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Relation of the Fourier series to the development of the functions of the real variable

Mary Vivian Huls

The University of Montana

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RELATION OF THE FOURIER SERIES
TO THE DEVELOPMENT
OF THE FUNCTIONS OF THE REAL VARIABLE

by

Sister Mary Vivian Huls
B.S., Creighton University, Omaha, 1924

presented in partial fulfillment of the requirement for the degree of Master of Arts.

State University of Montana
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Introduction

Much has been written concerning the Theory of Fourier Series so this paper will contain nothing new in regard to general properties. I have merely examined facts relative to these Series from material at my disposal and have found that up till the appearance of Fourier's memoir on the "Analytical Theory of Heat" the possibility of the expansion of an arbitrary function in a trigonometric series was not admitted by any mathematician. Also, that Fourier had a thorough grasp of the nature of such expansions and gave in broad outline, though not in such detail as its importance demanded, a sound proof of the expansion, so that from the time his memoir became known the validity of the expansion has never been questioned. Dirichlet was the first to give a proof in which the restrictions on the function to be expanded, in other words the limits of its arbitrariness, are carefully stated. The work of subsequent writers has consisted largely in extending the limits given by Dirichlet, while following in the main his methods.
Definition of Fourier Series

Series of the type
\[ \frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) \ldots \]

\[ = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

where \( a_n, b_n \) are independent of \( x \), are of great importance in many investigations. They are called trigonometrical series.

If there is a function \( f(t) \) such that \( \int_{-\pi}^{\pi} f(t) \, dt \) exists as a Riemann integral or as an improper integral which converges absolutely, and such that

\[ \pi a_n = \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad \pi b_n = \int_{-\pi}^{\pi} f(t) \sin nt \, dt, \]

then the trigonometrical series is called a Fourier Series.¹

By taking a larger and larger number of terms in our series we get better and better approximations to the function \( f(t) \). The question thus naturally suggests itself whether the infinite trigonometric series may not give us, no longer an approximate, but a perfect representation of \( f(t) \). This is the fundamental question which lies at the foundation of the whole theory of Fourier Series.

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Early Investigations

The controversy as to the possibility of expending an arbitrary function of one variable in a series of sines and cosines of multiples of the variable, arose about the middle of the 18th century in connection with the problem of vibrating chords. To appreciate properly the difficulty which the expansion presented to the mathematicians of that day one must bear in mind that their conception of a function was much more limited than ours. Euler says that curves may be divided into continuous and discontinuous or mixed; a curve is continuous when its nature can be expressed by one definite function of the variable; if on the other hand different portions of the curve require different functions to express them the curves are called discontinuous or mixed or irregular as not following the same law through their whole course but being composed of portions of continuous curves. Curves which are discontinuous in this sense seem to have been considered to be beyond the scope of analysis. As a consequence of this view it was supposed that if two functions of a variable, were equal for any definite range of values of the variable, they must be so for all values so that if the curves which represent them coincide for any interval they must do
so entirely. Thus the objection was constantly urged that an Algebraic function would not be represented by a trigonometric series for the latter gives a periodic curve while the former does not. Fourier was the first to see and state that when a function is defined for a given range of values of the argument its course outside that range is in no way determined. One obvious consequence of these views is that no one before Fourier could have properly understood the representation of an arbitrary function by a trigonometrical series.

D'Alembert in his memoirs discusses the problem of the vibrating chord. The origin of co-ordinates being at one end of the chord whose length is 1, the axis of $x$ in the direction of the chord and $y$ the displacement at time $t$, he shows that $y$ must satisfy the equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

He obtains the solution $y = f(at + x) + \phi(at - x)$ and since $y = 0$ for $x = 0$ and $x = 1$, he finds $y = f(at + x) - f(at - x)$ and shows that $f$ represents such a function that $f(z) = f(x + 2l)$. In a memoir immediately following this one in the same volume he seeks to find functions which satisfy this relation of periodicity. Euler discusses
this same problem. He observes that the motion of the
string will be completely determined if its form and
velocity of each point of it be known for any one
position. He deduces the equation $y = \phi(x+at) + \phi(x-at)$
where $\phi$ is such that $\phi(at)+\phi(at) = 0$ and $\phi(1+at)+\phi(1-at) = 0$
for every $t$; and from these equations which $\phi$ must satisfy
he concludes that every curve whether regular or irregular
which consists of repetitions alternately below and above
the axis of any given curve which the string may be sup­
posed to take is suitable for representing $\phi$. He then
shows how the ordinate of any point at any given time
may be determined by a simple geometrical construction.
He gives as a particular solution for $\phi(x)$ the equation

$$\phi(x) = \alpha \sin \frac{\pi x}{1} + \beta \sin 2 \frac{\pi x}{1} + \gamma \sin 3 \frac{\pi x}{1} \ldots .$$

