Riemannian geometry and the general theory of relativity

Marie McBride Vanisko

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RIEMANNIAN GEOMETRY  
AND  
THE GENERAL THEORY OF RELATIVITY  

By  
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**Notation Reference Page**
"The Great Architect of the Universe now begins to appear as a pure mathematician."\(^1\)

Mathematics enjoys special esteem above all the other sciences for several reasons. Its laws are absolutely certain and indisputable. This priority would not be momentous if the laws of mathematics referred only to objects of one's imagination. However, it is mathematics which affords the exact natural sciences a certain measure of security, which without mathematics they could not attain.

A fitting exemplification of this role of mathematics can be seen in the mathematics of relativity, one aspect of which is treated in this paper. The explanations of Riemannian geometry and of the general theory of relativity given herein are by no means complete. However, it is hoped that they will provide the reader with a general insight into the relation between geometry and physics.

Werner Heisenberg's statement regarding the law of

Mathematics adequately sums up this thesis:

The elemental particles of modern physics, like the regular bodies of Plato's philosophy, are defined by the requirements of mathematical symmetry. They are not eternal and unchanging, and they can hardly, therefore, strictly be termed real. Rather, they are simple expressions of fundamental mathematical constructions which one comes upon in striving to break down matter even further, and which provide the content for the underlying laws of nature. In the beginning, therefore, for modern science, was the form, the mathematical pattern, not the material thing.2

PART I

Riemannian Geometry

"A geometer like Riemann might almost have foreseen the more important features of the actual world."

INTRODUCTION TO GEOMETRY

Geometry was first studied because it was useful. Such studies date back to 3000 B.C. The word itself is derived from two Greek words meaning "earth" and "to measure". Geometrical conclusions were arrived at intuitively and then tested experimentally.

It was the ancient Greeks who first attempted a scientific approach to geometry. Greek interest in demonstrative geometry began with Thales of Miletus (600 B.C.), who has received the title of Father of Geometry because of the impetus he gave that made geometry the model for logical thought. By the time of Euclid (300 B.C.) the science of geometry had reached a well advanced stage, and from the accumulated material Euclid compiled his Elements - consisting of definitions, postulates and common notions,

---

3E. T. Bell, op. cit. (quote from E. S. Eddington), p. 484.
and propositions. The most remarkable feature of this work lies in furnishing logical proofs in logical order. It is a landmark in scientific progress.

After the progress made by the Greeks, the first decided advance in geometry came about the beginning of the nineteenth century. But before considering this, one should have a better idea of the very essence of geometry.

Geometry treats of entities which are denoted by the words point, straight line, plane, hyperplane. These entities presuppose only the validity of the axioms, which, in purely axiomatic geometry, are to be taken in a purely formal sense and, which, in practical geometry, are to be based upon intuition or experience. All other propositions of geometry are logical inferences from the axioms. Hence, the axioms determine the geometry.

Not until the nineteenth century did man fully comprehend the axiomatic view of geometry. Before this time Euclidean geometry was considered to be the only logically consistent geometry possible. However one postulate, the fifth, in Euclid's geometry perplexed men.

---

throughout the centuries, because of its complexity and lack of self-evidence. The fifth postulate stated that:

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely, meet on that side on which the angles are less than the two right angles. This has been described as "perhaps the most famous single utterance in the history of science." It is the starting point in the study of Non-Euclidean Geometry. Numerous and varied attempts made throughout many centuries to deduce the fifth postulate as a consequence of the other Euclidean postulates and common notions, stated or implied, all ended unsuccessfully. Today it is known that the postulate cannot be so derived.

It so happened that independently and about the same time the discovery of a logically consistent geometry in which the fifth postulate was denied was made by Bolyai

\[5\text{There have been several substitutions for the fifth postulate. Two are:}
\]

1. Playfair's axiom: Through a given point can be drawn only one parallel to a given line.

2. The sum of the angles of a triangle is always equal to two right angles.

\[6\text{Harold E. Wolfe, Introduction to Non-Euclidean Geometry (New York: The Dryden Press, 1945), p. 4.}\]
(1802-1860) in Hungary, Lobachevsky (1793-1852) in Russia, and probably by Gauss (1777-1855) in Germany. Each developed with satisfactory results a new geometry based on the assumption that the sum of the three angles of a triangle is less than 180°. This geometry retained all the other postulates and common notions of Euclid.

Shortly after this a new figure appeared — George F. Riemann (1826-1866). This man studied under Gauss and became the outstanding student in the long teaching career of that great mathematician. In a dissertation delivered before the Philosophical Faculty at Göttingen in 1854, he stated that: "However certain we may be of the unboundedness of space we need not as a consequence infer its infinitude. For if we assume independence of bodies from position and therefore ascribe to space constant curvature, it must necessarily be finite provided this curvature has ever so small a positive value." Riemann thus suggested a geometry in which the infinitude of the line is not assumed. Euclid assumed the infinitude of the line not only in his fifth postulate, but also, e.g., in the second postulate: "To produce a finite straight line continuously in a

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7Ibid., p.7.

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straight line." Furthermore, Riemann's notion of geometry would involve "straightest" lines rather than straight lines. The detailed investigations along these lines suggested by Riemann were carried out by others, especially Helmholtz, Lie, and Beltrami.  

Since these first discoveries of non-Euclidean geometries, many other non-Euclidean geometries have been developed, but these will not be discussed herein.

Riemannian Geometry

The modern theory of Riemannian geometry, based upon Riemann's characteristic postulate, was developed from the elementary differential geometry of surfaces in Euclidean space by the process of abstraction. The problem of formulating a geometry without the framework of straight lines and their Euclidean network of axioms and theorems is not so strange as it may first appear.

---


9For some of these developments one may refer to the works: S. Lie, *Verlesungen über Kontinuierliche Gruppen* (Leipzig: G. Scheffers, B.G. Teubner Verlagsgesellschaft, 1893), and H. von Helmholtz, "Über die Tatsachen, die der Geometrie zu Grunde Liegen," *Wissenschaftliche Abhandlungen* (Leipzig:1883).
Before the reader continues in this explanation of Riemannian Geometry he should familiarize himself with the theory of tensors given in the appendix. At all times in what is to follow the Einstein summation convention will be used, i.e., whenever in the same expression the same index appears twice, this will imply that this expression is to be summed with respect to that index for all admissible values of the index. Also, with regard to notation used, a function \( f(u', \ldots u^n) \) will be said to be of class \( C^n \) provided the first \( n \) partial derivatives of \( f \) with respect to the \( u^i(\ldots u^n) \) exist and are continuous.

A Riemann space is an open set in a Cartesian space \( E_n \) in which is defined a positive definite symmetric metric \( ds^2 = q_{ij}(r)dx^i dx^j \). A Riemann manifold is a manifold.

---

10 Definition of a manifold: First let \( F \) be a mapping of \( E \) into \( E_m(n, m \neq 4 \) for the purposes in this paper). \( F(p + tv) \) (real, \( c < t < e \)) is a mapping of \( p + tv \) into \( E_m \) when \( p \in E_n, v \in V_n(V_n \text{ is some set of vectors in } E_n) \). If \( F'(p + tv) \) exists at \( t = 0 \) and \( |F'(p + tv)| > 0 \), then the \( F \) is regular at \( p \), with respect to \( v \). If \( F \) is regular with respect to every \( p \in E_n, v \in V_n \), then \( F \) is regular. Now, a one-to-one regular map \( F: E_n \rightarrow E_m, \) restricted to a domain \( D(D \in E_n) \), is a patch if and only if \( D \) is open in \( E_n \). It is a proper patch if and only if \( F' \) is continuous. Finally, a \( k \)-dimensional manifold is a subset of \( E_m \) together with a set \( P \) of proper patches such that a) It is covered by the ranges of the patches in \( P \), b) \( f \in P \) implies the domain of \( f \in E_k \), and c) \( f, g \in P \) implies \( f'g \) and \( g'f \) are differentiable functions defined on open sets of \( E_k \).
on which is defined a positive definite symmetric twice covariant tensor field \( G = q_{ij}(u) \). The metric tensor \( q_{ij} \) is assumed to be \( C^4 \); coordinate transformations are admissible if they are at least \( C^5 \) and one-to-one. The resulting geometry is the Riemannian Geometry.\(^{11}\)

The simplest way of describing the geometrical properties of a Riemann space is to identify it locally, as far as possible, with a Euclidean space \( E^n \). This suggests an investigation of the properties of curves and surfaces in \( E_3 \).

**GEOMETRICAL VECTOR ANALYSIS**

Let a Cartesian system of axes be determined by a point (the origin) and an orthonormal basis \((\vec{b}_1, \vec{b}_2, \vec{b}_3)\). The \( \vec{e}_i (i=1,2,3) \) will be termed base vectors. Then the position vector \( \vec{r} \) of any point \( P(x^1, x^2, x^3) \) can be represented in the form \( \vec{r} = \vec{b}_i \cdot \vec{x}^i \). If \( P' = P'(y^1, y^2, y^3) \) is defined by the formula \( \vec{r}' = \frac{\partial \vec{r}}{\partial y^i} \cdot dy^i \), the displacement vector \( \vec{dr} \) from \( P \) to \( P' \) is defined by the formula \( \vec{dr} = \frac{\partial \vec{r}}{\partial y^i} \cdot dy^i \). Let the square of the element of arc between the points \( P \) and \( P' \) be given by \( ds^2 = (dr)^2 = \vec{dr} \cdot \vec{dr} \), then

\[
\begin{align*}
   ds^2 &= \vec{e}_i \cdot \vec{e}_j \cdot dy^i \cdot dy^j \\
   &= \sum_{ij} q_{ij} dy^i \cdot dy^j
\end{align*}
\]

\(^{11}\)The reason for this will be obvious in what is to follow.
Now, consider an admissible transformation of coordinates from the Cartesian system \( Y \) to a curvilinear system \( X \) given by \( x^i = x^i(y^1, y^2, y^3) \) (\( i = 1, 2, 3 \)).

The position vector \( \vec{r} \) can be thought of as a function of the coordinates \( x^i \). Hence

\[
\text{d} \vec{r} = \frac{\partial \vec{r}}{\partial x^i} \text{d}x^i;
\]

\[
\text{d}s^2 = \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j} \text{d}x^i \text{d}x^j = q_{ij} \text{d}x^i \text{d}x^j,
\]

where \( q_{ij} = \frac{\partial x^i}{\partial y^j} \frac{\partial x^j}{\partial y^i} \). Geometrically, the \( \frac{\partial \vec{r}}{\partial x^i} \) \( (i = 1, 2, 3) \) form a basis for the \( x^i \) coordinate curves at the point \( P \).

Letting \( \frac{\partial \vec{r}}{\partial x^i} = \hat{a}^i \) then

\[
\text{d} \vec{r} = \hat{a}^i \text{d}x^i;
\]

\[
q_{ij} = \hat{a}^i \cdot \hat{a}^j = \hat{a}^i \cdot \hat{a}^j = q_{ij}.
\]

Note that the base vectors \( \hat{a}^i \) are not necessarily unit vectors and are not independent of the coordinates \( (x^1, x^2, x^3) \) as \( \hat{b}^i \) are independent of \( (y^1, y^2, y^3) \).

The relation between the two sets of base vectors can be obtained by noting the expression for \( \text{d} \vec{r} \)

\[
\text{d} \vec{r} = \hat{b}^i \text{d}y^i = \hat{a}^j \text{d}x^j
\]

But from the transformation, \( \text{d}y^i = \frac{\partial y^i}{\partial x^j} \text{d}x^j \). So
And since $dx^1$ is arbitrary one sees that the base vectors transform according to the law for transformation of components of covariant tensors of rank one. And, consequently, from their definition the $q_{ij}$'s are seen to be the components of a symmetric covariant tensor of rank two.

