \) is a compact topological space and suppose \( X_0 \subseteq X \). Then \( W(X_0) \subseteq L(X_0) \).

**Proof:** By Theorem 3.9, \( W(X_0) = U(V_1(X_0)) \). Also \( V_1(X_0) = P(X_0) \), so \( U(V_1(X_0)) \subseteq U(P(X_0)) \). Hence by Theorem 3.14, \( W(X_0) = U(V_1(X_0)) \subseteq U(P(X_0)) = L(X_0) \), so \( W(X_0) \subseteq L(X_0) \).

**Theorem 3.16:** Suppose \( X \) is a compact topological space and suppose \( X_0 \subseteq X \). Then \( L(X_0) \) is the smallest collection which contains \( X_0 \) which is closed under addition, scalar multiplication, multiplication, the lattice operations, and the absolute value operation, and which is closed under uniform passage to the limit.

**Proof:** Suppose \( K(X_0) \) is the smallest collection which contains \( X_0 \) and is closed under addition, scalar multiplication, multiplication, the lattice operations, the absolute value operation, and uniform passage to the limit. We know \( L(X_0) \subseteq K(X_0) \). By Theorem 3.13, \( L(X_0) \) is also closed under the lattice operations and the absolute value operation. Thus since \( K(X_0) \) is the smallest collection with these properties containing \( X_0 \), then \( K(X_0) \subseteq L(X_0) \). Thus
$K(X_o) = L(X_o)$.

3.2 Theorem 3.17: Suppose $X$ is a compact topological space, and suppose $X_o \subseteq X$. Suppose $f \in X$. Then

$f \in U(X_o)$ if and only if $\varepsilon > 0$, $x \in X$, $y \in X$, $x \neq y$ implies that there exists $f_{x,y} \in U(X_o)$ such that $|f_{x,y}(x) - f(x)| < \varepsilon$, and $|f_{x,y}(y) - f(y)| < \varepsilon$.

Proof: The necessity of the theorem follows immediately from the definition of $U(X_o)$. For the sufficiency part, given any $x \in X$, for each $y \in X$, there exists $f_{x,y}$ such that $f(x) - f_{x,y} < \varepsilon$. Let $G_y = \{z | f(z) - f_{x,y}(z) < \varepsilon\}$. We have $f(z)$ continuous and $f_{x,y}(z)$ continuous, so $\emptyset(z) = f(z) - f_{x,y}(z)$ is continuous and $G_y$ is an open set. By hypothesis $x$ and $y$ are in $G_y$. Since $x$ is fixed, but $y$ is an arbitrary point, the union of all the sets $G_y$ is the entire space $X$. Since $X$ is compact, there exist points $y_1 \ldots y_n$ such that $\bigcup_{j=1}^{n} G_{y_j} = X$, $i = 1 \ldots n$. Now let

$g_x = f_{x,y_1} \cup f_{x,y_2} \cup \ldots \cup f_{x,y_n}$. If $z \in X$, then $z \in G_{y_k}$ for a suitable choice of $k$ and hence $g_x(z) > f_{x,y_k}(z) > f(z) - \varepsilon$. From the hypothesis, $f_{x,y}(x) < f(x) + \varepsilon$, so $g_x(x) < f(x) + \varepsilon$.

Let $H_x$ be the open set such that

$H_x = \{z | g_x(z) < f(z) + \varepsilon\}$. Since $x \in H_x$ and since $x$ is arbitrary, the union of all the open sets $H_x$ is the entire space $X$. The compactness of $X$ implies that there exists
points \( x_1 \ldots x_m \) such that the union of the sets \( H_{x_1} \ldots H_{x_m} \) is the entire space \( X \). Let \( h = \min(g_{x_1} \ldots g_{x_m}) \), and let \( z \in X \), then \( z \in H_{x_k} \) for a suitable choice of \( k \) and hence \( h(z) \leq g_{x_k}(z) < f(z) + \varepsilon \). Also since \( g_{x}(z) > f(z) - \varepsilon \), then \( h(z) > f(z) - \varepsilon \). Putting the above equations together, we see that \( f(z) - \varepsilon < h(z) < f(z) + \varepsilon \). Now since only the lattice operations have been used in constructing the functions \( g_x \) and \( h \) from the functions \( f_{x,y} \), these functions are all in \( U_1(X_0) \). Since \( h \in U_1(X_0) \), and \( |h(x) - f(x)| < \varepsilon \) for all \( x \in X \), and since \( \varepsilon > 0 \) is arbitrary, it follows that \( f \in U_2(U_1(X_0)) \), i.e., \( f \in U(X_0) \).

**Theorem 3.18:** Suppose \( X \) is a compact topological space having at least two points, and suppose \( X_0 \subseteq X \). Suppose \( x \in X \), \( y \in X \), \( x \neq y \), \( \alpha \in \mathbb{R} \), \( \beta \in \mathbb{R} \) implies that there is \( f \in X_0 \) such that \( f(x) = \alpha \) and \( f(y) = \beta \). Then \( U(X_0) = X \).

**Proof:** Suppose \( f \in X \), \( x \in X \), \( y \in X \), \( \varepsilon > 0 \), and \( x \neq y \). There exists \( g \in X_0 \) such that \( g(x) = f(x) \), \( g(y) = f(y) \). Then \( g \in U_1(X_0) \), \( |g(x) - f(x)| < \varepsilon \), \( |g(y) - f(y)| < \varepsilon \), and thus the conditions of Theorem 3.17 are satisfied. Therefore \( f \in U(X_0) \) and \( X \subseteq U(X_0) \). Since \( U(X_0) \subseteq X \), we have \( U(X_0) = X \).

**Theorem 3.19:** Suppose \( X \) is a compact topological
space, suppose \( f_n \in X, n = 1, 2 \ldots \), suppose \( f_n \leq f_{n+1} \) on \( X, n = 1, 2, \ldots \), \( f \in X \), and suppose \( p \in X \) implies \( \{f_n(p)\} \) converges to \( f(p) \). Then \( f_n \xrightarrow{U} f \) on \( X \).

Proof: Suppose \( X_o \) is the collection of functions \( f_n, n = 1, 2, \ldots \). Since \( f_n \) is a monotone sequence, then \( f_n \cup f_{n+1} = f_{n+1}, f_n \cap f_{n+1} = f_n \), and \( U_1(X_o) = X_o \). The assumption that \( \lim_{n \to \infty} f_n(p) = f(p) \) for every \( p \) shows that the conditions for Theorem 3.17 are satisfied. Hence \( f \) is in \( U(X_o) \). If \( \varepsilon > 0 \), there exists \( g \in U_1(X_o) \), such that \( |g(x) - f(x)| < \varepsilon \) for all \( x \in X \). But \( U_1(X_o) = X_o \), so \( g \in X_o \) and there exists \( N \) such that \( g = f_N \). Therefore there exists \( N \) such that \( |f(x) - f_N(x)| < \varepsilon \) for all \( x \in X \).

Since \( |f(x) - f_N(x)| \) decreases as \( n \) increases, it follows that \( n \geq N \) implies \( |f(x) - f_n(x)| < \varepsilon \) for all \( x \in X \).

3.3 Definition 3.17: Suppose \( X \) is a topological space, suppose \( x \in X, y \in X, x \neq y \), and suppose \( X_o \subset X \). Then \( X_o(x, y) = [\emptyset \mid \emptyset : \{x, y\}] \rightarrow \mathbb{R} \); there exists \( f \in X_o \) such that \( f(x) = \emptyset(x), f(y) = \emptyset(y) \). Note that \( X_o(x, y) \subset X(x, y) \subset Y \), where \( Y \) denotes the collection of all continuous real-valued functions on \( \{x, y\} \).

Theorem 3.20: Suppose \( X \) is a topological space, suppose \( x \in X, y \in X, x \neq y \), and suppose \( X_o \subset X \). Then \( A(X_o)(x, y) = A(X_o(x, y)), B(X_o)(x, y) = B(X_o(x, y)), U_1(X_o)(x, y) = U_1(X_o(x, y)), V_1(X_o)(x, y) = V_1(X_o(x, y)), V(X_o)(x, y) = V(X_o(x, y)), P(X_o)(x, y) = P(X_o(x, y)) \),
Proof: We first show that $A(\bar{x}_o)(x, y) = A(\bar{x}_o(x, y))$. We know that $\bar{x}_o \subseteq A(\bar{x}_o) \subseteq \bar{x}$, so $\bar{x}_o(x, y) \subseteq A(\bar{x}_o)(x, y)$. We now want to show that $A(\bar{x}_o)(x, y)$ is closed under the maximum operation. Suppose $\emptyset \in A(\bar{x}_o)(x, y)$ and $\psi \in A(\bar{x}_o)(x, y)$. Then there exists $f \in A(\bar{x}_o)$ such that $f(x) = \emptyset(x)$, $f(y) = \emptyset(y)$ and there exists $g \in A(\bar{x}_o)$ such that $f(x) = \psi(x)$, $g(y) = \psi(y)$. Since $A(\bar{x}_o)$ is closed under the max operation, then $(f \cup g) \in A(\bar{x}_o)$. Thus $(f \cup g)(x) = \max f(x), g(x) = \max \emptyset(x), \psi(x) = (\emptyset \cup \psi)(x)$, and $(f \cup g)(y) = (\emptyset \cup \psi)(y)$. Therefore $\emptyset \cup \psi \in A(\bar{x}_o)(x, y)$. This implies $A(\bar{x}_o)(x, y) \subseteq A(\bar{x}_o)(x, y)$. We now show that $A(\bar{x}_o)(x, y) \subseteq A(\bar{x}_o)(x, y)$. Suppose $\emptyset \in A(\bar{x}_o)(x, y)$, then there exists $f \in A(\bar{x}_o)$ such that $f(x) = \emptyset(x)$, $f(y) = \emptyset(y)$. Since $f \in A(\bar{x}_o)$, there exists $f_i \in \bar{x}_o$, $i = 1 \ldots n$, such that $f = \bigcup_{i=1}^n f_i$. Let $\emptyset_i : [x, y] \rightarrow \mathbb{R}$, $i = 1 \ldots n$, be defined by $\emptyset_i(x) = f_i(x)$ and $\emptyset_i(y) = f_i(y)$. Thus $\emptyset_i \in A(\bar{x}_o(x, y))$, $i = 1 \ldots n$, and $\emptyset(x) = f(x) = (\bigcup_{i=1}^n \emptyset_i)(x) = (\bigcup_{i=1}^n \emptyset_i)(x)$ and $\emptyset(y) = (\bigcup_{i=1}^n \emptyset_i)(y)$. Therefore since $\emptyset_i \in A(\bar{x}_o(x, y)) \subseteq A(\bar{x}_o(x, y))$ and $\bigcup_{i=1}^n \emptyset_i \in A(\bar{x}_o(x, y))$, we see that $\emptyset \in A(\bar{x}_o(x, y))$. Thus $A(\bar{x}_o)(x, y) = A(\bar{x}_o(x, y))$, and finally $A(\bar{x}_o)(x, y) = A(\bar{x}_o(x, y))$.

By similar reasoning $B(\bar{x}_o)(x, y) = B(\bar{x}_o(x, y))$.

To show that $U_1(\bar{x}_o)(x, y) = U_1(\bar{x}_o(x, y))$, we make use of Theorem 3.2. Thus $U_1(\bar{x}_o)(x, y) = A(B(\bar{x}_o))(x, y)$.
= A(B(X;))(x, y) = A(B(X;))(x, y) = U(x;)(x, y).

We now show that V(X;)(x, y) = V(X;)(x, y). We
know that X; C V(X;), so X; (x, y) C V(X;)(x, y). We
now want to show that V(X;)(x, y) is closed under addi-
tion and scalar multiplication. Let f C V(X;)(x, y)
and g C V(X;)(x, y), then there exists f C V(X;)
such that f(x) = f(x) and f(y) = f(y), and there exists
g C V(X;)(x, y) such that g(x) = g(x), g(y) = g(y). Thus
(f + g)(x) = (f + g)(x), (f + g)(y) = (f + g)(y), and
f + g C V(X;)(x, y). This implies f + g: [x, y] ! R and
f + g C V(X;)(x, y). Similarly cf C V(X;)(x, y) and so
V(X;)(x, y) C V(X;)(x, y). Now suppose f C V(X;)(x, y).
Then there exists f C V(X;)(x, y) such that f(x) = f(x),
f(y) = f(y). Since f C V(X;)(x, y), there exists c_i C R,
f_i : X; , i = 1, . . . , n, such that f = i=1 c_i f_i, i = 1, . . . , n.
Let f_i: [x, y] ! R, i = 1, . . . , n, be defined by f_i(x)
= f_i(x), f_i(y) = f_i(y). Thus f_i C X; (x, y), so
f_i C V(X;)(x, y), i = 1, . . . , n, and n i=1 c_i f_i C V(X;)(x, y).
Also f(x) = f(x) = i=1 c_i f_i(x) = i=1 c_i f_i(x)
= ( i=1 c_i f_i(x) ) and f(y) = ( i=1 c_i f_i(y) ). Therefore
f = i=1 c_i f_i, so f C V(X;)(x, y), which says that
V(X;)(x, y) C V(X;)(x, y). Since V(X;)(x, y) C V(X;)(x, y),
we have V(X;)(x, y) = V(X;)(x, y).

