1927

General configurations

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GENERAL CONFIGURATIONS

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Definition of a configuration: A figure is called a configuration if it consists of a finite number of points, lines, and planes, with the property that each point is on the same number a_{12} of lines and also on the same number a_{13} of planes; each line is on the same number a_{21} of points and the same number a_{23} of planes; and each plane is on the same number a_{31} of points and the same number a_{32} of lines.

A configuration can be conveniently described by a square matrix:

$$\begin{array}{l|ccc|}
 & \begin{matrix}1\\ \text{point}\end{matrix} & \begin{matrix}2\\ \text{line}\end{matrix} & \begin{matrix}3\\ \text{plane}\end{matrix} \\
\hline
1 \text{ point} & a_{11} & a_{12} & a_{13} \\
2 \text{ line} & a_{21} & a_{22} & a_{23} \\
3 \text{ plane} & a_{31} & a_{32} & a_{33} \\
\hline
\end{array}$$

In this notation, if we call a point an element of the first kind, a line an element of the second kind, and a plane one of the third kind, the number a_{ij} $(i\neq j)$ gives the number of elements of the jth kind on every element of the ith kind. The numbers a_{11}, a_{22}, a_{33} give the total number of points, lines, and planes respectively. Such a square matrix is called the symbol of the configuration.

A tetrahedron for example is a figure, consisting of four points, six lines, and four planes; on every line of the figure are two points of the figure, on every plane are three points, through every point pass three lines and also three planes, every plane contains three lines, and through every line pass two planes. A tetrahedron is therefore a configuration of the symbol

$$\begin{vmatrix} 4 & 3 & 3 \\ 2 & 6 & 2 \\ 3 & 3 & 4 \end{vmatrix}$$

The symmetry shown in this symbol is due to the fact that the figure in question is self-dual. A triangle evidently has the symbol

$$\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$$

Since all the numbers referring to planes are of no importance in case of a plane-figure, they are omitted from the symbol for a plane configuration.

In general, a complete plane n-point is of the symbol

n	$n-1$
2	$\frac{1}{2}n(n-1)$

and a complete space n-point of the symbol

n	$n-1$	$\frac{1}{2}(n-1)(n-2)$
2	$\frac{1}{2}n(n-1)$	$n-2$
3	3	$1/6\,n(n-1)(n-2)$

<u>The Desargues configuration.</u> A very important configuration is obtained by taking the plane section of a complete space five-point. The five-point is clearly a configuration whose symbol may be obtained from the one just given by removing the first column and the first row.

5	4	6
2	10	3
3	3	10

This is due to the fact that every line of the space figure gives rise to a point in the plane and every plane gives rise to a line. The configuration in the plane has then the symbol

10	3
3	10

We proceed to study in detail the properties of the configuration just obtained. It is known as the configuration of Desargue. The theorem of Desargue states that if two triangles in the same plane are perspective from a point, the three pairs of homologous sides meet in collinear points; i.e. the triangles are perspective from a line. ¶ Proof: Let the two triangles be $A_1B_1C_1$ and $A_2B_2C_2$, Plate I, the lines A_1A_2, B_1B_2, C_1C_2 meeting in the point O. Let B_1A_1, B_2A_2 intersect in the point C_3; A_1C_1, A_2C_2 in B_3; B_1C_1,

d to prove that $A_3B_3C_3$ are collinear. Consider any

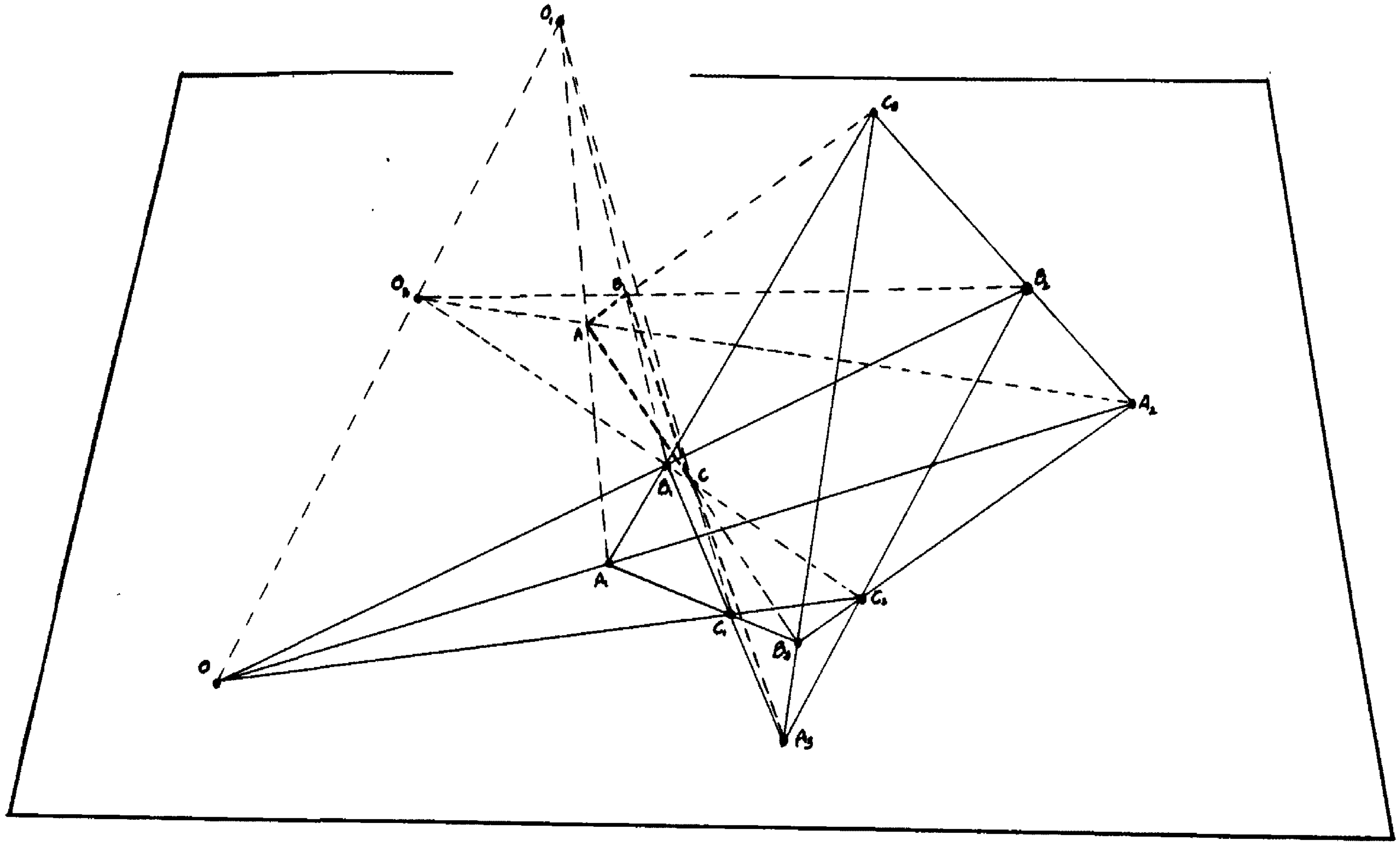


PLATE ONE.

DESARGUE CONFIGURATION.

line through O which is not in the plane of the triangles, and denote by $O_1 O_2$ any two distinct points on this line other than O. Since the lines A_2O_2 and A_1O_1 lie in the plane $(A_1A_2O_1O_2)$ they intersect in a point A. Similarly B_1O_1 and B_2O_2 intersect in a point B and likewise C_1O_1 and C_2O_2 in a point C. Thuse A B C O_1 O_2 together with thelines and planes determined by them form a complete five-point in space of which the perspective triangles form a part of a plane section. The theorem is proved by completing the plane section. Since A B lies in a plane with A_1B_1 and also in a plane with A_2B_2 the lines A_1B_1, A_2B_2, and A B meet in C_3. So also A_1C_1, A_2C_2, and AC meet in B_3; and B_1C_1, B_2C_2, B C meet in A_3. Since A_3, B_3, C_3, lie in the plane A B C and also in the plane of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ they are collinear.

As a corollary to the above we may state that if two triangles in the same plane are perspective from a point, the pairs of homologous sides intersect in collinear points; and conversely.

Perspective tetrahedra: If two tetrahedra are perspective from a point, the six pairs of homologous edges intersect in coplanar points, and the four pairs of homologous faces intersect in coplanar lines; i.e. the tetrahedra are perspective from a plane. Plate 2

Proof: Letthe two tetrahedra be $P_1P_2P_3P_4$ and $P'_1P'_2P'_3P'_4$ and let the lines $P_1P'_1$, $P_2P'_2$, $P_3P'_3$, $P_4P'_4$ meet in the center of perspectivity O. Two homologous edges P_iP_j and $P'_iP'_j$ then clearly intersect; call the point of intersection P_{ij}. The points P_{12}, P_{13}, P_{23} lie on the same line, since the triangles $P_1P_2P_3$ and $P'_1P'_2P'_3$ are perspective from O. By similar reasoning applied to the other pairs of perspective triangles we find that the following triples of points are collinear: $P_{12}P_{13}P_{23}$; $P_{12}P_{14}P_{24}$; $P_{13}P_{14}P_{34}$; $P_{23}P_{24}P_{34}$. The first two triples have the point P_{12} in common and hence determine a plane; each of the other two triples has a point in common with each of the first two. Hence all the points P_{ij} lie in the same plane. The lines of the four triples just given are the lines of intersection of the pairs of homologous faces of the tetrahedra.

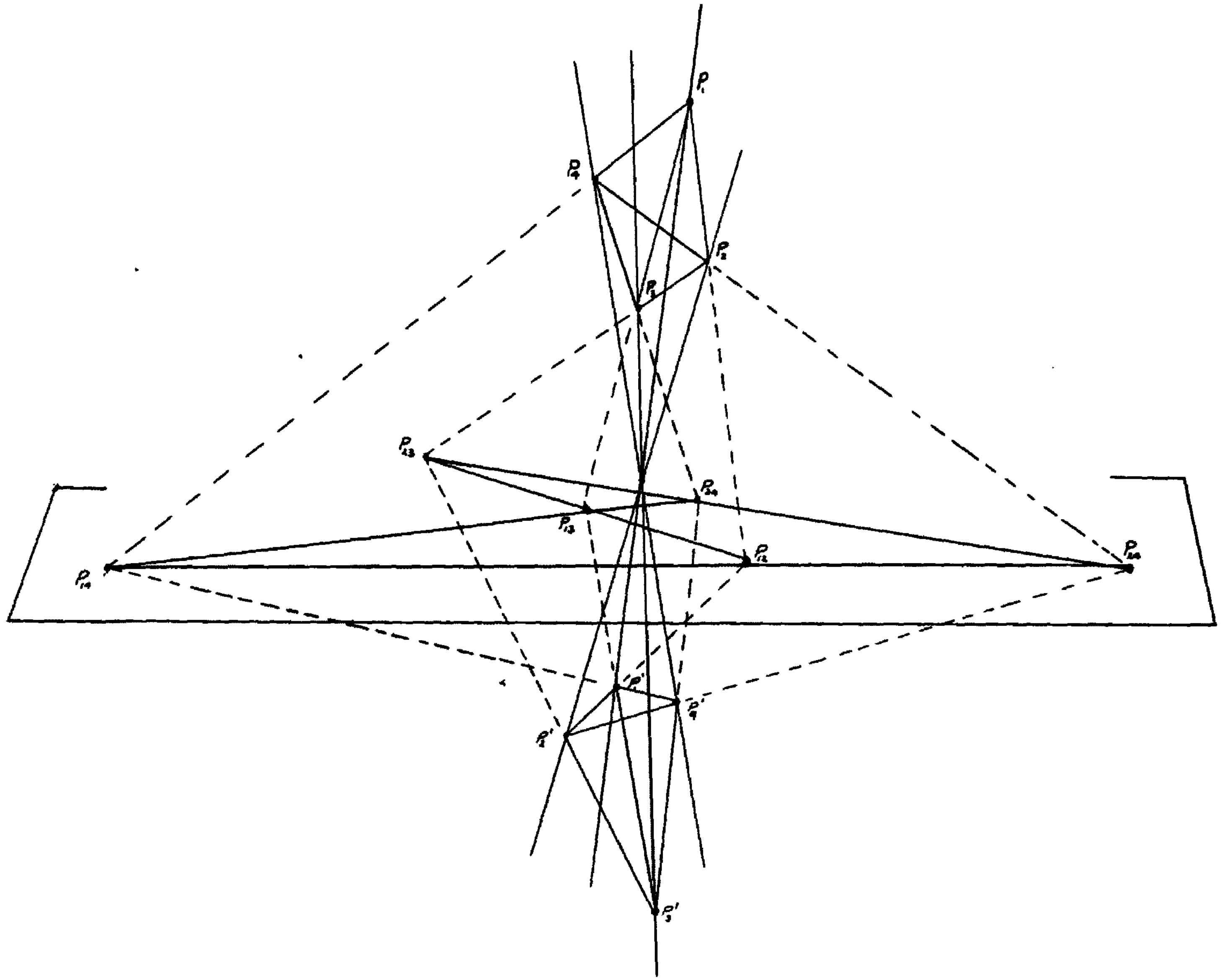


PLATE TWO.

PERSPECTIVE

TETRAHEDRA.

The quadrangle-quadrilateral configuration:

Definition: A complete plane four-point is called a complete quadrangle. It consists of four vertices and six sides. Two sides not on the same vertex are called opposite. The intersection of two opposite sides is called a diagonal point. If the three diagonal points are not collinear, the triangle formed by them is called the diagonal triangle of the quadrangle.

Definition: A complete plane four-line is called a complete quadrilateral. It consists of four sides and six vertices. Two vertices not on the same side are called opposite. The lines joining two opposite vertices is called a diagonal line. If the three diagonal lines are not concurrent, the triangle formed by them is calledthe diagonal triangle of the quadrilateral.

From plate six let $P_1P_2P_3P_4$ be the vertices of the given complete quadrangle, and let $D_{12}D_{13}D_{14}$ be the vertices of the diagonal triangle, D_{12} being on the side P_1P_2, D_{13} on the side P_1P_3 and D_{14} on the side P_1P_4. We observe first that the diagonal triangle is perspective with each of the four triangles formed by a set of three of the vertices of the quadrangle, the center of perspectivity being in each case the fourth vertex. This gives rise to four axes of perspectivity one corresponding to each vertex of the quadrangle. These four lines clearly form the sides of a complete quadrilateral whose diagonal triangle is $D_{12}D_{13}D_{14}$.

The Fundamental theorem on quadrangular sets:

If two complete quadrangles $P_1P_2P_3P_4$ and $P'_1P'_2P'_3P'_4$ correspond -- P_1 to P'_1, P_2 to P'_2, etc.-- in such a way that five of the pairs of homologous sides intersect in points of a line l, then the sixth pair of homologous sides will intersect in a point of l.

Proof: Suppose, first, that none of the vertices or sides of one of the quadrangles coincide with any vertex or side of the other. Let P_1P_2, P_1P_3, P_1P_4, P_2P_3, P_2P_4 be the five sides which, by hypothesis, meet their homologous sides $P'_1P'_2$, $P'_1P'_3$, $P'_1P'_4$, $P'_2P'_3$, $P'_2P'_4$ in points of l. We must show that P_3P_4 ar[illegible] of l. The triangles $P_1P_2P_3$ and $P'_1P'_2P'_3$ are by hypothesis, perspective from l; as also the triangles $P_1 P_2 P_4$ and $P_1'P_2'P_4'$. Each pair is therefore perspective from a point, and this point is in each case the intersection O of the lines P_1P_1' and P_2P_2'. Hence the triangles $P_2P_3P_4$ and $P_2'P_3'P_4'$ are perspective from O and their pairs of homologous sides intersect in the points of a line, which is evidently l, since it contains two points of L. But P_3P_4 and $P_3'P_4'$ are two homologous sides of these last two triangles. Hence they intersect in a point of the line L.