If $\hat{\mathbf{v}}$ is a fixed vector, then there does exist a $df$ such that $\hat{\mathbf{v}} = kdf$ where $k$ is a suitably chosen constant. Thus

$$\hat{\mathbf{v}} = \hat{a}_i \cdot kd\gamma^i = \hat{a}_i \cdot \lambda^i$$

where $\lambda^i = kdx^i$. Thus $\lambda^1, \lambda^2, \lambda^3$ are the components of $\hat{\mathbf{v}}$ in the $x^1, x^2, x^3$ directions respectively. Furthermore, they may be termed the contravariant components of $\hat{\mathbf{v}}$, since $\hat{\mathbf{v}}$ is an invariant and the $\hat{a}_i$'s are covariant vectors.

Projecting $\hat{\mathbf{v}}$ orthogonally on the directions of $\hat{a}_i, \hat{a}_j, \hat{a}_k$ one obtains from the expression $\hat{\mathbf{v}} = \lambda^i \hat{a}_i$, \n
$$\hat{v}_i \hat{a}_j = \lambda^i \hat{a}_i \cdot \hat{a}_j = \lambda^i q_{ij} \lambda^j$$

where $\lambda^i = q_{ij} \lambda^j$. Thus $\lambda^i$ are the covariant components of the vector. In the terminology of tensors one would say that $\lambda^i$ and $\lambda^j$ are associated components of the same tensor.

\[\text{References:}\]  \[\text{Refer to the appendix, p. 52.}\]
Now let \( q_{ij} = G_{ij} \) where \( G_{ij} \) is the cofactor of
\( q_{ij} \) in the matrix \( (q_{ij}) \). Then by the theory of determinants
\[ q_{ik} q_{ij} = \delta_k^i \]
and so
\[ q_{ik} \lambda^i = q_{ik} q_{ij} \lambda^j = \lambda^i. \]
Therefore one can write \( \vec{v} \) as
\[ \vec{v} = (q_{ik} \lambda^i) \vec{a}_k, \]
where \( \vec{a}_i \cdot q_{ik} \vec{a}_k \). Then
\[ \vec{a}_i \cdot \vec{a}_j = (q_{ik} \delta^k_j)(q_{ik} \vec{a}_k) = q_{ik} q_{ij} \delta^k_j \]
\[ = \delta^i_j. \]
And
\[ \vec{a}_i \cdot \vec{a}_j = q_{ik} \delta^k_j = q_{ij}. \]

Note that the \( q_{ij} \) and \( q_{ik} \) so defined are the fundamental
tensors.\(^{13}\)

Thus, it has been pointed out that geometrically the
covariant components \( \lambda^i \) of a vector \( \vec{v} \) are the components
of \( \vec{v} \) in the \( x^i \) directions. The geometrical representations
of the covariant components \( \lambda^i \) of \( \vec{v} \) are the orthogonal
projections of \( \vec{v} \) in the \( x^i \) directions.

\(^{13}\)Refer to the appendix, p. 58.
VECTOR FIELD DIFFERENTIATION

Now, consider some region of \( \mathbb{R}^3 \), say \( R \), covered by a Cartesian coordinate system \( Y : (y^1, y^2, y^3) \). Let there be uniquely defined at every point \( P \) of \( R \) a vector \( \vec{A} \). The set of all vectors \( \vec{A} \) in \( R \) then constitutes a vector field on \( R \). Assume that the components of \( \vec{A} \) are continuous and differentiable functions of \( y^i \) in \( R \). Introducing an admissible transformation to curvilinear coordinates:

\[
T: x^i = \xi^i (y^1, y^2, y^3) \quad (i = 1, 2, 3),
\]

and again letting \( (\hat{a}_i, \hat{a}_j, \hat{a}_k) \) where \( \hat{a}_i = \frac{\partial x^i}{\partial y^j} \) and \( \vec{r} \) is the position vector of a given point \( P \) in \( R \), be a basis for the curvilinear coordinate system at the point \( P \), one has for \( \vec{A} \) at \( P \)

\[
\vec{A} = A^i \hat{a}_i
\]

where the \( A^i \) are the components of \( \vec{A} \) in the \( x^i \) directions.

These components will be continuous and differentiable functions of the \( x^i \) at the point \( P \) with position vector \( \vec{r} \).

What will be the vector change from the point \( P (x^1, x^2, x^3) \) to the point \( P' (x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 + \Delta x^3) \)? Keep in mind that, not only will the contravariant components of \( \vec{A} \) change, but also the base vectors, since the position vector \( \vec{r} \) changes. Thus one has for the vector change

\[
\Delta \vec{A} = (A^i + \Delta A^i)(\hat{a}_i + \Delta \hat{a}_i) - A^i \hat{a}_i
\]

\[
= \Delta A^i \hat{a}_i + A^i \Delta \hat{a}_i + \Delta A^i \Delta \hat{a}_i
\]

and, consequently, the partial derivative of \( \vec{A} \) with respect to \( x^j \) is
One can calculate $\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}$ as follows. By definition $q_{ij} \mathbf{A}_i \cdot \mathbf{A}_j$

So

1. $\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \mathbf{A}_k \right) = \frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k$

Since $T$ is at least of class $C^2$

$\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = \mathbf{A}_k$, i.e., $\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \mathbf{A}_k \right)$

Now $[i, j, k] = \frac{1}{2} \left( \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j \partial x_i} - \frac{\partial^2}{\partial x_i \partial x_j} \right)$ so, substituting in 0, 2, and 3 one obtains

$\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = [i, j, k] \mathbf{A}_k$

and multiplying both sides by $\mathbf{A}_k$ and summing

or

$\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = [i, j, k] \mathbf{A}_k \cdot \mathbf{A}_k$

or

$\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = [i, j, k] \mathbf{A}_k \cdot \mathbf{A}_k$

so

$\frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \mathbf{A}_k \right) = \{i, j\} \mathbf{A}_k$

And from the formula for $\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}$ one obtains

$\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j} \mathbf{A}_k \right) = \left( \frac{\partial^2}{\partial x_i \partial x_j} + \{i, j\} \mathbf{A}_k \cdot \mathbf{A}_k \right) \mathbf{A}_k$

Note that the expression in parentheses is precisely $A_k^i$. 15

Hence

$\frac{\partial^2}{\partial x_i \partial x_j} \mathbf{A}_k = A_k^i \mathbf{A}_i$

---

14 Refer to appendix, p. 58.

15 Refer to appendix, p. 60.
That is to say, the covariant derivative $A^i_{ij}$ of the vector $A^i$ is a vector whose components are the components of $\frac{dA^i}{dx_j}$ with respect to a base system $\dot{a}_i$.

A similar expression can be derived for the case when $\dot{A}$ is expressed in terms of its covariant components.

$$\dot{A} = A_i \dot{a}^i$$

$$\frac{d\dot{A}}{dx} = \frac{dA^i}{dx_j} \dot{a}_j + \frac{d\dot{a}^i}{dx_j} A_j$$

The $\frac{d\dot{a}^i}{dx_j}$ can be found by differentiating the expression

$$\dot{a}_i \cdot \dot{a}_j = \delta^i_j$$

$$\frac{dA^i}{dx_j} \dot{a}_j + \frac{d\dot{a}^i}{dx_j} A_j = 0$$

$$\frac{dA^i}{dx_j} \dot{a}_j = -\dot{a}_i \cdot a_k \frac{dA^k}{dx_j}$$

$$= -\{j^i_k \dot{a}_j$$

Then

$$\frac{d\dot{A}}{dx} = \frac{dA^i}{dx_j} \dot{a}_j - \{j^i_k A_j \dot{a}^k = (\frac{dA^i}{dx_j} - \{j^i_k A_j \dot{a}^k = A^i_{ij} \dot{a}_j$$

**INTRINSIC DIFFERENTIATION**

Suppose that in a region $R$ of $E_3$ on which is defined a vector field $\dot{A}(x)$ there is a curve continuous and differentiable in $R$ given by the formula

$$C: x^i = x^i(t) \quad t_1 \leq t \leq t_2$$

Then for points on that curve one can consider

$$\frac{dA^i}{dt} = \frac{dA^i}{dx_j} \frac{dx_j}{dt}$$

$$= A^i_j \frac{dx_j}{dt} \dot{a}_j$$

$$= (\frac{dA^i}{dx_j} + \{j^i_k A_j \frac{dx_j}{dt} ) \dot{a}_j$$

$$= \frac{dA^i}{dt} \dot{a}_j$$
where one defines \( \frac{dA^i}{dt} + \{i, j \} A^i \frac{dx^j}{dt} \) to be the absolute or intrinsic derivative of \( A^i \) with respect to the parameter \( t \). In light of its relation to the covariant derivative one can see that it follows the familiar rules for the differentiation of sums and products, and that \( \frac{dA^i}{dt} = 0 \). Thus, the fundamental tensors \( q^i_j \) and \( q^i_j \) act as constants with respect to differentiation. This definition of the intrinsic derivative could obviously be extended to any general tensor.

Next, it is of interest to consider under what conditions the components \( \vec{\alpha} \) of the vector field are parallel, i.e., of equal magnitude and direction, along the curve \( C \). This will be the case provided \( \frac{dA^i}{dt} = 0 \) and since
\[
0 = \frac{dA^i}{dt} = \frac{dA^i}{dt} \frac{dx^i}{dt},
\]
and \( \frac{dx}{dt} \) is arbitrary, this implies that
\[
0 = \frac{dA^i}{dt} = \frac{dA^i}{dt} + \{i, j \} A^i \frac{dx^j}{dt} \quad (i = 1, 2, 3).
\]
It can be shown that any solution to this set of differential equations yields a parallel vector field along \( C \).

This idea of parallel vector fields can be extended to the whole region \( R \). Given any continuous and differentiable curve
\[
C: \; \chi^i = \chi^i(t) \quad (i = 1, 2, 3)
\]
in \( R \), one has the relation

\[\text{---}^{16}\text{---}
\]

and since $\frac{d\xi}{dt}$ and $\vec{a}$ are arbitrary, the parallel vector field in $\mathbb{R}$ satisfies the system $A^i_{;j} = 0$. Or, in terms of the covariant components $\widetilde{A}_{;j} = 0$. These conditions can be taken to be the definition of a parallel vector field.\textsuperscript{17}

**CURVILINEAR COORDINATES ON A SURFACE**

It is natural to inquire next about the properties of a surface. For convenience consider a surface as being imbedded in $\mathbb{E}_3$ and define a set of orthogonal Cartesian axes $Y: (y^1, y^2, y^3)$ in the space in which the surface is imbedded. A surface representation in Gaussian parameters, $u^i$ and $u'$, can be defined in this way as a two-dimensional subset of points in the three-dimensional Euclidean space. The equations for the surface can be thought of as a transformation from a set of three-dimensional Cartesian coordinates to a set of two-dimensional curvilinear coordinates. In this way the $u^i$ could be envisaged as the coordinates of the points on the surface. One can choose for the same

surface such systems of Gaussian surface coordinates in a
great many ways, by admissible transformations of the form
\[ v_i' = v_i (u^i, u^2) \quad (i = 1, 2) \]

The problem of the differential geometry is to express those laws which have an intrinsic geometric meaning, ie., properties of the surface itself, in a form which is independent of the accidental choice of surface coordinates.

A study of the intrinsic properties of a surface is made to depend on a certain quadratic differential form describing the metric character of the surface. This quadratic form will now be derived.