In an article by Damien Bernoulli, the problem of
the vibrating string is approached from the physical
rather than from the mathematical side. He maintains
that any position of the string may be given by the
equation $y = \alpha \sin \frac{\pi x}{1} + \beta \sin 2 \frac{\pi x}{1} + \gamma \sin 3 \frac{\pi x}{1} \ldots .$
He asserted that this represents the most general solu­
tion of the problem, and that the solutions of D'Alembert
and Euler must therefore be contained in it. Euler
admits that if it be general it is much better than his
own; but he does not admit its generality, for that
would be equivalent to admitting that every curve could
be represented by a trigonometric series and this proposition he considers to be certainly false, seeing that a curve given by a trigonometric series is periodic—a property not possessed by all curves. In seeking to establish his position he remarks that it might be argued that since there is an infinite number of disposable constants, \( \alpha, \beta, \gamma, \) etc., at disposal, it must be possible to make the proposed curve coincide with any given curve, but he states explicitly that Bernoulli himself has not used this argument. Bernoulli indeed does not seem in his memoir to have quite grasped the mathematical consequences of his solution; his results seemed so satisfactory in their explanation of the facts of observation that he was prepared to maintain the generality of his solution on that ground alone.

When the controversy was at this stage Lagrange\(^6\) writes concerning the methods of Euler, D'Alembert, and Bernoulli. He accepts Euler's solution as the most general, but objects to his mode of demonstration, and proposes to obtain a satisfactory solution by first considering the case of a finite number of vibrating particles and then seeking the limit for an infinite number, that is, for a chord. In this he showed that when the initial displacement of the string of unit length is
given by \( f(x) \) and the initial velocity by \( F(x) \), the displacement at time \( t \) is given by

\[
y = 2\int_0^t \sum \left( \sin nwx' \sin nwx \cos n\pi at \right) f(x') dx' + \frac{2}{a\pi} \int \sum \frac{1}{n} \left( \sin nwx' \sin nwx \sin n\pi at \right) F(x) dx.
\]

This seems undoubtedly to be a Fourier series in the proper sense of the term; yet it appears doubtful if Lagrange actually supposed it to be such. It could hardly have escaped his notice that for a definite value of \( t \) this is simply Bernoulli's solution. It was doubtless no part of Lagrange's purpose to determine the coefficients in Bernoulli's series, but rather to obtain the functional solution given by D'Alembert as he actually does by summing the series by trigonometric methods. It is hard to understand how near Lagrange came to the conception of expanding an arbitrary function in an infinite series without ever actually attaining to it, especially when we see him adopting the method of passing a trigonometric curve through a finite number of points on a given curve and succeeding in solving the necessary equations in the manner used later by Dirichlet. That he did not really solve the problem of expansion in trigonometric series is best understood from the circumstance that neither he nor any of his contemporaries believed such expansion to be possible.
For the next forty years there seems to have been almost no progress made towards a solution of the difficulties raised in these discussions. It is easy to see where the difficulties of the subject lay; they lay in the inadequacy of the notion of a function. Both Euler and Lagrange seem at times as if they had in part transcended the limits of their original conception; Euler in giving his geometrical constructions for the solution of the equation for the vibrating chord and Lagrange in his method of constructing the equation to a curve passing through the vertices of an inscribed polygon. Yet neither of them got beyond the old notion of continuity and its consequences in any of their writings on the subject of trigonometric series. But a great part of their work was of immense service to Fourier, as he himself indicates when he approached the consideration of the subject with his conception of a function as given graphically.
Fourier's Work

Fourier Series takes its name from Joseph Fourier the author of "La Theorie Analytique de la Chaleur."

Fourier's first investigations on the Theory of Heat were communicated to the Academy of Sciences on the 21st of December in 1807, though it is in his memoir of 1811 that is found the exposition of the representation of arbitrary functions by trigonometric series. In this treatise Fourier cleared away the difficulties which had puzzled his predecessors. Even before Dirichlet's proof of 1829 which has generally been considered to be the first satisfactory exposition from the mathematical standpoint, Fourier's results had been universally accepted; although criticized by some.

