Sturm-Liouville systems

Harry Bauer

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STURM-LIOUVILLE SYSTEMS

by

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H.B.
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CHAPTER I

EXISTENCE THEOREM

This thesis is primarily concerned with the system
\[ y'' + (\lambda - q(x)) y = 0, \]
\[ y'(0) - hy(0) = 0, \quad y'(\pi) - H y(\pi) = 0, \]
where \( h \) and \( H \) are real constants, \( \lambda \) is a complex parameter, and \( q(x) \) is a continuous, real-valued function of bounded variation on \([0, \pi]\).
Such a system is called a Sturm-Liouville system.

The values of \( \lambda \) for which the above system has a non-trivial solution are called eigenvalues of the system; the corresponding non-trivial solutions, determined uniquely to within a constant multiple, are called eigenfunctions.

It will be shown that the set of eigenvalues of the above system is an enumerably infinite set of real numbers without a limit point, bounded below, and unbounded above. Thus, the eigenvalues may be arranged as an increasing, unbounded sequence \( \lambda_1, \lambda_2, \lambda_3 \ldots \).

The eigenfunction corresponding to \( \lambda_n \) is uniquely determined after imposing the additional conditions
\[ \int_0^\pi (y(x))^2 \, dx = 1, \quad y(0) > 0. \]
This normalized, initially positive, eigenfunction corresponding to \( \lambda_n \) is denoted by \( u_n(x) \).

Approximations for \( \lambda_n \) and \( u_n(x) \) will be given. In particular, it will be shown that
\[ \lambda_n = n + \frac{c}{n} + O\left(\frac{1}{n^2}\right), \]
and
\[ u_n(x) = \sqrt{\frac{2}{\pi}} \cos nx + \frac{\beta(x)}{n} \sin nx + O\left(\frac{1}{n^2}\right) \]
(here \( n > 0 \)), where the constant \( c \) is determined by \( h \), \( H \), and \( q(x) \), and \( \beta(x) \) is a function with a first derivative of bounded variation also determined by \( h \), \( H \), and \( q(x) \). The term \( O\left(\frac{1}{n}\right) \), in the expression for \( \lambda_n \), is a constant depending only upon \( n \), while \( O\left(\frac{1}{n^2}\right) \), in the expression for \( u_n(x) \), is a function of \( x \), depending upon \( n \).

Further, it will be shown that if \( f(x) \) is a suitably restricted function defined on \( [a, \pi] \), then
\[ f(x) = \sum_{n=0}^{\infty} c_n u_n(x) \]
for all \( x \) in \( [a, b] \), where
\[ c_n = \int_0^\pi f(t)u_n(t)dt, \quad n = 0, 1, 2, \ldots. \]

**Definition 1.1.** The function \( f(x, y_1, y_2, \ldots, y_n) \)
defined on a set \( R \) satisfies the **Lipschitz condition in** \( y_1, y_2, y_3, \ldots, y_n \)
on \( R \) if there exists a constant \( k > 0 \) such that
\[ |f(x, y_{11}, y_{12}, \ldots, y_{1n}) - f(x, y_{21}, y_{22}, \ldots, y_{2n})| \]
\[ \leq k \sum_{i=1}^{n} |y_{i1} - y_{2i}| \]
for all \((x, y_{11}, y_{12}, \ldots, y_{1n}), (x, y_{21}, y_{22}, \ldots, y_{2n}) \) in \( R \).