In what is to follow Latin indices will be used with respect to the Y coordinate system and these will range from 1 to 3. Greek indices will be used for the curvilinear coordinate system and will take on values of 1 and 2. Let

\[ C: \quad u^* = u^* (t) \]

represent the equations of a curve C on the surface S. As viewed as a curve in \( \mathbb{E}_3 \), the square of an arc element on C is

\[ ds^2 = dy_i dy_i = \left( \frac{\partial y_i}{\partial u^r} \right) \left( \frac{\partial y_i}{\partial u^s} \right) du^r du^s \]

\[ = \frac{\partial y_i}{\partial u^r} \frac{\partial y_i}{\partial u^s} du^r du^s = a_{i\ell} du^i du^\ell \]

where

\[ a_{i\ell} = \frac{\partial y_i}{\partial u^r} \frac{\partial y_i}{\partial u^s} = \frac{\partial y_i}{\partial u^r} \frac{\partial y_i}{\partial u^s} = \frac{\partial y_i}{\partial u^r} \frac{\partial y_i}{\partial u^s} = a_{i\ell} \]

The expression \( a_{i\ell} du^i du^\ell \) is termed the first fundamental
quadratic form of the surface. The length of an arc of the curve is given by
\[ s = \int_{t_1}^{t_2} \sqrt{a_{\mu\nu} \frac{du^\mu}{dt} \frac{du^\nu}{dt}} \, dt. \]

As was encountered before, since \( ds^2 \) is an invariant and \( du^\alpha \), by definition, are the components of a contravariant tensor of rank one, it follows that \( a_{\mu\nu} \), so defined, is a covariant symmetric tensor of rank two. \( a_{\mu\nu} \) is termed the covariant metric tensor of the surface. The reciprocal contravariant metric tensor \( a^{\mu\nu} \) can be defined by the formula
\[ a^{\mu\nu} a_{\nu\gamma} = \delta^\mu_\gamma. \]

The matter to consider now is: Given two points \( t_1 \) and \( t_2 \) on the surface \( S \), what curve through these points should be chosen in order that the distance along the curve from \( t_1 \) to \( t_2 \) be a minimum? This is the problem of geodesics and this problem deals with some fundamental concepts of the calculus of variations.

Let \[ I = \int_{t_1}^{t_2} F(x, y, \frac{dy}{dx}) \, dx \]
where \( F \) denotes a given functional form. The functional relation between \( y \) and \( x \) is not known and the problem consists in finding this relation so that \( I \) is a maximum or minimum.\(^{18}\)

Let \( y \) be a single-valued continuous and differentiable

function of \( x \) in the interval \((a, b)\). Assume that \( F(x, y, \frac{dy}{dx}) \) possesses partial derivatives of \( x, y \) and \( \frac{dy}{dx} \) of at least the fourth order in an interval which includes \((a, b)\).

Let 
\[ y = s(x) \]
be the equation of the admissible curve passing through \( x = a \) and \( x = b \) for which \( I \) is a maximum or minimum.

Let 
\[ y = s(x) + \epsilon t(x) \]
where \( \epsilon \) is an arbitrary constant independent of \( x \) and \( y \), and \( t(x) \) denotes any arbitrary function of \( x \) independent of \( \epsilon \), be another curve passing through \( x = a \) and \( x = b \).

With this restriction on \( t(x) \) the ordinate \( y \) is said to be subjected to weak variations. Since both curves defined pass through the points \( x = a \) and \( x = b \), it follows that \( t(a) = t(b) = 0 \).

Differentiating the expression \( y = s(x) + \epsilon t(x) \) with respect to \( x \), one obtains 
\[ \frac{dy}{dx} = s'(x) + \epsilon t'(x) \, . \]

Also note that as \( \epsilon \to 0 \), \( y = s(x) + \epsilon t(x) \) approaches \( y = s(x) \).

Let 
\[ I_s = \int_a^b F(x, s, s') \, dx \]
and let 
\[ I_s + \delta I_s = \int_a^b F(x, s + \epsilon t, s' + \epsilon t') \, dx \, . \]

Since \( F \) is differentiable, the mean value theorem for functions of several variables may be applied. Thus 
\[ F(x, s + \epsilon t, s' + \epsilon t') = F(x, s, s') + \epsilon \left( t \frac{\partial F}{\partial s} + t' \frac{\partial F}{\partial s'} + \frac{\epsilon}{2} \left( t^2 \frac{\partial^2 F}{\partial s^2} + 2tt' \frac{\partial^2 F}{\partial s \partial s'} + t'^2 \frac{\partial^2 F}{\partial s'^2} \right) \right) \]
\[ + \theta(\epsilon^2) \]
where \( \theta(\epsilon^3) \) refers to a collection of terms each of which contains \( \epsilon \) raised to at least the third power. Therefore

\[
\delta I_s = \epsilon \int_a^b \left( t \frac{\partial f}{\partial s} + t' \frac{\partial f}{\partial s'} \right) dx
+ \epsilon^2 \int_a^b \left( t'' \frac{\partial f}{\partial s''} + 2tt' \frac{\partial^2 f}{\partial s \partial s'} + t' \frac{\partial f}{\partial s''} \right) dx + \theta(\epsilon^3).
\]

If \( I \) is a maximum (minimum) \( \delta I_s \) must be negative (positive) for all sufficiently small values of \( \epsilon \) whether positive or negative. Hence, in order for a maximum or minimum to occur, the coefficient of \( \epsilon \) (called the first variation) must be zero, and the coefficient of \( \epsilon^3 \) (called the second variation) must be less than zero for a maximum or greater than zero for a minimum.

In the case at hand one need only consider the expression

\[
\int_a^b \left( t \frac{\partial f}{\partial s} + t' \frac{\partial f}{\partial s'} \right) dx = 0.
\]

Integrating this expression by parts one obtains

\[
0 = \int_a^b \left( t \frac{\partial f}{\partial s} + t' \frac{\partial f}{\partial s'} \right) dx = \left[ \frac{t \partial f}{\partial s} \right]_a^b - \int_a^b \left( \frac{\partial f}{\partial s} + \frac{\partial f}{\partial s'} \right) dx
+ \left[ \frac{t' \partial f}{\partial s'} \right]_a^b - \int_a^b \left( \frac{\partial f}{\partial s'} \right) dx.
\]

It can easily be verified that since \( t(x) \) is an arbitrary continuous function and \( t(a) \cdot t(b) = 0 \) the above integral is zero if and only if

\[
\frac{\partial f}{\partial s} - \frac{\partial^2 f}{\partial s \partial s'} = 0.
\]

Therefore, solving this expression for \( y \cdot s(x) \) yields a solution for which the value of \( I \) is a maximum or minimum.

These ideas can readily be extended to the case where there are \( n \) parameters. One may state the following theorem.
THEOREM: Let the values of \( t_0 \) and \( t_1 \) and the functional form \( F \) be given. Then the integral
\[
\int_{t_0}^{t_1} F(q, \ldots, q_\ell; \dot{q}, \ldots, \dot{q}_\ell; t) \, dt
\]
where the \( q \)'s are arbitrary functions of \( t \), is a maximum or a minimum for weak variations when the \( q \)'s satisfy the \( n \) equations
\[
\frac{\partial F}{\partial q_m} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{q}_m} \right) = 0 \quad (m = 1, \ldots, n).
\]

GEODESICS IN \( R_n \)

Now problems of finding curves of minimum length joining a pair of given points on the surface can be discussed. Instead of restricting oneself to the two-dimensional manifold as was developed previously, one can carry out the calculations for the case of the \( n \)-dimensional Riemannian manifolds.

Let the square of an arc element on the \( n \)-dimensional Riemannian manifold \( R_n \) be determined by
\[
ds^2 = \sum q_{ij} \, dx^i \, dx^j
\]
where the \( q_{ij} = q_{ji} \) are specified functions of the variables \( x^i \). It will be assumed that \( ds^2 \) is positive definite in a certain region of \( R_n \), and the \( q_{ij} \) are of class \( C^2 \). The length of a curve \( C \) in \( R_n \) given by
\[
C: \quad \overrightarrow{x} = x^i(t) \quad t, s \leq t \leq t_a
\]
is
\[
s = \int_{t_s}^{t_a} \sqrt{\sum q_{ij} \dot{x}^i \dot{x}^j} \, dt
\]
where the dots denote differentiation with respect to \( t \.

The functional relationships yielding the minimum values

\[20\text{Ibid., pp.62-63.}\]
for the above integral are termed geodesics in $R^n$. If one applies the ideas from the calculus of variations to this case where

$$F = \sqrt{q_{kl} \dot{x}^k \dot{x}^l},$$

the determination of a geodesic connecting $t_1$ and $t_2$ requires the solution of the differential equations

$$\frac{dF}{dx} - \frac{d}{dt} \left( \frac{dF}{dx} \right) = 0 \quad (i = 1, \ldots, n).$$

Now

$$\frac{dF}{dx} = \left( \frac{\dot{x}^k \dot{x}^l}{2} \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\partial q_{kl}}{\partial x^j} \right) \right) \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right).$$

and

$$\frac{d}{dt} \left( \frac{dF}{dx} \right) = q_{kl} \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right) + q_{kl} \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right) - q_{kl} \left( \frac{d}{dt} \left( \frac{\partial q_{kl}}{\partial x^j} \right) \right).$$

Therefore

$$0 = \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right) - \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right) - \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right) + \left( \frac{\partial q_{kl}}{\partial x^j} \right) \left( \frac{\dot{x}^k \dot{x}^l}{2} \right);$$

$$q_{kl} \left( \frac{\partial q_{kl}}{\partial x^j} \right) \frac{\dot{x}^k \dot{x}^l}{2} = \frac{\partial q_{kl}}{\partial x^j} \frac{d}{dt} \left( \frac{\partial q_{kl}}{\partial x^j} \right);$$

$$q_{kl} \frac{\dot{x}^k \dot{x}^l}{2} = \frac{\partial q_{kl}}{\partial x^j} \frac{d}{dt} \left( \frac{\partial q_{kl}}{\partial x^j} \right).$$

If one chooses the parameter $t$ to be the arc length $s$,

$$\frac{d}{dt} \left( \frac{\partial q_{kl}}{\partial x^j} \right) = 0 \quad \text{and} \quad \frac{d^2 \dot{x}^c}{ds^2} = 0,$$

the equations of the geodesics then simplify to read

$$q_{kl} \left( \frac{\partial q_{kl}}{\partial x^j} \right) + \left( \frac{\partial q_{kl}}{\partial x^j} \right) \frac{\dot{x}^k \dot{x}^l}{2} = 0$$

where the dots denote differentiation with respect to the arc parameter $s$. Or, in alternate form, multiplying by

$$q_{ij} \text{ and summing one has }$$

$$\dddot{x}^j + \left\{ \dot{x}^i \right\} \dddot{x}^i = 0.$$ 

Since this is an ordinary second order differential equation, it possesses a unique solution when the values $x^i(s)$ and the first derivatives $\frac{d^2 x^i}{ds^2}$ are prescribed.

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arbitrarily at a given point $x^i(s_0)$. The uniqueness of the solution implies that through each point in the n-dimensional Riemann space and in any given direction there passes a unique geodesic.

Note that, if the manifold is Euclidean, the Christoffel symbols vanish and hence the geodesic equation becomes

$$\frac{d^2 x^i}{ds^2} = 0,$$

the solution of which is a straight line. Thus, the geodesics in $E^n$ are straight lines.

In the special case of a surface $S$ considered as a Riemannian two-dimensional manifold $R_2$, covered by Gaussian coordinates $u^a (u^1, u^2)$, the geodesic equation assumes the form

$$\frac{d^2 u^a}{ds^2} + \sum_{i,j} \Gamma^a_{ij} \frac{du^i}{ds} \frac{du^j}{ds} = 0,$$

and since

$$\frac{d^2 u^a}{ds^2} = -\sum_{i,j} \Gamma^a_{ij} \frac{du^i}{ds} \frac{du^j}{ds},$$

initial values of $u(s)$ and $(\frac{du}{ds})(s)$, i.e., for $s = 0$, yield initial values for $(\frac{du}{ds})(s)$ and all higher derivatives also, by differentiation of the above expression. Therefore, in accordance with the formation of the Taylor series, one can write

$$u^a = u^a_0 + \left(\frac{du^a}{ds}\right)_0 s + \frac{1}{2} \left(\frac{d^2 u^a}{ds^2}\right)_0 s^2 + \frac{1}{3!} \left(\frac{d^3 u^a}{ds^3}\right)_0 s^3 + \cdots$$

for all values of $s$ for which the series converge, the subscript 0 indicating initial values.\(^{21}\)

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Since the values \( \left( \frac{\partial u^*}{\partial s} \right)_{\mu_0} \) determine the direction of the geodesic at the initial point \( u^*(\mu_0) \) one can state the following theorem.