Fourier sums up his views on the nature of a function which admits of expansion; it is not necessarily continuous in the old sense of that word but may be composed of separate functions or parts of functions. By these phrases he means a function f(x) which has values while x lies between given limits but is zero for all other values of x. The function may even become infinite between the limits and in general the function need only be given graphically. Again, Fourier has accurate conceptions of the convergency of series, but an important
question still remains, namely, how far did Fourier succeed in his mathematical demonstration that the series which represents the function actually converges to the value of the function. In special cases which he gives, the convergency of the series and its equivalence with the function are, as he says, easily demonstrated; but it is usually maintained, especially by Riemann, that he gave no mathematical proof of the Theorem. However, it is a mistake to suppose that Fourier did not establish in a rigorous and conclusive manner that a quite arbitrary function (meaning by this any function capable of being represented by an arc of a continuous curve or by successive portions of different continuous curves) could be represented by the series we now associate with his name. In this discussion Fourier followed the line of argument which is now customary in dealing with infinite series. He proved that when the values

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx', \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx' \]

are inserted in the terms of the series

\[ a_0 + a_1 \cos x + a_2 \cos 2x \ldots \ldots \ldots \ldots \ldots \]

\[ b_1 \sin x + b_2 \sin 2x \ldots \]
the sum of the terms up to \( \cos nx \) and \( \sin nx \) is
\[
\sin(2n+1) \frac{x'-x}{2} \frac{1}{\sin x} \frac{x'-x}{2} \int_{-\pi}^{\pi} f(x) \frac{dx'}{\sin x}.
\]
He then discussed the limiting value of this sum as \( n \) becomes infinite, and deduced from this limit the sum of the infinite series.

Fourier made no claim to the discovery of the value of the coefficients
\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x') dx',
\]
\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \cos nx' dx', \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x') \sin nx' dx'.
\]
They were employed by Euler before this time. Still, there is an important difference between Fourier's interpretation of these integrals and that which was current among the mathematicians of the 18th century. The earlier writers by whom they were employed applied them to the representation of an entirely arbitrary function, in the sense in which we have explained this term. It should also be noted that he was the first to allow that the arbitrary function might be given by different analytical expressions in different parts of the interval; also that he asserted that the sine series could be used for other functions than odd ones, and the cosine series for other than even ones. Further, he was the first to see
that when a function is defined for a given range of the variable, its value outside that range is in no way determined and it follows that no one before him could have properly understood the representation of an arbitrary function by a trigonometrical series.
Later Developments

An attempt to prove Fourier's Theorem was next made by Poisson. As a matter of fact Poisson's proof is invalid and seems to have been recognized as such almost from the first. Dirichlet does not allude to it and Cauchy lays his finger on the weak point. However, the integral that Poisson makes use of is of great importance, and has played a fundamental part in many modern developments; but its value appears after the Fourier series has been established and not in the proof of the series itself. Poisson had treated the trigonometric series now dealt with in several places and always in practically the same way. His process is as follows:

When \( p < 1 \) \[
\frac{1 - p^2}{1 - 2p \cos(x - a) + p^2} = 1 + 2 \sum_{n=1}^{\infty} p^n \cos n(x - a)
\]

Multiplying by \( f(a) \) and integrating between \(-\pi, +\pi\),

he gets

\[
\frac{(1-p^2) \int f(a) \, da}{1 - 2p \cos(x - a) + p^2} = \int_{-\pi}^{\pi} f(a) \left\{ 1 + 2 \sum p^n \cos n(x - a) \right\} \, da
\]

When \( p = 1 \), the integral on the left has all its elements zero except when \( a = x \). Putting \( p = 1 - g \), where \( g \) is small, and \( x - a = z \), he gets for the value of the integral

\[
\int f(x) \left( \frac{q \, dz}{q^2 + z^2} \right)
\]

where \( \epsilon, \epsilon' \) are small; but no error will
be introduced by making the limits infinite, so that when
$p=1$, the integral is equal to $2f(x)$. Making $p=1$ on the
right side he deduces
\[ f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a) \left\{ 1 + 2 \sum p^n \cos n(x-a) \, da \right\} \]

The proof is usually extended so as to include the cases
in which $f(x)$ presents discontinuities. On this proof
there are two remarks to be made. In the first place if
the series \( \int_{-\pi}^{\pi} f(a) \left\{ 1 + 2 \sum p^n \cos n(x-a) \, da \right\} \)
be denoted by \( \sum A_n p^n \), and if we write \( F(p) = \sum A_n p^n \),
then we are justified in assuming \( F(1) = \sum A_n \) when the
series \( \sum A_n \) is convergent. This theorem is generally
quoted as Abel's Theorem. But in the present case this
procedure amounts to assuming that the trigonometric
series is convergent, and the convergency of the series
is not proved by Poisson. In other words one of the
greatest difficulties of the subject is tacitly passed
over. It may be added that unless the function $f(x)$
be very greatly restricted it does not seem possible
to prove the convergency of the series from a consider-
ation of the integrals which give the coefficients. In
the second place, the quantity $p$ has no natural connec-
tion with the series and is a source of ambiguity that
is not inherent in the series itself. However, valuable
Poisson's integral may be in other respects it does not
seem to furnish a satisfactory starting point for the investigation of the series in question.