The rest of the proofs are somewhat similar. We
present one more for illustration, namely, \( U_2(\mathbb{X}_o)(x, y) \) \( \subset U_2(\mathbb{X}_o(x, y)) \). Suppose \( \emptyset \in U_2(\mathbb{X}_o)(x, y) \), then there exist \( f \in U_2(\mathbb{X}_o) \) such that \( f(x) = \emptyset(x) \), \( f(y) = \emptyset(y) \). Since \( f \in U_2(\mathbb{X}_o) \), there exists \( f_n \in \mathbb{X}_o \) such that \( f_n \rightarrow f \) on \( X \).

Let \( \emptyset_n: \{x, y\} \rightarrow \mathbb{R} \), \( n = 1, 2, \ldots \), be defined by \( \emptyset_n(x) = f_n(x) \), \( \emptyset_n(y) = f_n(y) \). Thus \( \emptyset_n(x) \rightarrow \emptyset(x) \), \( \emptyset_n(y) \rightarrow \emptyset(y) \) and \( \emptyset_n \in \mathbb{X}_o(x, y) \). This implies \( \emptyset \in U_2(\mathbb{X}_o(x, y)) \) so \( U_2(\mathbb{X}_o)(x, y) \subset U_2(\mathbb{X}_o(x, y)) \).

We now show by an example that it may happen that \( U_2(\mathbb{X}_o(x, y)) \notin U_2(\mathbb{X}_o)(x, y) \). Let \( X = [0, 1] \) and define

\[
f_n(x) = \begin{cases} 
\frac{(2n^2+2)x - 1}{n} & \text{if } 0 \leq x \leq \frac{1}{2} \\
\frac{(2n^2+1) - (2n^2+2)x}{n} & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Suppose \( m > n \), then \(-\frac{1}{m} > -\frac{1}{n}\) so \( f_n(x) \leq f_m(x) \) on \([0, 1]\) for \( n < m \). Let \( \mathbb{X}_o = \{f_1, f_2, \ldots, f_n, \ldots\} \). Since \( f_n \cup f_m = f_m \), and \( f_m \cap f_n = f_n \), we see that \( \mathbb{X}_o \) is closed under \( \cap \) and \( \cup \), so \( U_1(\mathbb{X}_o) = \mathbb{X}_o \). Hence \( U(\mathbb{X}_o) = U_2(U_1(\mathbb{X}_o)) = U_2(\mathbb{X}_o) \).

Suppose \( f \in U_2(\mathbb{X}_o) \) and \( \varepsilon = \frac{1}{4} \), then there exists \( g \in \mathbb{X}_o \) such that \( |g(x) - f(x)| < \frac{1}{4} \) for all \( x \in [0, 1] \). Let \( g = f_N \), then \( |f_N(x) - f(x)| < \frac{1}{4} \) for all \( x \in [0, 1] \).

Suppose there exists \( x_o \) such that \( f_N(x_o) \neq f(x_o) \). Let \( \varepsilon = |f_N(x_o) - f(x_o)| \), then for \( \varepsilon > 0 \), there exists \( h \in \mathbb{X}_o \) such that \( |h(x) - f(x)| < \varepsilon \) for all \( x \in [0, 1] \).

Let \( h = f_M \), then \( |f_M(x) - f(x)| < \varepsilon \) for all \( x \in [0, 1] \).

Also \( |f_M(x_o) - f(x_o)| < \varepsilon \), therefore \( M \neq N \). Now look at
\[ |f_M(\frac{1}{2}) - f(\frac{1}{2})| < \varepsilon < \frac{1}{4}. \]

We see that \[ |M - f(\frac{1}{2})| = |f_M(\frac{1}{2}) - f(\frac{1}{2})| < \frac{1}{4}, \]
and \[ |N - f(\frac{1}{2})| = |f_N(\frac{1}{2}) - f(\frac{1}{2})| < \frac{1}{4} \]
and thus \[ |M - N| < \frac{1}{2}, \]
but this can't happen since \( M \) and \( N \) are integers. Hence \( f_N = f \), but \( f_N \in X_0 \) so \( f \in X_0 \) and \( X_0 \) is closed under uniform limit operations and therefore \( U(X_0) = X_0 \).

Now consider \( X_0(0, 1) \). We see that \( X_0(0, 1) = U(X_0)(0, 1) \).

The elements of \( X_0(0, 1) \) look like \( \phi_1, \phi_2, \ldots, \phi_n \ldots \) where \( \phi_n(0) = -\frac{1}{n}, \phi_n(1) = \frac{1}{n} \).

Then \( U(X_0(0, 1)) = U_2(U_1(X_0)(0, 1)) = U_2(X_0(0, 1)) \). Thus there exists \( \psi: [0, 1] \rightarrow \mathbb{R} \) such that \( \psi(0) = 0 \) and \( \psi(1) = 0 \). Hence \( \phi_n \xrightarrow{U} \psi \) on \( [0, 1] \). This implies \( \psi \in U_2(X_0(0, 1)) \) and hence \( \psi \in U(X_0(0, 1)) \) but

\[ \psi \notin U(X_0)(0, 1) \]
which says \( U_2(X_0(0, 1)) \neq U_2(X_0)(0, 1) \).

3.4 Definition 3.18: We define operations \( \land \) and \( \lor \) in \( \mathbb{R}_2 \) as follows: Suppose \((a, b) \in \mathbb{R}_2, (c, d) \in \mathbb{R}_2\). Then

\((a, b) \land (c, d) = (\min\{a, c\}, \min\{b, d\})\), and

\((a, b) \lor (c, d) = (\max\{a, c\}, \max\{b, d\})\).

Note that \( \land \) and \( \lor \) are associative and commutative operations in \( \mathbb{R}_2 \).

\( \land \) is distributive with respect to \( \lor \), and \( \lor \) is distributive with respect to \( \land \).

If \((a_i, b_i) \in \mathbb{R}_2\), for \( i = 1 \ldots n \), we write \( \bigwedge_{i=1}^n (a_i, b_i) \) for \((a_1, b_1) \land (a_2, b_2) \land \ldots \land (a_n, b_n)\), and \( \bigvee_{i=1}^n (a_i, b_i) \) for \((a_1, b_1) \lor (a_2, b_2) \lor \ldots \lor (a_n, b_n)\). Note that

\[ \bigwedge_{i=1}^n (a_i, b_i) = (\min_{i=1\ldots n} a_i, \min_{i=1\ldots n} b_i) \]
and \( \bigvee_{i=1}^n (a_i, b_i) = (\max_{i=1\ldots n} a_i, \max_{i=1\ldots n} b_i) \).
Definition 3.19: Suppose \( S \subseteq \mathbb{R}_2 \). We say that \( S \) is closed with respect to \( \sqrt{\cdot} \) if \((a, b) \in S, (c, d) \in S\) implies \((a, b) \sqrt{\cdot} (c, d) \in S\).

Definition 3.20: Suppose \( S \subseteq \mathbb{R}_2 \). We say that \( S \) is closed with respect to \( \land \) if \((a, b) \in S, (c, d) \in S\) implies \((a, b) \land (c, d) \in S\).

Definition 3.21: Suppose \( S \subseteq \mathbb{R}_2 \). We define \( \hat{S} \) to be the smallest set which contains \( S \) and which is closed with respect to \( \land \).

Theorem 3.21: Suppose \( S \subseteq \mathbb{R}_2 \). Then \( \hat{S} = \{(a, b) | (a, b) \in \mathbb{R}_2; \text{there exists } (a_i, b_i) \in S \text{ for } i = 1 \ldots n \text{ such that } (a, b) = \bigwedge_{i=1}^{n} (a_i, b_i)\} \).

Proof: Let \( Y = \{(a, b) | (a, b) \in \mathbb{R}_2; \text{there exists } (a_i, b_i) \in S, i = 1 \ldots n, \text{ such that } (a, b) = \bigwedge_{i=1}^{n} (a_i, b_i)\} \).

Suppose \((a, b) \in Y\), then \((a, b) \in \mathbb{R}_2\), and there exists \((a_i, b_i) \in S, i = 1 \ldots n, \text{ such that } (a, b) = \bigwedge_{i=1}^{n} (a_i, b_i)\).

Since \( S \subseteq \hat{S} \) and since \( \hat{S} \) is closed with respect to \( \land \), then \((a, b) \in \hat{S} \) and \( Y \subseteq \hat{S} \).

Now suppose \((a, b) \in S \) and \( a_1 = a = a_2, b_1 = b = b_2 \).

Then \((a, b) = (a_1, b_1) \land (a_2, b_2)\) which implies that \((a, b) \in Y\). Since \((a_1, b_1) \in S \) and \((a_2, b_2) \in S\), we have \( S \subseteq Y \). Now suppose \((a, b) \in Y, (c, d) \in Y\), then there exists \((a_i, b_i) \in S, i = 1 \ldots n, \text{ such that } (c, d) = \bigwedge_{j=1}^{m} (c_j, d_j)\). Thus \((a, b) \land (c, d) = (\bigwedge_{i=1}^{n} (a_i, b_i)) \land (\bigwedge_{j=1}^{m} (c_j, d_j)) \in Y\). Therefore \( Y \) is closed with respect.
to $\land$ and so $\hat{S} \subseteq Y$. Also since $Y \subseteq \hat{S}$, we have $\hat{S} = Y$.

**Definition 3.22:** Suppose $S \subseteq \mathbb{R}^2$. We say that $S$ is closed with respect to multiplication if $(a, b) \in S$, $(c, d) \in S$ implies $(ac, bd) \in S$. We say that $S$ is closed with respect to addition if $(a, b) \in S$, $(c, d) \in S$ implies $(a + c, b + d) \in S$. We say that $S$ is closed with respect to scalar multiplication if $(a, b) \in S$, $c \in \mathbb{R}$ implies that $(ca, cb) \in S$.

**Definition 3.23:** A space of functions, $T$, is called a linear space if it is closed under addition and scalar multiplication.

**Definition 3.24:** Suppose $S \subseteq \mathbb{R}^2$. We define $\bar{S}$ to be the smallest set which contains $S$ and which is closed with respect to $\lor$.

**Definition 3.25:** Suppose $S \subseteq \mathbb{R}^2$. We define $S^\sim$ to be the smallest set which contains $S$ and which is closed with respect to $\land$ and $\lor$.

**Theorem 3.22:** Suppose $S \subseteq \mathbb{R}^2$. Then $\bar{S} = \{(a, b) | (a, b) \in \mathbb{R}^2; \text{ there exists } (a_i, b_i) \in S \text{ for } i = 1 \ldots n \text{ such that } (a, b) = \sum_{i=1}^{n} (a_i, b_i)\}$.

Proof: The proof is similar to the proof of Theorem 3.20.

**Theorem 3.23:** Suppose $S \subseteq \mathbb{R}^2$. Then $S^\sim = \bar{S}$ and $S^\sim = \hat{S}$. Also $S^\sim = \{(a, b) | (a, b) \in \mathbb{R}^2; \text{ there exists } (a_i, j, b_i, j) \in S \text{ for } i = 1 \ldots n; j = 1 \ldots k_i \text{ such that }$
\[(a, b) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} (a_i, b_i, j)]\text{ and } S = \{(a, b) | (a, b) \in R_2; \text{ there exists } (a_i, j, b_i, j) \in S \text{ for } i = 1 \ldots n; j = 1 \ldots k_i \text{ such that } (a, b) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} (a_i, j, b_i, j)\}. \]

**Proof:** Let \( S = \{(a, b) | (a, b) \in R_2; \text{ there exists } (a_i, j, b_i, j) \in S \text{ for } i = 1 \ldots n; j = 1 \ldots k_i \text{ such that } (a, b) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} (a_i, j, b_i, j)\}. \)

Suppose \((a, b) \in Y\), then \((a, b) \in R_2 \text{ and there exists } (a_i, j, b_i, j) \in S \text{ such that } (a, b) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} (a_i, j, b_i, j), i = 1 \ldots n; j = 1 \ldots k_i. \)

Since \( S = S_\sim\) and since \( S_\sim\) is closed with respect to \( \land \) and \( \lor \), then \((a, b) \in S_\sim\) and \( Y \subseteq S_\sim\). Now suppose \((a, b) \in Y, (c, d) \in Y\), then there exists \((a_i, j, b_i, j) \in S, i = 1 \ldots n; j = 1 \ldots k_i, \text{ such that } (a, b) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{k_i} (a_i, j, b_i, j)\) and there exists \((c_q, r, d_q, r) \in S, q = 1 \ldots m; r = 1 \ldots p_q\), such that \((c, d) = \bigoplus_{q=1}^{m} \bigoplus_{r=1}^{p_q} (cqr, dqr)\). Thus \((a, b) = [(a_{11}, b_{11}) \lor \ldots \lor (a_{12}, b_{12}) \lor \ldots \lor (a_{1k_1}, b_{1k_1})] \land \ldots \land [(a_{n1}, b_{n1}) \lor \ldots \lor (a_{n2}, b_{n2}) \lor \ldots \lor (a_{nk_n}, b_{nk_n})]. (c, d)\)
could also be written out in this form. Hence \((a, b) \land (c, d) \in Y. \) Now let \((a_i, b_i) = [(a_{i1}, b_{i1}) \lor \ldots \lor (a_{i2}, b_{i2}) \lor \ldots \lor (a_{ik_i}, b_{ik_i})]\) and \((c_q, d_q) = [(c_{q1}, d_{q1}) \lor \ldots \lor (c_{qp_q}, d_{qp_q})]\) then \((a, b) = (a_1, b_1) \land (a_2, b_2) \land \ldots \land (a_n, b_n)\) and \((c, d) = (c_1, d_1) \land \ldots \land (c_m, d_m)\).
Then \((a, b) \bigvee (c, d) = ((a_1, b_1) \bigvee (c_1, d_1)) \bigwedge ((a_2, b_2) \bigvee (c_2, d_2)) \bigwedge \cdots \bigwedge ((a_n, b_n) \bigvee (c_n, d_n)) \bigwedge \cdots ((a_m, b_m) \bigvee (c_m, d_m)) \bigwedge \cdots \bigwedge ((a_1, b_1) \bigwedge (c_1, d_1)) \bigwedge ((a_n, b_n) \bigwedge (c_2, d_2)) \bigwedge \cdots \bigwedge ((a_n, b_n) \bigwedge (c_m, d_m)) \in Y.