**THEOREM:** Through each point in a surface and in any given direction there passes a unique geodesic.\(^{22}\)

Again consider the series solution form
\[
\tilde{u}^* = u^*_0 + \left( \frac{du^*}{ds} \right)_0 \xi + \frac{1}{2} \left( \frac{d^2u^*}{ds^2} \right)_0 \xi^2 + \ldots
\]
and make the substitution \( \tilde{u}^* = \left( \frac{du^*}{ds} \right)_0 \xi \). Then
\[
\tilde{u}^* - u^*_0 = \tilde{u}^* + a^* (\tilde{u}')^2 + b^* (\tilde{u}') (\tilde{u})^2 + c^* (\tilde{u})^3 + \ldots
\]
where \( a, b, c \) are functions of \( u^*_0 \) and \( u^* \). These series are convergent for values of \( \tilde{u}' \) and \( \tilde{u}^* \) in absolute value less than some fixed quantity. Since the Jacobian of \( u^* \) with respect to \( \tilde{u}^* = \tilde{u}^*(\xi, \eta, \lambda) \) for \( \tilde{u}^* = 0 \) is equal to 1, these series may be inverted giving \( \tilde{u}^* \) as power series in \( u^* - u^*_0 \) and \( u^* - u^*_0 \), which are convergent provided that \( u^* - u^*_0 \) and \( u^* - u^*_0 \) in absolute value are less than some fixed quantity.\(^{23}\) For such values of \( u^* - u^*_0 \) the values of \( \tilde{u}^* \) are uniquely determined, and consequently there passes only one geodesic through the points \( u^*_0 \) and \( u^* \). Furthermore, since
\[
\left( q_{\xi \xi} \right)_0 \frac{du^*}{ds} \frac{du^*}{ds} = 1
\]
and
\[
\tilde{u}^* = \left( \frac{du^*}{ds} \right)_0 \xi
\]
then
\[
\left( q_{\xi \xi} \right)_0 \tilde{u}^* \tilde{u}^* = s^2
\]


where \( s^2 \) denotes the square of the length of the arc of the geodesic between the points \( u^a \) and \( u^a \). Hence one can state the following theorem.

**Theorem:** Through two sufficiently near points on a surface there passes one and only one geodesic.\(^{24}\)

**Parallel Vector Fields in a Surface**

The concept of parallel vector fields along a curve imbedded in \( E^3 \) was generalized by Levi-Civita to curves imbedded in \( n \)-dimensional Riemann manifolds.\(^{25}\) Consider a surface \( S \) imbedded in \( E^3 \) and a curve \( C \) on \( S \) where

\[
C: u^a = u^a(t) \quad t_1 \leq t \leq t_2
\]

and suppose that the metric properties of \( S \) are governed by the tensor \( g_{ab} \). Let \( A^a \) be a surface vector field defined along \( C \). Then

\[
\frac{ds^2}{dt} = \frac{dA^a}{dt} + \{ \xi^a \} \frac{dA^b}{dt} \frac{du^b}{dt}
\]

and if \( A^a \) is to be parallel

\[
\delta = \frac{ds^2}{dt} - \frac{dA^a}{dt} + \{ \xi^a \} \frac{dA^b}{dt} \frac{du^b}{dt}
\]

and, choosing the parameter \( t \) to be the arc length \( s \)

\[
\frac{dA^a}{ds} + \{ \xi^a \} \frac{dA^b}{ds} = 0
\]

Now, taking \( A^a \) to be the unit tangent vector to \( C \), so that

\[
A^a = \frac{du^a}{ds} = \lambda^a
\]

one obtains

\[
\frac{d\lambda^a}{ds} + \{ \xi^a \} \frac{d\lambda^b}{ds} \frac{du^b}{ds} = 0
\]


which can be recognized as the equation of a geodesic.

This leads one to the following theorem.

**THEOREM:** The vector obtained by the parallel displacement of the tangent vector to a geodesic always remains tangent to the geodesic.\(^{26}\)

Now, consider the problem of the parallel displacement on a surface \(S\) of a vector with components \(\lambda^a (a=1,...,n)\) from a point \(P_1\) to a point \(P_2\) both on a given surface \(S\) defined by the metric coefficients \(g_{ab}\). Parallel displacements of this same vector along two different curves on the surface connecting \(P_1\) and \(P_2\) do not necessarily yield the same value for \(\lambda^a\) at \(P_2\).

Equivalently, if one displaces a vector parallel to itself around a closed path, he has no reason to expect \(\lambda^a\) to return to their initial values. The angle between the initial vector and the final vector measures another intrinsic property of \(S\), known as the Gaussian curvature of \(S\). This entity will be derived in a different manner in the next section. However, one can get some insight into this idea by actually calculating the differences between the final vectors when the parallel displacement is taken over two different paths connecting two very

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\(^{26}\)I.S. Sokolnikoff, *op. cit.*, p.166.
close points on the surface. This can be done in a very straightforward manner, simply applying the equation

\[ \frac{d\lambda^i}{dt} + \{\xi_j^i, \lambda^k\} \frac{d\lambda^k}{dt} = 0 \]

Consider two paths between the points \( P \) and \( R \). One path consists of a displacement along \( dx \) followed by a displacement along \( dy \) and the other is along the same vectors in reverse order.

The change in \( \lambda^k \) along both paths will be computed and compared. By the law of parallel displacement, the change of \( \lambda^k \) between \( P \) and \( Q \) is

\[ d\lambda^k(P, Q) = \{\xi_j^k, \lambda^\ell\} \lambda^\ell dx^\ell \]

(Unless explicitly noted, the vector \( \lambda^k \) and the Christoffel symbols are always evaluated at the point \( P \)). At \( Q \) the displaced vector is then given by

\[ \lambda^k(Q) = \lambda^k - \{\xi_j^k, \lambda^\ell\} \lambda^\ell dx^\ell \]

The Christoffel symbols at \( Q \) are, to the first order in a Taylor series expansion in the vector \( dx \)

\[ \{\xi_j^i\}_q = \{\xi_j^i\}_p + \frac{\partial}{\partial x^\ell} \{\xi_j^i\}_p dx^\ell \]

Next, applying the law of parallel displacement to the vector \( \lambda^k(Q) \), one obtains the change in \( \lambda^k \) between \( Q \) and \( R \) to the second order in the displacement vectors \( dx \) and \( dy \):

\[ d\lambda^k(Q, R) = -\{\xi_j^k, \lambda^\ell\} \lambda^\ell dx^\ell - \{\xi_j^k, \lambda^\ell\} \lambda^\ell dx^\ell dy^\ell \]

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or, rearranging terms
\[ d\lambda^\alpha (q, x) = - \left\{ \frac{\partial}{\partial x^\beta} \right\} \lambda^\alpha \, dq^\gamma - \frac{1}{2} \left\{ \partial_\beta \right\} \lambda^\alpha \, dx^\gamma \]
\[ + \left\{ \kappa^\alpha \right\} \left\{ \kappa^\beta \right\} \lambda^\alpha \, dx^\gamma \, dy^\gamma. \]

Thus, the value of \( \lambda^\alpha \) after traversing the entire path is, to second order displacement vectors \( dx \) and \( dy \)
\[ \lambda^\alpha (q, a, x) = \lambda^\alpha + d\lambda^\alpha (q, a) + d\lambda^\alpha (a, x) \]
\[ = \lambda - \left\{ \kappa^\alpha \right\} \lambda^\beta \, dx^\gamma - \left\{ \kappa^\alpha \right\} \lambda^\beta \, dy^\gamma \]
\[ - \frac{1}{2} \left\{ \partial_\beta \right\} \lambda^\alpha \, dx^\gamma \, dy^\gamma + \left\{ \kappa^\alpha \right\} \left\{ \kappa^\beta \right\} \lambda^\alpha \, dx^\gamma \, dy^\gamma. \]

The result along the second path \( P \rightarrow T \rightarrow R \) is obtained simply by interchanging \( dx \) and \( dy \):
\[ \lambda (q, r, x) = \lambda - \left\{ \kappa^\alpha \right\} \lambda^\beta \, dy^\gamma - \left\{ \kappa^\alpha \right\} \lambda^\beta \, dx^\gamma \]
\[ - \frac{1}{2} \left\{ \partial_\beta \right\} \lambda^\alpha \, dy^\gamma \, dx^\gamma + \left\{ \kappa^\alpha \right\} \left\{ \kappa^\beta \right\} \lambda^\alpha \, dy^\gamma \, dx^\gamma. \]

The difference between \( \lambda^\alpha \) obtained by parallel displacement along the two routes is therefore
\[ \Delta \lambda^\alpha = \frac{1}{2} \left\{ \partial_\beta \right\} \lambda^\alpha \, dy^\gamma \, dx^\gamma - \frac{1}{2} \left\{ \partial_\beta \right\} \lambda^\alpha \, dx^\gamma \, dy^\gamma \]
\[ + \left\{ \kappa^\alpha \right\} \left\{ \kappa^\beta \right\} \lambda^\alpha \, dx^\gamma \, dy^\gamma - \left\{ \kappa^\alpha \right\} \left\{ \kappa^\beta \right\} \lambda^\alpha \, dy^\gamma \, dx^\gamma. \]
But, by definition of the Riemann tensor, \( R \) this is precisely
\[ \Delta \lambda^\alpha = \kappa^\alpha \lambda^\beta \, dx^\gamma \, dy^\gamma. \]
Thus, the value of \( \lambda^\alpha \) at the nearby point is independent of path if and only if \( \kappa^\alpha \lambda^\beta \, dx^\gamma \, dy^\gamma = 0. \)

RIEMANN TENSOR AND GAUSSIAN CURVATURE

Once again consider the n-dimensional Riemannian

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27 Refer to appendix, p. 43.

manifold the first fundamental quadratic form of which is
\[ ds^2 = a_{\alpha \beta} du^\alpha du^\beta . \]
One can form the Christoffel symbols with respect to the surface and the corresponding Riemann tensor:
\[ R^{\alpha \beta \gamma \delta} = \frac{\partial^2 \xi^\alpha}{\partial \xi^\beta \partial \xi^\gamma} - \frac{\partial^2 \xi^\alpha}{\partial \xi^\gamma \partial \xi^\beta} + \frac{\partial \xi^\beta}{\partial \xi^\gamma} \frac{\partial \xi^\delta}{\partial \xi^\alpha} \frac{\partial \xi^\alpha}{\partial \xi^\delta} - \frac{\partial \xi^\beta}{\partial \xi^\delta} \frac{\partial \xi^\delta}{\partial \xi^\gamma} \frac{\partial \xi^\gamma}{\partial \xi^\alpha} . \]

Consider the invariant
\[ R = a^{\alpha \beta} R_{\alpha \beta} , \]
where \( R_{\alpha \beta} \) is the Ricci tensor given by
\[ R_{\alpha \beta} = \epsilon^{\gamma \delta} R_{\alpha \beta \gamma \delta} = a^{\gamma \delta} R_{\gamma \delta \alpha \beta} \]
then
\[ R = a^{\alpha \beta} a^{\gamma \delta} R_{\gamma \delta \alpha \beta} . \]

In the two-dimensional manifold there is only one unique non-vanishing component of the Riemann tensor, so in that case
\[ R = -2 R_{12,12} \left( a^{11} a^{22} - a^{12} a^{21} \right) , \]
and since by definition \( a^{11} = \frac{\partial x_1}{\partial \xi^1} , \ a^{22} = \frac{\partial x_2}{\partial \xi^2} , \ a^{12} = -\frac{\partial x_1}{\partial \xi^2} \),
\[ R = -2 k_{12,12} \left| a_{\xi^1 \xi^2} \right| , \]
and in this case the Gaussian curvature \( K \) is defined to be
\[ K = \frac{R_{12,12}}{\left| a_{\xi^1 \xi^2} \right|} . \]
Therefore
\[ R = -2 K . \]
The invariant \( R \) is sometimes called the Einstein curvature of \( S \).

Again, noting the construction of the Riemann tensor, it is clear that if the metric coefficients are constants
\[ R^{\alpha \beta \gamma \delta} = 0 . \]
On the other hand, if the Riemann tensor is zero at all points of the surface, it can be proved that there exists a
coordinate system \( X \) in which the \( \xi_a \) are constants, for if
\[
[\xi_i]' = 0, \quad \xi_i^j, k = \frac{\partial^2 \xi_i}{\partial x^k \partial x^j} \quad \text{but} \quad \xi_i^j, k = 0 \quad \text{hence} \quad \frac{\partial^2 \xi_i}{\partial x^k \partial x^j} = 0
\]
which implies that the \( \xi_i^j \) are constants. If the metric coefficients are constant, the tensorial formula for the square of the arc element reduces to the form
\[
ds^2 = dx_i dx_i'
\]
for some coordinate system \( X(x^1, x^2) \). This is the metric for the Euclidean plane.\(^{29}\) Thus, one may state a theorem.