**Theorem 1.1.** The system
\[ y'' + (\lambda - q(x)) y = 0, \]
\[ y(a) = c, \quad y'(a) = d, \]
where \( q(x) \) is a continuous, real-valued function on \( [a, b] \), \( |\lambda| \neq N \)
for some constant \( N \), and \( \lambda, c, d \) are real constants, has a unique solution.
Proof. Let \( y_1(x, \lambda) = c + d(x-a) \). Clearly \( y_1(a, \lambda) = c \), \( y_1'(a, \lambda) = d \). Let
\[
y_2(x, \lambda) = c + d(x-a) + \int_a^x (q(t) - \lambda) y_1'(t, \lambda) \, dt \, du,
\]
then the sequence \( \{ y_n(x, \lambda) \} \) is defined inductively. Define \( y_{n+1}(x, \lambda) \) as follows:
\[
y_{n+1}(x, \lambda) = c + d(x-a) + \int_a^x (q(t) - \lambda) y_n(t, \lambda) \, dt \, du, \quad n = 1, 2, 3, \ldots.
\]
Since \( y_1(x, \lambda) \) is continuous for \( a \leq x \leq b \), it follows immediately that
\( y_n(x, \lambda), \quad n = 2, 3, 4, \ldots \), is also continuous on \([a, b]\). Let
\[
A = |c| + |d|/(b-a) \quad \text{and choose } M \text{ so that } |q(x)| \leq M \text{ for } a \leq x \leq b.
\]
Then
\[
|y_2(x, \lambda) - y_1(x, \lambda)| \leq \int_a^x \int_a^x |q(t) - \lambda|/y_1(t, \lambda) \, dt \, du
\]
\[
\leq A (M + N) \frac{(x-a)^2}{2!}.
\]
Observe that \( y_2''(x, \lambda) = (q(x) - \lambda) y_1'(x, \lambda) \), \( y_2(a, \lambda) = c \), \( y_2'(a, \lambda) = d \). By induction it is readily established that
\[
|y_{n+1}(x, \lambda) - y_n(x, \lambda)| \leq A (M + N)^n \frac{(x-a)^{2n}}{(2n)!}, \quad n = 1, 2, 3, \ldots;
\]
also, \( y_{n+1}''(x, \lambda) = (q(x) - \lambda) y_n(x, \lambda) \), \( y_{n+1}(a, \lambda) = c \), \( y_{n+1}'(a, \lambda) = d \).

Note that
\[
y_n(x, \lambda) = y_1(x, \lambda) + \sum_{k=1}^{n-1} (y_{k+1}(x, \lambda) - y_k(x, \lambda)).
\]
In view of the preceding estimates, we have
\[
\sum_{k=1}^{\infty} |y_{k+1}(x, \lambda) - y_k(x, \lambda)| \leq \sum_{k=1}^{\infty} A (M + N)^k \frac{(x-a)^{2k}}{(2k)!}
\]
\[
= A \sum_{k=0}^{\infty} \frac{[(M + N)(b-a)^2]^k}{k!}
\]
\[
\leq A e^{(M + N)(b-a)^2}.
\]
Thus the uniform convergence of the sequence \( \{ y_n(x, \lambda) \} \)
of continuous functions to a continuous function, say \( y(x, \lambda) \), is established by showing the uniform convergence of the series.
\[
\sum_{k=1}^{\infty} (y_{k+1}(x, \lambda) - y_k(x, \lambda)). \text{ Also, the sequence } [(q(x) - \lambda) y_n(x, \lambda)] \text{ is uniformly convergent to } (q(x) - \lambda) y(x, \lambda) \text{ for } a \leq x \leq b \text{ and } |\lambda| \leq N. \\
\]
Thus, given \(\varepsilon > 0\), there exists \(N_0\) such that if \(n \geq N_0\), then
\[
|q(t) - \lambda| |y_n(t, \lambda) - y(t, \lambda)| < \varepsilon
\]
for \(a \leq t \leq b\) and \(|\lambda| \leq N\). Now
\[
\left| \int_a^x \int_a^u (q(t) - \lambda) y_n(t, \lambda) \, dt \, du - \int_a^x \int_a^u (q(t) - \lambda) y(t, \lambda) \, dt \, du \right|
\leq \varepsilon \frac{(x-a)^2}{2}
\]
for \(n \geq N_0\) and for \(a \leq x \leq b\) and \(|\lambda| \leq N\).

Hence,
\[
y(x, \lambda) = c + d(x-a) + \int_a^x \int_a^u (q(t) - \lambda) y(t, \lambda) \, dt \, du,
\]
\[
y''(x, \lambda) = (q(x) - \lambda) y(x, \lambda),
\]
y(a, \lambda) = c, and \(y'(a, \lambda) = d\).