Also \((a_1, b_1) \bigvee (c_q, d_q) = ((a_1, l, b_1, l) \bigvee \cdots \bigvee (a_1, k_i, b_i, k_i)) \bigvee ((c_q, l, d_q, l) \bigvee \cdots \bigvee (c_q, p_q, d_q, p_q)).

Thus \((a, b) \bigvee (c, d) \in Y\) and \((a, b) \bigwedge (c, d) \in Y\), and so \(S^\sim \subseteq Y\). Since \(Y \subseteq S^\sim\), we have \(Y = S^\sim\).

A similar argument can be used to show that
\[S^\sim = \{(a, b) \mid (a, b) \in R^2; \text{there exists } (a_i, j, b_i, j) \in S \text{ for } i = 1 \ldots n; j = 1 \ldots k_i, \text{ such that }\]
\[\mathbf{a, b} = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{k_i} (a_i, j, b_i, j).\]

**Definition 3.26**: Suppose \(S \subseteq R^2\). Then \(S\) is said to be topologically closed if \(\{p_n\} \in S, p \in R^2, p_n \rightarrow p \) implies \(p \in S\).

**Definition 3.27**: Suppose \(S \subseteq R^2, p \in R^2\). Then \(p \in \overline{S}\) if and only if there exist \(\{p_n\} \in S\) such that \(p_n \rightarrow p\). \(\overline{S}\) is called the closure of \(S\).

**Theorem 3.24**: Suppose \(S \subseteq R^2\). Then \(\overline{S}\), the closure of \(S\), is the smallest set which contains \(S\) and which is topologically closed.

**Proof**: Suppose \(p_n \in \overline{S}, p_n \in R^2, \) and \(p_n \rightarrow p\).

Since \(p_n \rightarrow p\), then \(d(p_n, p) \rightarrow 0\). Hence there exists \(q_{n,m} \in S\) such that \(q_{n,m} \rightarrow p_n\) or \(d(q_{n,m}, p_n) \rightarrow 0\). Therefore there exists \(k_n\) such that \(d(q_{n,k_n}, p_n) < \frac{1}{n}\). Let
\[ r_n = q_n, k_n, \text{ then } r_n \in S \text{ and } d(r_n, p_n) < \frac{1}{n} \text{ and } r_n \to p. \] But \( r_n \in S, \) so \( p \in S \) and \( S \) is closed. Also, if \( q \in S, \) let \( q_n = q \) for each \( n. \) Then \( q_n \to q, \) so \( q \in S \) and \( S = S. \)

Now suppose \( S \subseteq T \) and \( T \) is a closed set. Let \( p \in S, \) then there exists \( p_n \in S \) such that \( p_n \to p. \) Therefore \( p_n \in T, p_n \to p, \) and since \( T \) is closed \( p \in T. \) Therefore \( S \subseteq T \) and we are finished.

**Definition 3.28:** Suppose \( S \subseteq R_2. \) We define \( S^* \) to be the smallest set which contains \( S, \) which is closed with respect to \( \wedge \) and \( \vee, \) and which is topologically closed.

**Theorem 3.25:** Suppose \( S \subseteq R_2. \) Then \( S^* = S^{\circ}. \)

**Proof:** We know that \( S \subseteq S^* \) and \( S^* \) is closed under \( \vee \) and \( \wedge, \) so \( S^{\circ} \subseteq S^*. \) Since \( S^{\circ} \subseteq S^* \) and since \( S^* \) is topologically closed, we have \( S^{\circ} \subseteq S^{\circ} = S^*. \) We now want to show that \( S^* \subseteq S^{\circ}. \) Since \( S \subseteq S^{\circ} \subseteq S^{\circ} \) and since \( S^{\circ} \) is topologically closed, we must show that \( S^{\circ} \) is closed under \( \wedge \) and \( \vee \). Suppose \( (a, b) \in S^{\circ} \) and \( (c, d) \in S^{\circ}, \) then there exists \( \{(a_n, b_n)\} \) such that \( (a_n, b_n) \in S^{\circ} \) and \( (a_n, b_n) \to (a, b) \) and there exists \( (c_n, d_n) \in S^{\circ} \) such that \( (c_n, d_n) \to (c, d). \) Therefore \( a_n \to a, b_n \to b, c_n \to c, d_n \to d. \) We have shown that \( \max(a_n, c_n) \to \max(a, c) \) and \( \max(b_n, d_n) \to \max(b, d). \) Also \( (a_n, b_n) \vee (c_n, d_n) \in S^{\circ} \) and \( (a_n, b_n) \vee (c_n, d_n) = (\max(a_n, c_n, \max(b_n, d_n)) \) and \( (a, b) \vee (c, d) \) \( \max(a, c, \max(b, d). \) Hence \( (a_n, b_n) \vee (c_n, d_n) \to \)
(a, b) ∨ (c, d). Similarly (a_n, b_n) ∧ (c_n, d_n) → (a, b) ∧ (c, d) and (a_n, b_n) ∨ (c_n, d_n) ∈ S^- . This implies (a, b) ∨ (c, d) ∈ S^- and (a, b) ∧ (c, d) ∈ S^- . Hence S^* ⊆ S^- .

Also since S^- = S^*, we see that S^- = S^*.

**Definition 3.29:** Suppose X is a topological space, suppose x ∈ X, y ∈ X, x ≠ y. Suppose Y is the set of all real valued functions on {x, y} and suppose Z ⊆ Y. Let $Z^\Delta = \{(a, b) | (a, b) ∈ \mathbb{R}^2; \text{there exists } \emptyset ∈ Z \text{ such that } \emptyset(x) = a, \emptyset(y) = b\}$. Note that if $Z ⊆ W ⊆ Y$, then $Z^\Delta ⊆ W^\Delta$.

Note also that $(\mathcal{X}_0(x, y))^\Delta = \{(a, b) | (a, b) ∈ \mathbb{R}^2; \text{there exists } \emptyset ∈ \mathcal{X}_0(x, y) \text{ such that } \emptyset(x) = a, \emptyset(y) = b\}$.

**Theorem 3.26:** Suppose X is a topological space, $\mathcal{X}_0 ⊆ \mathcal{X}$, x ∈ X, y ∈ X, x ≠ y. Then $(U_1(\mathcal{X}_0(x, y)))^\Delta$

= \((\mathcal{X}_0(x, y))^\Delta)^\sim, (U_2(\mathcal{X}_0(x, y)))^\Delta = (\mathcal{X}_0(x, y))^\Delta, (U(\mathcal{X}_0(x, y)))^\Delta$

= \((\mathcal{X}_0(x, y))^\Delta)^* .

**Proof:** First we show that $(U_1(\mathcal{X}_0(x, y)))^\Delta = (\mathcal{X}_0(x, y))^\Delta)^\sim$.

We know $\mathcal{X}_0(x, y) ⊆ U_1(\mathcal{X}_0(x, y))$, so $(\mathcal{X}_0(x, y))^\Delta ⊆ (U_1(\mathcal{X}_0(x, y)))^\Delta$.

We now want to show that $(U_1(\mathcal{X}_0(x, y)))^\Delta$ is closed under $\wedge$ and $\vee$. Suppose $(a, b) ∈ (U_1(\mathcal{X}_0(x, y)))^\Delta$ and $(c, d) ∈ (U_1(\mathcal{X}_0(x, y)))^\Delta$, then there exists $\emptyset ∈ U_1(\mathcal{X}_0(x, y))$ such that $\emptyset(x) = a, \emptyset(y) = b$ and there exists $\psi ∈ U_1(\mathcal{X}_0(x, y))$ such that $\psi(x) = c, \psi(y) = d$. Also $\emptyset ∪ \psi ∈ U_1(\mathcal{X}_0(x, y))$, $(\emptyset ∪ \psi)(x) = \max \emptyset(x), \psi(x) = \max a, c, (\emptyset ∪ \psi)(y)$

= \max \emptyset(y), \psi(y) = \max b, d. Since $(a, b) \vee (c, d) = (\max a, c, \max b, d)$, this implies $(a, b) \vee (c, d) ∈$
\((U_1(\mathcal{X}_o(x, y)))^\Delta\). Similarly \((a, b) \land (c, d) \in (U_1(\mathcal{X}_o(x, y)))^\Delta\).

Therefore \(((\mathcal{X}_o(x, y))^\Delta) \subseteq (U_1(\mathcal{X}_o(x, y)))^\Delta\). Now we want to show that \((U_1(\mathcal{X}_o(x, y)))^\Delta \subseteq ((\mathcal{X}_o(x, y))^\Delta)^\sim\). Suppose \((a, b) \in (U_1(\mathcal{X}_o(x, y)))^\Delta\), then there exists \(\emptyset \in U_1(\mathcal{X}_o(x, y))\) such that \(\emptyset(x) = a, \emptyset(y) = b\). Since \(\emptyset \in U_1(\mathcal{X}_o(x, y))\), there exists \(\emptyset_{i,j} \in \mathcal{X}_o(x, y), i = 1 \ldots n; j = 1 \ldots k_i\), such that

\[\emptyset_{i,j} = \bigcup_{i=1}^{n} j \in \mathcal{X}_o(x, y), i = 1 \ldots n; j = 1 \ldots k_i\]

Define \(a_{i,j}, b_{i,j}\) by \(a_{i,j} = \emptyset_{i,j}(x), b_{i,j} = \emptyset_{i,j}(y)\), then \((a_{i,j}, b_{i,j}) \in (\mathcal{X}_o(x, y))^\Delta \subseteq ((\mathcal{X}_o(x, y))^\Delta)^\sim\). Let

\[a_{i,j} = \bigcup_{j=1}^{k_i} (a_{i,j}, b_{i,j}), \text{ then } a = \min(a_1 \ldots a_n),\]

\[b = \min(b_1 \ldots b_n). \text{ Since } (a_{i,j}, b_{i,j}) \in ((\mathcal{X}_o(x, y))^\Delta)^\sim, \text{ then } (a, b) = (\min a_1 \ldots a_n, \min b_1 \ldots b_n) = \bigcup_{i=1}^{n} (a_i, b_i) \text{ and } (a, b) \in (\mathcal{X}_o(x, y))^\Delta)^\sim. \text{ Therefore } (U_1(\mathcal{X}_o(x, y)))^\Delta \subseteq ((\mathcal{X}_o(x, y))^\Delta)^\sim \text{ and finally } (U_1(\mathcal{X}_o(x, y)))^\Delta = ((\mathcal{X}_o(x, y))^\Delta)^\sim.

Next we show that \((U_2(\mathcal{X}_o(x, y)))^\Delta = (\mathcal{X}_o(x, y))^\Delta\).

We know that \(\mathcal{X}_o(x, y) \subseteq U_2(\mathcal{X}_o(x, y))\) and so \((\mathcal{X}_o(x, y))^\Delta \subseteq (U_2(\mathcal{X}_o(x, y)))^\Delta\). We now want to show that \((U_2(\mathcal{X}_o(x, y)))^\Delta\) is closed. Suppose \((a, b) \in (U_2(\mathcal{X}_o(x, y)))^\Delta\), then there exists \((a_n, b_n) \in (U_2(\mathcal{X}_o(x, y)))^\Delta\) such that \((a_n, b_n) \to (a, b)\).
Then \( a_n \rightarrow a, b_n \rightarrow b \). Since \( (a_n, b_n) \in (U_2(x_0(x, y)))^\Delta \), there exists \( \varnothing_n \in U_2(x_0(x, y)) \) such that \( \varnothing_n(x) = a_n \), \( \varnothing_n(y) = b_n \). Now let \( \varnothing : \{x, y\} \rightarrow \mathbb{R} \) be defined by \( \varnothing(x) = a, \varnothing(y) = b \). Then \( \varnothing_n(x) \rightarrow \varnothing(x) \) and \( \varnothing_n(y) \rightarrow \varnothing(y) \) so \( \varnothing_n \uparrow \varnothing \) on \( \{x, y\} \) and \( \varnothing \in U_2(x_0(x, y)) \). Since \( \varnothing(x) = a \) and \( \varnothing(x) = b \), then \( (a, b) \in (U_2(x_0(x, y)))^\Delta \) and \( (U_2(x_0(x, y)))^\Delta \) is closed.