**THEOREM:** A necessary and sufficient condition that a surface \( S \) be isometric with the Euclidean plane is that the Riemann tensor (or the Gaussian curvature) be identically zero.\(^{30}\)

**CONCLUSION**

This then should provide the reader with an insight into some of the basic concepts of Riemannian geometry. With the aid of tensors, geometric significance was given to notions of vectors, curves, and surfaces imbedded in \( E_3 \). However, the properties of these entities so derived were seen not to depend on the Euclidean space in which they were imbedded, but rather upon a particular metric in the manifold. For the greater part of a century multi-dimensional differential geometry was studied for its own intrinsic interest; but its importance has been emphasised by its application to general theories of relativity.

\(^{29}\)I.S. Sokolnikoff, *op. cit.*, p. 45.

\(^{30}\)Ibid., p. 168.
PART II

Einstein's General Theory of Relativity

"The theory of relativity represents the greatest advance in our understanding of nature that philosophy has yet witnessed."31

INTRODUCTION TO CLASSICAL PHYSICS

The history of physics records many attempts to explain physical phenomena by geometric arguments, and the problem of space has entered the foundations of Newtonian mechanics from the beginning. The law of inertia which states that a material point which is not affected by any force must have uniform motion is basic for discovering forces in nature. Every time that a nonuniform motion occurs in nature one can be sure that forces are involved. But, it is evident that a uniform motion relative to one observer will not be uniform for a second observer who is himself in nonuniform motion with respect to the first. Which one of the observers has the right to claim that the law of inertia is valid in his frame of reference?

The heliocentric theory of Copernicus would lead to a referende system in which Newtonian mechanics is valid.

But Newtonian mechanics endowed space with physical significance and introduced a distinguished coordinate system in which it is valid. Thus, absolute space was postulated by this theory of mechanics. Yet, no person whose mode of thought is logical can be satisfied with this condition of things. Why are certain reference frames given priority over other reference frames? What is the reason for this preference? For quite some time this objection was ignored.

EINSTEIN'S PRINCIPLE OF EQUIVALENCE

As mechanics developed, forces were distinguished as being actual or apparent. Apparent forces, e.g., centrifugal force associated with rotational motion, occurred only because a wrong coordinate system was used; they were the penalty for the use of an incorrect geometry. There is one criterion which distinguishes apparent forces from actual forces. Since apparent forces are all of an inertial nature and since inertia is mass-proportional, apparent forces should always be mass-proportional. If one were to observe a universal effect on all bodies considered which was precisely proportional to their mass, one should then suspect that the coordinate system was wrong and that, by a

32 Adler, Bazin, and Schiffer, op. cit., p.3.
proper choice of coordinates, this universal effect could be transformed away. Obviously this is the case for inertial forces and centrifugal forces. And there is another well-known universal force which effects every material point mass-proportionally, namely the force of gravity. One is not accustomed to calling gravity an apparent force; however, it is not difficult to show that it can indeed be transformed away by proper choice of a reference system.

Einstein's box experiment readily demonstrates this possibility. Consider an observer in a closed box who feels that he and all apparatus in the box possess a downward acceleration. He cannot look out the box, and he wishes to ascertain the reason for this acceleration by measurement inside. There are at least two possible interpretations: (1) There may be a heavy mass affixed to the bottom of the box, and the attraction by that mass on all matter in the box may be the reason for the downward acceleration; or (2) the box may be in accelerated upward motion due to a pull on a pole which is attached to the roof of the box. In mechanics there is no known effect which would allow one to distinguish between these two alternatives. Thus, intuition suggests the equality between the gravitational mass and the inertial mass.
To extend this idea even further, consider a ray of light passing horizontally through this box. It would be curved downward in the shape of a parabola if the box were accelerating upward. Thus Einstein posited that light rays are curved by a gravitational field. This hypothesis was verified by Eddington in 1919 by measuring the deflection by the sun of light from a star.33

The axiom of indistinguishability between gravity and inertia is called the principle of equivalence; and upon this axiom rests most of the fundamental concepts of the general theory of relativity.34 This important law had hitherto been recorded in mechanics, but it had not been interpreted. A satisfactory explanation can be obtained only by recognition of the following facts: The same quality of a mass manifests itself according to circumstances as "inertia" or as "weight". This fact enables one to investigate further the laws satisfied by the gravitational field itself and thus to formulate general laws of nature into equations which hold good for all systems of coordinates, i.e., are covariant with respect to any substitution whatever. It is this generally covariant property of the equations representing the laws of nature that constitutes the basic


34 This will be demonstrated in Part III.
postulate of the general theory of relativity.

RELATION BETWEEN THE SPECIAL AND GENERAL THEORIES OF RELATIVITY

Throughout this discussion there has been reference made to the general theory of relativity. At this point there should be some clarification made as to the differences between the general and the special theories of relativity.

According to Newtonian mechanics, if a mass $m$ is moving uniformly in a straight line with respect to a coordinate system $K$, then it will also be moving uniformly and in a straight line relative to a second coordinate system $K'$, provided that the latter is executing a uniform translatory motion with respect to $K$. Now this was found to be valid except in the case where one of the coordinate systems was moving at a very great velocity, approaching that of the speed of light. In such a case it was found, e.g., that moving rods are shortened and that moving clocks run slow. The exact alterations are defined by the Lorentz transformations:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vy}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

35The derivation of these Lorentz transformations can be found in Appendix I of the book *Relativity* by Albert Einstein (included in the bibliography).
where the primed terms indicate the coordinates in the K' frame, the unprimed terms the coordinates in the K frame and v the velocity in the x direction of the K' frame relative to the K frame of reference. Thus, according to these transformations, time and space are inextricably interwoven. And, taking these transformations into account, one can then say that: If, relative to K, K' is a uniformly moving coordinate system devoid of rotation, then natural phenomena run their course with respect to K' according to exactly the same general laws as with respect to K.

The general theory is, in a sense, an extension of the special theory, however much more comprehensive. For according to the general theory of relativity: Natural laws are expressed in such a way that they hold true with respect to any frame of reference whatever--even in the case where K and K' are moving in nonuniform motion relative to one another.

It should be noted here that from the standpoint of method there is an interesting difference between the special theory and the general theory of relativity. The special theory consists in the coordination of certain known experimental results, chiefly electromagnetic. The general theory on the other hand is a work of rationalization which
was in no way imposed by the facts of observation that were known at the time. Its creation is due to the genius of Einstein. Only subsequently was the general theory confirmed by the discovery of new facts. 

CRITICISM OF THE GENERAL THEORY OF RELATIVITY

It should be noted at this point that Einstein's general theory of relativity, as well as its confirmations, do not stand without criticism. One of the major verifications of the general theory involves the motion of the planet Mercury around the sun. Classical mechanics was unable to account for the shift in Mercury's orbit about the sun. The orbit is an ellipse, but the place at which Mercury is closest to the sun changes and thus the ellipse itself rotates. Part of this rotation of the ellipse can be accounted for by the gravitational pull of the other planets. But there is a slight discrepancy. Forty-three seconds of a degree every one-hundred years of this rotation cannot be explained by classical mechanics. According to Einstein's general theory of relativity this discrepancy was accounted for exactly. Now, Dr. Robert H. Dicke, a Princeton physicist, claims that 8% of this forty-three second shift is due to the oblateness of the sun.

36 A. d'Abro, op. cit., p. 462.
(the flattening of the poles caused by a strongly rotating core deep within the sun) and the consequent distortion of its gravitational field. If this claim is correct it could imply that Einstein's estimate of the amount of curvature of light is not quite correct or that something more fundamental in the theory is wrong. Present knowledge of the shape of the sun and of its internal distribution of mass is not precise enough to either verify or rule out Dicke's interpretation. However, the numerical agreement between the prediction of the general theory of relativity and the observed motion of the shift of Mercury's orbit seems too close to be only accidental. Thus, to this date, Einstein's explanation still stands.

CONCLUSION

Einstein's relativity theory thus arose in the early part of this century from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties, using only a few very convincing assumptions.


Part III

Riemannian Geometry as Applied to the General Theory of Relativity

"Had it not been for Riemann's work and for the considerable extension it has conferred upon our understanding of space, Einstein's general theory of relativity could never have arisen." 40

Geometry and Gravitation

The choice lying at the basis of the theory of general relativity is to treat gravitation on the same footing as the classical inertial forces. Since these latter forces were best understood by geometric considerations, it was natural to suspect that gravitation had a closer connection with geometry than had been realized before.

At this point the question arises as to what the object of geometric concepts is anyway. From the previous consideration of geometry in the first part of this paper, it can be said that the objects of practical geometry must be preconstructed forms of pure intuition which are the base of the judgements that one makes about real objects in empirical situations. The objects of the geometry which are actually applied to the world of things are thus these things themselves regarded from a definite point of

40A. d'Abro, op. cit., p. xiv.
Without this interpretation of geometry as being in a certain sense a natural science, Einstein himself said that he would have been unable to formulate the theory of relativity. For without it the following reflection would have been impossible: In a system of reference moving in non-uniform motion relative to an inert system, the laws of disposition of rigid bodies do not correspond to the rules of Euclidean geometry on account of the Lorentz contractions; thus, if non-inert systems are admitted, Euclidean geometry must be abandoned. The decisive step in the transition to general covariant equations would certainly not have been taken if the above interpretation had not served as a stepping stone.

By way of analogy, consider the axioms of classical mechanics from the point of view of a geometric interpretation. All fixed stars and galaxies of the universe determine a Euclidean geometry such that a free material point moves along a shortest line, i.e., a geodesic or straight line. Geometry becomes a physical reality.

Geometrizing the theory of gravitation, one would say that a heavy body modifies the geometry around it in

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such a way that the geodesics in this geometry are the curved trajectories of the attracted particles. After finding the law by which the matter affects the geometry, the actual calculation of motion would be reduced to the well studied mathematical problem of determining the geodesics of a given geometry.

In order to represent a universe with gravitation, Einstein considered Riemannian space–times, the metrics of which were supposed to determine the basic law of gravitation. He postulated:

**THE GEODESIC PRINCIPLE:** For any distribution of mass and energy the geodesics of the line element of $V_4$ define the motions of material test bodies and the paths of light rays.\[42\]

Einstein conceived the universe to be represented by a four-dimensional Riemannian space $V_4$ with the metric coefficients $g_{ij}(x^1, x^2, x^3, x^4)$ and the fundamental quadratic form

$$ds^2 = g_{ij} dx^i dx^j.$$

(In the special case of the restricted theory this reduces to

$$ds^2 = c^2 (dt)^2 - dy^i dy^i.$$  

The coefficients $g_{ij}$ in the general theory are termed the gravitational potentials. The essential problem in the


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general theory of relativity is the determination of the gravitational potentials which correspond to various states of matter, i.e., the determination of the gravitational potentials in such a way that the trajectory of particles satisfy the equations of geodesics

\[ \frac{d^2 x^i}{ds^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0. \]

EINSTEIN'S GRAVITATIONAL FIELD EQUATIONS

Observing the fundamental quadratic form for the special theory of relativity, one notes that since the metric coefficients are constants, the Christoffel symbols and the Riemann tensor are zero. Hence, the geodesics determined by this theory are simply straight lines and thus do not reflect the presence of a gravitational field. Therefore, if the manifold with the quadratic form

\[ ds^2 = g_{ij} dx^i dx^j \]

is to account for a gravitational field, the Riemann curvature tensor must not vanish.

Now, Einstein was led to the partial differential equations limiting the generality of the gravitational potentials by two essential requirements: These equations must generalize the equations of Laplace and Poisson which govern the Newtonian potential; and they must be expressible in the form of relations between tensors in \( V_4 \).
The developments in the section on Riemannian geometry would lead one to expect that the equations in tensor form to describe the gravitational field should in some way involve the Riemann tensor \( R_{\mu \nu \rho} \) since this tensor appears to contain a great deal of information about the geometric structure of space.