After Poisson, Cauchy attacked the problem. He starts with the series

\[ \int_0^a f(\mu) \, d\mu + 2 \sum_{n=1}^{\infty} e^{\frac{2\pi i}{a} (x - \mu)} \, f(\mu) \, d\mu \]

To prove that this has for sum a \( f(x) \) he replaces it by another series

\[ \int_0^a f(\mu) \, d\mu + \sum_{n=1}^{\infty} \theta^{n-1} \int_0^a e^{\frac{2\pi i}{a} (x - \mu)} \, f(\mu) \, d\mu + \]

\[ \sum_{n=1}^{\infty} \theta^{n-1} \int_0^a e^{\frac{2\pi i}{a} (x - \mu)} \, f(\mu) \, d\mu \]

where \( \theta = 1 - \epsilon \) and \( \epsilon \) is a small quantity. The series when summed gives

\[ \int_0^a \left\{ 1 + \frac{1}{e^{-\frac{2\pi i}{a} (x - \mu) - \theta}} + \frac{1}{e^{\frac{2\pi i}{a} (x - \mu) - \theta}} \right\} f(\mu) \, d\mu \]

and this integral being evaluated in Poisson's manner is equal to a \( f(x) \), But Cauchy recognizes one of the faults of Poisson's proof and tries to prove the convergence of the series when \( \theta = 1 \). To do this he throws it into the form
This equation, as Cauchy remarks later may be deduced by integration of the function

\[
\frac{f(z)}{e^{\pm \frac{2\pi n z}{a} (z - x)}}
\]

round a properly selected boundary. As to the function \(f(z)\) it must remain finite for all real or imaginary values of \(x\). He now, instead of examining the integral in its closed form, throws it again into a series of which the general term is, \(z = \frac{2n \pi}{a}\)

\[
\frac{1}{2\pi n i} e^{-\frac{2n \pi x}{a}} \int_0^\infty e^{-z} \left\{ f(a + \frac{a i}{2n \pi} z) - f\left(\frac{a i}{2n \pi} z\right) \right\} dz
\]

\[-\frac{1}{2n \pi i} e^{\frac{2n \pi x}{a}} \int_0^\infty e^{-z} \left\{ f(a - \frac{a i}{2n \pi} z) - f\left(-\frac{a i}{2n \pi} z\right) \right\} dz
\]

so that when \(n\) is very large the general term approximates to \(f(0) - f(a)\) \(\frac{1}{\frac{2\pi n x}{a}} \sin 2n \pi x\). The series of which this is the general term is convergent and he therefore concludes the trigonometric series to be convergent. Now in regard to this proof two points in particular require notice. First, as Dirichlet noticed there may be two series whose terms differ infinitely little from each other when \(n = \infty\), and yet the one series diverges while
the other converges; for example $\sum \frac{(-1)^n}{r^n}$ converges while $\sum \frac{(-1)^n}{r^n}(1 + \frac{(-1)^n}{r^n})$ diverges.

Cauchy's proof of the convergence of the series thus fails.
Dirichlet's Investigations

The first investigations of Dirichlet appeared in Crelle's Journal in 1829, the second which has become a model for nearly every discussion on the series is dated 1837.12

Dirichlet saw that the convergence of the series does not depend solely on the decrease of the terms, but is due also to the presence of negative terms. Hence, he adopts the method, which Fourier had employed, of summing to $n$ terms and finding the limit for $n \to \infty$.

The first $2n+1$ terms of the series for $f(x)$ may be written

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \, d\alpha \phi(a) \sin \left( \frac{a-x}{2} \right)$$

and this integral divided into

$$\frac{1}{\pi} \int_{-\pi}^{\pi} d\beta \phi(x-2\beta) \frac{\sin(2n+1)\beta}{\sin \beta} + \frac{1}{\pi} \int_{-\pi}^{\pi} d\beta \phi(x+2\beta) \frac{\sin(2n+1)\beta}{\sin \beta}$$

and the limit for $n \to \infty$ has to be found. The investigation hinges upon the limit for $k \to \infty$ of

$$\int \frac{\sin k\beta}{\sin \beta} f(\beta) \, d\beta$$

where $k = 2n+1$ and $0 < h < \frac{\pi}{2}$. The function $f(\beta)$ is supposed in the first place to be continuous, positive, and not increasing, while $\beta$ goes from 0 to $h$. The integral is decomposed into a series
of partial integrals with limits
\[ 0, \frac{\pi}{K}; \frac{\pi}{K}, \frac{2\pi}{K}; \text{ etc.}; \frac{r\pi}{K}, h \]
where \( \frac{r\pi}{K} \) is the greatest multiple of \( \frac{\pi}{K} \) contained in \( h \).