Therefore \( (x_0(x, y))^\Delta \subset (U_2(x_0(x, y)))^\Delta \). Now we wish to show that \( (U_2(x_0(x, y)))^\Delta \subset (x_0(x, y))^\Delta \). Suppose \( (a, b) \in (U_2(x_0(x, y)))^\Delta \), then there exists \( \varnothing \in U_2(x_0(x, y)) \) such that \( \varnothing(x) = a, \varnothing(y) = b \). Since \( \varnothing \in U_2(x_0(x, y)) \), there exists \( \varnothing_n \in x_0(x, y) \) such that \( \varnothing_n \uparrow \varnothing \) on \( \{x, y\} \), so \( \varnothing_n(x) \rightarrow \varnothing(x) \) and \( \varnothing_n(y) \rightarrow \varnothing(y) \). Let \( \varnothing_n(x) = a_n \), and \( \varnothing_n(y) = b_n \), then \( (a_n, b_n) \in (x_0(x, y))^\Delta \) and since

\( a_n \rightarrow a, b_n \rightarrow b \) then \( (a_n, b_n) \rightarrow (a, b) \) and \( (a, b) \in (x_0(x, y))^\Delta \). Therefore \( (U_2(x_0(x, y)))^\Delta \subset (x_0(x, y))^\Delta \).

Also since \( (x_0(x, y))^\Delta \subset (U_2(x_0(x, y)))^\Delta \), we have the desired result that \( (x_0(x, y))^\Delta = (U_2(x_0(x, y)))^\Delta \).

We now show that \( (U(x_0(x, y)))^\Delta = ((x_0(x, y))^\Delta)^* \).

To show this we make use of previous theorems and the above two results. We observe that \( (U(x_0(x, y)))^\Delta = (U_2(U_1(x_0(x, y)))^\Delta = (U_2(U_1(x_0(x, y)))^\Delta = (U_1(x_0(x, y)))^\Delta = (x_0(x, y))^\Delta = ((x_0(x, y))^\Delta)^* \).

Theorem 3.27: Suppose \( x \in X, y \in X, x \neq y \). Then
$$(V_1(\mathcal{H}_o(x, y)))^\Delta$$ is the smallest linear space which contains $(\mathcal{H}_o(x, y))^\Delta$ and which is contained in $R_2$.

**Proof:** Since $\mathcal{H}_o(x, y) \subset V_1(\mathcal{H}_o(x, y))$, then $(\mathcal{H}_o(x, y))^\Delta \subset (V_1(\mathcal{H}_o(x, y)))^\Delta$. Now suppose $T$ is a linear space which contains $(\mathcal{H}_o(x, y))^\Delta$ and which is contained in $R_2$. Let $(a, b) \in (V_1(\mathcal{H}_o(x, y)))^\Delta$, then there exists $\phi \in V_1(\mathcal{H}_o(x, y))$ such that $\phi(x) = a$ and $\phi(y) = b$. Since $\phi \in V(\mathcal{H}_o(x, y))$ there exists $c_i \in R, \phi_i \in \mathcal{H}_o(x, y), i = 1 \ldots n$, such that $\phi = \sum c_i \phi_i$.

Now let $a_i = \phi_i(x), b_i = \phi_i(y), i = 1 \ldots n$. Then $(a_i, b_i) \in (\mathcal{H}_o(x, y))^\Delta \subset T, i = 1 \ldots n$. Therefore $\sum_i c_i(a_i, b_i) \in T$ or $(\sum_i c_i a_i, \sum_i c_i b_i) \in T$. Also $\sum_i c_i a_i = \sum_i c_i \phi_i(x)$

$$= (\sum_i c_i \phi_i)(x) = \phi(x) = a$$

likewise $\sum_i c_i b_i = b$ which implies that $(a, b) \in T$. Therefore $(V_1(\mathcal{H}_o(x, y)))^\Delta$ is contained in $T$. Now if we can show $(V_1(\mathcal{H}_o(x, y)))^\Delta$ is a linear space, then we are finished. Suppose $(a, b) \in (V_1(\mathcal{H}_o(x, y)))^\Delta$ and $c \in R$, then there exists $\phi \in V_1(\mathcal{H}_o(x, y))$ such that $\phi(x) = a, \phi(y) = b$. Since $\phi \in V_1(\mathcal{H}_o(x, y))$, then $c \phi \in V_1(\mathcal{H}_o(x, y)), (c\phi)(x) = c \phi(x) = ca, (c\phi)(y) = cb$.

Hence $(a, b) = (ca, cb) \in (V_1(\mathcal{H}_o(x, y)))^\Delta$. Now suppose $(d, e) \in (V_1(\mathcal{H}_o(x, y)))^\Delta$. Then there exists $\psi \in V_1(\mathcal{H}_o(x, y))$ such that $\psi(x) = d, \psi(y) = e$. Then $\phi + \psi \in V_1(\mathcal{H}_o(x, y))$, 

$$((\phi + \psi)(x) = \phi(x) + \psi(x) = a + d, (\phi + \psi)(y) = b + e.$$ 

Hence $(a + d, b + e) \in (V_1(\mathcal{H}_o(x, y)))^\Delta$ and so $(V_1(\mathcal{H}_o(x, y)))^\Delta$ is closed under addition and scalar multiplication, so

$$(V_1(\mathcal{H}_o(x, y)))^\Delta$$ is a linear space.
Theorem 3.28: Suppose \( x \in X, y \in X, x \neq y \). Then 
\( (V(X_o(x, y)))^\Delta \) is the smallest linear space which contains 
\( (X_o(x, y))^\Delta \) and which is closed with respect to \( \land \) and \( \lor \).

Proof: In the previous theorem, we have established 
that \( (V_1(X_o(x, y)))^\Delta \) is the smallest linear space which 
contains \( (X_o(x, y))^\Delta \) and which is contained in \( R_2 \). Therefore we can characterize \( (V_1(X_o(x, y)))^\Delta \) as one of the following:

1.) \([ (a, b) | (a, b) \in R_2, a = 0, b = 0 ] \)

2.) \([ (a, b) | (a, b) \in R_2, b = 0 ] \)

3.) \([ (a, b) | (a, b) \in R_2, a = 0 ] \)

4.) \([ (a, b) \in (a, b) \in R_2 ] \)

5.) \([ (a, b) | (a, b) \in R_2, b = \lambda a \) for \( \lambda \in R, \lambda > 0 \)

6.) \([ (a, b) | (a, b) \in R_2, b = \lambda a \) for \( \lambda \in R, \lambda < 0 \).

We also notice that \( (X_o(x, y))^\Delta \subseteq (V_1(X_o(x, y)))^\Delta \)
\( \subseteq (V(X_o(x, y)))^\Delta = (U_1(V_1(X_o(x, y)))^\Delta = (U_1(V_1(X_o)(x, y)))^\Delta \)
\( = ((V_1(X_o)(x, y))^\Delta) = ((V_1(X_o(x, y)))^\Delta) \). Therefore
\( (V(X_o(x, y)))^\Delta \) contains \( (X_o(x, y))^\Delta \) and is closed under \( \land \)
and \( \lor \). Next we need to know that \( (V(X_o(x, y)))^\Delta \) is a
linear space. This follows from the fact that in the first
five cases listed above \( (V(X_o(x,y)))^\Delta = ((V_1(X_o(x, y))^\Delta) \)
\( = (V_1(X_o(x, y))^\Delta and (V_1(X_o(x, y))^\Delta is a linear space.

In the sixth case, there is \( \lambda < 0 \) such that \( (V_1(X_o(x, y))^\Delta = [(a, b) | (a, b) \in R_2, b = \lambda a] \), and so \( (V(X_o(x, y)))^\Delta = ((V_1(X_o(x, y)))^\Delta) \)
\( = [(a, b) | (a, b) \in R_2, b = \lambda a] = R_2, \)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
which is again a linear space. Therefore $(V(X^o(x, y)))^\Delta$ is a linear space. We must now show that $(V(X^o(x, y)))^\Delta$ is the smallest linear space containing $(X^o(x, y))^\Delta$ which is closed under $\land$ and $\lor$. Suppose $T$ is a linear space which contains $(X^o(x, y))^\Delta$ and which is closed under $\land$ and $\lor$. From the previous theorem, $(V_1(X^o(x, y)))^\Delta \subseteq T$. Hence in the first five cases listed above $(V(X^o(x, y)))^\Delta \subseteq T$. In the sixth case, there is $\lambda < 0$ such that $[(a, b) | (a, b) \in \mathbb{R}^2, b = \lambda a] = (V_1(X^o(x, y)))^\Delta \subseteq T$ and $T$ is closed under $\land$ and $\lor$ so $T$ must be the whole space, therefore $(V(X^o(x, y)))^\Delta \subseteq T$.

From the discussion in the proof of Theorem 3.28, we can formulate the following theorem.

Theorem 3.29: Suppose $x \in X, y \in X, x \neq y$. Then $(V(X^o(x, y)))^\Delta$ is one of the following:

1.) $[(a, b) | (a, b) \in \mathbb{R}^2]$
2.) $[(a, b) | (a, b) \in \mathbb{R}^2, a = 0]$
3.) $[(a, b) | (a, b) \in \mathbb{R}^2, b = 0]$
4.) $[(a, b) | (a, b) \in \mathbb{R}^2, a = 0, b = 0]$
5.) $[(a, b) | (a, b) \in \mathbb{R}^2, a = b]$
6.) $\lambda \in \mathbb{R}, 0 < \lambda < 1$ and $[(a, b) | (a, b) \in \mathbb{R}^2, a = \lambda b]$
7.) $\lambda \in \mathbb{R}, 0 < \lambda < 1$ and $[(a, b) | (a, b) \in \mathbb{R}^2, b = \lambda a]$

Theorem 3.30: Suppose $x \in X, y \in X, x \neq y$. Then $(W(X^o(x, y)))^\Delta = (V(X^o(x, y)))^\Delta$.

Proof: To prove this theorem, we notice that $(W(X^o(x, y)))^\Delta = (U_2(V(X^o(x, y)))^\Delta = (U_2(V(X^o(x, y)))^\Delta$
\[ (V(X_o)(x, y))_\Delta = (V(X_o(x, y)))_\Delta = (V(X_o(x, y)))_\Delta. \]

**Theorem 3.31**: Suppose \( x \in X, y \in X, x \neq y \). Let \( Y \) be the collection of all continuous real-valued functions on \([x, y]\) and let \( Z \subseteq Y, F \subseteq Y \). Then \( Z_\Delta = F_\Delta \) if and only if \( Z = F \).

**Proof**: Suppose \( Z_\Delta = F_\Delta \) and let \( \emptyset \in Z \) and let \( \emptyset(x) = a, \emptyset(y) = b \), then \( (a, b) \in Z_\Delta = F_\Delta \). Since \( (a, b) \in F_\Delta \), there exists \( \psi \in F \) such that \( \psi(a) = a, \psi(y) = b \). Hence \( \psi(x) = a = \emptyset(x), \psi(y) = b = \emptyset(y) \) and thus \( \psi = \emptyset \). Since \( \psi \in F \), then \( \emptyset \in F \) and \( Z \subseteq F \). The argument is similar for \( F \subseteq Z \), so \( F = Z \). If \( F = Z \), clearly \( F_\Delta = Z_\Delta \).

**Theorem 3.32**: Suppose \( x \in X, y \in X, x \neq y \). Then \( (P(X_o(x, y)))_\Delta \) is the smallest linear space containing \( (X(x, y))_\Delta \) that is closed under multiplication.

**Proof**: First we must show that \( (P(X_o(x, y)))_\Delta \) is closed under multiplication. Suppose \( (a, b) \in (P(X_o(x, y)))_\Delta \), \( (c, d) \in (P(X_o(x, y)))_\Delta \), then there exists \( \emptyset \in P(X_o(x, y)) \), \( \psi \in P(X_o(x, y)) \) such that \( \emptyset(x) = a, \emptyset(y) = b, \psi(x) = c, \psi(y) = d \). From closure, \( \emptyset \psi \in P(X_o(x, y)) \). Also we see that \( (\emptyset \psi)(x) = \emptyset(x) \psi(x) = ac, (\emptyset \psi)(y) = bd \) which implies that \( (ac, bd) \in (P(X_o(x, y)))_\Delta \) and so \( (P(X_o(x, y)))_\Delta \) is closed under multiplication. We now show that \( (P(X_o(x, y)))_\Delta \) is a linear space. Suppose \( (a, b) \in (P(X_o(x, y)))_\Delta \) and \( c \in \mathbb{R} \), then there exists \( \emptyset \in P(X_o(x, y)) \) such that \( \emptyset(x) = a, \emptyset(y) = b \). Since \( \emptyset \in P(X_o(x, y)) \), then \( c \emptyset \in P(X_o(x, y))), (c\emptyset)(x) = c\emptyset(x) = ca, (c\emptyset)(y) = cb \). Hence \( c(a, b) = (ca, cb) \in (P(X_o(x, y)))_\Delta \).
Now suppose \((d, e) \in (P(\mathbb{X}_o(x, y)))^A\), then there exists \(\psi \in P(\mathbb{X}_o(x, y))\) such that \(\psi(x) = d, \psi(y) = e\). Then
\[
\psi + \psi \in P(\mathbb{X}_o(x, y)), (\psi + \psi)(x) = \psi(x) + \psi(x) = a + d, \\
(\psi + \psi)(y) = b + e.
\]
Hence \((a + d, b + e) \in (P(\mathbb{X}_o(x, y)))^A\) and we have closure under addition. Therefore
\( (P(\mathbb{X}_o(x, y)))^A \) is a linear space. Also \(\mathbb{X}_o(x, y) = P(\mathbb{X}_o(x, y))\), so \((\mathbb{X}_o(x, y))^A \subset (P(\mathbb{X}_o(x, y)))^A\).