The uniqueness of the above solution is established as follows:
Suppose there exists a second solution \(Y(x, \lambda)\) for a fixed \(\lambda, \, |\lambda| \neq N\). Then
\[
Y(x, \lambda) = c + d(x-a) + \int_a^x \int_a^u (q(t) - \lambda) Y(t, \lambda) \, dt \, du
\]
and
\[
|Y(x, \lambda) - y(x, \lambda)| \leq \int_a^x \int_a^u |q(t) - \lambda| |Y(t, \lambda) - y(t, \lambda)| \, dt \, du
\leq (M+N) \int_a^x \int_a^u |Y(t, \lambda) - y(t, \lambda)| \, dt \, du.
\]
Let \(K = \max.|Y(x, \lambda) - y(x, \lambda)| \text{ for } a \leq x \leq b\). Hence,
\[
|Y(x, \lambda) - y(x, \lambda)| \leq (M+N) K \frac{(x-a)^2}{2!}
\]
Repeating the argument and applying the immediately preceding results, one has
\[
|Y(x, \lambda) - y(x, \lambda)| \leq (M+N)^2 K \frac{(x-a)^4}{4!}.
\]
It follows inductively then that for \( a \leq x \leq b \) and \( \lambda \leq N \),
\[
|Y(x, \lambda) - y(x, \lambda)| \leq (M + N)^n \frac{(x-a)^{2n}}{(2n)!} + \cdots
\]
Since the series \( \sum_{k=0}^{\infty} k \frac{(M + N)^n}{(2n)!} (x-a)^{2n} \) converges to \( e^{(M+N)(x-a)} \),
\[
\lim_{n \to \infty} \frac{(M + N)^n}{(2n)!} (x-a)^{2n} = 0.
\]
Thus, it is necessary that \( Y(x, \lambda) = y(x, \lambda) \) for \( a \leq x \leq b \). The solution \( y(x, \lambda) \) is real-valued because \( q(x) \) is real-valued and \( \lambda, c, d \) are real constants.

We shall conclude this chapter with a derivation of several estimates which will yield the uniform continuity of \( y(x, \lambda), y'(x, \lambda), \) and \( y''(x, \lambda) \) in \( x \) and \( \lambda \) simultaneously, where \( a \leq x \leq b, \lambda \leq N \).

(1) Estimates for \( y(x, \lambda) \) and \( y_n(x, \lambda) \).

Since
\[
y_n(x, \lambda) = y_1(x, \lambda) + \sum_{k=1}^{n-1} (y_{k+1}(x, \lambda) - y_k(x, \lambda)),
\]
\[
|y_n(x, \lambda)| \leq A + \sum_{k=1}^{n-1} A (M + N)^k \frac{(b-a)^{2k}}{(2k)!},
\]
\[
\leq A \sum_{k=0}^{\infty} \frac{(M + N)(b-a)^{2k}}{k!}
\]
\[
\leq A e^{(M+N)(b-a)^2}
\]
for \( a \leq x \leq b \) and \( \lambda \leq N \), and
\[
|y(x, \lambda)| \leq A e^{(M+N)(b-a)^2},
\]
because of the convergence of the sequence \( \{y_n(x, \lambda)\} \) to \( y(x, \lambda) \).

(2) Estimate for \( y(x_1, \lambda) - y(x_0, \lambda) \).

Suppose \( a \leq x_0 \leq x_1 \leq b \). Then
\[
|y(x_1, \lambda) - y(x_0, \lambda)| \leq |d/| (x_1-x_0) + \int_{x_0}^{x_1} \int_a^u |q(t) - \lambda|/|y(t, \lambda)|/|dt du
\]
\[
\leq |d/| (x_1-x_0) + A (M+N) e^{(M+N)(b-a)^2} \int_{x_0}^{x_1} \int_a^u dt du
\]
\[
\leq |d/| (x_1-x_0) + A(M+N) e^{(M+N)(b-a)^2} (b-a)(x_1-x_0)
\]
\[
= B (x_1-x_0)
\]
for \( a \leq x_0 \leq x_1 \leq b \) and \( |\lambda| \leq N \), where \( B = \frac{d}{d+1} + A \left( \frac{M+N}{b-a} \right) e^{(M+N)(b-a)^2} \).