The plane dual of this theorem is : If two complete quadrilaterals $a_1a_2a_3a_4$ and $a_1'a_2'a_3'a_4'$ correspond-- a_1 to a_1', a_2 to a_2', etc-- in such a way that five of the lines joining homologous vertices pass through a point P the line joining the sixth pair of the homologous vertices will also pass through P.

Exercise 3-- If two sets of three points A B C and A'B'C' on two coplanar lines l and l' respectively, are so related that the lines A A', B B', C C', are concurrent then the points of intersection of the pairs of lines A B', and B A', B C' and C B', C A' and A C' are collinear with ll'. The line thus determined is called the polar of the point (A A', B B') with respect to ll'

Given: A A', B B', C C' concurrent to prove ll', K,M,N are collinear.
See Plate 5'.

Proof: Triangles A'B' C and A B C' are perspective. Therefore P,M,N are collinear by the Desargue theorem. Triangles A'BC and A B'C' are perspective Therefore P,K,M are collinear. The points P and M determine both lines and therefore represent the same line, hence K,M,N are collinear.

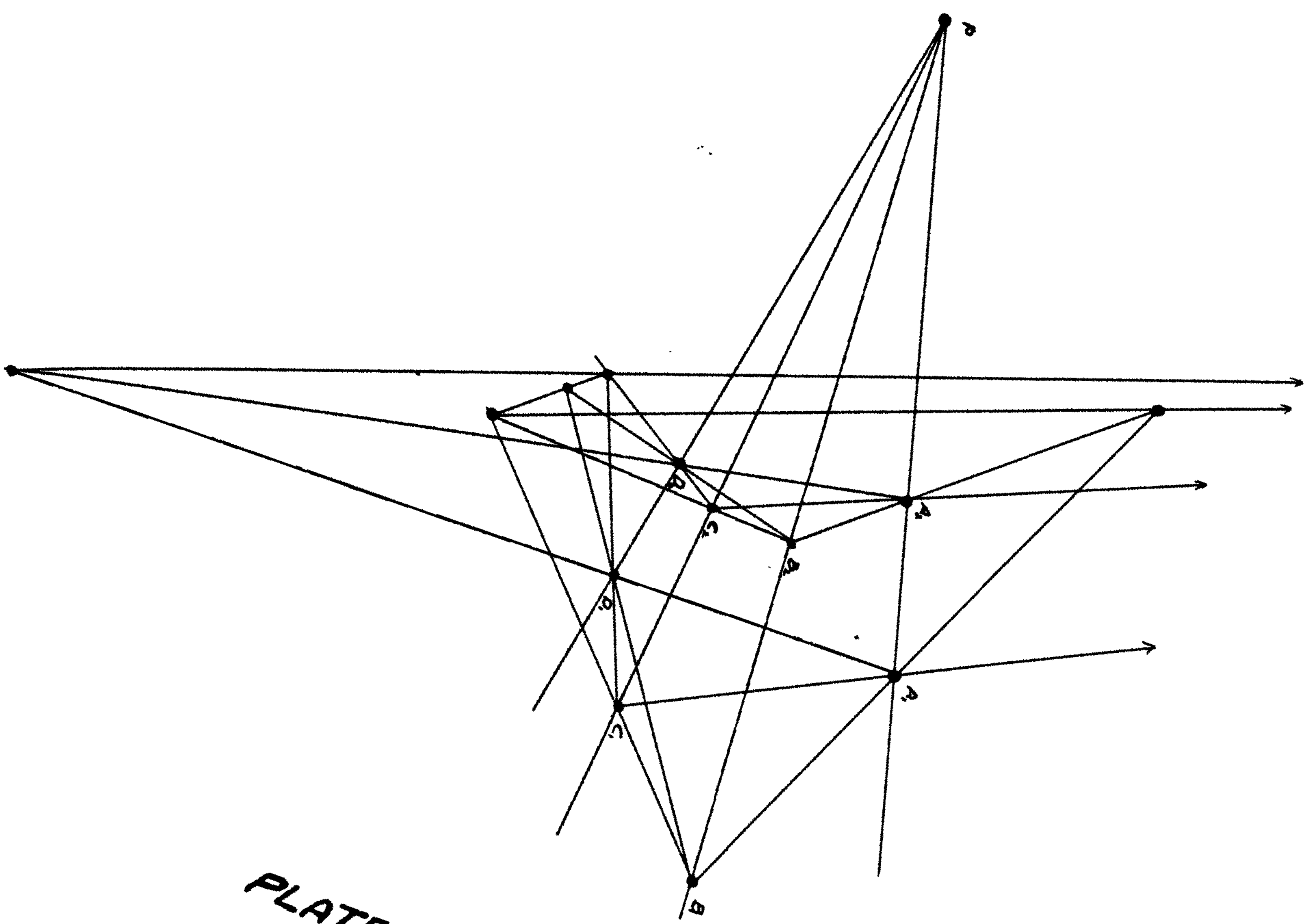
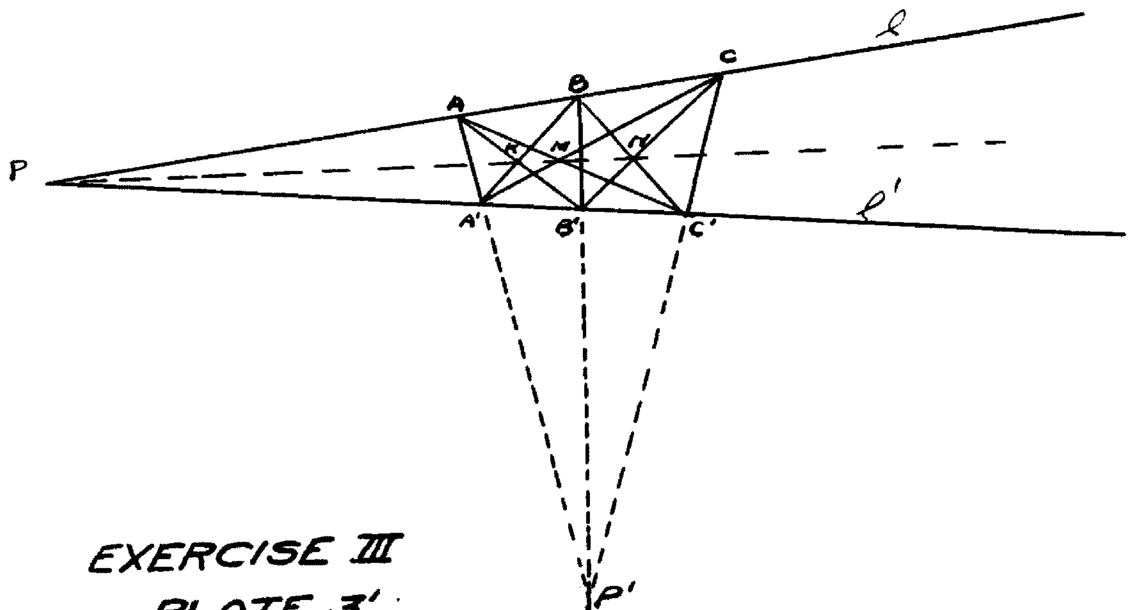
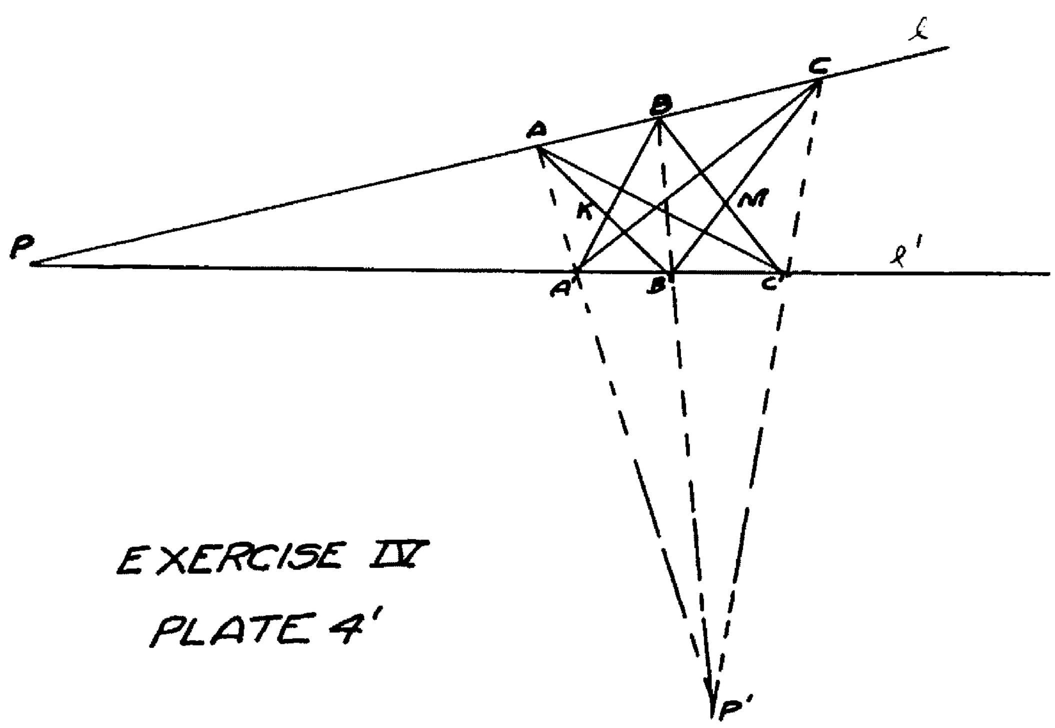


PLATE THREE.



EXERCISE III
PLATE 3'

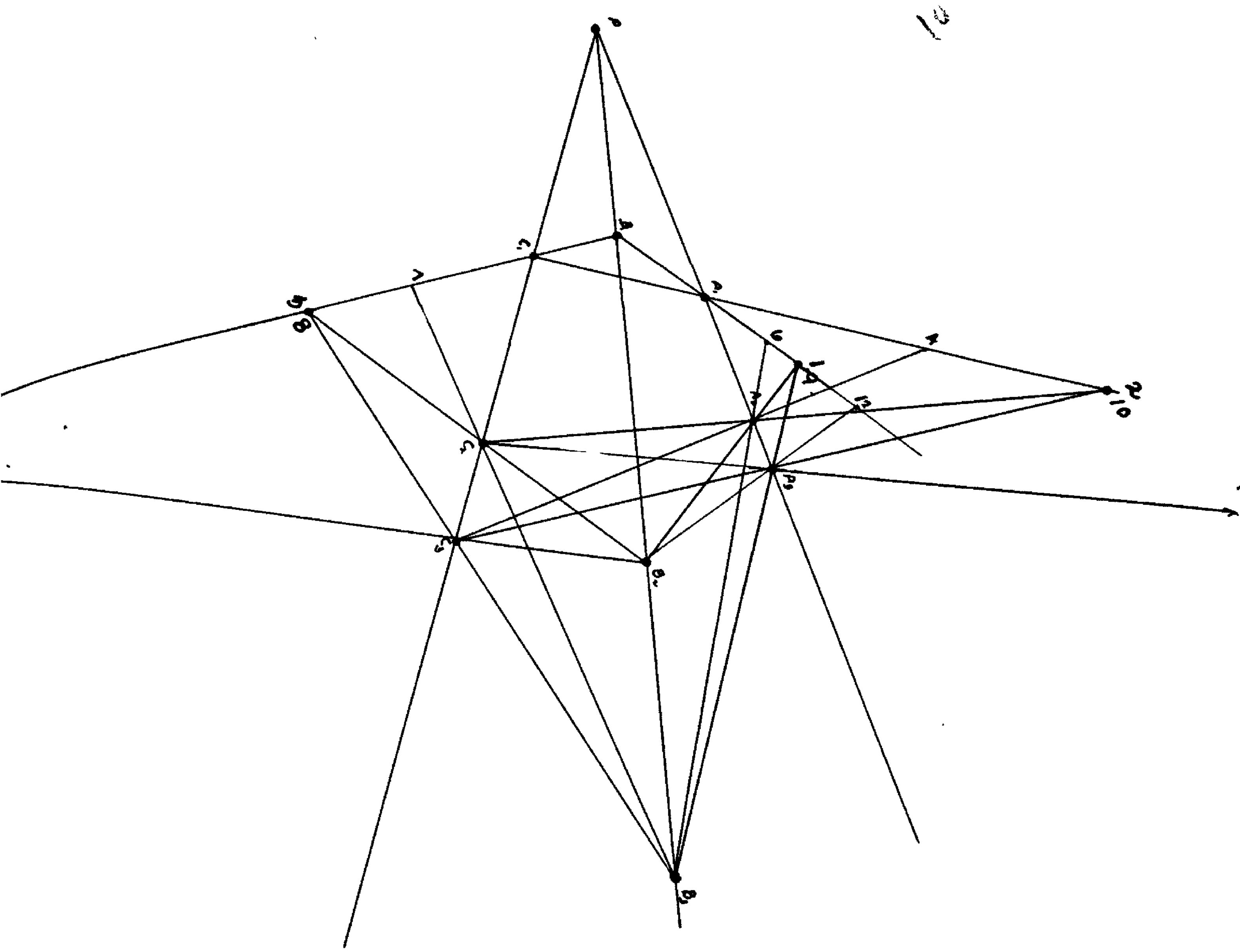


EXERCISE IV
PLATE 4'

xercise 4. Plate 4'

Using the theorem of exercise three, give a construction for a line joining any given point in the plane of the two given lines l and l' to the point of intersection of l and l' without making use of the latter point.

Solution: Let M be the given point. Draw any two lines through M intersecting l and l' in points B and B', C and C' respectively. Draw B B' and C C' intersecting at P'. Draw any line AP' intersecting lines l and l' in points A and A' respectively. Draw AB' and A'B intersecting at K. Then KM is the required line, by theorem preceding.



EXERCISE X.

PLATE FOUR.

7--

Exercise:5: Plate 5'

Using the definition of exercise 3 show that if the point P' is on the polar p of a point P with respect to the two line 1 and 1'. Then the point P is on the polar p' of P'with respect to 1 and 1'.

Given the point P3 on the polar p, to prove that P is on the polar P'.

Proof: Let the points E B D on line 1 and points C'A'D' on line 1' be respectively perspective from P and P'be any point on the polar of P. Draw A'P' intersecting line 1 at A. Draw line AP intersecting line 1' in B,. Since A'BPAB'P' and A'BPDD'K are two complete quadrangle B, = B' by the quadrangle theorem, B' being the intersection of BP' and AP). Draw any line through P' such as E E'. Draw EB' and BE' meeting at N. Since triangles B'AE and E'A'B are perspective P, N and 11' are collinear by Desargue theorem. But points N and 11' determine the polar of P' with respect to the lines 1 and 1'. Therefore P lies on the polar of P', and the theorem is proved.