We now show that \((P(\mathbb{X}_o(x, y)))^A\) is the smallest linear space containing \((\mathbb{X}_o(x, y))^A\) and is closed under multiplication. Suppose \(T\) is a linear space which contains \((\mathbb{X}_o(x, y))^A\) and is closed under multiplication. Let \((a, b) \in (P(\mathbb{X}_o(x, y)))^A\), then there exists \(\psi \in P(\mathbb{X}_o(x, y))\) such that \(\psi(x) = a, \psi(y) = b\). Since \(\psi \in P(\mathbb{X}_o(x, y))\), there exists \(c_i \in \mathbb{R}, i = 1 \ldots n, \psi_{i, j} \in \mathbb{X}_o(x, y), i = 1 \ldots n; j = 1 \ldots k_i\) such that \(\psi = \sum_{i=1}^n \sum_{j=1}^{k_i} c_i \psi_{i, j}\). Define \(a_{i, j}\) and \(b_{i, j}\) by
\[
\psi_{i, j}(x) = a_{i, j}, \psi_{i, j}(y) = b_{i, j}; \\
then (a_{i, j}, b_{i, j}) \in \\
(P(\mathbb{X}_o(x, y)))^A \text{ and } (a_{i, j}, b_{i, j}) \in T. \text{ Hence } (\sum_{i=1}^n \sum_{j=1}^{k_i} c_i a_{i, j}, \sum_{i=1}^n \sum_{j=1}^{k_i} c_i b_{i, j}) \in T \text{ and so } (a, b) \in T. \text{ Thus } (P(\mathbb{X}_o(x, y)))^A \subset T.
\]

**Theorem 3.33:** Suppose \(x \in \mathbb{X}, y \in \mathbb{X}, x \neq y\). Then
\( (L(\mathbb{X}_o(x, y)))^A = (P(\mathbb{X}_o(x, y)))^A \). Also \(L(\mathbb{X}_o(x, y)) = P(\mathbb{X}_o(x, y))\), and \(W(\mathbb{X}_o(x, y)) = V(\mathbb{X}_o(x, y))\).

**Proof:** To show the first part, we note that
The second part of the theorem follows immediately from the first part and from Theorem 3.30 and Theorem 3.31.

**Theorem 3.34:** Suppose \( x \in X, y \in X, x \neq y. \) Then \((P(x_o(x, y)))^\Delta\) is one of the following:

1. \( [(a, b) \mid (a, b) \in \mathbb{R}_2, a = 0, b = 0] \)
2. \( [(a, b) \mid (a, b) \in \mathbb{R}_2, a = 0] \)
3. \( [(a, b) \mid (a, b) \in \mathbb{R}_2, b = 0] \)
4. \( [(a, b) \mid (a, b) \in \mathbb{R}_2, a = b] \)
5. \( [(a, b) \mid (a, b) \in \mathbb{R}_2] \).

**Proof:** \((P(x_o(x, y)))^\Delta\) is the smallest linear space which contains \((x_o(x, y))^\Delta\) that is closed under multiplication. Suppose \( T \) is a linear space. If \( T \) consists of the origin, the horizontal or vertical axes, the bisector of the angle between the positive coordinate axis, or the whole plane, then we have closure under multiplication. If \( T \) is a linear space closed under multiplication and none of the above cases hold, then \((a, b) \in T\) implies \( a \neq 0, b \neq 0, a \neq b \). By the closure of multiplication, \((a^2, b^2) \in T\). Thus we have two different points on two different lines, so we have the whole plane.

**Lemma 3.1:** Suppose \( x \in X, y \in X, x \neq y. \) Then \( U(x_o(x, y)) \subseteq x(x, y). \)

**Proof:** We first show that \( x(x, y) = \{\emptyset \mid \emptyset : \{x, y\} \to \mathbb{R}\} \)
on $\mathbb{X}(x, y) = [\emptyset \mid \emptyset :\{x, y\} \rightarrow \mathbb{R}$ such that $\emptyset(x) = \emptyset(y)$].

Suppose $c \in \mathbb{R}$, and define $f: \mathbb{X} \rightarrow \mathbb{R}$ by $f(z) = c$ for all $z \in \mathbb{X}$. We see that $f$ is continuous since $f$ is constant, so $f \in \mathbb{X}$. Define $\emptyset: \{x, y\} \rightarrow \mathbb{R}$ by $\emptyset(x) = f(x) = c$, $\emptyset(y) = f(y) = c$, so $\emptyset \in \mathbb{X}(x, y)$. Therefore $(c, c) \in (\mathbb{X}(x, y)^\Delta$ and so $[(a, b) \mid (a, b) \in \mathbb{R}^2, a = b] \subset \mathbb{X}(x, y)$.

Thus either $(\mathbb{X}(x, y)^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}^2, a = b]$, or $(\mathbb{X}(x, y))\Delta = [(a, b) \mid (a, b) \in \mathbb{R}^2]$ since $(\mathbb{X}(x, y))\Delta$ is a linear space. Suppose $\mathbb{X}_0 \subset \mathbb{X}$. If $(\mathbb{X}(x, y))^\Delta \subset \mathbb{R}^2$, then $(U(\mathbb{X}_0(x, y)))\Delta \subset (\mathbb{X}(x, y))^\Delta$, so $U(\mathbb{X}_0(x, y)) \subset \mathbb{X}(x, y)$. If $(\mathbb{X}(x, y))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}^2, a = b]$ then $(\mathbb{X}_0(x, y))^\Delta = (\mathbb{X}(x, y))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}^2, a = b]$, so $(U(\mathbb{X}_0(x, y)))\Delta = ((\mathbb{X}_0(x, y))^\Delta)^\sim = [(a, b) \mid (a, b) \in \mathbb{R}^2, a = b]^\sim$, so $U(\mathbb{X}_0(x, y)) \subset \mathbb{X}(x, y)$.

**Theorem 3.35:** Suppose $X$ is a compact topological space and suppose $\mathbb{X}_0 \subset \mathbb{X}$. Then $U(\mathbb{X}_0) = \mathbb{X}$ if and only if $x \in \mathbb{X}$, $y \in \mathbb{X}$, $x \neq y$ implies $U(\mathbb{X}_0(x, y)) = \mathbb{X}(x, y)$.

**Proof:** Suppose $U(\mathbb{X}_0) = \mathbb{X}$ and $x \in \mathbb{X}$, $y \in \mathbb{X}$, $x \neq y$. Then $U(\mathbb{X}_0)(x, y) = [\emptyset \mid \emptyset :\{x, y\} \rightarrow \mathbb{R}$; there exists $f \in U(\mathbb{X}_0)$ such that $f(x) = \emptyset(x)$, $f(y) = \emptyset(y)] = [\emptyset \mid \emptyset :\{x, y\} \rightarrow \mathbb{R}$; there exists $f \in \mathbb{X}$ such that $f(x) = \emptyset(x)$, $f(y) = \emptyset(y)] = \mathbb{X}(x, y)$. Therefore $\mathbb{X}(x, y) = U(\mathbb{X}_0)(x, y) \subset U(\mathbb{X}_0(x, y))$. Also by Lemma 3.1, $U(\mathbb{X}_0(x, y)) \subset \mathbb{X}(x, y)$, so we have $U(\mathbb{X}_0(x, y)) = \mathbb{X}(x, y)$. Now suppose $x \in \mathbb{X}$, $y \in \mathbb{X}$, $x \neq y$.
implies $U(\mathbb{X}_o(x, y)) = \mathbb{X}(x, y)$. We know $U(\mathbb{X}_o) \subseteq \mathbb{X}$ so let $f \in \mathbb{X}$, and define $\emptyset : \{x, y\} \rightarrow \mathbb{R}$ by $\emptyset(x) = f(x)$, $\emptyset(y) = f(y)$, then $\emptyset \in \mathbb{X}(x, y)$. Since $\emptyset \in \mathbb{X}(x, y) = U(\mathbb{X}_o(x, y))$, then $\emptyset \in U_2(U_1(\mathbb{X}_o(x, y)))$ and there exists $\psi \in U_1(\mathbb{X}_o(x, y))$ such that $|\psi(x) - \emptyset(x)| < \varepsilon$ and $|\psi(y) - \emptyset(y)| < \varepsilon$. Since $\emptyset \in U_1(\mathbb{X}_o(x, y))$ then $\psi \in U_1(\mathbb{X}_o)(x, y)$ and there exists $g \in U_1(\mathbb{X}_o)$ such that $g(x) = \psi(x)$, $g(y) = \psi(y)$. Hence $|g(x) - f(x)| < \varepsilon$ and $|g(y) - f(y)| < \varepsilon$. Since $g \in U_1(\mathbb{X}_o)$, the conditions for Theorem 3.17 are satisfied and $f \in U(\mathbb{X}_o)$ and $\mathbb{X} \subseteq U(\mathbb{X}_o)$. Therefore $U(\mathbb{X}_o) = \mathbb{X}$.

**Theorem 3.36:** Suppose $X$ is a compact topological space and suppose $\mathbb{X}_o \subseteq \mathbb{X}$. Then $U(\mathbb{X}_o) = \mathbb{X}$ if and only if $x \in X, y \in X, x \neq y$, implies $(U(\mathbb{X}_o(x, y)))^\Delta = (\mathbb{X}(x, y))^\Delta$

i.e. if and only if $x \in X, y \in X, x \neq y$, implies $((\mathbb{X}_o(x, y))^\Delta)^* = (\mathbb{X}(x, y))^\Delta$.

**Proof:** The proof depends on Theorem 3.31 and Theorem 3.35. From Theorem 3.35, we see that $U(\mathbb{X}_o) = \mathbb{X}$ if and only if $x \in X, y \in X, x \neq y$ implies $U(\mathbb{X}_o(x, y)) = \mathbb{X}(x, y)$.

Also $U(\mathbb{X}_o(x, y)) = \mathbb{X}(x, y)$ if and only if $(U(\mathbb{X}_o(x, y)))^\Delta = (\mathbb{X}(x, y))^\Delta$. From Theorem 3.36, $(U(\mathbb{X}_o(x, y)))^\Delta = ((\mathbb{X}_o(x, y))^\Delta)^*$, so $U(\mathbb{X}_o) = \mathbb{X}$ if and only if $x \in X, y \in X, x \neq y$ implies $((\mathbb{X}_o(x, y))^\Delta)^* = (\mathbb{X}(x, y))^\Delta$.

**Theorem 3.37:** Suppose $X$ is a compact topological space, and suppose $\mathbb{X}_o \subseteq \mathbb{X}$. If $f \in \mathbb{X}$ then $f \in U(\mathbb{X}_o)$ if and only if $x \in X, y \in X, x \neq y$ implies $(f(x), f(y)) \in ((\mathbb{X}_o(x, y))^\Delta)^*$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Proof: Suppose \( f \in U(X_o) \) and \( x \in X, y \in X, x \neq y \), and define \( \emptyset: \{ x, y \} \rightarrow \mathbb{R} \) by \( \emptyset(x) = f(x), \emptyset(y) = f(y) \). Then \( \emptyset \in U(X_o)(x, y) \) and so \( \emptyset \in U(X_o(x, y)) \) with \( \emptyset(x) = f(x), \emptyset(y) = f(y) \) which implies that \((f(x), f(y)) \in (U(X_o(x, y)))^\Delta \) which in turn implies that \((f(x), f(y)) \in ((X_o(x, y)))^\Delta \).

Now suppose \( x \in X, y \in X, x \neq y \) implies \((f(x), f(y)) \in (X_o(x, y))^\Delta \), then \((f(x), f(y)) \in (U(X_o(x, y)))^\Delta \). Since \((f(x), f(y)) \in (U(X_o(x, y)))^\Delta \), there exists \( \emptyset \in U(X_o(x, y)) \) such that \( \emptyset(x) = f(x), \emptyset(y) = f(y) \). Since \( \emptyset \in U(X_o(x, y)) = U_2(U_1(X_o(x, y))) \) there exists \( \emptyset \in U_1(X_o(x, y)) \) such that \(| \emptyset(x) - \emptyset(x) | < \varepsilon \) and \(| \emptyset(y) - \emptyset(y) | < \varepsilon \). Since \( \emptyset \in U_1(X_o(x, y)) \), then \( \emptyset \in U_1(X_o(x, y)) \) and there exists \( g \in U_1(X_o) \) such that \( g(x) = \emptyset(x), g(y) = \emptyset(y) \). Hence \(| g(x) - f(x) | < \varepsilon \) and \(| g(y) - f(y) | < \varepsilon \). Since \( g \in U_1(X_o) \), the conditions for Theorem 3.17 are satisfied and \( f \in U(X_o) \).