Recall from the theory of tensors the development and definition of the Einstein tensor

\[
G_{i}^{j} = R_{i}^{j} + \frac{1}{2} g_{i}^{j} R
\]

where \( R_{\mu \nu} \), \( R_{\mu} \) where \( R_{\mu} \) is the Ricci tensor, obtained by the contraction from the Riemann tensor.\(^{43}\) Recall that \( G_{i}^{j} \) was defined in such a way that

\[
G_{i}^{j} ; \kappa = 0
\]

Also, note that \( G_{i}^{j} = 0 \) implies, on contraction of indices that \( R = 0 \) and thus \( R_{\mu} = 0 \); obviously if \( R_{\mu} = 0 \), \( G_{i}^{j} = 0 \). All these ideas are very important in the formation of the gravitational field equations.

It is beyond the scope of this thesis to give either a precise derivation of or a solution of Einstein's gravitational field equations. They will simply be stated.

\(^{43}\)Refer to the appendix, pp. 64-65.
1) Gravitational equations for free space, replacing Laplace's equation, \( \nabla^2 \phi - 0 \), must satisfy the differential equations \( \nabla^2 \phi = 0 \) or equivalently \( G_{ij} = 0 \).

2) Gravitational equations for non-empty space, replacing Poisson's equation, \( \Delta \phi = 4\pi \rho \), must satisfy the differential equations \( G_{ij} = (\text{constant}) \ T_{ij} \) where \( T_{ij} \) represents the properties of the space geometry which encompasses all physical qualities except gravitation; and \( \nabla^2 \phi = 0 \). Note that this energy principle of matter, \( G_{ij, k} = 0 \), is a mathematical consequence of the definition of \( G_{ij} \).

The procedure for determining the motion for a test particle in a given physical situation is to describe the distribution of matter and fields by means of the energy momentum tensor, to calculate the metric field from the Einstein field equations by integrating the Einstein tensor, and to find the trajectory of the test particle as a geodesic of the Riemannian geometry. Two basic different laws are used: Einstein's field equations and requirements for geodesic motion. However, one would expect that the motion of a test particle should be contained in the field equations, since they lead to equations which determine the

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44 Adler, Bazin, and Schiffer, op. cit., p. 159.

behavior of the test particle in time and space. In 1927, Einstein and Grommer established that the postulate of geodesic motion could indeed be deduced from the field equations instead of being axiomatically required. 46

In the consideration of sufficiently great regions of space interesting results of the theory of relativity can be seen. The most interesting is the conception that this space-time system is closed upon itself (based partially on the theory that the mean density of matter is finite), just as a sphere is closed upon itself in threedimensional space. It would thus have no real limits and yet would not be infinite. Light could travel all around such a universe and come back to its source. 47 As yet there are no actual means of testing this conclusion, but other equally startling conclusions of the theory of relativity have been tested and found true. The theory of relativity has thus been widely accepted, together with the conclusion that the universe of space and time, which includes all matter, is finite and yet unbounded, owing to its peculiar geometrical structure.

46 Adler, Bazin, and Schiffer, op. cit., p. 297.

CONCLUSION

The objective of this thesis has now been realized. Einstein's gravitational equations, upon which the general theory of relativity is based, have been shown to utilize the basic ideas of differential geometry with the aid of the tensor calculus. It is wise at this point for one to recall what Einstein himself said about his general theory of relativity: "The possibility of explaining the numerical equality of inertia and gravitation by the unity of their nature gives to the general theory of relativity, according to my conviction, such a priority over the conceptions of classical mechanics, that all the difficulties encountered in development must be considered small in comparison."^48

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CONCLUSION

"Number rules the universe."^{49}

Relativity has brought about the fusion of two realms of knowledge which had hitherto been developed independently of each other — geometry and physics. This fusion is illustrated by the fundamental role that is played by the geometrical quantities $g_{ij}$ in the laws describing physical phenomena. This synthesis may be thought of as:

$$\text{Pythagorus} + \text{Newton} \rightarrow \text{Einstein}.$$^{50}

One cannot help but realize that this step in men's knowledge of nature is of an importance which it would be difficult to overestimate.

^{49}Robert W. Marks (ed.), op. cit., p. xix.

^{50}A. d'Abro, op. cit., p. 464.
APPENDIX

Elements of Tensor Analysis

INTRODUCTION

Tensor analysis is a study of abstract objects, called tensors, the properties of which are independent of the reference frames used to describe the objects. In a particular reference frame a tensor is represented by a set of functions, termed its components. It is the law of transformation of these functions from one coordinate system to another that determines whether a given set of functions represents a tensor.

TRANSFORMATION OF COORDINATES

Throughout this discussion only those functional transformations \( T : y^i = y^i(x^1, x^2, ..., x^n) \) \((i = 1, ..., n)\) will be dealt with which possess the following properties.

1) The functions \( y^i(x) \) are continuous together with their first partial derivatives in some region \( R \) of the \( n \)-dimensional manifold \( V_n \).

2) The Jacobian determinant \( J = \left| \frac{\partial y^i}{\partial x^j} \right| \) does not vanish at any point of the region \( R \).

It would follow then that a single-valued inverse exists:

\[ T^{-1} : \quad x^i = x^i(y^1, y^2, ..., y^n) \quad (i = 1, ..., n), \]
and the functions \( x^i(\eta) \) are also of class \( C^1 \) in \( R \). Transformations possessing these properties will be termed admissible transformations.

It is important to note that the product of two admissible transformations is an admissible transformation. This follows immediately from the fact that the Jacobian of the product transformation equals the product of the Jacobians of the transformations entering in the product. Also, the product transformation possesses an inverse, since the transformations appearing in the product have inverses. The identity transformation \( x^i = \eta^i \) surely exists. The associative law \( T_1(T_2T_3) = (T_1T_2)T_3 \) obviously holds. Thus, the set of all admissible transformations of coordinates forms a group.

**DEFINITIONS**

Suppose \( F(p) \) is a real valued continuous and differentiable function in some region \( R \) of an \( n \)-dimensional manifold \( V_n \). The values then of \( F(p) \) depend on the point \( p \), not on the coordinate system used to represent \( p \). In a reference frame \( \lambda(x^1, \ldots, x^n) \) \( F(p) \) may have the form \( f(x^1, \ldots, x^n) \) and in a reference frame \( \gamma(\eta^1, \eta^2, \ldots, \eta^n) \), obtained by the admissible transformation

\[ T: \quad x^i = x^i(\eta^1, \eta^2, \ldots, \eta^n) \quad (i = 1, \ldots, n), \]
$F(p)$ will have the form

$$(1) \quad f(x'(y), x^2(y), \ldots, x^n(y)) = g(y', y^2, \ldots, y^n).$$

Suppose that $f(x', \ldots, x^n)$ is continuous and differentiable, one may form the set of $n$ partial derivatives: $(\frac{\partial}{\partial y'}, \ldots, \frac{\partial}{\partial y^n})$. The corresponding derivatives of $g(y', \ldots, y^n)$ can be gotten from $(1)$ by the chain rule for differentiation of composite functions:

$$(2) \quad \frac{\partial g}{\partial y'} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial y'} \quad (i', j = 1, \ldots, n).$$

One may think of these sets of functions $\{\frac{\partial}{\partial y'}\}$ and $\{\frac{\partial}{\partial y^n}\}$ as representing in different frames of reference the same entity which transforms according to $(2)$. Recall that the repeated index $j$ in the right side of $(2)$ implies that that term is to be summed with respect to $j$ for all admissible values of $j$. (In this case $j = 1, \ldots, n$).

Now consider a set of $n$ differentials $(dx', \ldots, dx^n)$, determining the displacement vector from $P(x', \ldots, x^n)$ to $P'(x' + dx', \ldots, x^n + dx^n)$. When referred to the $Y$ coordinate system as given by $T$, the displacement has for its components

$$(3) \quad dy^i = \frac{\partial y^i}{\partial x^j} dx^j \quad (i, j = 1, \ldots, n).$$

Thus, the sets of differentials $\{dx^i\}$ and $\{dy^i\}$ may be thought of as representing in different coordinate systems the same entity which transforms according to $(3)$. 

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The laws of transformations as stated in (1), (2), and (3) are fundamental in the development of tensor analysis. On the basis of these the following definitions will be stated.51

DEFINITION I: A tensor of rank zero is the entire class of sets of quantities \( \{\eta(i; x)\}, \{\mu(i; y)\}, \ldots \), related to one another by the transformation form
\[
E(i; y) = A(i; x) (i; x, \ldots, n),
\]
where \( A(i; x) \) and \( E(i; y) \) are the representations of the tensor in the X and Y coordinate systems respectively, related to one another by the admissible transformation \( T: x \rightarrow x'(y', \ldots, y^n) \). A tensor of rank zero is termed a scalar or invariant.

DEFINITION II: A covariant tensor of rank one is the entire class of sets of quantities \( \{\Lambda(i; x)\}, \{\nu(i; y)\}, \ldots \), related to one another by the transformation form
\[
E(i; y) = \partial_{y^j} A(i; x) (i, j; x, \ldots, n),
\]
where \( A(i; x) \) and \( E(i; y) \) are the representations of the tensor in the X and Y coordinate systems respectively, related to one another by the admissible transformation \( T: x \rightarrow x'(y', \ldots, y^n) \). Components of a covariant tensor are denoted by subscripts, e.g., \( E_{ij} \). Covariant tensors of rank one are termed covariant vectors.

DEFINITION III: A contravariant tensor of rank one is the entire class of quantities \( \{\Lambda(i; x)\}, \{\nu(i; y)\}, \ldots \), related to one another by the transformation forms of the form
\[
E(i; y) = \partial_{x^j} A(i; x) (i, j; x, \ldots, n),
\]
where \( A(i; x) \) and \( E(i; y) \) represent the tensor in the X and Y coordinate systems respectively, related to one another by the admissible transformation \( T: x \rightarrow x'(y', \ldots, y^n) \).

51 S. Sokolnikoff, op. cit., pp. 61-65.
Components of a contravariant tensor are denoted by superscripts, e.g., $E_i^j \frac{\partial}{\partial x^j}$.

Contravariant tensors of rank one are termed contravariant vectors.

These definitions can be generalized to include tensors of any contravariant or covariant rank. One can define a mixed tensor as follows:

**DEFINITION IV:** The totality of sets of quantities, typified in the $X$ coordinate system by the expressions $A^i_{j...r}(x)$ is a mixed tensor, covariant of rank $r$ and contravariant of rank $s$, provided that the corresponding quantities $B^i_{j...r}(y)$ in the $Y$ coordinate system (related to $X$ by the admissible transformation $T: x^i \rightarrow y^i$) are given by the law:

$$B^i_{j...r} = \frac{\partial x^i}{\partial y^j} \frac{\partial x^j}{\partial y^r} A^i_{j...r}. $$

It can easily be shown that the set of tensor transformations so defined forms a group.

Given two admissible transformations

$$T_i: y^i = y^i(x^1, ..., x^n),$$

$$T_2: y^i = y^i(y^1, ..., y^n),$$

then

$$T_3 = T_1 T_2: y^i = y^i(y^1(x), ..., y^n(x)).$$

Suppose $A^i_j(x)$ are the components of a mixed tensor, covariant of rank one and contravariant of rank one, in the $X$ coordinate system. Then, by the law for transformation of such a tensor one has

$$G_i: B^i_j(y) = \frac{\partial x^i}{\partial y^j} A^i_j(x),$$

$$G_2: C^i_j(y) = \frac{\partial x^i}{\partial y^j} B^i_j(y).$$

Therefore

$$G_2 G_1: C^i_j(y) = \frac{\partial x^i}{\partial y^j} \frac{\partial x^j}{\partial y^r} \frac{\partial x^r}{\partial y^p} A^i_p(x).$$
and performing the summation with respect to $p$ and $q$,

$$C^i_j(q) = \partial_y^{\frac{p}{q}} C^i_j(r) A^p_q(x).$$

Hence, the tensor transformation is such that the product of two admissible transformations $T_3, T_2$ corresponds to the product of two corresponding transformations $G_3, G_2$ with respect to the tensor. When such a relation exists between any two groups of transformations, the groups are said to be isomorphic. This concept can be used to define tensors in a broader sense than has been done in this paper, but such an extended definition will not be considered here.