(3) Estimate for \( y(x, \lambda_1) - y(x, \lambda_0) \).

\[
y_{n+1}(x, \lambda) = c + d(x-a) + \sum_{a}^{x} \left( q(t) - \lambda \right) y_n(t, \lambda) \, dt \, du, \quad n = 1, 2, 3, \ldots
\]

\[
y_1(x, \lambda_1) - y_1(x, \lambda_0) = 0.
\]

\[
|y_2(x, \lambda_1) - y_2(x, \lambda_0)| \leq \sum_{a}^{x} \left( q(t) - \lambda_1 \right) y_1(t, \lambda_1) - (q(t) - \lambda_0) y_1(t, \lambda_0) \, dt \, du
\]

\[
\leq \sum_{a}^{x} \left( |q(t) - \lambda_1| |y_1(t, \lambda_1) - y_1(t, \lambda_0)| + |\lambda_1 - \lambda_0| |y_1(t, \lambda_0)| \right) \, dt \, du
\]

\[
\leq \left| \lambda_1 - \lambda_0 \right| A \frac{e^{(M+N)(b-a)^2} (x-a)^2}{2!}.
\]

for \( a \leq x \leq b \) and \( |\lambda_1/| \leq N, |\lambda_1| \leq N \). By repeating this process and applying the immediately preceding estimates, one establishes, by induction, that

\[
|y_n(x, \lambda_1) - y(x, \lambda_0)| \leq A e^{(M+N)(b-a)^2} \left| \frac{|\lambda_1 - \lambda_0|}{(M+N)} \right| \sum_{k=1}^{n} \frac{(M+N)(x-a)^2}{(2k)!}
\]

\[
\leq A e^{(M+N)(b-a)^2} \left| \frac{|\lambda_1 - \lambda_0|}{(M+N)} \right| \sum_{k=1}^{n} \frac{(M+N)(b-a)^2}{(2k)!}
\]

for \( a \leq x \leq b \) and \( |\lambda_1/| \leq N, |\lambda_1| \leq N \). Clearly,

\[
\sum_{k=1}^{n} \frac{(M+N)(b-a)^2}{k!} \leq e^{(M+N)(b-a)^2},
\]

so

\[
|y_n(x, \lambda_1) - y_n(x, \lambda_0)| \leq A e^{2(M+N)(b-a)^2} \left| \frac{|\lambda_1 - \lambda_0|}{(M+N)} \right|
\]

\[
= C \left| \lambda_1 - \lambda_0 \right|
\]

for \( a \leq x \leq b \) and \( |\lambda_1/| \leq N, |\lambda_1| \leq N \), where \( C = A e^{2(M+N)(b-a)^2} / (M+N) \).

Since the sequence \( \{y_n(x, \lambda)\} \) converges to \( y(x, \lambda) \) for \( a \leq x \leq b \) and \( |\lambda| \leq N \), \( |y(x, \lambda_1) - y(x, \lambda_0)| \leq C \left| \lambda_1 - \lambda_0 \right| \) for \( a \leq x \leq b \) and \( |\lambda_1/| \leq N, |\lambda_1| \leq N \).
(4) Estimate for $y(x_1, \lambda_1) - y(x_0, \lambda_0)$.

Suppose $a \leq x_0 \leq x_1 \leq b$. Then
\[
|y(x_1, \lambda_1) - y(x_0, \lambda_0)| \leq |y(x_1, \lambda_1) - y(x_1, \lambda_0)| + |y(x_1, \lambda_0) - y(x_0, \lambda_0)|
\leq C|\lambda_1 - \lambda_0| + B(x_1 - x_0)
\leq D \left\{ |\lambda_1 - \lambda_0| + (x_1 - x_0) \right\}
\]
for $a \leq x \leq b$ and $|\lambda_0| \leq N$, $|\lambda_1| \leq N$, where $D = \max. (B, C)$. Hence, we conclude that $y(x, \lambda)$ satisfies a Lipschitz condition in $x$ and $\lambda$ simultaneously in the region defined by $a \leq x \leq b$ and $|\lambda| \leq N$.