Each of these integrals is less in absolute value than its predecessor and the signs of them are alternately positive and negative. The integral is thus found to lie between limits which for \( n = \infty \), coincide in value \( \frac{3}{2} \pi f(0) \).

The restrictions on \( f(b) \) are then partly removed; it may either be constant or negative or a non-decreasing function as \( \beta \) goes from \( a \) to \( h \). It follows immediately that
\[ \lim_{K \to \infty} \int_{0}^{h} \frac{\sin K\beta}{\sin \beta} f(\beta) d\beta = 0 \text{ if } 0 < q < h < \omega = \frac{\pi}{2} \]
By this last result it is possible to extend the first theorem to all continuous functions which have a finite number of maxima and minima, while if \( f(\beta) \) be discontinuous for \( \beta = 0 \) the limit is \( \frac{1}{2} \pi f(0) \) if \( h \) be positive, but \(-\frac{1}{2} \pi f(-0) \) if \( h \) be negative. The limit for \( n = \infty \) of the sum of the first \( 2n+1 \) terms of the trigonometric series is thus
\[ \frac{1}{2\pi} \left\{ \phi(x+0) + \phi(x-0) \right\} \text{ if } x \neq \pm \pi \text{ but } \frac{1}{2\pi} \left\{ \phi(\pi-0) + \phi(\pi+0) \right\} \]
if \( x = \pm \pi \)

The results may therefore be summed up as follows: The limit for \( n = \infty \) of the series
\[
\frac{1}{2} \sum_{m=1}^{\infty} \left( a_m \cos mx + b_m \sin mx \right)
\]

where

\[
a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(a) \cos ma \, da, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(a) \sin ma \, da,
\]

is

\[
\frac{1}{2} \left\{ \phi(x-a) + \phi(x+a) \right\} \quad \text{if } x \neq \pm \pi,
\]

\[
\frac{1}{2} \left\{ \phi(\pi-a) + \phi(\pi+a) \right\} \quad \text{if } x = \pm \pi
\]

provided that while \(-\pi < x < \pi\), \(\phi(x)\) has a finite number of maxima and minima, a finite number of discontinuities, and does not become infinite. Of course if \(\phi(x)\) is continuous near \(x\), the value is simply \(\phi\). These conditions with another regarding infinite values of \(\phi(x)\) are usually called Dirichlet's conditions.

The definite form which Dirichlet gives to the sum of the trigonometric series suggests that the phrases "the function \(\phi(x)\) can be expanded in a series" or "the series represents the function \(\phi(x)\) should be precisely defined, for where there is a breach of continuity in the function, the series has a definite value while the function has not. The natural definition seems to be that adopted by Sachse, namely, a series represents a function in a given interval if its values coincide with those of the function for all points in the interval."
with the exception of a limited number of known points. A Fourier series, therefore, represents a function which satisfies Dirichlet's conditions.

There is one point in Dirichlet's demonstration which has been subjected to criticism in some quarters. According to Dirichlet the value of the series at a point of discontinuity in the functions is the arithmetic mean of the values of the function at that point. It has been contended on the other hand by Schlafli\textsuperscript{13} and Du-Bois Reymond\textsuperscript{14} that the value is really indeterminate and that the sum may have all values between the two values of the function at the point.

The conditions given by Dirichlet in his first memoir as those which a function satisfy if it is to be represented by a trigonometric series, are certainly very general, and in an addition to his memoir on the representation of an arbitrary function by a series of Spherical Harmonics he shows that the function φ(β) may become infinite at a finite number of points provided that ∫φ(β)dβ remain finite and continuous. But Dirichlet believed that a function, with fewer restrictions than those implied in his conditions, could be represented by a trigonometric series.
It should be noticed that Dirichlet's conditions so not include all continuous functions, since they exclude every function with an infinite number of maxima and minima; but if a function have an infinite number of oscillations in the neighborhood of a point it may be continuous when the amplitude of the oscillations is infinitely small. Thus the function \( x \cos \frac{1}{x} \) between \(-\pi, \pi\), on the understanding that it is 0 for \( x = 0 \), yet this would be excluded from Dirichlet's conditions.

One of the main objects of later investigations has been to extend the limits of the arbitrariness allowable to a function which may still be represented by a trigonometric series, but it is a somewhat striking fact that the conditions do not yet include all continuous functions, and Du Bois-Reymond\(^1\) has even proved that there are continuous functions such that the trigonometric series which represent them become infinite at certain points, that is, cease to represent them at these points.