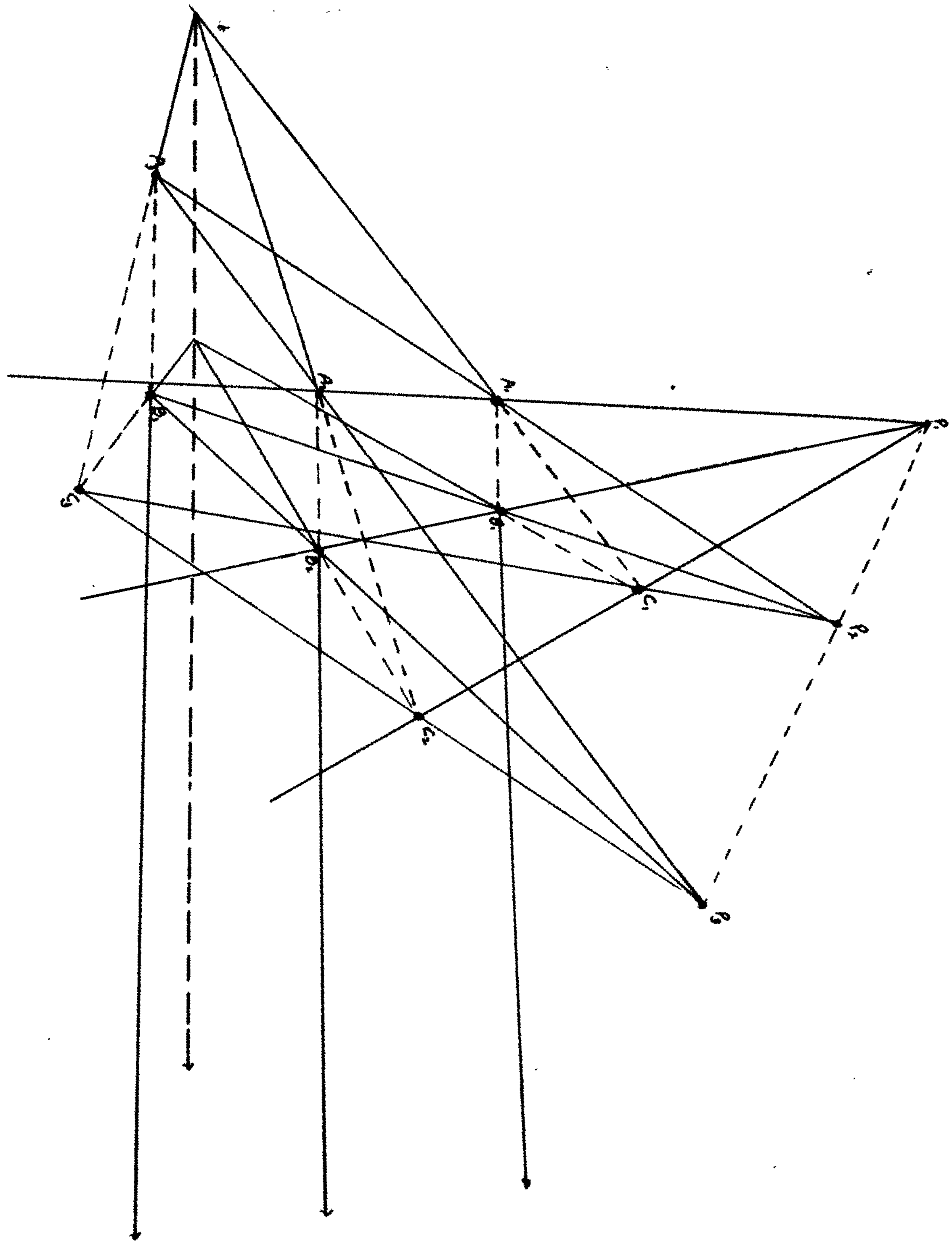
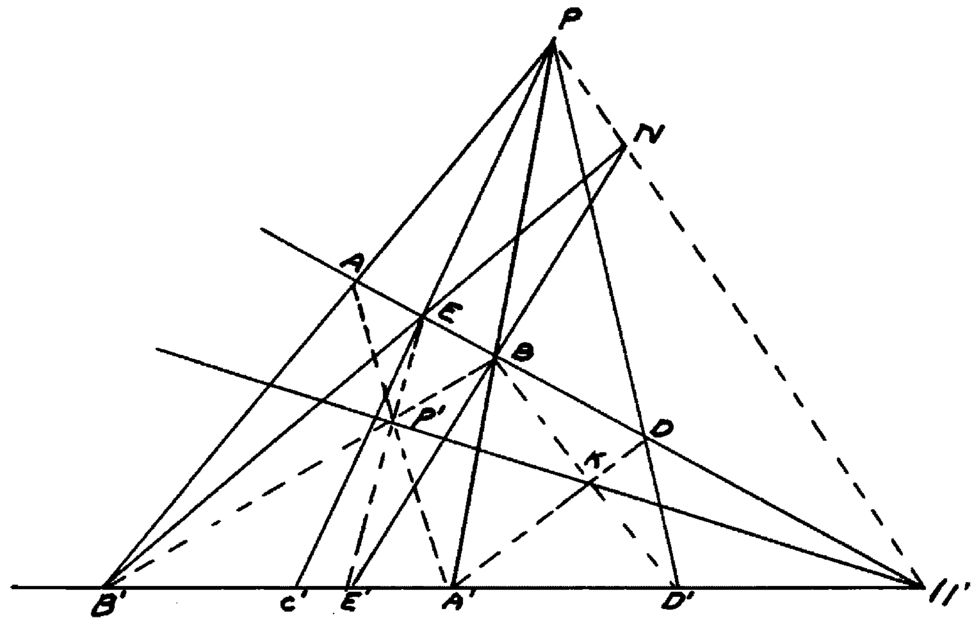


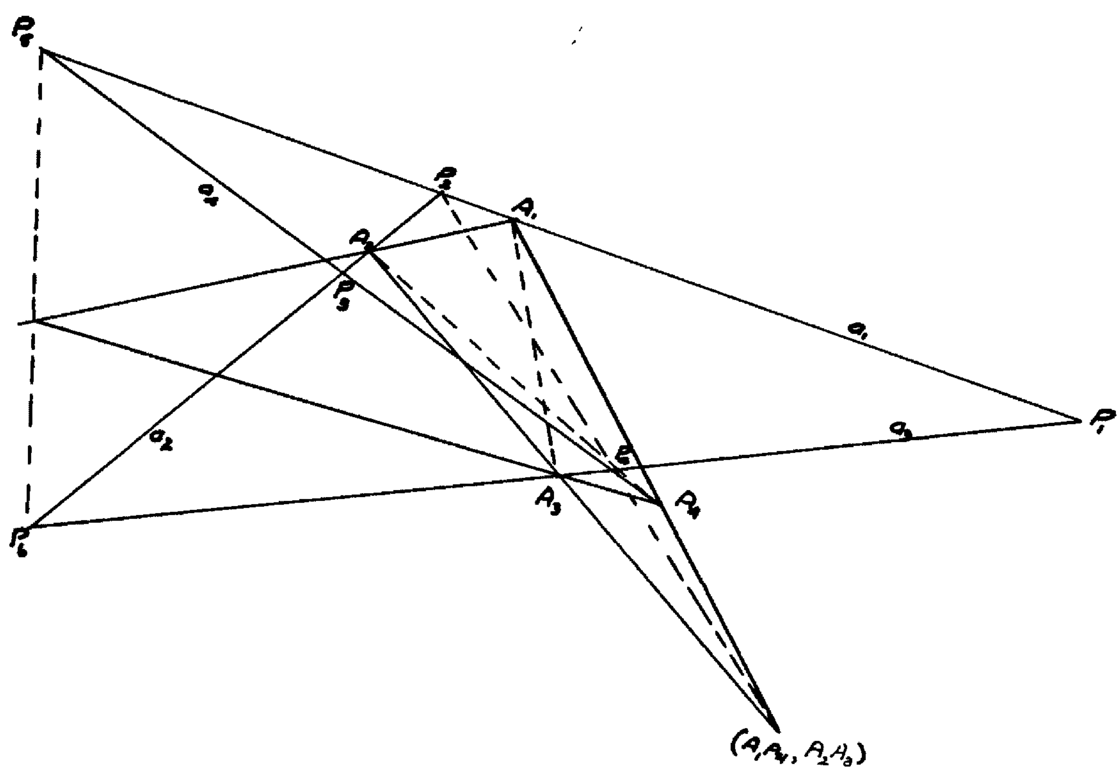
PLATE FIVE.

PLANE SECTION
6-POINT
IN SPACE.



EXERCISE V

PLATE 5'



EXERCISE VI

PLATE 6'

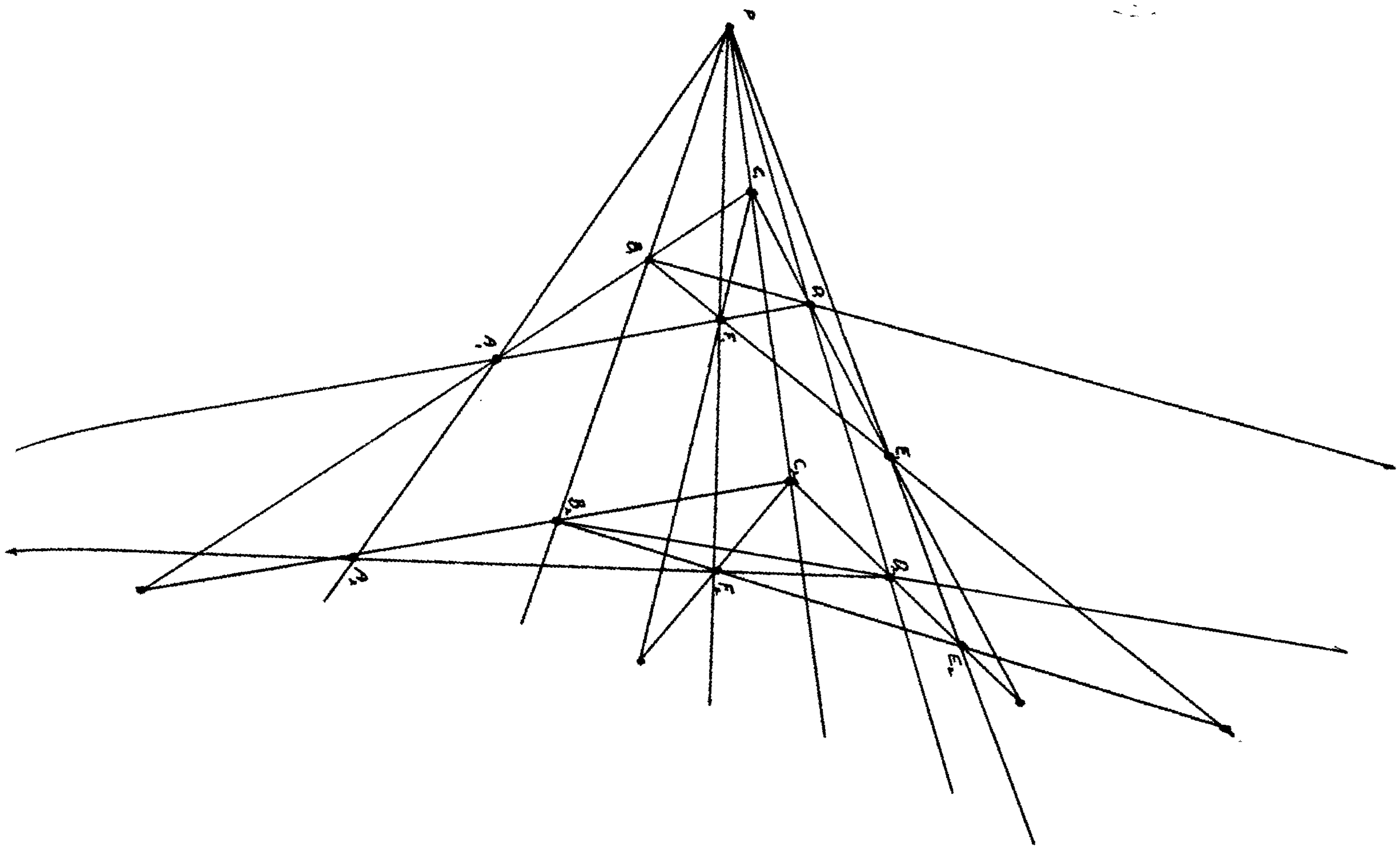


PLATE SIX.

PLANE SECTION

6-POINT

IN SPACE

Plate 6'

Exercise 6-- If the vertices $A_1A_2A_3A_4$ of a simple plane quadrangle are respectively on the sides $a_1a_2a_3a_4$ of a simple plane quadrilateral and if the intersection of the pair of opposite sides $A_1A_2A_3A_4$ is on the line joining the pair of opposite points a_1a_4, a_2a_3 the remaining pair of opposite sides of the quadrangle will meet in the line joining the remaining pair of opposite vertices of the quadrilateral.

Given: the vertices $A_1A_2A_3A_4$ of a simple plane quadrangle on the sides $a_1a_2a_3a_4$ of the simple plane quadrilateral and the intersection of the sides $A_1A_2A_3A_4$ on line joining the pair of points a_1a_4, a_2a_3. To prove that the remaining pair of opposite sides of the quadrangle will meet on the line joining the remaining pair of opposite vertices of the quadrilateral.

Proof: Connect A_1A_3 and A_2A_4. The two triangles $A_1A_2P_2$ and $A_4A_3P_4$ have their corresponding sides meeting in three collinear points. The lines joining the corresponding vertices are copointal by the converse of the Desargue theorem. Thence, P_2P_4 and A_1A_4, A_2A_3 are collinear.

Exercise 7-- If two complete plane n-points are so related that the side A_1A_2 and the remaining 2(n-2) sides passing through A_1 and A_2 meet the corresponding sides of the other n-point in points of a line l and the two n-points are perspective from a point.

Given: Two complete n-points: $A_1A_2A_3 \ldots\ldots A_n$ $A_1'A_2'A_3' \ldots\ldots A_n'$ and line l.

To prove: n-points perspective, from a point

Proof:

Lines passing through A_1 and A_2 in one n-point and A_1' and A_2' in another n-point all meet in a line l. Triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$, $A_1A_2A_m$ and $A_1'A_2'A_m'$; $A_1A_2A_n$ and $A_1'A_2'A_n'$ are all persepctive. Also they are perspective from the same point. This makes A_1A_1' A_2A_2' A_3A_3' A_mA_m' A_nA_n' copointal. Hence triangles $A_3A_mA_n$ and $A_3'A_m'A_n'$ have their sides meeting on l as they are perspective from the pole of l. Hence all the remaining 2(n-2) sides can be shown to meet on l and and perspective from a point.