(5) Estimate for $y''(x_1, \lambda) - y''(x_0, \lambda)$.

Suppose $a \leq x_0 \leq x_1 \leq b$. Since $y''(x, \lambda) = (q(x) - \lambda) y(x, \lambda)$,
\[
|y''(x_1, \lambda) - y''(x_0, \lambda)| \leq |q(x_1) - \lambda|/ y(x_1, \lambda) - y(x_0, \lambda)|
+ |q(x_1) - q(x_0)|/ y(x_0, \lambda)|
\leq (M+N) B (x_1 - x_0) + |q(x_1) - q(x_0)|/ A e^{(M+N)(b-a)}
\]
for $a \leq x \leq b$ and $|\lambda| \leq N$.

(6) Estimate for $y'(x_1, \lambda) - y'(x_0, \lambda)$.

Suppose $a \leq x_0 \leq x_1 \leq b$. Since $y'(x, \lambda) = d + \int_a^x (q(t) - \lambda) y(t, \lambda) \, dt$,
\[
|y'(x_1, \lambda) - y'(x_0, \lambda)| \leq \int_{x_0}^{x_1} |q(t) - \lambda|/ y(t, \lambda) \, dt
\leq (M+N) A e^{(M+N)(b-a)} (x_1 - x_0)
= E (x_1 - x_0)
\]
for $a \leq x \leq b$ and $|\lambda| \leq N$, where $E$ is a suitable constant.

(7) Estimate for $y''(x, \lambda_1) - y''(x, \lambda_0)$.

Since $y''(x, \lambda) = (q(x) - \lambda) y(x, \lambda)$,
\[
|y''(x, \lambda_1) - y''(x, \lambda_0)| \leq |q(x) - \lambda_1|/ y(x, \lambda_1) - y(x, \lambda_0)|
+ |\lambda_1 - \lambda_0|/ y(x, \lambda_0)|
\leq |\lambda_1 - \lambda_0| \left\{ (M+N) C + A e^{(M+N)(b-a)} \right\}
= F |\lambda_1 - \lambda_0|
\]
for $a \leq x \leq b$ and $|\lambda_0| \leq N$, $|\lambda_i| \leq N$, where $F = (M+N)C + Ae^{(M+N)(b-a)^2}$.

(8) Estimate for $y'(x, \lambda_1) - y'(x, \lambda_0)$.

Since $y'(x, \lambda) = d + \int_a^x y''(t, \lambda) \, dt$,

$$|y'(x, \lambda_1) - y'(x, \lambda_0)| \leq \int_a^x |y''(x, \lambda_1) - y''(t, \lambda_0)| \, dt$$

$$\leq F(b-a) / |\lambda_1 - \lambda_0|$$

for $a \leq x \leq b$ and $|\lambda_0| \leq N$, $|\lambda_i| \leq N$.

(9) Estimate for $y'(x_1, \lambda_1) - y'(x_0, \lambda_0)$.

Suppose $a \leq x_0 \leq x_1 \leq b$. Then

$$|y'(x_1, \lambda_1) - y'(x_0, \lambda_0)| \leq |y'(x_1, \lambda_1) - y'(x_0, \lambda_1)| + |y'(x_0, \lambda_1) - y'(x_0, \lambda_0)|$$

$$\leq E(x_1 - x_0) + F(b-a) / |\lambda_1 - \lambda_0|$$

$$\leq G \left\{ (x_1 - x_0) + |\lambda_1 - \lambda_0| \right\}$$

for $a \leq x \leq b$ and $|\lambda_0| \leq N$, $|\lambda_1| \leq N$, where $G$ is a suitable constant.

Also, observe that $y'(x, \lambda)$ satisfies a Lipschitz condition in $x$ and $\lambda$ simultaneously in the region defined by $a \leq x \leq b$ and $|\lambda| \leq N$.