R. Lipschitz.

The first published attempt to show that a function having an infinite number of maxima and minima may be represented by a trigonometric series is that of Lipschitz.\(^2\) His proof depends on the evaluation of two integrals noticed as fundamental in Dirichlet's method.
and he shows that these still maintain their validity if in the neighborhood of those points \( \beta \) for which \( f(\beta) \) oscillates, \( \{f(\beta + \delta) - f(\beta)\} \) is less in absolute value than \( \beta \delta^a \) where \( a \) is positive and \( \beta \) a constant. As an extension of Dirichlet's conditions the result is important, but it is to be observed that there may be continuous functions not satisfying this condition. \( f(\beta) \) will be continuous near \( \beta \) if, given an arbitrarily small quantity \( \epsilon \), a value \( h \) can be found such that for all values of \( \delta \) less numerically than \( h \), \( \{f(\beta + \delta) - f(\beta)\} \) is less than \( \epsilon \). Lipschitz' conditions implies that \( \epsilon = o(\delta) < \beta h^a \) or \( h = o(\delta) < \frac{\sqrt[\delta]}{\beta} \), a relation not necessary for continuity. Lipschitz' results would hold if \( L \log \delta \{f(\beta + \delta) - f(\beta)\} = 0 \).
Riemann's Investigations

Riemann's investigations as contained in his great memoir is divided into three main sections. The first is historical, the second, contains a thorough investigation of the fundamental principles of definite integrals, and in particular determines in what cases a function has an integral. We see here the great extension of meaning which the word function has gained in modern times, chiefly under the guidance of Fourier, Dirichlet, and Riemann himself, and which is essential to the modern function theory. The third section completes the memoir and is devoted to the representation of a function by a trigonometric series without special suppositions as to the nature of a function. The problem proposed for solution is the following:—If a function can be represented by a trigonometric series, what follows respecting the march of the function, respecting the change in its value for a continuous change in the argument? The preceding investigations argued from the function to the series; here the series is supposed given and the conclusion is the nature of the function. Riemann denotes the Series

\[ A_0 + A_1 + A_2 + \cdots + A_n \]

where \( A_0 = \frac{1}{2} b_0 \).
\[ A_n = a_n \sin nx + b_n \cos nx \] and when it is convergent its value is denoted by \( f(x) \) so that \( f(x) \) only exists for those values of \( x \) for which the series is convergent. He first supposes \( \Omega \) to be such that for every value of \( x \), \( A_n \) becomes infinitely small when \( n \) becomes infinitely great. If the series \( \Omega \) be integrated twice and the series thus formed be denoted by \( F(x) \) so that \( F(x) = C + C'x + \frac{1}{2}A_0x^2-A_1+\frac{1}{n^2}A_n \ldots \ldots \) he shows that \( F(x) \) is convergent for every value of \( x \), is continuous, and is integrable. He then proves—

1. That when the series \( \Omega \) converges, the expression \[ \frac{F(x+a+\beta)-F(x+a-\beta)-F(x-a+\beta)+F(x-a-\beta)}{4\alpha\beta} \] converges to the value \( f(x) \) when \( \alpha \) and \( \beta \) becomes infinitely small but such that their ratio remain finite;

2. That \( F(x+2\alpha)+F(x-2\alpha)-2F(x) \) becomes infinitely small with \( \alpha \); and

3. That the integral \[ \mu^2 \int_{b}^{c} F(x) \cos \mu (x-a) \lambda(x) \, dx \] becomes infinitely small with \( \frac{1}{\mu^2} \) where \( b, c \), denote two arbitrary constants \( (c > b) \) a function which with its first derivative is continuous between \( b \) and \( c \) and vanishes at the limits and whose second derivative has not an infinite number of maxima and minima.

By means of these theorems he proves that if a
periodic function \( f(x) \) of period \( 2\pi \), can be represented by a trigonometric series whose terms become ultimately indefinitely small there must exist a continuous function \( F(x) \) such that

\[
F(x+a+\beta)-F(x+a-\beta)-F(x-a+\beta)+F(x-a-\beta)
\]

converges to the value \( f(x) \) when \( a, \beta \) converge to zero, their ratio remaining finite. Further the integral of (3) subject to the conditions there given, must become infinitely small with \( \frac{1}{\pi} \).