**Theorem 3.38:** Suppose \( X \) is a compact topological space and suppose \( X_o \subseteq X \). If \( X_o = U(X_o) \), then \( x \in X, y \in X, x \neq y \) implies \((U(X_o(x, y)))^\Delta = ((X_o(x, y))^\Delta)^* = (X_o(x, y))^\Delta \).

Proof: From Theorem 3.26, \((U(X_o(x, y)))^\Delta = ((X_o(x, y))^\Delta)^* \). Since \( X_o \subseteq U_1(X_o) \subseteq U(X_o) \) and \( X_o \subseteq U_2(X_o) \subseteq U(X_o) \), then \( U_1(X_o) = X_o, U_2(X_o) = X_o \). Hence we see that \((U(X_o(x, y)))^\Delta = ((X_o(x, y))^\Delta)^* = ((X_o(x, y))^\Delta)^* = (X_o(x, y))^\Delta \).

**Theorem 3.39:** Suppose \( X \) is a compact topological
space, and suppose $X_o = X$, $X_1 = X$, $\bar{X}_o = U(X_o)$, $\bar{X}_1 = U(X_1)$. Then $X_o = X_1$ if and only if $x \in X$, $y \in X$, $x \neq y$ implies $((X_o(x, y))^\Delta)^* = ((X_1(x, y))^\Delta)^*$.  

Proof: Suppose $X_o \subseteq X$, $X_1 \subseteq X$, $\bar{X}_o = U(X_o)$, and $\bar{X}_1 = U(X_1)$. Suppose $X_o = X_1$, then $X_o(x, y) = X_1(x, y)$ and so $U(X_o(x, y)) = U(X_1(x, y))$. From Theorem 3.31  

$(U(X_o(x, y)))^\Delta = (U(X_1(x, y)))^\Delta$. From Theorem 3.26,  

$((X_o(x, y))^\Delta)^* = (U(X_o(x, y)))^\Delta = (U(X_1(x, y)))^\Delta$  

$= ((X_1(x, y))^\Delta)^*$. Now suppose $((X_o(x, y))^\Delta)^* = ((X_1(x, y))^\Delta)^*$. Let $f \in X_o$, then $f \in U(X_o)$. By Theorem 3.37 $f \in U(X_o)$ if and only if $x \in X$, $y \in X$, $x \neq y$ implies $(f(x), f(y)) \in (X_o(x, y))^\Delta$. Hence by the same theorem, $f \in U(X_1) = X_1$, and so $X_o \subseteq X_1$. In the same fashion, we find that $X_1 \subseteq X_o$, so we have that $X_o = X_1$.

**Theorem 3.40:** Suppose $X$ is a compact topological space and suppose $X_o \subseteq X$. Then $W(X_o) = X$ if and only if $x \in X$, $y \in X$, $x \neq y$ implies $W(X_o(x, y)) = X(x, y)$, i.e., $V(X_o(x, y)) = X(x, y)$, i.e., $V(X_o)(x, y) = X(x, y)$.

Proof: Suppose $x \in X$, $y \in X$, $x \neq y$, and suppose $W(X_o) = X$. Then by Theorem 3.9, $U(V_1(X_o)) = X$, by Theorem 3.35, $X(x, y) = U(V_1(X_o)(x, y)) = U(V_1(X_o(x, y))) = W(X_o(x, y))$. Now suppose $x \in X$, $y \in X$, $x \neq y$ implies $W(X_o(x, y)) = X(x, y)$. Then $X(x, y) = W(X_o(x, y)) = U(V_1(X_o(x, y))) = U(V_1(X_o)(x, y))$. By Theorem 3.35 $U(V_1(X_o)) = X$ which implies $W(X_o) = X$. By Theorem 3.32
\[ W(\mathbf{x}_o(x, y)) = V(\mathbf{x}_o(x, y)), \text{ so } W(\mathbf{x}_o) = \mathbf{x} \text{ if and only if } x \in X, y \in X, x \neq y, \text{ implies } \mathbf{x}(x, y) = V(\mathbf{x}_o(x, y)) = V(\mathbf{x}_o)(x, y). \]

**Theorem 3.41:** Suppose \( X \) is a compact topological space and suppose \( \mathbf{x}_o \subseteq X \). Then \( W(\mathbf{x}_o) = X \) if and only if \( x \in X, y \in X, x \neq y \) implies \( (W(\mathbf{x}_o(x, y)))^\Delta = (\mathbf{x}(x, y))^\Delta \), i.e., \( (V(\mathbf{x}_o(x, y)))^\Delta = (\mathbf{x}(x, y))^\Delta \).

**Proof:** The proof follows immediately from the previous theorem and Theorem 3.31.

**Theorem 3.42:** Suppose \( X \) is a compact topological space and suppose \( \mathbf{x}_o \subseteq X \). If \( f \in \mathbf{x} \), then \( f \in W(\mathbf{x}_o) \) if and only if \( x \in X, y \in X, x \neq y \) implies \( (f(x), f(y)) \in (V(\mathbf{x}_o(x, y)))^\Delta \).

**Proof:** Suppose \( f \in \mathbf{x} \) and \( f \in W(\mathbf{x}_o) \). Then \( f \in W(\mathbf{x}_o) \) if and only if \( f \in U(V_1(\mathbf{x}_o)) \). By Theorem 3.37, \( f \in U(V_1(\mathbf{x}_o)) \) if and only if \( x \in X, y \in X, x \neq y \) implies \( (f(x), f(y)) \in (U(V_1(\mathbf{x}_o))(x, y))^\Delta = (U(V_1(\mathbf{x}_o)(x, y)))^\Delta = (W(\mathbf{x}_o(x, y)))^\Delta = (V(\mathbf{x}_o(x, y)))^\Delta \).

**Theorem 3.43:** Suppose \( X \) is a compact topological space, and suppose \( \mathbf{x}_o \subseteq X \). If \( f \in \mathbf{x} \), then \( f \in W(\mathbf{x}_o) \) if and only if \( x \in X, y \in X, x \neq y \), implies

1. \( f(x) = 0 \) if \( a) \ g \in \mathbf{x}_o \) implies \( g(x) = 0 \)
   b) there is \( h \in \mathbf{x}_o \) such that
   \[ h(y) \neq 0 \] (i.e., \( (V(\mathbf{x}_o)(x, y))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}_2, a = 0] \)).

2. \( f(y) = 0 \) if \( a) \ g \in \mathbf{x}_o \) implies \( g(y) = 0 \)
   b) there is \( h \in \mathbf{x}_o \) such that
\[ h(x) \neq 0 \text{ (i.e., } (V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}_2, b = 0]. \]

(3) \( f(x) = 0, f(y) = 0 \) if \( g \in \mathbb{X}_o \) implies \( g(x) = 0, \)
\( g(y) = 0 \) (i.e., \( (V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}_2, a = 0, b = 0]). \)

(4) \( f(x) = f(y) \) if a) \( g \in \mathbb{X}_o \) implies \( g(x) = g(y) \)
b) there is \( h \in \mathbb{X}_o \) such that
\( h(x) \neq 0 \neq h(y) \) (i.e.,
\( (V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}_2, a = b]). \)

(5) \( f(x) = \lambda f(y) \) if a) \( 0 < \lambda < 1 \)
b) \( g \in \mathbb{X}_o \) implies \( g(x) = \lambda g(y) \)
c) there is \( h \in \mathbb{X}_o \) such that
\( h(x) \neq 0 \neq h(y) \) (i.e.,
\( (V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}_2, a = b]). \)

(6) \( f(y) = \lambda f(x) \) if a) \( 0 < \lambda < 1 \)
b) \( g \in \mathbb{X}_o \) implies \( g(y) = \lambda g(x) \)
c) there is \( h \in \mathbb{X}_o \) such that
\( h(x) \neq 0 \neq h(y) \) (i.e.,
\( (V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in \mathbb{R}_2, b = \lambda a]). \)

Proof: Suppose conditions (1) - (6) hold. Then no matter what \( (V(\mathbb{X}_o(x, y)))^\Delta \) turns out to be, we see that
\( (f(x), f(y)) \in (V(\mathbb{X}_o(x, y)))^\Delta \). Then by Theorem 3.42
\[(f(x), f(y)) \in (\mathcal{V}(X_0(x, y)))^A\] if and only if \(f \in W(X_0)\).

Now suppose \(f \in W(X_0)\) and \(x \in X, y \in X,\) and \(x \neq y\). If all the functions in \(X_0\) satisfy relations of the kind enumerated in (1) - (6), then every function in \(W(X_0)\) must do likewise; for the sums, scalar multiplies, absolute values, and uniform limits of functions which satisfy a condition of any one of these types must satisfy the same condition. For example, if all the functions in \(X_0\) vanish at a point \(x\), then all the functions in \(W(X_0)\) vanish at the point \(x\).

3.6. **Theorem 3.44**: Suppose \(X\) is a compact topological space, and suppose \(X_0 \subseteq X\). If \(f \in X\), then \(f \in W(X_0)\) if and only if

1.) \(x \in X\) implies \(f(x) = 0\) if \(g \in X_0\) implies \(g(x) = 0\),
2.) \(x \in X, y \in X, x \neq y\), implies \(f(x) = u f(y)\) if
   a) \(u > 0\)
   b) \(g \in X_0\) implies \(g(x) = u g(y)\).

**Proof**: Suppose conditions (1) and (2) of the theorem hold, then no matter how \((\mathcal{V}(X_0(x, y)))^A\) is characterized, we see that \((f(x), f(y)) \in (\mathcal{V}(X_0(x, y)))^A\). Hence by Theorem 3.42 \(f \in W(X_0)\). If \(f \in W(X_0)\), then conditions (1) and (2) follow from the properties of \(W(X_0)\).

**Theorem 3.45**: Suppose \(X\) is a compact topological space, and suppose \(X_0 \subseteq X\). Define \(\emptyset: X \rightarrow \mathbb{R}\) by \(\emptyset(x) = 1\) for all \(x \in X\). Then \(\emptyset \in W(X_0)\) if and only if \(x \in X, y \in X, x \neq y\) implies either

A. 1.) \(g \in X_0\) implies \(g(x) = g(y)\)
2.) there is \( h \in \mathbb{X}_o \) such that \( h(x) \neq 0 \neq h(y) \) (i.e., \((V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2, a = b])\).

or B. 1.) there are \( g \in \mathbb{X}_o \) and \( h \in \mathbb{X}_o \) such that \( g(x)h(y) \neq g(y)h(x) \) (i.e., \((V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2])\).

Proof: Define \( \mathcal{O}:X \rightarrow R \) by \( \mathcal{O}(x) = 1 \) for all \( x \in X \) and suppose conditions A or B hold. Suppose \( x \in X \), \( y \in X \), \( x \neq y \). If \((\mathcal{O}(x), \mathcal{O}(y)) \in (V(\mathbb{X}_o(x, y)))^\Delta \), then \((1, 1) \in (V(\mathbb{X}_o(x, y)))^\Delta \). If condition A holds, then \((V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R, a = b] \), and so \((\mathcal{O}(x), \mathcal{O}(y)) \in (V(\mathbb{X}_o(x, y)))^\Delta \). If condition B holds, then \((V(\mathbb{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2], \) so \((\mathcal{O}(x), \mathcal{O}(y)) \in (V(\mathbb{X}_o(x, y)))^\Delta \).

In either case, we have \((\mathcal{O}(x), \mathcal{O}(y)) \in (V(\mathbb{X}_o(x, y)))^\Delta \), so \( \mathcal{O} \in W(\mathbb{X}_o) \). Now suppose \( \mathcal{O} \in W(\mathbb{X}_o) \), then \( x \in X \), \( y \in X \), \( x \neq y \) implies \((\mathcal{O}(x), \mathcal{O}(y)) \in (V(\mathbb{X}_o(x, y)))^\Delta \), or \((l, l) \in (V(\mathbb{X}_o(x, y)))^\Delta \). If \((l, l) \in (V(\mathbb{X}_o(x, y)))^\Delta \), then either condition A or B must hold for Theorem 3.29.

**Theorem 3.46:** Suppose \( X \) is a compact topological space and suppose \( \mathbb{X}_o \subseteq X \). Suppose \( x \in X \), \( y \in X \), \( x \neq y \) implies that there exists \( g \in \mathbb{X}_o \) and \( h \in \mathbb{X}_o \) such that \( g(x)h(y) \neq g(y)h(x) \). Then \( W(\mathbb{X}_o) = X \).