The following notation concerning intervals will be employed throughout this thesis: The set of all numbers $x$ such that $a \leq x \leq b$ defines the open interval $(a, b)$. The corresponding closed interval is written $[a, b]$ and contains all $x$ with $a \leq x \leq b$. The notations for $(a, b)$ and $[a, b]$ are then apparent.
CHAPTER II

COMPARISON THEOREMS

Section I

This chapter deals mainly with differential equations of the form \( y'' - Gy = 0 \), where \( G \) is any continuous, real-valued function of a real variable \( x \), \( a \leq x \leq b \).

We will consider two distinct differential equations of the above form and compare the zeros of their solutions. Then the number of zeros possessed by a solution to such a differential equation will be determined, and finally, it will be shown that the system

\[
y'' + (\lambda - q(x))y = 0.
\]

\[
y(a) = 1, \quad y'(a) = h, \quad y'(b) - Hy(b) = 0,
\]

with the usual assumption on \( \lambda, h, H, \) and \( q(x) \), has a solution for certain values of \( \lambda \).

The next two theorems from elementary differential equations theory will be employed in the proof of the subsequent theorems.

**Theorem 2.1.** If \( \varphi_1(x) \) and \( \varphi_2(x) \) are solutions of

\[
y'' + P(x)y' + Q(x)y = 0,
\]

where \( P(x) \) and \( Q(x) \) are continuous functions on an interval \( I \), then a necessary and sufficient condition that \( \varphi_1(x) \) and \( \varphi_2(x) \) be linearly independent on the interior of \( I \) is that the Wronskian

\[
W[\varphi_1, \varphi_2] = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix}
\]

of \( \varphi_1(x) \) and \( \varphi_2(x) \) be non-vanishing on the interior of \( I \).
Theorem 2.2. The differential equation \( y'' + P(x)y' + Q(x)y = 0 \), where \( P(x) \) and \( Q(x) \) are continuous functions on an interval \( I \), subject to the initial conditions \( y(a) = c \), and \( y'(a) = d \), for \( a \) in \( I \) and arbitrary constants \( c \) and \( d \), possesses a unique solution on a subinterval of \( I \) containing \( a \).

Theorem 2.3 (Separation Theorem). A non-trivial solution of \( y'' - G(y) = 0 \) cannot have an infinite number of zeros in \( [a, b] \).

Proof. Suppose a solution \( y(x) \) has infinitely many zeros in \( [a, b] \). Then, according to the Bolzano-Weierstrass Theorem, there exists a limit point \( c \) of the set of zeros of \( y(x) \) in \( [a, b] \). In particular, there exists a sequence \( \{ \alpha_n \} \) such that \( \alpha_n \) is in \( [a, b] \), for \( n = 1, 2, 3, \ldots \), \( \alpha_n \neq \alpha_m \) for \( n \neq m \), \( y(\alpha_n) = 0 \) for each \( n \), and \( \lim_{n \to \infty} \alpha_n = c \). From the continuity of \( y(x) \), it follows that \( y(c) = 0 \).

Now \( y'(c) = \lim_{n \to \infty} \frac{y(\alpha_n) - y(c)}{\alpha_n - c} = 0 \).

If \( y(c) = y'(c) = 0 \), the solution \( y(x) \) is trivial, by the uniqueness of the solution satisfying these initial conditions.

Theorem 2.4. The zeros of two linearly independent real solutions \( y_1(x) \) and \( y_2(x) \), defined on \( [a, b] \), of a linear second order differential equation separate one another, i.e., between any two consecutive zeros of one solution there exists a zero of the other solution.

Proof. Because of the linear independence of \( y_1(x) \) and \( y_2(x) \), their Wronskian differs from zero for \( a \leq x \leq b \). Suppose \( x_1 \) and \( x_2 \), \( x_1 < x_2 \), are two consecutive zeros of \( y_1(x) \) in \( [a, b] \). It is to be
shown that $y_2(x)$ has at least one zero in $(x_1, x_2)$. Suppose $y_2(x)$ has no zero in $[x_1, x_2]$. That $y_2(x_1) \neq 0 \neq y_2(x_2)$ is a consequence of the non-vanishing of the Wronskian of $y_1(x)$ and $y_2(x)$ on $[a, b]$.