Conversely, when these conditions are satisfied, there exists a trigonometric series whose terms become infinitely small and which is such that, where it converges, it represents the function. For, determining \( C' A_0 \) so that \( F(x)-C'x-\frac{1}{2}A_0x^2 \) has the period \( 2\pi \), and then developing this function by the Fourier method, the term \( A_n \), where

\[
A_n = \frac{n^2}{\pi} \int_{-\pi}^{\pi} \left\{ F(t)-C't-\frac{1}{2}A_0t^2 \right\} \cos n(x-t) \, dt
\]

will become infinitely small with \( \frac{1}{n} \) and therefore the series \( A_0 + A_1 + A_2 \ldots \) will, whenever it converges, converge to \( f(x) \).

Riemann then shows that the convergence of the series for a definite value of \( x \) depends only on the behavior of the function in the neighborhood of that value.

It will have been observed that as yet, Riemann has given no criterion for determining when the coefficients
of the series \( \sum \) will in fact become infinitely small. Later he comes to this point, and he there states that in many cases this question cannot be settled by consideration of their expression as definite integrals, but must be determined in other ways. For the very important case in which \( f(x) \) is integrable, finite throughout the range of the variable, and has only a finite number of maxima and minima, he proves that the coefficients do become infinitely small and therefore that the series represents \( f(x) \) whenever it is convergent.

Next he takes up the case in which the terms of \( \sum \) do not become ultimately indefinitely small for every value of \( x \), and shows that the series can converge only for those values of \( x \) which are symmetrically placed with respect to those for which the integral

\[
\int_{b}^{c} f(x) \cos \frac{\pi}{\rho} (x-a) \lambda(x) \, dx.
\]

does not become infinitely small with \( \frac{1}{\rho} \).

His next step is to consider the possibility of the function becoming infinite, and gives as necessary and sufficient conditions that when \( f(x) \) is infinite for \( x = a \), \( tf(a-t) \) and \( tf(a+t) \) becomes infinitely small for \( t = 0 \) and \( f(a+t) + f(a-t) \) be integrable up to \( t = 0 \), it being understood that \( f(x) \) has not an infinite number
of maxima and minima.

Lastly, he deals with functions having an infinite number of maxima and minima. In this connection he first shows by an example that there may be integrable functions having an infinite number of maxima and minima which are yet not capable of representation by a Fourier Series. He here takes $f(x) = \frac{d}{dx}(x^2 \cos 1)$ where

He shows in the second place by examples that there may be functions having a finite number of maxima and minima and not integrable which nevertheless may be represented by a trigonometric series.

Riemann has thus given a very general solution of the problem of representation of functions by trigonometric series and his theorems are of fundamental importance in the subsequent investigation of Heine, Cantor, and Reymond. But other methods than those he gives must in many cases be resorted to to determine when the series is convergent, and as a matter of fact, Dirichlet's integrals seem indispensable for this purpose. 18

D. D. Stokes

The investigations of Stokes is important in the development of series, for he there draws attention to what has since been called the uniform convergence of
series, though this honor is usually attributed to Seidel, whose paper did not appear till 1848. In the first section Stokes discusses the expansion of a function in a series of sines and also in a series of cosines, and adopts the method of Poisson as that which he employed when he first began the investigations and which best harmonized with the rest of the paper.
Contributions Subsequent to Riemann.

The course of the Fourier Series now takes a new departure. In the preceding work it has been seen that under certain circumstances the series will converge to the value of the function, but in more recent times it has been recognized that mere convergence is not sufficient for most of the applications for which the series is needed; the convergence must be uniform. Suppose for instance, that we have for \( f(x) \) the series

\[
  f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

and we wish to evaluate \( \int_{\alpha}^{\beta} f(x) \phi(x) \, dx \) by means of the series: then we can only safely assert the equation

\[
  \int_{\alpha}^{\beta} f(x) \phi(x) \, dx = \frac{1}{2}a_0 \int_{\alpha}^{\beta} \phi(x) \, dx + \sum_{n=1}^{\infty} \int_{\alpha}^{\beta} (a_n \cos nx + b_n \sin nx) \phi(x) \, dx
\]

if the series is uniformly convergent. Unless then the series is to be shorn of much of its value its uniform convergence must be established. Another difficulty that this conception of uniform convergence raises is that the old proof for the uniqueness of the expansion becomes invalid, as resting upon an integration, the legitimacy of which is not proved.
The first to call attention to the points just mentioned was Heine, in a paper contributed to Crelle's Journal. He gives the definition of uniform convergence and shows that the Fourier series cannot converge uniformly in the neighborhood of a point at which the function is discontinuous, and establishes the following theorems:

(1) The Fourier Series for a finite function \( f(x) \) with a finite number of maxima and minima converges uniformly if \( f(x) \) be continuous for \((-\pi = \omega < x = \omega < \pi)\) and \( f(-\pi) = f(\pi) \); in all other cases it is only uniformly convergent in general, that is, it converges uniformly for every interval which does not include a point of discontinuity, these points being supposed finite in number. The points \( \pm \pi \) are to be considered points of discontinuity if \( f(-\pi) \neq f(\pi) \).