Proof: Suppose \( f \in \mathbb{X} \). Suppose \( x \in X \), \( y \in X \), \( x \neq y \) implies that there exists \( g \in \mathbb{X}_o \) and \( h \in \mathbb{X}_o \) such that \( g(x)h(y) \neq g(y)h(x) \). Hence \((g(x), g(y)) \neq (0, 0) \) and \((h(x), h(y)) \neq (0, 0) \). Also \((g(x), g(y)) \) and \((h(x), h(y)) \)
are not on the same line that coincides with the vertical or horizontal axes since \( g(x)h(y) \neq g(y)h(x) \). Also \((g(x), g(y))\) and \((h(x), h(y))\) are not on the same line passing through the origin. If they were on the same line through the origin, then for \( k \neq 0 \in \mathbb{R} \), \( g(y) = k g(x) \), \( h(y) = kh(x) \) which implies \( kh(x) g(y) = g(y)h(y) = k g(x)h(y) \). But \( g(x)h(y) \neq g(y)h(x) \) so the points are not on the same line through the origin. Hence we have two different points on two different lines, so we have the whole plane. Define \( \phi:[x, y] \rightarrow \mathbb{R} \) by \( \phi(x) = g(x) \), \( \phi(y) = g(y) \), and define \( \psi:[x, y] \rightarrow \mathbb{R} \) by \( \psi(x) = h(x) \), \( \psi(y) = h(y) \). Hence \( (\phi(x), \phi(y)) \in (X^\Delta(x, y)) \) and \( (\psi(x), \psi(y)) \in (X^\Delta(x, y)) \). We know that \( (X^\Delta(x, y)) \subseteq (V(X^\Delta(x, y))) \). Thus \( (f(x), f(y)) \in (V(X^\Delta(x, y))) \) since \( (V(X^\Delta(x, y))) \) is the whole plane. By Theorem 3.42, if \( f \in X \), then \( f \in W(X^\Delta) \) and so \( X \subseteq W(X^\Delta) \). Since \( W(X^\Delta) \subseteq X \), we have \( W(X^\Delta) = X \).

**Definition 3.30:** Suppose \( X \) is a compact topological space, and suppose \( X^\Delta \subseteq X \). Then \( X^\Delta \) is said to be a separating family for \( X \) if \( x \in X \), \( y \in X \), \( x \neq y \), implies that there is \( g \in X^\Delta \) such that \( g(x) \neq g(y) \).

**Theorem 3.47:** Suppose \( X \) is a compact topological space, and suppose \( X^\Delta \subseteq X \). Suppose \( x \in X \), \( y \in X \), \( x \neq y \) implies that 1.) there is \( g \in X^\Delta \) such that \( g(x) \neq g(y) \) 2.) there is \( h \in X^\Delta \) such that \( h(x) \neq h(y) \neq 0 \). Then \( W(X^\Delta) = X \).
Proof: Suppose $x \in X$, $y \in X$, $x \neq y$, and suppose there is $g \in X_0$ such that $g(x) \neq g(y)$ and there is $h \in X_0$ such that $h(x) = h(y) \neq 0$. Then $g(x)h(y) \neq g(y)h(x)$ and by Theorem 3.46, $W(X_0) = X$.

Theorem 3.48: Suppose $X$ is a compact topological space, and suppose $X_0 = X$, suppose $X_0$ is a separating family for $X$, and suppose there is $c \in \mathbb{R}$, $c \neq 0$ such that $\phi_c(x) = c$ for all $x \in X$. Then $W(X_0) = X$.

Proof: Since $X_0$ is a separating family, for $x \in X$, $y \in X$, $x \neq y$, there exists $g \in X_0$ such that $g(x) \neq g(y)$. Suppose $c \in \mathbb{R}$, $c \neq 0$, and define $\phi_c : X \to \mathbb{R}$ by $\phi_c(x) = c$ for all $x \in X$. Then $\phi_c(x) = \phi_c(y) = c \neq 0$. Hence by Theorem 3.47, $W(X_0) = X$.

Theorem 3.49: Suppose $X$ is a compact topological space and suppose $X_0 = X$. If $X_0$ is a separating family for $X$, then $X$ is a separating family for $X$.

Proof: Since $X_0$ is a separating family for $X$, if $x \in X$, $y \in X$, $x \neq y$, then there is $g \in X_0$ such that $g(x) \neq g(y)$. Since $X_0 = X$, then $g \in X$ and so $X$ is a separating family.

Theorem 3.50: Suppose $X$ is a compact topological space, suppose $X_0 = X$, and suppose $U(X_0) = X$. Then $X_0$ is a separating family for $X$ if and only if $X$ is a separating family for $X$.

Proof: Suppose $X_0$ is not a separating family for $X$.
if \( x \in X, y \in X, x \neq y \), then there is \( g \in X_0 \) such that \( g(x) \neq g(y) \). Since \( X_0 \subseteq X \), then \( g \in X \) and so \( X \) is a separating family.

**Theorem 3.50:** Suppose \( X \) is a compact topological space, suppose \( X_0 \subseteq X \), and suppose \( U(X_0) = X \). Then \( X_0 \) is a separating family for \( X \) if and only if \( X \) is a separating family for \( X \).

Proof: Suppose \( X_0 \) is not a separating family for \( X \). Then there exists \( x \in X, y \in X, x \neq y \), such that \( g \in X_0 \) implies \( g(x) = g(y) \). Then \( f \in U(X_0) \) implies \( f(x) = f(y) \), so \( U(X_0) \) is not a separating family for \( X \), i.e., \( X \) is not a separating family for \( X \). From the previous theorem, if \( X_0 \) is a separating family, then \( X \) is a separating family.

**Theorem 3.51:** Suppose \( X \) is a compact topological space and suppose \( X_0 \subseteq X \). Then \( L(X_0) = X \) if and only if \( x \in X, y \in X, x \neq y \) implies \( L(X_0(x, y)) = X(x, y) \).

Proof: Suppose \( x \in X, y \in X, x \neq y \) implies \( L(X_0(x, y)) = X(x, y) \). Recall that \( L(X_0(x, y)) = U(P(X_0(x, y))) = U(P(X_0)(x, y)) \). By Theorem 3.35, \( x \in X, y \in X, x \neq y \) implies \( L(X_0(x, y)) = X(x, y) \) if and only if \( U(P(X_0)) = L(X_0) = X \).

**Theorem 3.52:** Suppose \( X \) is a compact topological space, and suppose \( X_0 \subseteq X \). Then \( L(X_0) = X \) if and only if \( x \in X, y \in X, x \neq y \) implies \( (L(X_0(x, y)))^\Delta = (X(x, y))^\Delta \), i.e., \( (P(X_0(x, y)))^\Delta = (X(x, y))^\Delta \).

Proof: The proof follows immediately from Theorems...
3.31 and 3.52 and the fact that \((L(\mathcal{X}_o(x, y)))^\Delta = (P(\mathcal{X}_o(x, y)))^\Delta\).

**Theorem 3.53:** Suppose \(X\) is a compact topological space, and suppose \(\mathcal{X}_o \subseteq \mathcal{X}\). If \(f \in \mathcal{X}_o\), then \(f \in L(\mathcal{X}_o)\) if and only if \(x \in X, y \in X, x \neq y\) implies \((f(x), f(y)) \in (P(\mathcal{X}_o(x, y)))^\Delta\).

**Proof:** Suppose \(f \in L(\mathcal{X}_o) = U(P(\mathcal{X}_o))\), then by Theorem 3.37, \(f \in L(\mathcal{X}_o)\) if and only if \(x \in X, y \in X, x \neq y\) implies \((f(x), f(y)) \in ((P(\mathcal{X}_o(x, y)))^\Delta)^* = (U(P(\mathcal{X}_o(x, y))))^\Delta = (U(P(\mathcal{X}_o(x, y))))^\Delta = (L(\mathcal{X}_o(x, y)))^\Delta = (P(\mathcal{X}_o(x, y)))^\Delta\).

**Theorem 3.54:** Suppose \(X\) is a compact topological space, and suppose \(\mathcal{X}_o \subseteq \mathcal{X}\). If \(f \in \mathcal{X}_o\), then \(f \in L(\mathcal{X}_o)\) if and only if \(x \in X, y \in X, x \neq y\) implies.

1.) \(f(x) = 0\) if \(a)\) \(g \in \mathcal{X}_o\) implies \(g(x) = 0\)

\(b)\) there is \(h \in \mathcal{X}_o\) such that

\(h(y) \neq 0\) (i.e., \((P(\mathcal{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2, a = 0]\)).

2.) \(f(y) = 0\) if \(a)\) \(g \in \mathcal{X}_o\) implies \(g(y) = 0\)

\(b)\) there is \(h \in \mathcal{X}_o\) such that

\(h(x) \neq 0\) (i.e., \((P(\mathcal{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2, b = 0]\)).

3.) \(f(x) = 0, f(y) = 0\) if \(g \in \mathcal{X}_o\) implies \(g(x) = 0, g(y) = 0\) (i.e., \((P(\mathcal{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2, a = 0, b = 0]\)).

4.) \(f(x) = f(y)\) if \(a)\) \(g \in \mathcal{X}_o\) implies \(g(x) = g(y)\)

\(b)\) there is \(h \in \mathcal{X}_o\) such that

\(h(x) \neq 0 \neq h(y)\), (i.e.,

\((P(\mathcal{X}_o(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2, a = b]\)).
Proof: By the previous theorem, if \( f \in X \) then
\[ f \in L(X_0) \]
if and only if \( x \in X, y \in X, x \neq y \) implies
\[ (f(x), f(y)) \in (P(X_0(x, y)))^\Delta. \]
The proof then follows as Theorem 3.43.

**Theorem 3.55:** Suppose \( X \) is a compact topological space and suppose \( X_0 \subset X \). If \( f \in X \), then \( f \in L(X_0) \) if and only if 1.) \( x \in X \) implies \( f(x) = 0 \) if \( g \in X_0 \) implies \( g(x) = 0 \) 2.) \( x \in X, y \in X, x \neq y \) implies \( f(x) = f(y) \) if \( g \in X_0 \) implies \( g(x) = g(y) \).

Proof: Suppose \( f \in X \) and suppose 1.) and 2.) hold. Then it is easy to see that \( x \in X, y \in X, x \neq y \) implies
\[ (f(x), f(y)) \in (P(X_0(x, y)))^\Delta, \]
and so \( f \in L(X_0) \).

Now suppose \( f \in L(X_0) \). Clearly, if \( x \in X \) implies \( g(x) = 0 \) for all \( g \in X_0 \), then \( f(x) = 0 \). Also, if \( x \in X, y \in X, x \neq y \) implies \( g(x) = g(y) \) for all \( g \in X_0 \), then \( f(x) = f(y) \).

**Definition 3.51:** Suppose \( X \) is a compact topological space, and suppose \( X_0 \subset X \). Define a relation \( \equiv_{X_0} \) in \( X \) as follows: If \( x \in X, y \in X \), then \( x \equiv_{X_0} y \) if and only if \( g \in X_0 \) implies \( g(x) = g(y) \).

**Theorem 3.56:** Suppose \( X \) is a compact topological space, and suppose \( X_0 \subset X \), then \( \equiv_{X_0} \) is an equivalence relation on \( X \).

**Definition 3.52:** Define \( S_{X_0} \) to be the collection of all equivalence classes in \( X \) with respect to \( \equiv_{X_0} \).
Definition 3.33: Define \( X_x = \{ y \mid y \in X, y \equiv_{X_o} x \} \) for \( x \in X \). \( X_x \) is the equivalence class in \( X \) with respect to \( \equiv_{X_o} \) to which \( x \) belongs.

Definition 3.34: Define \( X_{x,f} = \{ y \mid y \in X, f(y) = f(x) \} \) for \( x \in X, f \in X_o \).

Theorem 3.37: Suppose \( X \) is a compact topological space, and suppose \( X_o \subseteq X \), then \( X_{x,f} \) is topologically closed. Also \( X_x = \bigcap_{f \in X_o} X_{x,f} \) and \( X_x \) is topologically closed.

Proof: Suppose \( t \in X - X_{x,f} \), then \( t \in X \) and \( t \notin X_{x,f} \). This implies that \( f(t) \neq f(x) \) for \( x \in X \). Let \( \varepsilon = |f(t) - f(x)| \), then \( \varepsilon > 0 \). Since \( f \in X_o \), \( f \) is continuous. Thus there exists an open set \( G \) such that \( t \in G \) and such that if \( y \in G \), then \( |f(t) - f(y)| < \varepsilon = |f(t) - f(x)| \), so \( f(y) \neq f(x) \). So \( y \in X - X_{x,f} \) and \( G \subseteq X - X_{x,f} \). Hence \( X_{x,f} \) is closed.

Now suppose \( y \in X_x \), then \( y \equiv_{X_o} x \) for \( x \in X, y \in X \).

Since \( y \equiv_{X_o} x \), if \( f \in X_o \), then \( f(x) = f(y) \) for \( x \in X, y \in X \).

This implies \( y \in X_{x,f} \) and so \( y \in \bigcap_{f \in X_o} X_{x,f} \). Hence \( X_x = \bigcap_{f \in X_o} X_{x,f} \).

The argument is reversible so we see that \( X_x = \bigcap_{f \in X_o} X_{x,f} \). Also \( X_x \) is closed since it is the intersection of closed sets.

Theorem 3.58: Suppose \( X \) is a compact topological space, and suppose \( X_o \subseteq X \), then \( X_{x,f} \) is topologically closed. Also \( X_x = \bigcap_{f \in X_o} X_{x,f} \) and \( X_x \) is topologically closed.
space, suppose $X_0 \subseteq X$, suppose $x \in X$, and suppose $g \in X_0$ implies $g(x) = 0$. If $y \in X$, then $y \in X_x$ if and only if $g \in X_0$ implies $g(y) = 0$.