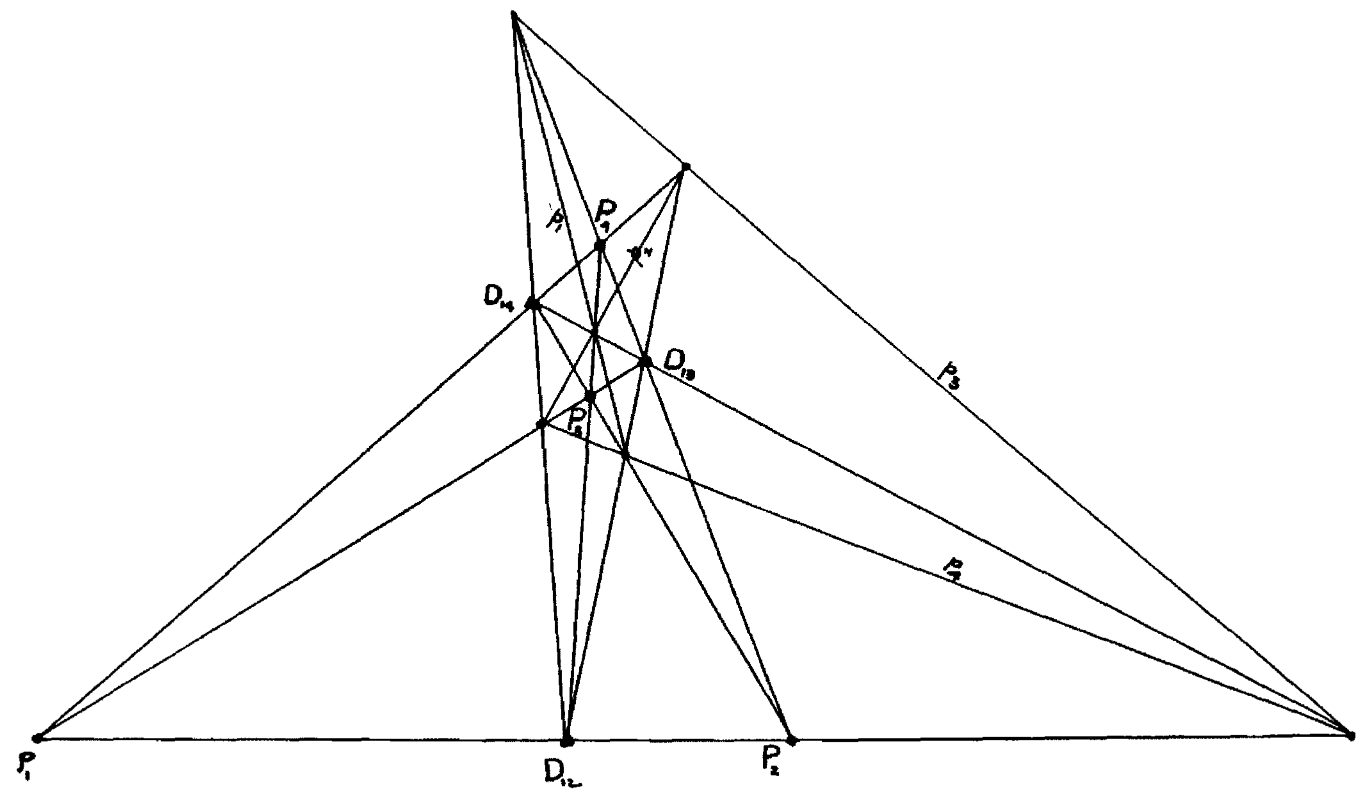


PLATE SIX

QUADRANGLE - QUADRILATERAL

CONFIGURATION.

Exercise --8. Plate 7'

If 5 sides of a complete quadrangle $A_1 A_2 A_3 A_4$ pass through five vertices of a complete quadrilateral $a_1 a_2 a_3 a_4$ in such a way that $A_1 A_2$ is on $a_3 a_4$, $A_2 A_3$ on $A_4 a_1$ etc., then the sixth side of the quadrangle passes through the sixth vertex of the quadrilateral.

Given: the complete quadrilateral $a_1 a_2 a_3 a_4$ whose vertices are $P_1 P_2 P_3 P_4 P_5 P_6$ and a complete quadrangle $A_1 A_2 A_3 A_4$ with

$A_1 A_2$ on point $a_3 a_4$, $A_2 A_3$ on point $a_1 a_4$, $A_1 A_3$ on point $a_2 a_4$, $A_1 A_4$ on point $a_2 a_3$ and $A_2 A_4$ on point $a_1 a_3$

To prove: That $A_3 A_4$ is on the point $a_1 a_2$

Proof: The sides of the triangles $A_3 P_3 P_4$ and $A_4 P_2 P_6$ meet in three collinear points $A_1 P_5 A_2$. Therefore the lines joining the corresponding vertices are copointal, by converse of Desargue theorem. Therefore line $A_3 A_4$ passes through point $a_1 a_2$ and the theorem is proved. Theorem is self dual.

 Plate 8'

Exercise 9-- If on each of three concurrent lines a b c two points are given $A_1 A_2$ on a; $B_1 B_2$ on b; $C_1 C_2$ on c -- there can be formed four pairs of triangles $A_i B_j C_k (i, j, k = 1, 2)$ and the pairs of corresponding sides meet in sixe points which are the vertices of a complete quadrilateral (Veronese, Atti dei Lincei, 1876--1877, p. 649)

Given: 3 concurrent lines two points to the line.

To Prove: That the pairs of corresponding sides meet in six points which are the vertices of a complete quadrilateral.

Proof: Triangles can be arranged in the following manner:

$A_1 B_1 C_1$ and $A_2' B_2' C_2'$
$A_1 B_2 C_2$ and $A_2' B_1' C_1'$
$A_1 B_1 C_2$ and $A_2' B_2' C_1'$
$A_1 B_2 C_1$ and $A_2' B_1' C_2'$

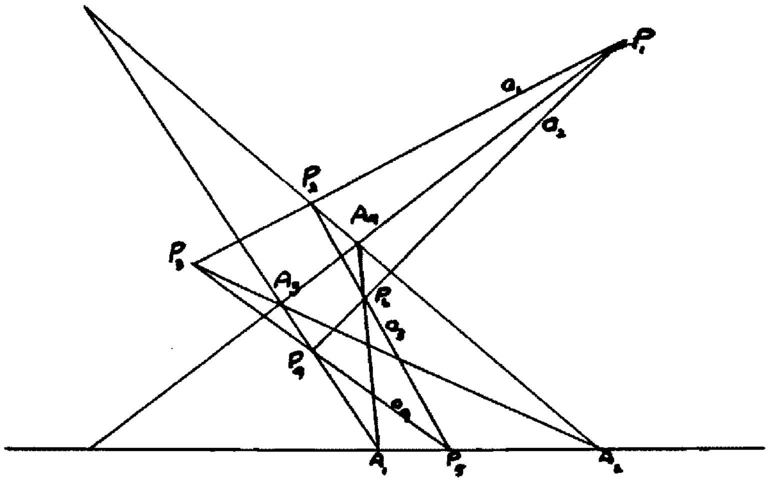
$A_1 B_1$ and $A_2 B_2$ meet in a point L
$B_1 C_1$ and $B_2 C_2$ meet in a point M
$A_1 C_1$ and $A_2 C_2$ meet in a point N

$A_1 B_2$ and $A_2 B_1$ meet in a point O
$A_1 B_2$ and $A_2 C_1$ meet in a point P
$B_2 C_2$ and $B_1 C_1$ meet in a point M

$A_1 B_1$ and $A_2 B_2$ meet in a point B
$A_1 C_2$ and $A_2 C_1$ meet in a point P
$B_1 C_2$ and $B_2 C_1$ meet in a point Q

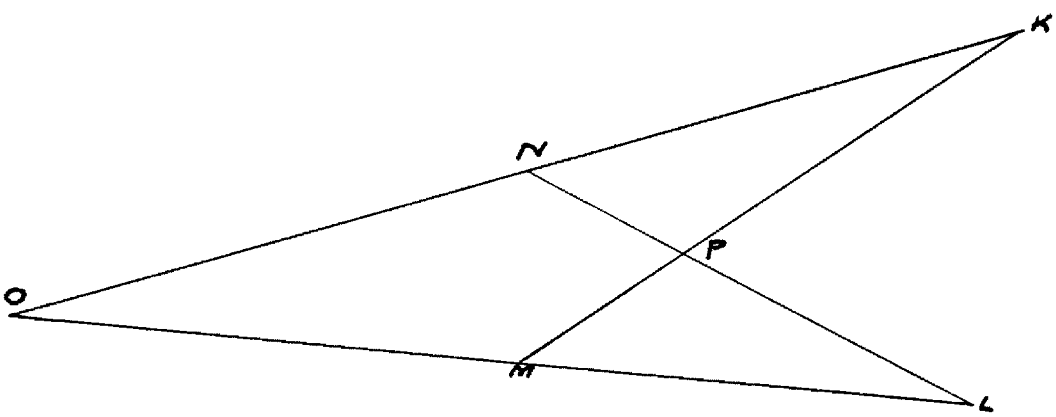
$A_1 B_2$ and $A_2 B_1$ meet in a point O
$A_1 B_1$ and $A_2 C_2$ meet in a point N
$B_2 C_1$ and $B_1 C_2$ meet in a point Q

Thus we have sixe points L M N O P Q which are collinear in groups of three as given in the above four groups. These points may be regarded as the points of a complete quadrilateral as each line has a point in common with each of the other lines.



EXERCISE VIII

PLATE 7'



EXERCISE IX

PLATE 8'

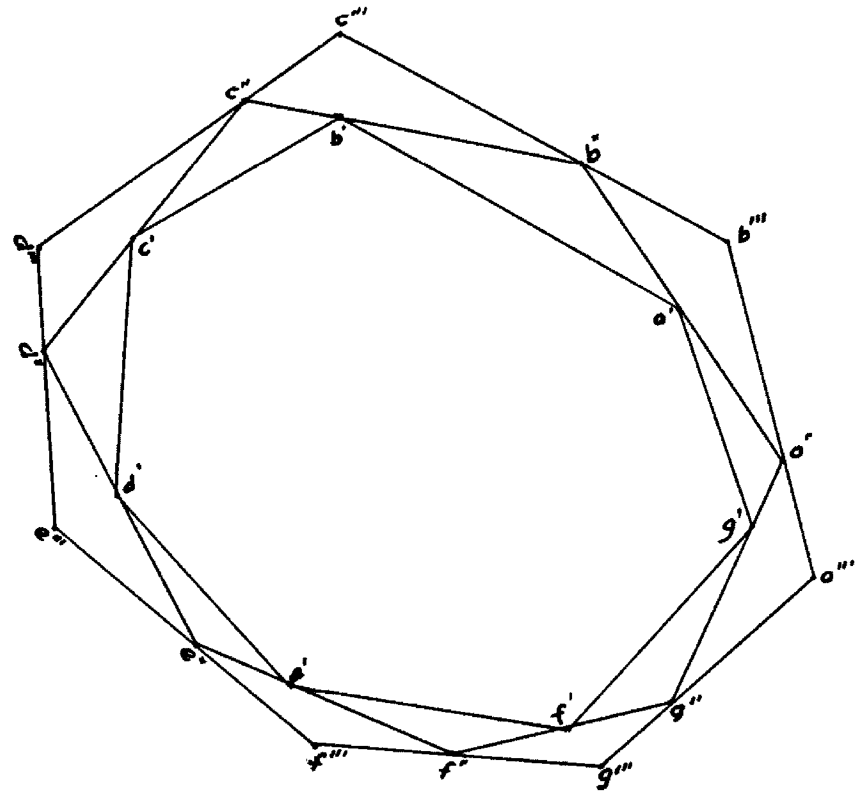


PLATE EIGHT.

PLANE SECTION 7-POINT IN SPACE.

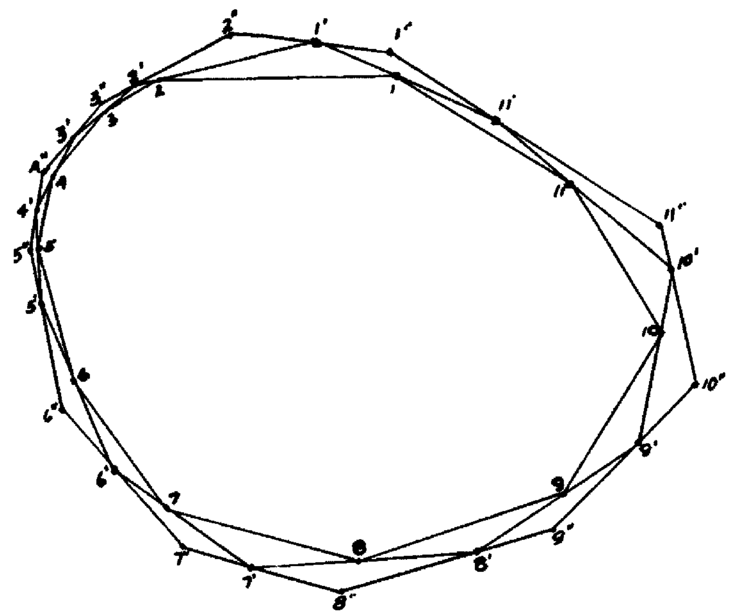


PLATE NINE.

PLANE SECTION 11-POINT IN SPACE.

Exercise 10-- With nine points situated in sets of three on three concurrent lines are formed 36 sets of three perspective triangles. For each set of three distinct triangles the axes of perspectivity meet in a point; and the 36 points thus obtained from the 36 sets of traiangles lie in sets of four on 27 lines giving a configuration

36	3
4	27

Given: $A_1A_2A_3$ on one line; $B_1B_2B_3$ on another line and $C_1C_2C_3$ on the other. Plate 4.

To prove: the above configuration is true and that the axes of perspectivity of the triangles all meet in a point.

Proof:

36 represents the number of points in the configuration.

$A_1B_1C_1$	$A_2B_1C_1$	$A_3B_1C_1$
$A_1B_1C_2$	$A_2B_1C_2$	$A_3B_1C_2$
$A_1B_1C_3$	$A_2B_1C_3$	$A_3B_1C_3$
$A_1B_2C_1$	$A_2B_2C_1$	$A_3B_2C_1$
$A_1B_2C_2$	$A_2B_2C_2$	$A_3B_2C_2$
$A_1B_2C_3$	$A_2B_2C_3$	$A_3B_2C_3$
$A_1B_3C_1$	$A_2B_3C_1$	$A_3B_3C_1$
$A_1B_3C_2$	$A_2B_3C_2$	$A_3B_3C_2$
$A_1B_3C_3$	$A_2B_3C_3$	$A_3B_3C_3$

From these 27 triangles can be chosen 36 sets of 3 distinct triangles with no side or vertex common in any set. The first triangle can be chosen in 27 different ways. For any triangle chosen there are 6 points left, 2 on each line. These 6 points can be formed into 8 sets of two triangles. We, therefore, have $\frac{27\times8}{6}$ combinations of three or 36 sets of three. For each triangle in the first column above there are four combinations of the others, having no vertex or side common in the three as:

$A_1B_1C_1$	$A_2B_2C_2$	$A_3B_3C_3$
	$A_2B_2C_3$	$A_3B_3C_2$
	$A_2B_3C_2$	$A_3B_2C_3$
	$A_2B_3C_3$	$A_3B_2C_2$
$A_1B_1C_2$	$A_2B_2C_3$	$A_3B_3C_1$
	$A_2B_2C_1$	$A_3B_3C_3$
	$A_2B_3C_3$	$A_3B_2C_1$
	$A_2B_3C_1$	$A_3B_2C_3$
$A_1B_2C_1$	$A_2B_1C_2$	$A_3B_3C_3$
	$A_2B_1C_3$	$A_3B_3C_2$
	$A_2B_3C_2$	$A_3B_1C_3$
	$A_2B_3C_3$	$A_3B_1C_2$

Ex. 10

$A_1B_1C_3$ $A_2B_2C_2$ $A_3B_3C_1$
$A_2B_2C_1$ $A_3B_3C_2$
$A_2B_3C_1$ $A_3B_2C_2$
$A_2B_3C_2$ $A_3B_2C_1$ And so on through the list. For each set of three distinct triangles the axes of perspectivity meet in a point, by converse of Desargue theorem. See Plate 4 / To prove that these points of intersection of the axes lie in sets of 4 on 27 lines let the sides of the triangles meet as follows:

A_1B_1	and	A_2B_2	meet	in	point	1	
A_1C_1	and	A_2C_2	"	"	"	2	in triangles $A_1B_1C_1$ and $A_2B_2C_2$
B_1C_1	"	B_2C_2	"	"	"	3	
A_1B_1	"	A_2B_2	"	"	"	1	
A_1C_1	"	A_2C_3	"	"	"	4	in triangles $A_1B_1C_1$ and $A_2B_2C_3$
B_1C_1	"	B_2C_3	"	"	"	5	
A_1B_1	"	A_2B_3	"	"	"	6	
A_1C_1	"	A_2C_2	"	"	"	2	in trainglea $A_1B_1C_1$ and $A_2B_3C_2$
B_1C_1	"	B_3C_2	"	"	"	7	
A_1B_1	"	A_2B_3	"		"	6	
A_1C_1	"	A_2C_3	"	"	"	4	in triangles $A_1B_1C_1$ and $A_2B_3C_3$
B_1C_1	"	B_3C_3	"	"	"	8	
A_1B_1	"	A_3B_3	"	"	"	9	
A_1C_1	"	A_3C_3	"	"	"	10	in triangles $A_1B_1C_1$ and $A_3B_3C_3$
B_1C_1	"	B_3C_3	"	"	"	8	
A_1B_1	"	A_3B_3	"	"		9	
A_1C_1	"	A_3C_2	"	"	"	11	in triangles $A_1B_1C_1$ and $A_3B_3C_2$
B_1C_1	"	B_3C_2	"	"	"	7	
A_1B_1	"	A_3B_2	"	"	"	12	
A_1C_1	"	A_3C_3	"	"	"	10	in triangles $A_1B_1C_1$ and $A_3B_2C_3$
B_1C_1	"	B_2C_3	"	"	"	5	
A_1B_1	"	A_3B_2	"	"	"	12	
A_1C_1	"	A_3C_2	"	"	"	11	in triangles $A_1B_1C_1$ and $A_3B_2C_2$
B_1C_1	"	B_2C_2	"	"	"	3	

In the same way the points of intersections of the sides of the second a and third triangles may be found in each set of four. This however is unnecessary for we would get points on the lines which pass throught the points of interaetion of the axes and these have been determined by the first two lines, since the three axes are copointal for each set of three. At the points obtained above collinear by threes, by Desargues theorem, as 1,2,3 are collinear and 12,11,3 are collinear etc. Also we have the following collinear points 1,6,9,12;2,4,10,11; and 3,5,7,8. The first set is A B. The second set is on A C. The third set of collinear points is on B C. From these sets of collinearities we obtain the figure 4. In this figure Y and Z are collinear with B by the theorem B'. Then the points U&V, W and X are collinear by Desargues theorem.

t these points are the points of intersection of the axes of persepo-
ivity of the first four groups of three triangles as grouped above. There-
ore the points of intersections of the axes of perspectivity are collinear
sets of four. By following the grouping outlined above we get 36 points
9 lines. If instead of taking each of the first column of the 27 triangles
ave with all the combinations of the others, not having a vertex or side
ommon, we had taken each of the second with all the combinations of the
thers, we would have obtained the same 36 points on 9 other lines, and like-
ise each of the third with the combinations of the others, would give the
ame 36 points on 9 other lines. We therefore get the required configuration
or the points of intersection of the axes of perspectivity.

 Plate 5

Exercise 11. A plane section of a 6-point in space may be considered as three triangles perspective in pairs from three collinear points with corresponding sides meeting in three collinear points.

Given: Three triangles perspective in pairs from three collinear points and the corresponding sides meeting in three collinear points.

To prove: that the figure may be considered as a plane sections of a 6-point in space.

Proof: A complete space 6 point is of the configuration

6	5	10
2	15	4
3	3	20

A plane section of a complete 3-space 6-point is of the configuration

15	4
3	20

The figure of three triangles perspective in pairs form three collinear points and whose sides meet in three collinear points is of this configuration. See Figure 5.

 Plate 6

Exercise 18. A plane section of a 6-point in space can be considered as 2 perspective complete quadrangles with the corresponding sides meeting in the vertices of a complete quadrilateral.

Given: The figure of 2 perspective quadrangles with the corresponding sides meeting in the vertices of a complete quadrilateral.

To prove that this figure is of the configuration of a plane section of a 6-point in space.

Proof: Such a figure can be constructed. We can prove that A B C are collinear; A F D collinear; C D E collinear; E F B collinear by Desargue theorem, as for instance triangles A'B'C' and A''B''C'' are peespective and therfore the corresponding sides meet in three collinear points. In the same way the others sets may be proved collinear. This figure is of the configuration

15	4
3	20

This is the configuration of a plane section of a 6-point in space and the theorem is proved. See figure 6.

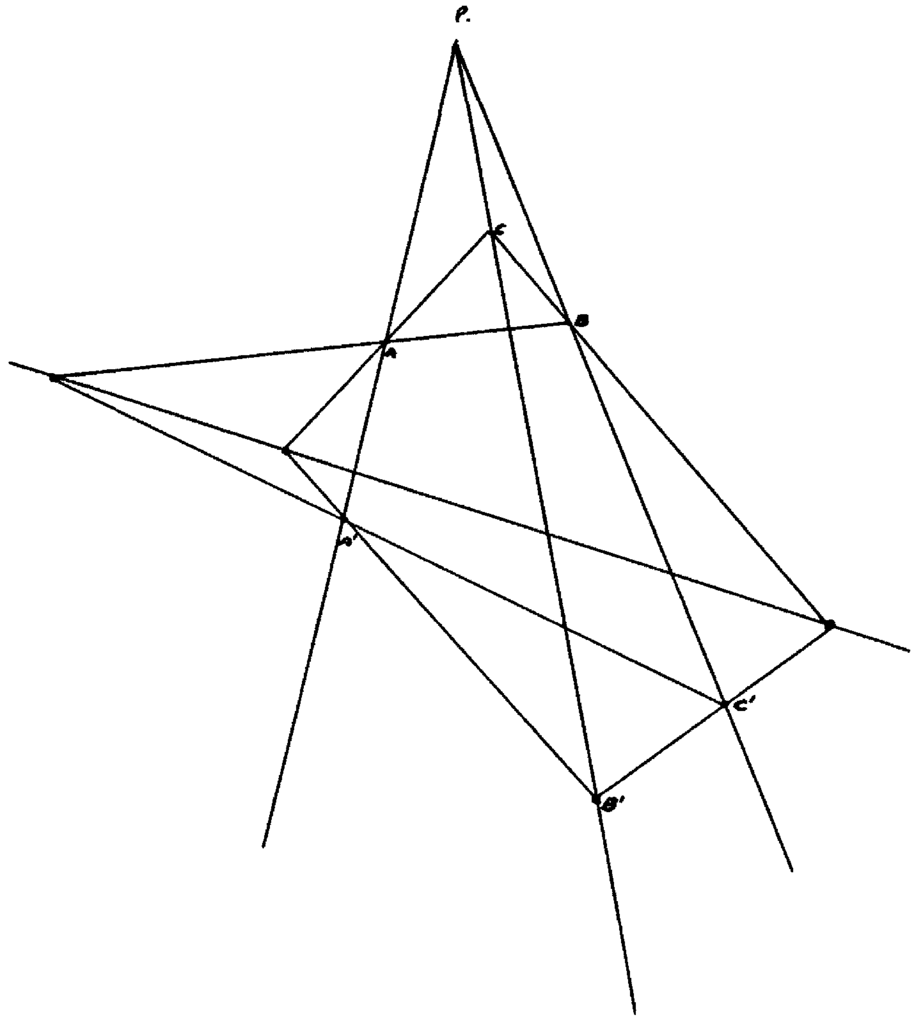


PLATE ELEVEN.

TWO PERSPECTIVE TRIANGLES FROM P

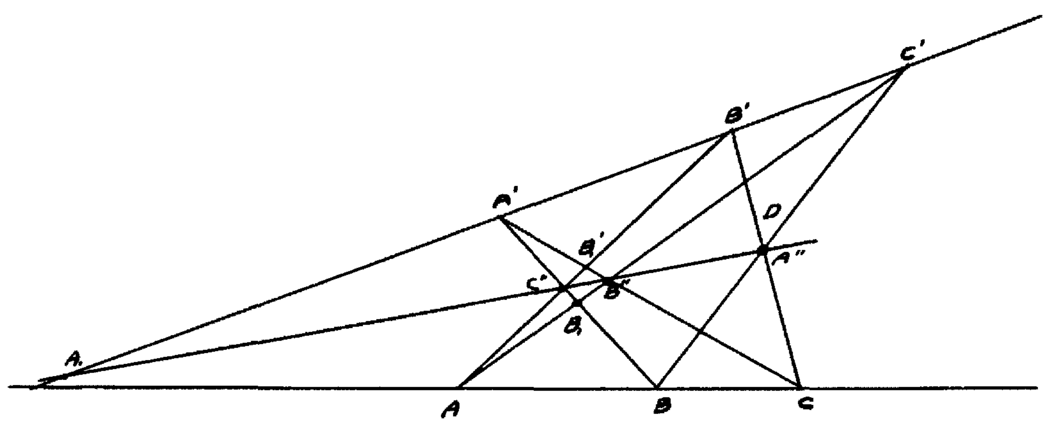


PLATE TWELVE.

CONFIGURATION OF PAPPUS.

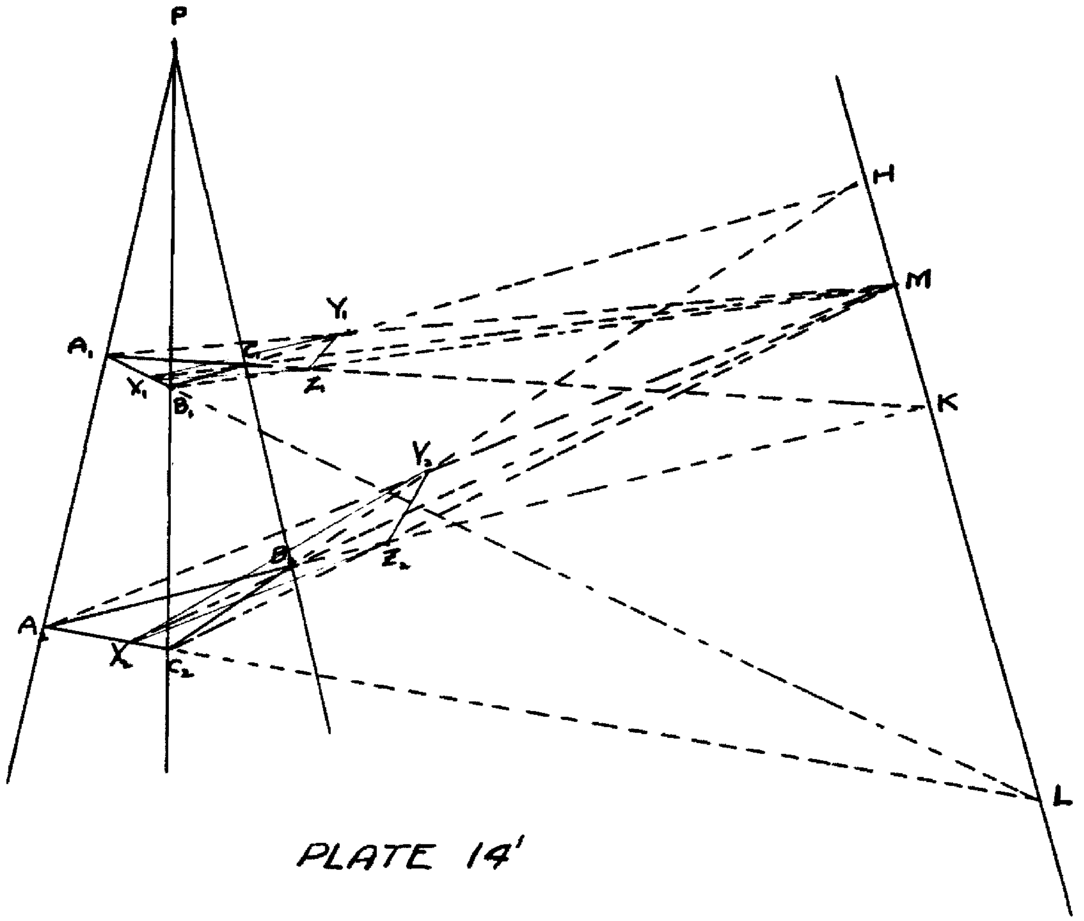


PLATE 14′

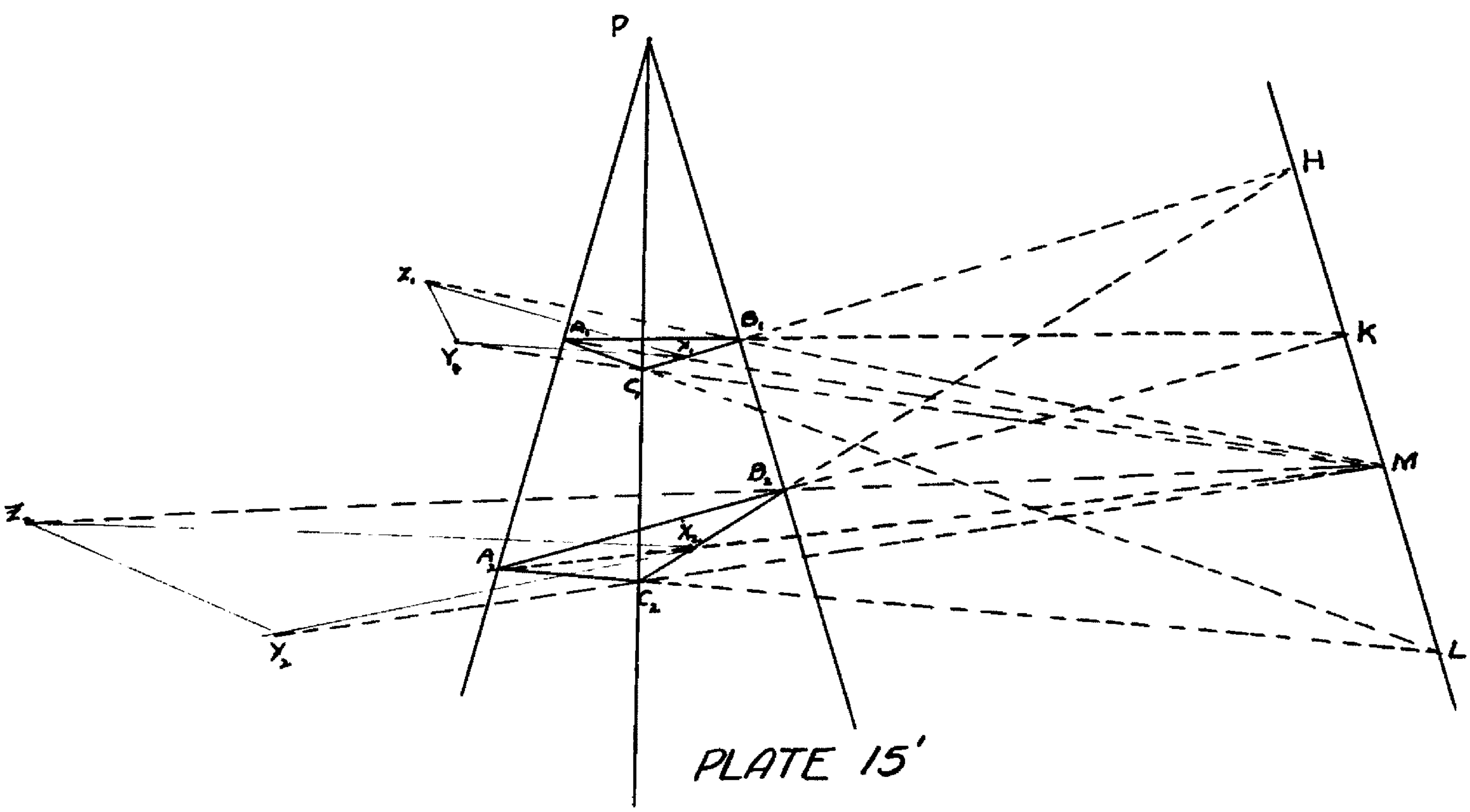


PLATE 15′

EXERCISE XXII

A plane section of an n-point in space gives the configuration

$$\begin{array}{|ll|} {}_nC_2 & n-2 \\ 3 & {}_nC_3 \end{array}$$

Which may be considered (in
ways) as a set of (n-k) k-points perspective in pairs from ${}_{n-k}C_2$ points,
ich form the configurations

$$\begin{array}{|ll|} {}_{n-k}C_2 & n-k-2 \\ 3 & {}_{n-k}C_3 \end{array}$$

d the points of intersections of the corresponding sides form the con-
guration

$$\begin{array}{|ll|} {}_kC_2 & k-2 \\ 3 & {}_kC_3 \end{array}$$

Given: a plane section of an n-point in space, to prove: that it is the above configurations.