Now $y_1(x)$ and $y_2(x)$ are continuous on $[a, b]$; hence, under the above circumstances, $\frac{y_1(x)}{y_2(x)}$ is continuous on $[x_1, x_2]$. Applying Rolle's Theorem to $\frac{y_1(x)}{y_2(x)}$, we see that

$$\left[ \frac{y_1(x)}{y_2(x)} \right]' = \frac{y_2(x) y_1'(x) - y_1(x) y_2'(x)}{y_2^2(x)}$$

vanishes at least once on $(x_1, x_2)$. But this is clearly impossible; hence, $y_2(x)$ has at least one zero on $(x_1, x_2)$.

**Theorem 2.5.** Let $u(x)$ and $v(x)$ be non-trivial, real solutions of $u'' - G_1 u = 0$ and $v'' - G_2 v = 0$, respectively, where $G_1$ and $G_2$ are real-valued, continuous functions of $x$ on $[a, b]$, not identically equal on any subinterval of $[a, b]$, and $G_1 \geq G_2$. Then, between every two consecutive zeros of $u(x)$ in $[a, b]$, there is at least one zero of $v(x)$.

**Proof.** Suppose the conclusion is false. Multiplying the first equation by $v$ and the second by $u$ and subtracting, gives

$$vu'' - uv'' = (G_1 - G_2) uv.$$  

Integration from $x_1$ to $x_2$, where $x_1$ and $x_2$ are consecutive zeros of $u(x)$ in $[a, b]$, yields

$$\int_{x_1}^{x_2} (u'v - u'u) \, dx = \int_{x_1}^{x_2} (G_1 - G_2) uv \, dx.$$  

Without loss of generality, $u$ and $v$ may be assumed positive on $(x_1, x_2)$. Then the right side of the above equation is positive, while the left side is non-positive, which is a contradiction. Hence,
v must vanish at least once in \((x_1, x_2)\).

**Theorem 2.6 (First Comparison Theorem).** Suppose \(u(x)\) and \(v(x)\) are non-trivial, real solutions of \(u'' - G_1 u = 0\) and \(v'' - G_2 v = 0\), respectively, where \(G_1\) and \(G_2\) are real-valued, continuous functions of \(x\) on \([a, b]\), not identically equal on any subinterval of \([a, b]\), and \(G_1 \neq G_2\) on \([a, b]\). Further, suppose \(u(a) \neq 0, v(a) \neq 0\), and \(\frac{u'(a)}{u(a)} \geq \frac{v'(a)}{v(a)}\). Then, if \(u\) has \(m\) zeros on \((a, b]\), \(v\) has at least \(m\) zeros on \((a, b]\), and the \(i\) th zero of \(v\) is less than the \(i\) th zero of \(u\), for \(i = 1, \ldots, m\).

**Proof.** Suppose \(x_1, x_2, x_3, \ldots x_m\) are the zeros of \(u\) in \((a, b]\) and \(a < x_1 < x_2 < \ldots x_m \leq b\). Between two consecutive zeros of \(u\) there exists at least one zero of \(v\). Suppose \(v\) has no zeros in \((a, x_1)\). As in the proof of theorem (2.5), we have

\[
\int_a^{x_1} (u'v - v'u) \, dx = \int_a^{x_1} (G_1 - G_2) u \, v \, dx.
\]

There are four cases to be considered, depending on the sign of \(u\) and \(v\) in \((a, x_1)\), but we will consider only the case where \(u > 0\) and \(v > 0\) on \((a, x_1)\), since the remaining cases may be carried through in a similar manner.

If \(u > 0\) on \((a, x_1)\), then \(u'(x_1) < 0\). Hence,

\[
\int_a^{x_1} (u'v - v'u) \, dx = \left[ u'(x_1) \, v(x_1) - v'(x_1) \, u(x_1) \right] - \left[ u'(a) v(a) - v'(a) u(a) \right]
\]

\[
= u'(x_1) \, v(x_1) - \left[ u'(a) \, v(a) - v'(a) \, u(a) \right].
\]

The first term on the right is non-positive in this case; the term in brackets is non-negative, according to the last hypothesis of this theorem.
Thus,
\[ 0 \geq (u'v - v'u) \int_a^{x_1} \, dx = \int_a^{x_1} (G_1 - G_2) \, u \, v \, dx > 0. \]
This is a contradiction, and hence \( v \) must vanish at least once in \((a, x_1)\). Consequently, the \( i \)th zero of \( v \) is less than the \( i \)th zero of \( u \).