(2) If a trigonometric series is in general uniformly convergent, and is in general equal to zero for \((-\pi = \omega < x = \omega < \pi)\) then will every coefficient be zero. For the proof of this theorem, he falls back on Riemann's proposition regarding \( \lim_{a \to 0} \frac{F(x+a) + F(x-a) - 2F(x)}{a} = 0 \)

Heine's second theorem shows that there cannot be two different expansions of a function if these are to be in general uniformly convergent.
Cantor has proved the more general theorem that even if uniform convergence be not demanded there can be but one convergent expansion in a trigonometric series and it is that of Fourier. Cantor’s memoirs also appear in Crelle’s Journal. In the first of these he proves that if two infinite series, \( a_1, a_2, \ldots, b_1, b_2, \) are such that \( L_n \left( a_n \sin nx + b_n \cos nx \right) = 0 \) where \( x \) is real and lies in a given interval \( a_1, b_1, \) then \( L_{a_n} = 0, L_{b_n} = 0 \) for \( n = \infty. \) In the second memoir he takes the function \( F(x) \) of Riemann, the conditions imposed on it being shown, by the proposition just stated, to be satisfied and forms the quotient \( \frac{F(x+a) - 2F(x) + F(x-a)}{a^2}. \) This quotient is zero for \( a = 0 \) and \( F(x) \) is continuous; and it now follows that \( F(x) \) must be a linear function of \( x. \) Giving to \( F(x) \) a linear value and adopting the notation of Riemann, we have \( \frac{1}{n^2} A_0 x^2 + C_1 x + C_2 = A_1 + \frac{1}{n^2} A_n. \) The right hand member being periodic, it follows that \( A_0 = 0 = C_1 \) and then multiplying by \( \sin nx \) or \( \cos nx \) and integrating between \( -\pi, \pi, \) it is seen that \( a_n = 0 = b_n \) for every value of \( n. \) Hence a convergent trigonometric series can represent zero only if every coefficient is zero, from which the uniqueness of the trigonometric expansion at once follows.
Du Bois-Reymond's contributions to the theory of series in general and of the Fourier series in particular have been both numerous and important. They contain notices of the work of predecessors and full references to his own papers bearing on the subject; he proves that the coefficients of the series

\[ f(x) = \sum_{p=0}^{\infty} (a_p \cos px + b_p \sin px) \]

have the values

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a) \, da, \]

\[ a_p = \frac{1}{\pi} \int_{-\pi}^{\pi} f(a) \cos pa \, da, \quad b_p = \frac{1}{\pi} \int_{-\pi}^{\pi} f(a) \sin pa \, da \]

whenever these integrals are finite and determinate.

This proposition includes of course the theorem that \( f(x) \) can be expanded in only one way in a Fourier series.

The contributions of Du Bois-Reymond may be said in a sense to include all the results of previous writers and to push the inquiry as to the nature of the functions which can be represented by a Fourier series, when the coefficients are determined as definite integrals, very near its utmost limits.
NOTES


2. L. Euler, Memoirs of Berlin Academy, (1748) p.68 quoted in Hobson, The Theory of Functions of a Real Variable, (Cambridge, 1907) p. 536


4. Ibid. vol. 11, p. 140

5. Daniel Bernoulli, Memoirs of Berlin Academy, (1748) p.84, quoted in Hobson, op. cit. 538


7. H. Burkhardt, Uber Trigonometrische Reihen und Integrals, in Enzyklopädie der Wissenschaften, Band 11 Heft 8, (Leipzig, 1924) p.956

8. Ibid. p. 959


10. S.D. Poisson, in Gibson, op. cit. p. 145

11. Whitaker, op. cit. p. 57


NOTES (continued)

15. Ibid. p. 244

16. R. Lipschitz, Ueber ein Integral der Differentialgeichung, Crelle, (1850) vol. 56 p. 189-195
De explicacione per series trigonometricas
Crelle, (1884) vol. 63 p. 408

17. Gibson, History of the Fourier Series, in


19. Ibid. p. 1045

20. E. Heine, Beitrag zur Theorie der Anziehung und
der Warme, Crelle (1845) vol 29, p. 185-207

21. G. Cantor, Uber trigonometrische Reihen, Crelle, (1871) vol. 72 p. 130

22. Paul Du Bois Reymond, Werthänderungen analytischer
Function, Math. Ann. (Leipzig, 1874) vol 7, p. 244
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23. G. Burkhardt, op. cit. p. 1001-1036
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