Proof: Suppose $x \in X, y \in X$, and suppose $g \in X_0$ implies $g(x) = 0$. Let $y \in X$ and $y \in X_x$, then $y \not\equiv_{X_0} x$ for $x \in X, y \in X$. Since $y \not\equiv_{X_0} x$, $g \in X_0$ implies $g(x) = g(y)$.

By hypothesis $g \in X_0$ implies $g(x) = 0$, so $g(y) = g(x) = 0$.

Now suppose $g \in X_0$ implies $g(y) = 0$. From hypothesis, $g \in X_0$ implies $g(x) = 0$. Hence $g \in X_0$ implies $g(x) = 0$ and thus $x \equiv_{X_0} y$ and so $y \in X_x$.

**Theorem 3.59:** Suppose $X$ is a compact topological space, suppose $X_0 \subseteq X$, and suppose $x \in X$ implies that there exists $g \in X_0$ such that $g(x) \neq 0$. If $f \in X$, then $f \in L(X_0)$ if and only if $A \in S^X_{X_0}$ implies $f$ is constant on $A$.

Proof: Suppose $x \in X, y \in X, x \neq y$. Let $A$ be the equivalence class to which $x$ belongs. If $y \in A$, then since $f$ is constant on $A$, we have $f(x) = f(y)$. Also, we have $(P(X_0(x, y)))^\Delta = [(a, b) \mid (a, b) \in R_2, a = b]$, so $(f(x), f(y)) \in (P(X_0(x, y)))^\Delta$. If $y \notin A$, there exists $g \in X_0$ such that $g(x) \neq g(y)$. Also, there exists $h_1 \in X_0$ such that $h_1(x) \neq 0$, and there exists $h_2 \in X_0$ such that $h_2(y) \neq 0$, therefore $(P(X_0(x, y)))^\Delta = R_2$, so $(f(x), f(y)) \in (P(X_0(x, y)))^\Delta$, and thus $f \in L(X_0)$.

Now suppose $f \in L(X_0)$ and suppose $A \in S^X_{X_0}$. Suppose
\[ x \in A, \ y \in A. \] Then \( g \in X_0 \) implies \( g(x) = g(y) \). Clearly, then \( f(x) = f(y) \). Hence \( f \) is constant on \( A \).

**Theorem 3.60:** Suppose \( X \) is a compact topological space, suppose \( X_0 \subset X \), suppose \( L(X_0) = X_0 \), and suppose \( x \in X \) implies there exists \( g \in X_0 \) such that \( g(x) \neq 0 \). If \( f \in X \), then \( f \in X_0 \) if and only if \( A \in S_{X_0} \) implies \( f \) is constant on \( A \).

**Proof:** The proof follows immediately from the previous theorem since \( L(X_0) = X_0 \).

**Theorem 3.61:** Suppose \( X \) is a compact topological space, suppose \( X_0 \subset X \), and suppose there exists \( x \in X \) such that \( g \in X_0 \) implies that \( g(x) = 0 \). If \( f \in X \), then \( f \in L(X_0) \) if and only if 1.) \( f = 0 \) on \( X \) 2.) \( A \in S_{X_0} \) implies \( f \) is constant on \( A \).

**Proof:** Suppose there exists \( x \in X \) such that \( g \in X_0 \) implies \( g(x) = 0 \) and suppose \( f \in L(X_0) \). Let \( z \in X_x \). Then \( z \notin X_0 \), so \( g \in X_0 \) implies \( g(z) = g(x) = 0 \). Since \( g(z) = 0 \) for \( g \in X_0 \), it is clear that \( f(x) = 0 \). Hence \( f = 0 \) on \( X_x \).

Now suppose \( A \in S_{X_0} \), \( u \in A, \ v \in A \). Then \( u \notin X_0 \), \( v \notin X_0 \), so \( g \in X_0 \) implies \( g(u) = g(v) \). Hence \( f(u) = f(v) \), so \( f \) is constant on \( A \).

Now suppose 1.) and 2.) hold. Suppose \( y \in X \), and suppose \( g \in X_0 \) implies \( g(y) = 0 \). Then \( g \in X_0 \) implies
\( g(y) = g(x) \), so \( y \equiv_x^y x \), and so \( y \in X_x \). Thus, \( f(y) = 0 \).

Further, suppose \( z \in X, y \in X, z \neq y \) and suppose \( g \in X_x \) implies \( g(z) = g(y) \). Then \( z \equiv_x^y y \). Let \( A \) be the equivalence class to which \( z \) and \( y \) belong. Since \( f \) is constant on \( A \), then \( f(z) = f(y) \). By Theorem 3.55, it follows that \( f \in L(X_x^0) \).

**Theorem 3.62:** Suppose \( X \) is a compact topological space, suppose \( X_x^0 \subseteq X \), suppose \( L(X_x^0) = X_x^0 \) and suppose there exists \( x \in X \) such that \( g \in X_x^0 \) implies \( g(x) = 0 \). If \( f \in X \), then \( f \in X_x^0 \) if and only if 1.) \( f = 0 \) on \( X_x \) 2.) \( A \in S_{X_x^0} \) implies \( f \) is constant on \( A \).

**Proof:** The proof follows immediately from Theorem 3.61 since \( L(X_x^0) = X_x^0 \).

**Theorem 3.63:** Suppose \( X \) is a compact topological space, and suppose \( X_x^0 \subseteq X \). Let \( \emptyset : X \rightarrow \mathbb{R} \) be defined by \( \emptyset(x) = 1 \) if \( x \in X \). Then \( \emptyset \in L(X_x^0) \) if and only if \( x \in X \) implies that there is \( g \in X_x^0 \) such that \( g(x) \neq 0 \).

**Proof:** Let \( \emptyset : X \rightarrow \mathbb{R} \) be defined by \( \emptyset(x) = 1 \) if \( x \in X \). Suppose \( \emptyset \in L(X_x^0) \), then by Theorem 3.55, \( x \in X \) implies \( \emptyset(x) = 0 \) if \( g \in X_x^0 \) implies \( g(x) = 0 \). But \( x \in X \) implies \( \emptyset(x) \neq 0 \), so \( x \in X \) implies that there is \( g \in X_x^0 \) such that \( g(x) \neq 0 \). Now suppose \( x \in X \) implies that there is \( g \in X_x^0 \) such that \( g(x) \neq 0 \). Thus condition (1) of Theorem 3.55 is satisfied. Also, since \( x \in X, y \in X \) implies \( \emptyset(x) = 1 = \emptyset(y) \), condition (2) is satisfied.
Hence by Theorem 3.55, \( \emptyset \in L(\mathcal{X}) \).

**Theorem 3.64**: Suppose \( X \) is a compact topological space, suppose \( \mathcal{X}_o \subset \mathcal{X} \), and suppose \( \mathcal{X}_o \) is a separating family for \( X \). Then either \( L(\mathcal{X}_o) = \mathcal{X} \) or else there exists \( x_o \in X \) such that \( L(\mathcal{X}_o) = \{ f \mid f \in \mathcal{X}, f(x_o) = 0 \} \).

**Proof**: Suppose \( \mathcal{X}_o \) is a separating family; then so is \( \mathcal{X} \) and so is \( L(\mathcal{X}_o) \). We divide the proof in two cases:

1. \( x \in X \) implies there exists \( g \in \mathcal{X}_o \) such that \( g(x) \neq 0 \),
2. there exists \( x_o \in X \) such that \( g \in \mathcal{X}_o \) implies \( g(x_o) = 0 \).

We first examine case (1). Since \( \mathcal{X}_o \) is a separating family for \( X \), there is \( h \in \mathcal{X}_o \) such that \( h(x) \neq h(y) \). Then \( (h(x), h(y)) \in (\mathcal{X}_o(x, y))^{\Delta} \). Also, there exists \( h_1 \in \mathcal{X}_o \) such that \( h_1(x) \neq 0 \), so \( (h_1(x), h_1(y)) \in (\mathcal{X}_o(x, y))^{\Delta} \). Also, there exists \( h_2 \in \mathcal{X}_o \) such that \( h_2(y) \neq 0 \), so \( (h_2(x), h_2(y)) \in (\mathcal{X}_o(x, y))^{\Delta} \). From this it follows that \( (P(\mathcal{X}_o(x, y)))^{\Delta} = R_2 \) for all \( x \in X \), \( y \in X \). Thus \( L(\mathcal{X}_o) = \mathcal{X} \).

We now examine case (2). Suppose there exists \( x_o \in X \) such that \( g \in \mathcal{X}_o \) implies \( g(x_o) = 0 \). Since \( \mathcal{X}_o \) is a separating family, it follows that if \( x \in X \), \( y \in X \), \( x \neq y \), then \( x \not\sim_{\mathcal{X}_o} y \). Therefore \( \mathcal{X}_x = \{x\} \) and \( \mathcal{X}_{x_o} = \{x_o\} \).

If \( y \in X \), \( y \neq x_o \), there exists \( g \in \mathcal{X}_o \) such that \( g(y) \neq g(x_o) = 0 \). Since \( g \in \mathcal{X}_o \) implies \( g(x_o) = 0 \), it follows that if \( f \in L(\mathcal{X}_o) \), then \( f(x_o) = 0 \). Hence \( L(\mathcal{X}_o) \subseteq \{ f \mid f \in \mathcal{X}, f(x_o) = 0 \} \).

Now suppose \( f \in \mathcal{X} \), \( x_o \in X \), \( f(x_o) = 0 \) and consider...
first the case when \( x \in X, y \in X, x \neq x_0, y \neq x_0, x \neq y \).

Since \( X \) is a separating family, there exists \( g \in X_0 \) such that \( g(x) \neq g(y) \). Also there exists \( h_1 \in X_0 \) such that \( h_1(x) \neq 0 \), and there exists \( h_2 \in X_0 \) such that \( h_2(y) \neq 0 \) since \( x \neq x_0, y \neq x_0 \). Therefore if \( x \in X, y \in X, x \neq x_0, y \neq x_0, x \neq y \), then \( (P(X_0(x, y)))^\Delta = R_2 \). Hence if \( f \in X \), we have \((f(x), f(y)) \in (P(X_0(x, y)))^\Delta \) in this case. Now consider the case where \( x_0 \in X, y \in X \) and \( x_0 \neq y \). Then there exists \( g \in X_0 \) such that \( g(y) \neq g(x_0) = 0 \). Since \( h \in X_0 \) implies \( h(x_0) = 0 \), it is clear that 
\[
(P(X_0(x, y)))^\Delta = [(a, b) | (a, b) \in R_2, a = 0].
\]
Thus if \( f \in X \) and \( f(x_0) = 0 \), we have \((f(x_0), f(y)) \in (P(X_0(x, y)))^\Delta \). Hence, in every case, if \( f \in X_0 \) and if \( f(x_0) = 0 \), we have \((f(x), f(y)) \in (P(X_0(x, y)))^\Delta \) for \( x \in X, y \in X, x \neq y \). Therefore \( f \in L(X_0) \), and \([f|f \in X, f(x_0) = 0] \subset L(X_0) \). Since we have containment both ways, we conclude that there exists \( x_0 \in X \) such that \( L(X_0) = [f|f \in X, f(x_0) = 0] \).

**Theorem 3.63**: Suppose \( X \) is a compact topological space, suppose \( X_0 \subset X \), suppose \( X \) is a separating family for \( X \). Then \( X_0 \) is a separating family for \( X \) if and only if either \( L(X_0) = X \) or else there is \( x_0 \in X \) such that 
\( L(X_0) = [f|f \in X, f(x_0) = 0] \).

**Proof**: Suppose \( X \) is a separating family and either \( L(X_0) = X \) or else there is \( x_0 \in X \) such that \( L(X_0) = [f|f \in X, f(x_0) = 0] \). Also, suppose \( L(X_0) \) is not a separating family and hence \( L(X_0) \neq X \). Then for all
f \in L(\mathcal{X}^o), f(x^o) = 0. Also since L(\mathcal{X}^o) is not a separating family, there exists points x \in X, y \in X, x \neq y such that 
\hat{f}_o(x) = \hat{f}_o(y) for every \hat{f}_o \in L(\mathcal{X}^o). Let f be an arbitrary function in \mathcal{X}, and define the function \hat{f}_o : X \to \mathbb{R} by 
\hat{f}_o(z) = f(z) - f(x^o) for z \in X. Then \hat{f}_o is continuous and vanishes at x^o. Thus \hat{f}_o \in L(\mathcal{X}^o). Since \hat{f}_o(x) = \hat{f}_o(y), we have \hat{f}_o(x) = f(x) - f(x^o) and \hat{f}_o(y) = f(y) - f(x^o) and so 
f(x) = f(y) for all f \in \mathcal{X}. But this can't happen since \mathcal{X} is a separating family. Hence L(\mathcal{X}^o) is a separating family. Since L(\mathcal{X}^o) is a separating family, then so is \mathcal{X}^o. Now suppose \mathcal{X}^o is a separating family; then the conclusion follows from the previous theorem.
REFERENCES