Proof: Every pair of points of the n-point in space determined a line.
ese lines intersect the plane section in points. There are ${}_nC_2$ of these
ints in the plane section. Every combinations of three points in the
point in space determine a plane. These planes intersect the plane section
lines. There are ${}_nC_3$ of these lines in the plane section. Each of the
anes spoken of above which intersect the plane section in points. There-
re there are 3 points on a line in the plane section. Any two points of
e n-point in space taken with each of the other points determine n-2
anes all intersecting on the same line. This gives n-2 lines on a point
the plane section. This proves the first configuration of the theorem.
prove the second confiuration, let us consider the n-point in space. In
e n-point in space we may select any set of (k) points. There are left
-k points. Select any two of these n-k points, such as f & g. These two
ints with each one of the k points determine a plane. These planes inter-
ect the plane section in k copointal lines, upon which there are two sets
f k points; those formed by the intersection, with the plane section, of
ines joining the point f to each of the k points in space, and the other
he points formed by lines from point g to the k points in space intersect-
ng the plane section. These two k-points are perspective from the point
f intersection of line f g with the plane section. Suppose we take any

her pair of points of the n-k points, other than f and g , using no point ice. There will be formed in this manner $\frac{n-k}{2}$ sets of 2 k-points, perspective by pairs, making in all (n-k) k-points. The k points and the n-k ints in space may be selected in ${}_nC_n$ ways and therefore we get the set of -k) k-points in the section, in ${}_nC_n$ ways. Now if instead of selecting our t of (n-k) k-points in the manner we did, we take the points of the n-k ints in space by pairs in other combinations, we would have the same set (n-k) k-points perspective by pairs from $\frac{n-k}{2}$ other centers. In short we ll get this set of (n-k) k-points perspective in pairs from as many centrs as there are combinations of 2 in n-k, namely ${}_{n-k}C_2$ centers. For example, we connect the point g to each of the other n-k-1 points in space we get e k-point formed by lines from g to the k points in space intersecting the ane section, perspective with each of the other (n-k-1) k -points by pairs. kewise we may show that any k-point of the set is perspective with each of e others, in pairs. Since these centers of perspectivity of the k-points e the intersections of the lines joining the n-k points in space, by pairs, th the plane section, their configuration in the plane section is simply plane section of an (n-k)- point in space. Which proves the second configration of the theorem.

Since the k-points are perspective by pairs, every pair of triangles rmed by three corresponding points of the two perspective k-points are rspective and by Desargues theorem their corresponding sides will meet in ree collinear points. There will be ${}_kC_3$ of these corresponding triangles d consequently ${}_kC_3$ lines and 3 points on a line. The two k-points will have rresponding sides for every pair of corresponding vertices, or ${}_kC_2$ sides, ich intersect each other in ${}_kC_2$ points on the ${}_kC_3$ lines. To find the total mber of points, which is $\frac{3\cdot\frac{1}{6}k(k-1)\,(k-2)}{\frac{1}{2}k(K-1)} = k-2$ lines on a point. erefore the configuration

$$\boxed{\begin{array}{cc} {}_kC_2 & k-2 \\ 3 & {}_kC_3 \end{array}}$$

follows and the theorem proved.

Exercise 14. *Plate 8*

A plane section of a seven point in space can be sonsidered (19 ways) as composed of 3 simple heptagons cyclically circumscribing each other.

Given: A plane section of a 7-point in space. To prove that it may be considered as composed of 3 simple heptagons cyclically circumscribing each other.

Proof: The configuration of a 7-point in space is

7	6	15
2	21	5
3	3	35

The plane section is

21	5
3	35

. Let the points of the 7-points in sp space be numbered 1,2,3,4,5,6,7, respectively. In a plane section there are 21 points made by the intersection with the plane section of lines joining the points of the 7-point in space, by pairs. These points we shall designate by the numbers of the points connected in the 7-point in space; as, the line connecting points 3 & 6 of the 7-point in space is the line 36 and the line intersects with the plane section in point 36. The 21 points are:

12	23	34	45	56	67
13	24	35	46	57	
14	25	36	47		
15	26	37			
16	27				
17					

The lines joining these points shall be named according to the 2 points connected: as, the line joining points 23 & 37 shall be line 23 37 etc. The three heptagons are chosen with the following vertices:
12,23,34,45,56,67,71. 13,35,57,27,24,46,61. 14,47,37,36,26,25,51.
Points 1,2,3, of the 7-point in space determine a plane. This plane cuts the plane section in a line. ON this line are points 12,13,23. In the same way 23,24,34;34,35,45;45,46,56;56,57,67;67,71,61;71,21,27;13,15,35; 35,37,57;25,27,57;24,27,47;24,26,46;24,46,61;13,56,61;14,47,17;34,47,37; 36,37,67;23,26,36;25,26,56;12,15,25;14,15,45 are all collinear in sets of 3. Therefore each vertex of the second heptagon lies on a side of the first and each vertex of the third lies on a side of the second, and each vertex of the first lies on a side of the third. The heptagons, therefore, cyclically circumscribe each other. The first heptagon may be chosenin 5 or 120 different ways, as will be shown in exercise 16. The second and third are fixed when the first is chosen.

Exercise 15:

A plane section of an 11-point in space can be considered (in $\lfloor 9$ different ways) as five simple 11-points cyclically circumscribing each other.

Given: A plane section of an 11-point in space. To prove: that it contains five simple 11-points cyclically circumscribing each other.

Proof: Let the lines and points of the 11-points in space and the plane section be numbered as in exercise 14. Then the five simple 11-points may by chosen with the following vertices:

$1(1+2^0),2(2+2^0),3(3+2^0),4(4+2^0),5(5+2^0),6(6+2^0),7(7+2^0),8(8+2^0)9(9+2^0),10(10+2^0)11(11$

$1(1+2^1),2(2+2^1),3(3+2^1),4(4+2^1),5(5+2^1),6(6+2^1),7(7+2^1),8(8+2^1)9(9+2^1),10(10+2^1)11(11$

$1(1+2^2),2(2+2^2),3(3+2^2),4(4+2^2),5(5+2^2),6(6+2^2),7(7+2^2),8(8+2^2)9(9+2^2),10(10+2^2)11(11$

$1(1+2^3),2(2+2^3),3(3+2^3),4(4+2^3),5(5+2^3),6(6+2^3),7(7+2^3),8(8+2^3)9(9+2^3),10(10+2^3)11(11$

$1(1+2^4),2(2+2^4),3(3+2^4),4(4+2^4),5(5+2^4),6(6+2^4),7(7+2^4),8(8+2^4)9(9+2^4),10(10+2^4)11(11$

Which may be written as follows, the numbers in the parentheses being taken in cyclic order:

12,23,34,45,56,67,7,8, 89,9 10,10 11, 11 1.
13,24,35,46,57,68,79,8 10,9 11,10 1,11 2.
15,26,37,48,59,6 10,7 11,81,92,10 3, 11 4.
19,2 10, 3 11,41, 52,63,74,85,96,10 7, 11 8.
16,27,38,49,5 10,6 11, 71,82,93, 10 4, 11 5.

Written in the order of adjacent vertices:

12,23,34,45,56,67,78,89,9 10,10 11, 11 1.
13,35,57,79,9 11,11 2,24,46,68,8 10,10 1.
15,59,92,26,6 10, 10 3, 37 7 11, 11 4, 48,81.
19,96,63,3 11, 11 8, 85, 52,2 10, 10 7, 74, 41.
16,6 11, 11 5, 5 10, 10 4, 49,93, 38,82, 27,71.

The first of these simple 11-points may be chosen in $\lfloor 9$ different ways for since there are 9 lines one a point after the first vertex is chosen the second may be chosen in 9 ways. Likewise the third may be chosen in 8 ways, since the 9thline goes back to the second. The 4th may be chosen in 7 ways: the 5th in 6; the 6th in 5; the 7th in 4 the 8th in three; the 9th in 2; and 10 in one and the 11the in one way , making factorial nine ways thefirst may be chosen. After the first is chosen the others are all fixed.

Exercise--16

A plane section of an n-point in space for n prime can be sonsidered (in $n-2$ ways) as $\frac{n-1}{2}$ simple n-points cyclically circumscribing each other.

Given: A plane section of an n-point in space. To prove that it can be considered as $\frac{n-1}{2}$ simple n-points cyclically circumscribing each other.

Proof: A plane section of an n-point in space is of the configuration

$\boxed{\begin{matrix} \frac{1}{2}n(n-1) & n-2 \\ 3 & \frac{1}{6}n(n-1)(n-2) \end{matrix}}$ Since there are $\frac{1}{2}n(n-1)$ points in the configuration we have exactly the number of points called for in the theorem. We will only have to construct the $\frac{n-1}{2}$ simple n-points to prove the theorem. Let the points of the n-points in space be numbered 1,2,3,4,5,6,--.... .n-3,n-2,n-1, n. In the plane section there are $\frac{1}{2}n(n-1)$ points formed by the interestion with the plane section of lines joining the points of the n-point in space, by pairs. Let us designate these points by the numbers of the points which the corresponding lines connect, as in exercise 14. Each set of three points of the n-point in space determine a plane which cuts the plane section in a line, so there will be three points on a line. Therefore there can be but one vertex of a second simple n-point on a side of the first. Let the simple n-points be chosen with the following vertices:

12,34,45,56,67,78,89,9 10, 10 11 ....n-2)(n-1), n-1)n, n 1.
13,24,35,46,57,68,79,8 10,9 11,10 12...............n-2 n ,(n-1)1, n 2.
14,25,36,47,58,69,7 10,8 11,9 12,....................n-2 1, n-1)2, n 3.
15,26,37,48,59,6 10, 7 11,8 12,9 13 -...............n-2 2 n-1)3, n 4.
16,27,38,49,5 10, 6 11,7 12,8 13......................n-2 3 n-1)4, n 5.
....
....
....
1 $\frac{n-3}{2}$, 2 $\frac{n-1}{2}$, 3 $\frac{n+1}{2}$, 4 $\frac{n+3}{2}$..............................n-1 $\frac{n-7}{2}$, n $\frac{n-5}{2}$
1 $\frac{n-1}{2}$, 2 $\frac{n+1}{2}$, 3 $\frac{n+3}{2}$ ....n-1 $\frac{n-5}{2}$, n $\frac{n-3}{2}$
1 $\frac{n+1}{2}$, 2 $\frac{n+3}{2}$, 3 $\frac{n+5}{2}$...................................n-1 $\frac{n-3}{2}$, n $\frac{n-1}{2}$

Ex 16

The simple n-point may be writeny in the order of the adjacent vertices if desired as: 13,35,57,79,9 11,11 13,13 15 ....etc. Let the simple n-points be numbered 1,2,3,4,5,6,7........to $\frac{n-1}{2}$ in the order written on the preceding page. Then the simple n-points are circumscribed as follows:

Number 1 is circumscribed by $\frac{n-1}{2}$

" 2 " " " 1

" 3 " " " $\frac{n-3}{2}$

" 4 " " " 2

" 5 " " " $\frac{n-5}{2}$

" 6 " " " 3

" 7 " " " $\frac{n-7}{2}$

" 8 " " " 4

" 9 " " " $\frac{n-9}{2}$

....

.... and so to .

Number $\frac{n-1}{2}$ is circumscribed by $\frac{n-1}{4}$ or $\dfrac{n - \frac{n-1}{2}}{2} = \frac{n+1}{4}$ depending on whether $\frac{n-1}{2}$ is even or odd. By the nature of the notation used in writing the $\frac{n-1}{2}$ simple n-points it is evident that every point of the plane section of the n-points in space has been used, and but once. It is also evident that the theorem does not hold for (n) an even number, for then $\frac{n-1}{2}$ is a fraction and not integral. Suppose (n) is any odd number not prime. Then some simple n-point may be circumscribed only by itself as, if n = 33, then the eleventh simple 33-point is circumscribed by $\frac{33-11}{2}$ which is itself. Let us show that this is impossible for (n) a prime number. Suppose k a positive integer less than, or equal to $\frac{n-1}{2}$, represents the number of any simple n-point. Now if k is an even number it will be circumscribed by number k/2. If k is an odd number it will be circumscribed by $\frac{n-k}{2}$. Now $\frac{n-k}{2}$ cannot be equal to k for then n would equal 3 k which is contrary to the hyporthesis that n is a prime number.

It remains to be proven that the simple n-points can be chosen in n-2 ways. We will select the first vertex of the first simple n-point as the intersection with the plane section of a line joining any two points of the n-point in space, such as f & g. It will be remembered that all the points in the plane section are formed in this way. To determine the second vertex we may take the intersection with the plane section, of the line from g to any of the n-2 remaining points. Let us select h. The third vertex will be the intersection with the plane section of the line from h to any of the remining n-3 points, etc. After the first vertex is chosen the second may be chosen in n-2 ways, the third in n-3 ways, the fourth in n-4 ways, the 5th, in n-5 ways........................the n-st in n(n-1) or 1 way and the last will be determined by the line from the the nth point chosen in the n-point in space, back to f. Thus we have the first simple n-point in the plane section determined in n-2 ways. Since there are only 3 points on a line, in the plane section, and since each simple n-point has to be circumscribed by one of the others, after the first simple n-point is chosen the others are fixed.

Ex 17--- Plate 13'

A plane section of a 6-point in space gives (in 6 ways) a five point whose sides pass through the poiꭓts of the configuration $\boxed{\begin{array}{cc}10 & 3\\ 3 & 10\end{array}}$

Given: A plane section of a 6-point in space, to Prove that it gives in six ways a five point whose sides pass through the configuration of Desargues.

Proof: To prove this theorem it is only necessary to construct a figure which fulfills the conditions, as the figure 13'. This figure is a plane section of a six point in space. To show that it satisfies the conditions of the theorem I have drawn a 5-point in the ~~red ink~~ pencil and Desargue configuration, satisfying the given conditions, in black ink. These two configurations were chosen after the 6-point section was drawn. We hav the conditions satisfied in the following six ways:

Desargues										Plane 5-point				
A_1	B_1	C_1	D_1	E_1	F_1	H_1	I_1	J_1	K_1	A_2	B_2	C_2	D_2	E_2
E_1	D_1	K_1	D_2	B_1	C_1	C_2	B_2	A_2	A_1	H_1	F_1	I_1	J_1	E_2
B_2	C_2	D_2	E_2	C_1	D_1	I_1	E_1	F_1	H_1	A_2	A_1	B_1	K_1	J_1
A_1	K_1	J_1	A_2	D_1	I_1	B_2	H_1	D_2	E_2	B_1	C_1	E_1	F_1	C_2
B_1	K_1	J_1	A_2	E_1	F_1	C_2	E_1	D_2	E_2	A_1	C_1	D_1	I_1	B_2
A_2	B_2	C_2	E_2	A_1	B_1	J_1	C_1	I_1	F_1	D_2	K_1	D_1	E_1	H_1

WHICH proves the conditions of the theorem in six ways.