It should be noted that there also exist quantities termed as relative tensors which transform according to the formula

$$B^{i_1 \ldots i_r}(q) = \frac{\partial^{i_1 \ldots i_r}}{\partial y^{i_1 \ldots i_r}} |^W_{x^{i_1 \ldots i_r}} \partial x^{i_1 \ldots i_r} \partial y^{i_1 \ldots i_r} \partial z^{i_1 \ldots i_r} A^{i_1 \ldots i_r}(x).$$

The set of quantities $A^{i_1 \ldots i_r}(x)$ obeying this law of transformation are called the components of a relative tensor of weight $W$.

**ALGEBRA OF TENSORS**

Now, given a tensor whose components are $A^{i_1 \ldots i_r}$, in the $X$ coordinate system and the corresponding components

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52 I bid., p. 57.

in the $Y$ coordinate system are $\mathbf{E}^{\ast \ast \ast \ast}$, then one can
write the equations of transformation as
$$
\mathbf{E}^{\ast \ast \ast \ast} = \frac{\partial y_1}{\partial y_i} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \\
or as
\mathbf{E}^{\ast \ast \ast \ast} = \frac{\partial y_1}{\partial y_i} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} \mathbf{E}^{\ast \ast \ast \ast} .
$$
Hence, if all components of a tensor vanish in one coordinate system, then they necessarily vanish in all other coordinate systems. This concept is fundamental in deriving many properties of tensors as will be seen in what is to follow.

From the definitions of tensors it should be clear that any linear combination of tensors of the same type and rank is again a tensor of the same type and rank.

Furthermore, if the tensor equation
$$
\mathbf{A}^{\ast \ast \ast \ast} (v) = \mathbf{A}^{\ast \ast \ast \ast} (v)
$$
is true in one coordinate system, it is true in all admissible coordinate systems. This follows from the fact that $\mathbf{A}^{\ast \ast \ast \ast} - \mathbf{A}^{\ast \ast \ast \ast} = 0$. Some particular examples of such tensor equations arise in considering symmetric and antisymmetric tensors. A tensor $\mathbf{A}^{\ast \ast \ast \ast}$ is said to be symmetric with respect to two indices, say $i_1$ and $i_2$, if
$$
\mathbf{A}^{i_1 i_2 \cdots i_r} = \mathbf{A}^{i_2 i_1 \cdots i_r} ;
$$
or skew-symmetric with respect to those indices if
$$
\mathbf{A}^{i_1 i_2 \cdots i_r} = - \mathbf{A}^{i_2 i_1 \cdots i_r} .
$$
One can define the outer product of two tensors as the set consisting of the product of each element of the set
\( A^i_{\ldots j} (x) \), representing a tensor \( A \), by each element of the set \( \bar{A}^i_{\ldots j} (x) \), representing a tensor \( \bar{A} \). Then say
\[
\alpha^i_{\ldots j} = \bar{A}^i_{\ldots j} A^i_{\ldots j}
\]
and \( \bar{A} \) is a tensor contravariant of rank \( q^*s \) and covariant of rank \( p^*r \). One can also define the operation of contraction: If, in a mixed tensor contravariant of rank \( s \) and covariant of rank \( r \), a contravariant and covariant index are equated and the sum is taken with respect to that index, the resulting set of \( r^*s - 2 \) sums is a mixed tensor covariant of rank \( r-1 \) and contravariant of rank \( s-1 \). The result of application of the operation of contraction to the outer product of the two tensors is called the inner product. These operations are preserved under tensor transformations.

THE METRIC TENSOR

Consider now an \( n \)-dimensional space and in that space a displacement vector \( d y^i (i = 1, \ldots, n) \) determined by a pair of points \( P(y', \ldots, y^n) \) and \( P'(y'+dy', \ldots, y^n + dy^n) \) where the coordinates \( y^i \) are orthogonal Cartesian. Then, by the Pythagorean formula for the square of the distance between \( P \) and \( P' \),
\[
d s^2 = dy^i dy^i (i = 1, \ldots, n)
\]
Under an admissible change of coordinate systems
\[
y^i = y^i (x', \ldots, x^n)
\]
the square of the element of arc can be written
\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \]
since \( dy^i = \frac{\partial y^i}{\partial x^\alpha} dx^\alpha \). And letting
\[ g_{\alpha\beta} = \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta} = \frac{\partial y^i}{\partial x^\beta} \frac{\partial y^j}{\partial x^\alpha} = g_{\beta\alpha} \]
one has
\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \]

From the definition of \( g_{\alpha\beta} \) and the fact that \( ds^2 \) is an
invariant and the differentials \( dx^i \) are components of a
contravariant tensor of rank one, it follows that \( g_{\alpha\beta} \) rep­
resents a symmetric tensor covariant of rank two. This
particular tensor \( g_{\alpha\beta} \) is called the metric tensor. It
will be shown that all essential metric properties of
space are completely determined by this tensor.

Let \( g^{\alpha\beta}(y) \) represent a symmetric tensor covariant
of rank two such that the \( g^{\alpha\beta} \) belong to class \( C^1 \) and
\( |g_{\alpha\beta}| \neq 0 \) at any point of the region under consideration.
Then one can define a corresponding symmetric tensor
contravariant of rank two as follows. Let
\[ g^{\alpha\beta} = \frac{G^{\alpha\beta}}{|g_{\alpha\beta}|} \]
where \( G^{\alpha\beta} \) is the cofactor of the element \( g_{\alpha\beta} \) in the matrix
\( (g_{\alpha\beta}) \). The symmetry of \( g^{\alpha\beta} \) follows from the symmetry of
\( g_{\alpha\beta} \). And by the laws for determinants
\[ g_{ij} g^{ij} = \delta^i_j \]
It can readily be proved that \( |g_{\alpha\beta}|^2 g^{\alpha\beta} \) (i is fixed, j is
summed from 1 to n) is a relative tensor of rank zero of
weight two, and thereby that \( G^{\alpha\beta} \) is a relative tensor
contravariant of rank two and weight two, and hence \( g^{ij} \) is a contravariant tensor of rank two.

The tensors \( g^{ij}(v) \) and \( g_{ij}(v) \) will play an essential role in all that is to follow. Hence they are termed the fundamental tensors. A tensor obtained by the process of inner multiplication of any tensor with either of the fundamental tensors is called the tensor associated with the given tensor. Thus, e.g.,

\[
\begin{align*}
& g^{j_{k}} A_{i}^{j_{k}} = A_{i}^{j_{k}} \quad \text{and} \quad g_{i}^{j_{k}} A^{i_{k}} = A^{i_{k}},
\end{align*}
\]

are associated with the tensor \( A^{i_{k}} \).

**CHRISTOFFEL SYMBOLS**

The following definitions with respect to the fundamental tensors will be useful.

**DEFINITION A:** The Christoffel symbol of the first kind is

\[
\Gamma^i_{jk} = \frac{1}{2} \left( \frac{\partial g^{ik}}{\partial x^j} + \frac{\partial g^{jk}}{\partial x^i} - \frac{\partial g^{ij}}{\partial x^k} \right).
\]

**DEFINITION B:** The Christoffel symbol of the second kind is

\[
\gamma^i_{jk} = g^{ik} \left[ \Gamma^j_{ik} \right].
\]

This is sometimes denoted by \( \Gamma^i_{jk} \).

From these definitions it is clear that both Christoffel symbols are symmetric with respect to the indices \( i \) and \( j \). Expressions for the partial derivatives of the fundamental tensors can readily be deduced.

\[
\frac{\partial g^{ij}}{\partial x^k} = g^{ik} \Gamma^j_{ik} + g^{jk} \Gamma^i_{jk},
\]

and differentiating the identity \( g_{ij} g^{ij} = \delta^i_j \) one has
In general these Christoffel symbols do not represent tensors. Under a coordinate transformation $y^i = y^i(x^1, \ldots, x^n)$ belonging to the class $C^2$, the fundamental tensor $g_{ij}(x)$ would transform to the $y$ system as $h_{ij}(y)$ as follows:

$$h_{ij}(y) = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} g_{kl}.$$ 

Denote the Christoffel symbols relative to the $y$ system as $\gamma^i_{\ j\ ;k}$ and $\gamma^i_{\ j\ |k}$. Then

$$\gamma^i_{\ j\ ;k} = \frac{1}{2} \left( \frac{\partial^2 x^i}{\partial y^j \partial y^k} + \frac{\partial^2 x^i}{\partial y^k \partial y^j} - \frac{\partial^2 x^i}{\partial y^j \partial y^j} \right).$$

But by the previous relation for $h_{ij}$ in terms of $g_{ij}$

$$\frac{\partial^2 x^i}{\partial y^j \partial y^k} = g_{ij} \left( \frac{\partial^2 x^k}{\partial y^j \partial x^i} + \frac{\partial^2 x^i}{\partial y^k \partial x^j} - \frac{\partial^2 x^k}{\partial y^j \partial x^j} \right)$$

and since $g_{ij} g_{jk} = g_{ik}$ this can be written

$$\frac{\partial^2 x^i}{\partial y^j \partial y^k} = g_{ij} \left( \frac{\partial^2 x^k}{\partial y^j \partial x^i} + \frac{\partial^2 x^i}{\partial y^k \partial x^j} - \frac{\partial^2 x^k}{\partial y^j \partial x^j} \right)$$

Carrying out this process for $\frac{\partial^2 x^i}{\partial y^j \partial y^k}$ and $\frac{\partial^2 x^i}{\partial y^j \partial x^k}$ one obtains

(1) $\gamma^i_{\ j\ ;k} = \frac{\partial^2 x^i}{\partial y^j \partial x^k} + \frac{\partial^2 x^i}{\partial y^k \partial x^j} - \frac{\partial^2 x^i}{\partial y^j \partial x^j} = \gamma^i_{\ j\ |k}$

Furthermore, since $\gamma^i_{\ j\ ;k} = \lambda^m_{\ ij} \gamma^i_{\ j\ ,m}$ where $\lambda_{ij}^m = \frac{\partial x^m}{\partial y^i} \frac{\partial x^j}{\partial y^j}$, so,

(2) $\gamma^i_{\ j\ ;k} = \frac{\partial^2 x^i}{\partial y^j \partial x^k} + \frac{\partial^2 x^i}{\partial y^k \partial x^j} - \frac{\partial^2 x^i}{\partial y^j \partial x^j}$

From (1) and (2) it is clear that the Christoffel symbols of the second kind are not tensors unless the coordinate
transformation is linear. Also, from (1) and (2) one has an expression for the second partial derivatives:
\[ \frac{\partial^2 x^k}{\partial y^i \partial y^j} = \left\{ \frac{\partial x^m}{\partial y^i} \frac{\partial y^m}{\partial y^j} - \frac{\partial y^m}{\partial y^i} \frac{\partial x^m}{\partial y^j} \right\} \frac{\partial^2 y^m}{\partial y^i \partial y^j}. \]

COVARIANT DIFFERENTIATION OF TENSORS

Consider the covariant tensor of rank one represented by \( A_i \) in the X coordinate system and \( B_i \) in the Y coordinate system where \( y^i = y^i(x^1, ..., x^n) \) \((i = 1, ..., n)\) is of class \( C^2 \). Then
\[ B_i = \frac{\partial x^i}{\partial y^j} A_j. \]
Differentiating this expression:
\[ \frac{\partial B_i}{\partial y^j} = \frac{\partial^2 x^i}{\partial y^j \partial y^k} + \frac{\partial x^i}{\partial y^j} \frac{\partial A_j}{\partial x^k} \text{ and since } \frac{\partial^2 x^i}{\partial y^j \partial y^k} = \frac{\partial x^i}{\partial y^k} \frac{\partial x^k}{\partial y^j} \text{ one can obtain } \]
\[ \frac{\partial B_i}{\partial y^j} = \left\{ \frac{\partial x^i}{\partial y^j} \right\} B_j = \left( \frac{\partial A_j}{\partial y^j} - \left\{ \frac{\partial x^i}{\partial y^j} \right\} A_j \right) \frac{\partial x^i}{\partial y^j} \frac{\partial x^k}{\partial y^j}. \]

Therefore the law of transformation of the \( n^2 \) quantities 
\[ \left( \frac{\partial A_i}{\partial y^j} - \left\{ \frac{\partial x^i}{\partial y^j} \right\} A_k \right) \]
obeys the law of transformation for a covariant tensor of rank two.