Ex 18--

A Plane section of an n-point in space gives a complete n-1 point whose sides pass through the points of the configuration $\boxed{\begin{matrix} {}_{n}C_{2} & n-3 \\ 3 & {}_{n-1}C_{2} \end{matrix}}$

Given: A plane section of a n-point in space; to Prove: that it gives a complete (n-1) point whose sides pass through the points of the given configuration .

Proof: A plane section of a complete n-point in space is of the configuration $\boxed{\begin{matrix} \frac{1}{2}(n-1)n & n-2 \\ 3 & \frac{1}{6}n(n-1)(n-2) \end{matrix}}$ The given configuration $\boxed{\begin{matrix} \frac{1}{2}(n-1)(n-2) & n-3 \\ 3 & \frac{1}{6}(n-1)(n-2)(n- \end{matrix}}$

A complete plane n-1 point is of the configuration $\boxed{\begin{matrix} n-1 & n-2 \\ 2 & \frac{1}{2}(n-1)(n-2) \end{matrix}}$

If we subtract the points of the plane section of the (n-1)point in space from the points of the plane section of the n-point in space, we get $\frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-2) = n-1$ points. If we go to the n-point in space, we find that these (n-1) points are formed by the intersections, of lines drawn from one of the points, k, of the 3-space n-point to the other n-1 points with the plane section. The point k, chosen, lies in the ${}_{n-1}C_{2}$ planes of the 3 space n-point of $\frac{1}{2}(n-1)(n-2)$ planes. These planes cut the plane section in lines so that the n-1 points above lie on the $\frac{1}{2}(n-1)(n-2)$ lines or the number of lines required by the complete plane (n-1) point in space, or $n(n-1)(n-2) - (n-1)(n-2)(n-3) = \frac{1}{2}(n-1)(n-2)$ the number of lines left for the complete plane n-1 point. Therefore we choose these lines in the manner spoken of above. Since any one of the n-1 points are obtained by lines from k to the other n-1 points of the three space n-point intersecting the plane section, it is evident that any line and each of the other n-2 points of the 3 -space n-point form planes which cut the plane section on the given point making n-2 lines on a point. Since lines were drawn from k to each of the other n-1 points of the 3-space n-points, to determine n-1 points, spoken of above (but the other points connected by pairs), there are just two points on each line, in this configuration, thus giving a complete plane (n-1)point whose $\frac{1}{2}(n-1)(n-2)$ sides lie on the vertices of the plane section of the (n-1) point in space.

Exercise 19

The section by a three space of ah n-point in a 4-space is of the configuration

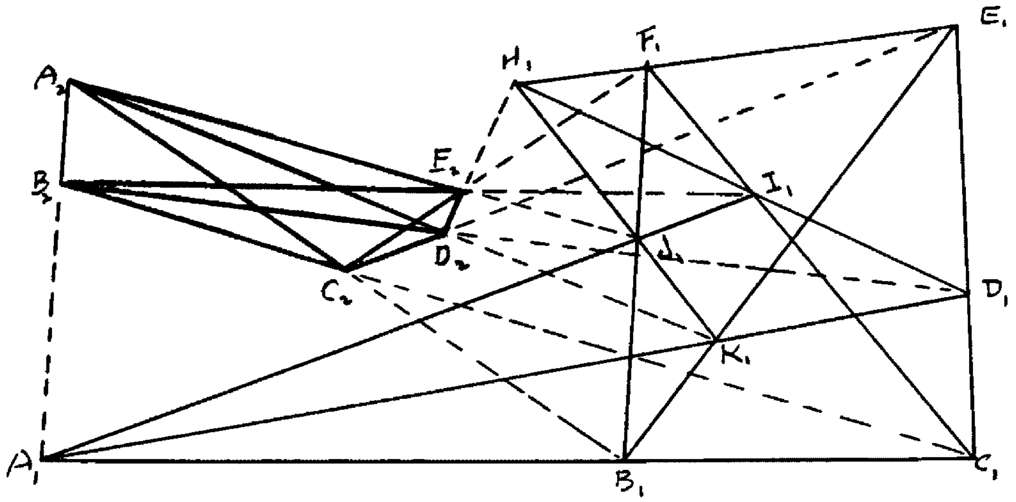
$_nC_2$	n-2	$_{n-2}C_2$
3	$_nC_3$	n-3
6	4	$_nC_4$

The plane section is

$_nC_3$	n-3
4	$_nC_4$

Proof: In a 4-space there are n-points, no 3 of which are collinear, no 4 of which are coplanar and no 5 of which are in the same 3-space. Every combination of 2 points determine a line, so there $_nC_2$ lines. Every combination of three points determine a plane so there are $_nC_3$ planes. Every combination of 4 points determine a 3-space so there are $_nC_4$ 3-spaces in the four-space n-point. A 3-space section of a 4 space cuts each of these 3-spaces in a 2-space so there are $_nC_4$ planes in the section. The 3-space section of the 4-space cuts the 2-spaces of the 4-space in l-spaces so there are $_nC_3$ lines in the section. The section cuts the lines of the 4-space in points so there are $_nC_2$ points in the section.

In the 4-space there are 3 points on every plane. These points are connected in parts by three lines, so there are three points on a line in the section. In the 4-space there are 4 points on every 3-space. These may be connected, in parts, by 6 lines, so, there are 6 points on every plane in the 3-space section. In the 4-space there are 4 points on every 3-space. These 4 points determine 4 planes so there are 4 planes on every 3-space, which makes 4 lines in the section on every plane. In the 4-space every 3 points taken with each of the other points determined (n-3) 3-spaces on every plane. These make n-3 planes on a line in the section. In the 4-space each 2 points with each of the n-2 others determine n-2 planes on a line. These give n-2 lines on a point in the 3-space section. Therefore the figure of the 3-space section of an n-point in a 4-space is of the configuration. In the 4-space there are 2 points on a line. These 2 points with each of the other $_{n-2}C_2$ points taken by pairs determine a three-space. There are $_{n-2}C_2$ 3-spaces on a line. In the section, there are $_{n-2}C_2$ planes on a point.



EXERCISE XVII

PLATE 13'

Exercise 19

A plane section of the configuration

${}_nC_2$	n-2	${}_{n-2}C_2$
3	${}_nC_3$	n-3
6	4	${}_nC_4$

Has ${}_nC_3$ points for the plane section cuts the ${}_nC_3$ lines in ${}_nC_3$ points. It has ${}_nC_4$ lines for the section cuts the ${}_nC_4$ planes of the configuration in ${}_nC_4$ lines. There are 4 points on a line, for in the configuration there are 4 lines on a plane and these become 4 points on a line in the plane section of the configuration. There are n-3 planes on a line in the configuration; these give n-3 lines on a point in the plane section of the configuration. The plane section of the given configuration is of the configuration

${}_nC_3$	n-3
4	${}_nC_4$

Exercise 20--

The configuration of the 2 perspecti ve tetrahedra of theorem A can be obtained as the section of a 3-space of a complete 6-point in a 4-space.

Given: The 2 perspective tetrahedra of theorem A. To prove that they can be obtained as a section by a 3-space of a comp.ete 6-points in a 4-space.

Proof: By theorem preceding the section by a three space of an n-point in a 4-space is of the configuration $\begin{bmatrix} {}_nC_2 & n-2 & {}_{n-2}C_2 \\ 3 & {}_nC_3 & n-3 \\ 6 & 4 & {}_nC_4 \end{bmatrix}$

If n is equal to 6 then the configuration is $\begin{bmatrix} 15 & 4 & 6 \\ 3 & 20 & 3 \\ 6 & 4 & 15 \end{bmatrix}$

The two perspective tetrahedra are of this configuration.The intersection of the 4 planes by 3's in each of the tetrahedra form 8 points. The center of perspectivity forms a ninth point and 6 points called for in the theorm A make, in all, 15 points. There are 6 lines to each tetrahedron and 4 co-planar lines formed by the intersection of the homologous faces, and the four lines of perspectivity make 20 lines in all. Each tetrahedron has 4 faces. The intersections of the homologous faces are coplanar and the lines of perspectitiy form 6 planes, and there are 8 plane faces in the 2 tetra-hedra, making 15planes in all. There are 3 points on each line, 6 points on each plane, 4 lines on each plane, etc. Three points on a line becausre there are 2 vertices of the tetrahedra and a point of intersection of the homologous edges or the center of peespectivity on every line. Six points on a plane because there are on each plane 3 vertices of one of the tetra-hedra and three collinear points of intersection of the 3 edges in that plane with the corresponding edges of the other tetrahedron, 4 vertices of the tetrahedra the center of perspectivity and one point of intersection of the homologous edges of the tetrahedra or the plane containing the 6 points of intersection of the homologous edges of the two per spective tetrahedra. 4 lines on each plane because there is one each plane two lines of perspect ivity and two edges of the tetrahedra or three edges of the tetrahedra and a

Ex 20
line of intersection of the homologous faces of the tetrahedra or the 4 lines of intersection of the homologous faces of the tetrahedra etc. Therefore the theorem is proved.

Exercise 21--

If the two five points in a 4-space are perspective from a point the corresponding edges meet in the vertices, the corresponding plane faces meet in the lines, and the corresponding 3-space faces in the planes of a complete 5-plane in a 3-space.

Given: The two 5-points in a 4-space perspective from a point. To prove that the corresponding edges meet in the vertices, the corresponding plane faces in the lines and the corresponding 3-space faces in the planes of complete 5-plane in a three-space.

Proof: The 5-point in the four-space is assumed to be such that no 3 points are on a line; no 4 points on a plane; and no 5 points on a 3-space. Let us consider any four corresponding points of the 2 perspective 5-points of the 4-space. These sets of points determine a 3-space, and by theroem A the 6 pairs of homologous faces meet in coplanar lines. This holds true for all the combinations of corresponding 4 points of the tet rehedra. If follow then that the corresponding 3-space faces meet in planes the edges in points and the plane faces in lines. In order to show that these points, lines and planes lie on the same 3-space, let the points of the perspective 5-points be A_1 A_2 A_3 A_4 A_5 and A'_1 A'_2 A'_3 A'_4 A'_5. First consider $A_1A_2A_3A_4$ and $A'_1A'_2A'_3A'_4$ Let:

A_1A_2 meet $A'_1A'_2$ at P_{12}	A_1A_3 meet $A'_1A'_3$ at P_{13}	A_2A_3 meet $A'_2A'_3$ at P_{23}
A_1A_3 " $A'_1A'_3$ " P_{13}	A_1A_4 " $A'_1A'_4$ " P_{14}	A_2A_4 " $A'_2A'_4$ " P_{24}
A_2A_3 " $A'_2A'_3$ " P_{23}	A_3A_4 " $A'_3A'_4$ " P_{34}	A_3A_4 " $A'_3A'_4$ " P_{34}

Each set of 3 points of intersections of the corresponding edges are collinear. All three lines are coplanar by theorem A. and with them $P_{12}P_{24}P_{14}$. Using the same motation considering $A_1A_2A_3A_4$ and $A'_1A'_2A'_3A'_4$; $A_1A_3A_4A_5$ and $A'_1A'_3A'_4A'_5$; $A_1A_2A_4A_5$ and $A'_1A'_2A'_4A'_5$; $A_2A_3A_4A_5$ and $A'_2A'_3A'_4A'_5$ in turn

$P_{12}P_{24}P_{14}$	$P_{13}P_{14}P_{34}$	$P_{12}P_{24}P_{14}$	$P_{23}P_{24}P_{34}$
$P_{12}P_{25}P_{15}$	$P_{13}P_{35}P_{15}$	$P_{12}P_{25}P_{15}$	$P_{23}P_{25}P_{35}$
$P_{13}P_{35}P_{15}$	$P_{14}P_{45}P_{15}$	$P_{14}P_{45}P_{15}$	$P_{24}P_{45}P_{25}$ are coplanar in groups
$P_{23}P_{35}P_{25}$	$P_{24}P_{25}P_{45}$	$P_{24}P_{25}P_{45}$	$P_{34}P_{45}P_{35}$ of four and coll in sets of three.

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Ex 21--

Let the first two planes (or any two) determine a thr e space, i ethat planes $P_{12}P_{13}P_{14}$ and $P_{13}P_{14}P_{34}$. Then the third plane is on~~e~~ that 3-space
$P_{12}P_{24}P_{14}$ " $P_{12}P_{23}P_{13}$.
for its determined by two lines one on each of the two planes determining the 3-space as $P_{13}P_{35}P_{15}$ are points of a line on the second and third planes. $P_{13}P_{14}P_{34}$ are on the first and third. Like wise $P_{12}P_{24}P_{14}$ are on the first and fourth planes and $P_{12}P_{25}P_{15}$ are on the second and fourth. And $P_{23}P_{24}P_{34}$ are on planes one and five. And $P_{23}P_{35}P_{25}$ are on the second and fifth. Since all the othee planes have a line common~~s~~ with each of the first two they all lie in the same 3-space. Each plane of intersection of the 3-space faces has one line and three points common with each of the other planes. We therefore have a complete 5-plane in a three-space. This proves the theorem.

Exercise 22. Plates $14'$ $15'$

If two triangles are perspective then are perspective also the two triangles whose vertices are points of intersection of each side of the given triangle with the line joining a fixed point of the axis of perspectivity to the opposite vertex.

Given: the two perspective triangles, To prove: that the two triangles whose vertices are the points of intersection of each side of the given triangle with the line joining a fixed point of the axis of perspectivity to the opposite vertex.

Proof: Considering the complete quadrangles $M\,Y_1B_1Z_1$ and $M\,Y_2B_2Z_2$, lines Y_2Z_2 and Y_1Z_1 meet in the line $M\,L$, by theorem B. Therefore triangles $A_1Y_1Z_1$, $A_2Y_2Z_2$ are perspective; $B_1Y_1Z_1$, $B_2Y_2Z_2$ are perspective; $C_1Y_1Z_1$; $C_2Y_2Z_2$ are perspective. Therefore since lines Y_1Y_2 and Z_1Z_2 meet on A_1A_2, on B_1B_2 and C_1C_2 they must meet at P. Consider the triangles $X_1B_1C_1$ and $X_2B_2C_2$. The corresponding sides of these triangles meet in three collinear points and therefore the triangles are perspective by the converse of Desargue theorem. Therefore line X_1X_2 passes through P, which proves the theorem. See Figures $14'$ and $15'$. In case M coincides with K, H, or L the triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ degenerate into three collinear points each and one point is common to both sets, in which case the theorem holds but is trivial, for two of the points are the vertices of the given perspective triangles and the common points is the fixed point on the axis of perspectivity.

Exercise 23..

The n-space section of an m-point ($m \geq n-2$) in an (n+1) space can be considered in the n-space as (m-k) k-points-points (in $_{m}C_{m-k}$ ways) perspective in pairs from the vertices of the n-space section of one (m-k)point; the r-spaces of the k-point figures meet in r-1-spaces (r--1,2,3....,n-1) which form the n-space section of the k-point.