In a similar manner it can be shown that the set of \( n^2 \) quantities \( \left( \frac{\partial A_i}{\partial y^j} + \left\{ \frac{\partial x^i}{\partial y^j} \right\} A_k \right) \) forms a mixed tensor of rank two. These ideas lead one to formulate the following definitions.

**DEFINITION 1:** The set of \( n^2 \) functions \( \left( \frac{\partial A_i}{\partial y^j} - \left\{ \frac{\partial x^i}{\partial y^j} \right\} A_k \right) \) defines the covariant \( i^j \) derivative (with respect to \( \partial y^j \)) of the covariant tensor \( A_i \). This is denoted by \( A^i_{;j} \).
DEFINITION 2: The set of $n^2$ functions 
\( (3A)_{ij} + \{ \xi^k \} \ A^k \) defines the covariant 
\( \gamma \) derivative (with respect to \( \gamma^\gamma \)) of 
the contravariant tensor \( A^i \). This is 
denoted by \( A^i_{,j} \).

These definitions can be extended to include mixed tensors 
of any rank, e.g.,
\[ A^i_{,k} = \frac{\partial A^i}{\partial x^k} - \{ \xi^k \} A^i_{,\gamma} - \ldots - \{ \xi^l \} A^i_{,\gamma} + \{ \xi^k \} A^i_{,\gamma} + \ldots + \{ \xi^k \} A^i_{,\gamma} \]

Note that if the \( \gamma^k \)'s are constants (as, e.g., in a Cartesian 
reference frame in Euclidean space), then the Christoffel 
symbols will vanish and then the covariant derivative re­
duces to the ordinary derivative. It is not difficult to 
show that the rules for covariant differentiation of sums 
and products of tensors are identical with those used in 
ordinary differentiation.\(^{54}\)

Consider the covariant derivative of some familiar 
tensors. 
\[ q_{i,j} = \partial q_{i,j} - \{ \xi^k \} q_{i,j} + \{ \xi^k \} q_{i,j} \]
\[ = 0 - \{ \xi^k \} + \{ \xi^k \} \]
\[ = 0. \]
And 
\[ g_{i,j} = \partial g_{i,j} - g_{i,j} \{ \xi^k \} - g_{i,j} \{ \xi^k \} \]
\[ = (g_{i,j} \{ \xi^k \} + g_{i,j} \{ \xi^k \}) - (g_{i,j} \{ \xi^k \} + g_{i,j} \{ \xi^k \}) \]
\[ = 0. \]
Here use was made of a previous calculation for \( g_{i,j}^{,k} \).

\(^{54}\)I.S. Sokolnikoff, \textit{op. cit.}, p. 88.
Since \( g_{ij} g^{ik} = \delta^i_j \) differentiating this one has
\[
g_{ij, k} g^{ik} + g_{ij} g^{ik, k} = \delta^i_j,
\]
\[
g_{ij} g^{ik, k} = 0.
\]
Therefore, since \( l g_{ij} \neq 0 \)
\[
g_{ik, k} = 0.
\]
Thus, the Kronecker deltas and the fundamental tensors behave like constants with respect to covariant differentiation.

RIEMANN — CHRISTOFFEL TENSOR

It is convenient to consider the conditions under which the order of covariant differentiation is immaterial.

Differentiate the covariant derivative \( A_{ij} = \frac{\partial A_j}{\partial x_i} - \{e^i_j\} A_k \) with respect to \( x^k \),
\[
A_{i j k} = \frac{\partial A_{i j}}{\partial x^k} - \{\xi^k_i\} A_{j k} - \{\xi^k_j\} A_{i k} - \{\xi^k_i\} \frac{\partial A_k}{\partial x^j} - \{\xi^k_j\} \frac{\partial A_k}{\partial x^i} - \{\xi^k_i\} \frac{\partial A_k}{\partial x^j} \frac{\partial A_k}{\partial x^i} - \{\xi^k_j\} \frac{\partial A_k}{\partial x^i} \frac{\partial A_k}{\partial x^j}.
\]
In a similar way one could calculate \( A_{ij} \) to be
\[
A_{i j k} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial}{\partial x^j} \frac{\partial A_i}{\partial x^k} - \frac{\partial}{\partial x^k} \frac{\partial A_i}{\partial x^j} - \frac{\partial^2 A_i}{\partial x^k \partial x^j} + \{\xi^k_i\} \frac{\partial A_k}{\partial x^j} + \{\xi^k_j\} \frac{\partial A_k}{\partial x^i}.
\]
Then
\[
A_{i j k} - A_{i k j} = \{\xi^k_i\} \{\xi^k_j\} A_k - \frac{\partial}{\partial x^j} \frac{\partial A_i}{\partial x^k} A_k - \frac{\partial}{\partial x^k} \frac{\partial A_i}{\partial x^j} A_k - \frac{\partial^2 A_i}{\partial x^k \partial x^j} A_k - \frac{\partial}{\partial x^j} \frac{\partial A_i}{\partial x^k} A_k - \frac{\partial}{\partial x^k} \frac{\partial A_i}{\partial x^j} A_k,
\]
or equivalently
\[
A_{i j k} - A_{i k j} = \left[ \frac{\partial}{\partial x^j} \frac{\partial A_i}{\partial x^k} - \frac{\partial}{\partial x^k} \frac{\partial A_i}{\partial x^j} + \{\xi^k_i\} \{\xi^k_j\} - \{\xi^k_j\} \{\xi^k_i\} \right] A_k.
\]
Since $A_i$ is an arbitrary covariant tensor of rank one and the difference of the two tensors $(A_{ijk} - A_ikj)$ is a covariant tensor of rank three, it follows that the expression in brackets is a mixed tensor of rank four, which will be denoted by $R^r_{ijk}$. Thus, one can arrive at the conclusion that the order of covariant differentiation is immaterial if and only if the tensor $R^r_{ijk}$ vanishes identically. The tensor

$$R^r_{ijk} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial^2 A_i}{\partial x^k \partial x^j} + \frac{\partial A_i}{\partial x^j} \frac{\partial A_i}{\partial x^k} - \frac{\partial A_i}{\partial x^k} \frac{\partial A_i}{\partial x^j}$$

is called the Riemann or the Riemann–Christoffel tensor of the second kind. The associated tensor

$$R^{ij}_{jkl} = g_{ik} R^r_{ijkl}$$

is known as the Riemann or the Riemann–Christoffel tensor of the first kind, which can be written

$$R^{ij}_{jkl} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} + \frac{\partial A_i}{\partial x^j} \frac{\partial A_i}{\partial x^k} - \frac{\partial A_i}{\partial x^k} \frac{\partial A_i}{\partial x^j}$$

or, carrying out the differentiation

$$R^{ij}_{jkl} = \frac{1}{2} \left( \frac{\partial^2 A_i}{\partial x^j \partial x^k} + \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial^2 A_i}{\partial x^k \partial x^j} \right) + g_{ik} \left( \frac{\partial A_j}{\partial x^j} \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^j} \frac{\partial A_k}{\partial x^j} \right),$$

from which it is obvious that

$$R^{i}_{jkl} = -R^{i}_{jkl},$$
$$R^{i}_{jlk} = -R^{i}_{jlk},$$
$$R^{i}_{ik} = R^{i}_{ik},$$
$$R^{i}_{jkl} + R^{i}_{ikl} + R^{i}_{ikj} = 0.$$
there is only one distinct non-vanishing component: $\delta_{\alpha\beta}$. The importance of this tensor in the study of surfaces is seen in the section on Riemannian geometry.

A particular contraction of the Riemann - Christoffel tensor of the second kind will be found to be useful:

$$\mathcal{R}_{ij} = \mathcal{R}^a_{ij} = \left| \begin{array}{cc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array} \right| + \left| \begin{array}{cc} \{a_i\} & \{a_j\} \\ \{c_i\} & \{c_j\} \end{array} \right|.$$  

Note that $\mathcal{R}_{ij} = \mathcal{R}_{ji}$. This symmetric tensor which plays a fundamental role in relativistic gravitational theory is known as the Ricci tensor. Contracting the Ricci tensor one obtains the invariant

$$R = \mathcal{R}_{ij} = g^{ij} \mathcal{R}_{ij}$$

which is called the scalar Riemann curvature of the space $V_n$.

There exist other identities with respect to the Riemann tensor of the second kind which will also be of importance. In order to establish these one may use a Cartesian system of coordinates at an arbitrary point $P$ of $V_n$. The Christoffel symbols are then zero at $P$ and so differentiating covariantly $\mathcal{R}_{ijkl}$ at the point $P$ one obtains

$$\mathcal{R}_{ijkl,m} = \mathcal{R}^{a}_{ijkl,m} = \frac{\partial}{\partial x^m} \mathcal{R}_{ijkl}.$$  

A cyclic permutation of $k$, $l$, and $m$ yields

$$\mathcal{R}_{jkm,l} = \mathcal{R}^{a}_{jkm,l} = \frac{\partial}{\partial x^l} \mathcal{R}_{jkm},$$

$$\mathcal{R}_{imk,l} = \mathcal{R}^{a}_{imk,l} = \frac{\partial}{\partial x^l} \mathcal{R}_{imk}.$$  

From these one can obtain

$$\mathcal{R}_{jkl,m} + \mathcal{R}_{jkm,l} + \mathcal{R}_{imk,l} = 0.$$  

From their tensorial form these identities are obviously
valid in any coordinate system for any point of \( V_n \). They are known as the Bianchi identities. Double contraction leads to an important consequence relating to the Ricci tensor. Letting \( i = 1 \), one obtains
\[
R_{j,m} - R_{j,m} + R_{j,m,k,l} = 0,
\]
and contracting \( j \) and \( k \)
\[
R_{m} - R_{m,k} - R_{m,k} = 0,
\]
or in alternative form
\[
(R_{m} - \frac{1}{2} \delta_{m} R)_{,k} = 0.
\]
The tensor
\[
G_{j} = G_{j}
\]
is known as the Einstein tensor.

CONCLUSION

The preceding development of tensor theory is not meant to be a complete analysis of the subject. There are many books on tensor analysis, some of which are listed in the bibliography, which could be used as a supplement by the reader. With the aid of the tensor definitions and tensor calculus given the development of the basic concepts of Riemannian Geometry and of General Relativity can be set forth in a systematic and relatively simple manner.
Einstein Summation Convention

An index appearing twice in the same term implies summation with respect to that index for all admissible values of the index.

\[ c^n \]

A function \( f(x^1, \ldots, x^n) \) is said to be of class \( c^n \) if the first \( n \) partial derivatives of \( f \) with respect to the \( x^i \) exist and are continuous.

\[ (g_{ij}) \]

This denotes the matrix \( \begin{pmatrix} g_{i1} & \cdots & g_{in} \\ \vdots & \ddots & \vdots \\ g_{i1} & \cdots & g_{in} \end{pmatrix} \).

\[ (g^{ij}) \]

This denotes the matrix \( \begin{pmatrix} g^{i1} & \cdots & g^{in} \\ \vdots & \ddots & \vdots \\ g^{i1} & \cdots & g^{in} \end{pmatrix} \).

\[ |g_{ij}| \]

This denotes the determinant of \( (g_{ij}) \).

\[ |g^{ij}| \]

This denotes the determinant of \( (g^{ij}) \).

\[ \delta^i_j \text{ and } \delta_j^i \]

These are the familiar Kronecker deltas:

\[ \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \]

\[ \delta_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}. \]

\[ |\vec{dr}| \]

This is the magnitude of the vector \( \vec{dr} \).
BIBLIOGRAPHY