Proof: For every k points in the m-point in the (n+1) space there are left m-k points. Let us consider a particular set these k points and the corresponding set of m-k remaining points. Any two of the m-k points determine a line, such as, points f & g determine the line f g. Now lines joining the points f & g to each of the k points in the (n+1) space will intersect the n-space section in points and there will be formed in this manner, two sets of k-points, perspective from the point of intersection of the line fg with the n-space section. Likewise we may take the other points of the m-k points in the (n+1) space by pairs using each point but once, and we get $\frac{m-k}{2}$ of the pairs of k-points in the n-space section making in al (m-K) k-points. If we take point f with each one of the other m-k points in turn we will get one k-point in the n-space section perspective with each of the others. Likewise taking the point g and then each of the others in turn, we get each k-point perspective with each of the others. These points of perspectivity of the k-points are the intersection points of lines joining the m-k points in the (n+1) space, with the n-space section, therefor they are the vertices of the n-space section of the (m-k)point in the (n+1) space. Since the m-k points may be selected in all combinations of m-k in m, we have (m-k) k points in the n-space in $_{m}C_{m-k}$ ways, perspective in pairs. The m point in the n+1 space is assumed to be such that no three points are on a line, no four points on a plane, no 5 points on a 3-space, and no 6 points on a 4-space, no p points on a (p-2) space etc. Figures in the n-space can be shown to possess this property

Ex.23

It remains to prove that the r-spaces of any k-point meet in r-1 spaces. Since any three points of a k-point determine a plane, 4 points determine 4 planes intersecting by pairs. Since each pair of these planes have two points common they intersect in a line. Likewise 5 points determine ${}_5C_4$ 3-spaces. Since these 3-spaces have 3 common points they meet in a plane. Likewise 6 points determine ${}_6C_5$ 4-spaces. Since the 4-spaces have 4 points common they meet in three spaces. In short, r+2 points determine ${}_nC_{r+1}$ r-points. Since these r-points have r points common they meet ina r-1 space

As we shall prove in following theorem, since any two k points are perspective, their r-spaces meet in r-1 spaces. In the r-1 spaces there will be 3 points on a line. There will be ${}_kC_2$ of these points of intersection of the corresponding sides of the 2 perspective k-points in the n-space section. There will be ${}_kC_3$ lines in the n-space section formed by the intersection of corresponding planes. There will be ${}_kC_4$ planes in the n-space section formed by the intersection of corresponding 3-spaces of the perspective k-points. Likewise there will be ${}_kC_5$ 3-spaces, ${}_kC_6$ 4-spaces, ${}_kC_7$ 5-spaces etc. In the n-space section, formed by the intersection of the corresponding 4-spaces,5-spaces,6-spaces etc. of the two perspective k-points. We notice also that there are 3 points on the line and k-2 lines on the point; 4 lines on the plane and k-3 planes on the line; 5 planes on the 3-space and k-4 3-spaces on the planes; 6 3-spaces on the 4-space and k-5 4-spaces on the 3-space;etc. in the n-space section. Therefore the corresponding r-spaces of the k-points meet in the r-1 spaces which form the n-space section of a k-point.

Exercise 24

If $2(n+1)$-points in an n-space are perspective from a point, their corresponding-r-spaces meet in r-1-spaces which lie in the same n-1 space ($r = 1, 2, \ldots, n-1$) and form ~~a~~ complete configuration of $(n+1)(N-2)$-spaces in the n-1 space.

Proof: It follows as a corallary to Desargues theorem that 2 triangles not in the same plane, perspective from a point, have their corresponding sides meet in points of a line. Since 3 corresponding points of the 2 perspective (n-1) points form perspective triangles their corresponding sides and consequently their planes, meet in a line. Therefor the corresponding lines meet in points and the corresponding planes meet in lines. Likewise every 4 corresponding points determined corresponding 3-spaces. The planes of these 3-spaces meet in lines, and since the ${}_4C_3$ planes have 2 points in common, the lines of intersection of the corresponding planes intersect by pairs. Let $A_1A_2A_3A_4$ & $A_1'A_2'A_3'A_4'$ be two corresponding 3-spaces and let their corresponding planes meet as follows:

A_1A_2 meet $A_1'A_2'$ at P_{12}	in planes $A_1A_2A_3$ & $A_1'A_2'A_3'$	A_1A_3 meet $A_1'A_3'$ at P_{13}	in planes $A_1A_3A_4$ & $A_1'A_3'A_4'$
A_1A_3 meet $A_1'A_3'$ at P_{13}		A_1A_4 meet $A_1'A_4'$ at P_{14}	
A_2A_3 meet $A_2'A_3'$ at P_{23}		A_3A_4 meet $A_3'A_4'$ at P_{34}	

A_1A_2 meet $A_1'A_2'$ at P_{12}	In planes $A_1A_2A_4$ & $A_1'A_2'A_4'$	A_2A_3 meet $A_2'A_3'$ at P_{23}	In planes $A_2A_3A_4$ & $A_2'A_3'A_4'$
A_1A_4 meet $A_1'A_4'$ at P_{14}		A_2A_4 meet $A_2'A_4'$ at P_{24}	
A_2A_4 meet $A_2'A_4'$ at P_{24}		A_3A_4 meet $A_3'A_4'$ at P_{34}	

Since any tw~~w~~ of these lines of perspectivity of 2 corresponding pairs of triangles have a point in common they intersect. Let the lines $P_{12}P_{13}P_{23}$ and $P_{12}P_{14}P_{24}$ determine a plane. Since each of the other 2 lines $P_{13}P_{14}P_{34}$ and $P_{23}P_{24}P_{34}$ have two points each on this plane, they lie wholly within the plane. Therefore the t~~ww~~ corresponding 3 spaces meet in planes.

Ex 25

In a like manner we may consider any 5 corresponding points of the perspective (n + 1) points which determine two corresponding 4-spaces. These 5 corresponding points determine 5 corresponding 3-spaces which meet in planes. Letthe 3-spaces meet as follows:

$A_1A_2A_3$ meet $A'_1A'_2A'_3$ at P_{123}	$A_1A_2A_4$ meet $A'_1A'_2A'_4$ at P_{124}
$A_1A_2A_4$ meet $A'_1A'_2A'_4$ at P_{124}	$A_1A_2A_5$ meet $A'_1A'_2A'_5$ at P_{125}
$A_1A_3A_4$ meet $A'_1A'_3A'_4$ at P_{134}	$A'_1A_4A_5$ meet $A'_1A'_4A'_5$ at P_{145}
$A_2A_3A_4$ meet $A'_2A'_3A'_4$ at P_{234}	$A_2A_4A_5$ meet $A'_2A'_4A'_5$ at P_{245}
$A_1A_3A_4$ meet $A'_1A'_3A'_4$ at P_{134}	$A_1A_2A_3$ meet $A'_1A'_2A'_3$ at P_{123}
$A_1A_3A_5$ emeet $A'_1A'_3A'_5$ at P_{135}	$A_1A_2A_5$ meet $A'_1A'_2A'_5$ at P_{125}
$A_1A_4A_5$ meet $A'_1A'_4A'_5$ at P_{145}	$A_1A_3A_5$ meet $A'_1A'_3A'_5$ at P_{135}
$A_3A_4A_5$ meet $A'_3A'_4A'_5$ at P_{345}	$A_2A_3A_5$ meet $A'_2A'_3A'_5$ at P_{235}

$A_2A_3A_4$ meet $A'_2A'_3A'_4$ at P_{234}

$A_2A_3A_5$ meet $A'_2A'_3A'_5$ at P_{235}

$A_2A_4A_5$ meet $A'_2A'_4A'_5$ at P_{245}

$A_3A_4A_5$ meet $A'_3A'_4A'_5$ at P_{345}

Therefore the following lines are coplanar:

P_{123}, P_{124} P_{134} P_{234} $P_{124}P_{125}$ P_{145} P_{245} $P_{234}P_{135}$ P_{145} P_{345} P_{123} $P_{125}P_{135}P_{235}P_{234}$ P_{235} P_{345} P_{345}

Since any ~~tow~~ two of these planes have a line common, the two planes determine a 3-space (two intersecting planes determine a 3-space). Each of the other planes have a line common with each of these two and therefore lie in the same 3-space. Therefore the corresponding 4-spaces meet in a 3-space.

In a like manner we may consider any 6 corresponding points of the perspective (n+1) points which determine 2 corresponding 5 spaces. These 6 corresponding points determine 6 corresponding 4-spaces which meet in 3-spaces. Let the corresponding 4-spaces meet in the following three-spaces:

Ex 25

$A_1A_2A_3A_4$ meet $A_1'A_2'A_3'A_4'$ in P_{1234}
$A_1A_2A_3A_5$ meet $A_1'A_2'A_3'A_5'$ in P_{1235}
$A_1A_2A_4A_5$ meet $A_1'A_2'A_4'A_5'$ in P_{1245}
$A_1A_3A_4A_5$ meet $A_1'A_3'A_4'A_5'$ in P_{1345}
$A_2A_3A_4A_5$ meet $A_2'A_3'A_4'A_5'$ in P_{2345}

$A_1A_2A_3A_5$ meet $A_1'A_2'A_3'A_5'$ in P_{1235}
$A_1A_2A_3A_6$ meet $A_1'A_2'A_3'A8$ in P_{1256}
$A_1A_2A_5A_6$ meet $A_1'A_2'A_5'A_6'$ in P_{1256}
$A_1A_3A_5A_6$ meet $A_1'A_3'A_5'A_6'$ in P_{1356}
$A_2A_3A_5A_6$ meet $A_2'A_3'A_5'A_6'$ in P_{2356}

$A_1A_3A_4A_5$ meet $A_1'A_3'A_4'A_5'$ in P_{1345}
$A_1A_3A_4A_6$ meet $A_1'A_3'A_4'A_6'$ in P_{1346}
$A_1A_3A_5A_6$ meet $A_1'A_3'A_5'A_6'$ in P_{1356}
$A_1A_4A_5A_6$ meet $A_1'A_4'A_5'A_6'$ in P_{1456}
$A_3A_4A_5A_6$ meet $A_3'A_4'A_5'A_6'$ in P_{3456}

$A_1A_2A_3A_4$ meet $A_1'A_2'A_3'A_4'$ in P_{1234}
$A_1A_2A_3A_6$ meet $A_1'A_2'A_3'A_6'$ in P_{1236}
$A_1A_2A_4A_6$ meet $A_1'A_2'A_4'A_6'$ in P_{1246}
$A_1A_3A_4A_6$ meet $A_1'A_3'A_4'A_6'$ in P_{1346}
$A_2A_3A_4A_6$ meet $A_2'A_3'A_4'A_6'$ in P_{2346}

$A_1A_2A_4A_5$ meet $A_1'A_2'A_4'A_5'$ in P_{1245}
$A_1A_2A_4A_6$ meet $A_1'A_2'A_4'A_6'$ in P_{1246}
$A_1A_2A_5A_6$ meet $A_1'A_2'A_5'A_6'$ in P_{1256}
$A_1A_4A_5A_6$ meet $A_1'A_4'A_5'A_6$ in P_{1456}
$A_2A_4A_5A_6$ meet $A_2'A_4'A_5'A_6'$ in P_{2456}

$A_2A_3A_4A_5$ meet $A_2'A_3'A_4'A_5'$ in P_{2345}
$A_2A_3A_4A_6$ meet $A_2'A_3'A_4'A_6'$ in P_{2346}
$A_2A_3A_5A_6$ meet $A_2'A_3'A_5'A_6'$ in P_{2356}
$A_2A_4A_5A_6$ meet $A_2'A_4'A_5'A_6'$ in P_{2456}
$A_3A_4A_5A_6$ meet $A_3'A_4'A_5'A_6'$ in P_{3456}

Therefore the following planes are on the same 3-space:

P_{1234} P_{1235} P_{1245} P_{1345} P_{2345} P_{1234} P_{1236} P_{1246} P_{1346} P_{2346}

P_{1235} P_{1256} P_{1256} P_{1356} P_{2356} P_{1246} P_{1246} P_{1256} P_{1456} P_{2456}

P_{1345} P_{1346} P_{1356} P_{1456} P_{3456} P_{2345} P_{2346} P_{2356} P_{2456} P_{3456}

Let $P_{1235}P_{1245}$ of the first set determine the 3-space and P_{1234} & P_{1236} of the second set determine a 3-space. Since the 2 3-spaces have a common plane the two 3-spaces determine a 4-space. Or points $P_{12}P_{13}P_{14}P_{23}P_{24}P_{34}$ all are on the first 3-space. These points together with the point P_{36} determine a 4-space. The second 3-space is in this 4-space. Point P_{35} P_{25} P_{15} are also on this 4-space. Since three non-collinear points of the plnes $P_{1256}P_{1356}P_{1456}P_{2356}P_{2456}P_{3456}$ (which are the planes of intersection of the corresponding 3-spaces which make up the 5-space are in this 4-space , and since three non-collinear points etermine a plane these planes are all in the 4-space. Therefore two corres-ponding 5-spaces meet in a 4-space. In the same manner it may be proved hat any two corresponding r-spaces meet in an (r-1) space and that the two erspective (n+1)-points in the n-space meet in an (n-1) space. Therefore ll the r-1 spaces are in an n-1-space. There are ${}_{n+1}C_{n}$ corresponding n-1-spaces hich meet in the n-2 spaces. This makes (n+1)(n-2) spaces in the (n-1) space. ow every corresponding n points of the n+1 perspective points in the n-space orm 2 corresponding n-1-spaces which meet in an n-2 space. Let us consider second pair of corresponding n-1-spaces. n-1 corresponding points of the irst and second sets will be common. These corresponding n-1 points form wo corresponding n-2 spaces which will meet in an n-3 space, in the n-1 space. herefore the (n+1)(n-2) spaces meet by pairs in n-3 space in the n-1 space. here fore the (n-1)(n-2) spaces meet by pairs in the (n-3) spaces of the n-1 pace and we have a complete configuration of the (n+1)(n-2)spaces in an (n-1 pace and the theorem is proved. In the same manner the ${}_{n+1}C_{n-1}$ correspondihg N-2 paces will meet in n-3 spaces. But these corresponding n-2-spaces will have n n-2 points common to three corresponding n-2-spaces and there form n-3 paces common to three correspondng n-2 spaces and therefore the correspond-ng n-3 spaces meet, by triples in n-4 spaces of the n-1 space, and so on.

The configuration of Pappus. This important configuration was discovered by Pappus of Alexandria, who lived about 340 A.D. Plate 12. If A,B,C are any three distinct points of a line l, and A',B',C' any three distinct points of another line l' meeting l, the three points of intersection of the pairs of lines AB' and A'B, BC' and B'C, CA' and C'A are collinear.

Proof: Let the three points of intersection referred to in the theorm be denoted by C'',A'', B'' respectively. Let the line B''C'' meet the line B'C in a point D; also let B''C'' meet l' in A_1 , the line A'B meet AC' in B_1, the line AB' meet A'C in B_1' . We then have the following perspectivities: $A'C''B_1B \overset{A}{\barwedge} A'B_1' B'' C \overset{B'}{\barwedge} A_1C'' B''D$.

By the principle of projectivity then, since in the projectivity thus establihed C'' is self-corresponding, we conclude that the three lines A_1A', $B''B_1$, DB meet in the point C'. Hence D is identical with A'', and A'',B''?C'' are collinear.

The configuration of the figure as given is:

9	3
3	9

This configuration may be considered as a simple plane hexagon inscribed in two intersecting lines, If the sides of such a hexagon be denoted in order 1,2,3,4,5,6 , and if we call the sides 1 and 4 opposite, likewise the sides 2 and 5, and the sides 3 and 6, the last theorem may be stated in the following form: If a simple hexagon be inscribed in two intersecting lines, the threee pairs of opposite sides will intersect in collinear points.