**Theorem 2.7 (Second Comparison Theorem).** Suppose that \( u, v \) satisfy the hypothesis of theorem (2.6), that \( c \) is a point such that \( a < c \leq b \), \( u(c) \neq 0 \), \( v(c) \neq 0 \), and \( u \) and \( v \) have the same number of zeros in \((a, c)\). Then \( \frac{u'(c)}{u(c)} > \frac{v'(c)}{v(c)} \).

**Proof.** If \( x_1 \) is the zero of either \( u \) or \( v \) closest to \( c \) in \((a, c)\), then \( u(x_1) = 0 \), according to the last theorem. Furthermore,
\[ (u'v - v'u) \int_{x_1}^{c} \, dx = \int_{x_1}^{c} (G_1 - G_2) \, u \, v \, dx. \]
As in theorem (2.7), there are four cases to be considered, but we will consider only the case where \( u > 0 \) and \( v > 0 \) on \((x_1, c)\), since the remaining cases may be carried through in a similar manner.

If \( u > 0 \) on \((x_1, c)\) and \( u(x_1) = 0 \), then \( u'(x_1) > 0 \). Hence,
\[ (u'v - v'u) \int_{x_1}^{c} \, dx = [u'(c)v(c) - v'(c)u(c)] - [u'(x_1)v(x_1)], \]
\[ u'(c)v(c) - v'(c)u(c) = u'(x_1)v(x_1) + \int_{x_1}^{c} (G_1 - G_2) \, u \, v \, dx, \]
and
\[ u'(c)v(c) - v'(c)u(c) = u'(x_1)v(x_1) + \int_{x_1}^{c} (G_1 - G_2) \, u \, v \, dx. \]
Considering the last equation, the first term on the right is non-negative, while the second term is positive. Thus,
\[ u'(c)v(c) - v'(c)u(c) > 0, \]
and \( \frac{u'(c)}{u(c)} > \frac{v'(c)}{v(c)} \), because \( u > 0 \) and \( v > 0 \).
on \((x_1, c)\) by assumption. The case of interest to us will be when \(c = b\).

Section II

Theorem 2.8. The system

\[ y'' - (q(x) - \lambda) y = 0, \]
\[ y(a) = 1, \quad y'(a) = h, \]
has for \(\lambda < \left(\frac{\pi}{b-a}\right)^2 m_1\), at most \(n\) zeros in \([a, b]\), and for \(\lambda > \left(\frac{\pi}{b-a}\right)^2 M_1\) at least \(n\) zeros in \([a, b]\), where \(n \geq 1\), \(m_1 < m = \min q(x)\) on \([a, b]\), \(M_1 > M = \max q(x)\) on \([a, b]\), and \(q(x)\) is a real-valued, continuous function of \(x\) on \([a, b]\).

Proof. Consider the systems

(i) \[ y'' - (m_1 - \lambda)y = 0, \quad y(a) = 1, \quad y'(a) = h, \]

(ii) \[ y'' - (q(x) - \lambda)y = 0, \quad y(a) = 1, \quad y'(a) = h, \]

(iii) \[ y'' - (M_1 - \lambda)y = 0, \quad y(a) = 1, \quad y'(a) = h. \]

If \(\lambda < m_1\), the solution to (i) is

\[ y(x, \lambda) = \frac{h}{\sqrt{m_1 - \lambda}} \sinh \sqrt{m_1 - \lambda} (x-a) + \cosh \sqrt{m_1 - \lambda} (x-a). \]

Suppose \(a < c \leq b\) and \(y(c, \lambda) = 0\). Then

\[ \tanh \sqrt{m_1 - \lambda} (c-a) = \frac{-\sqrt{m_1 - \lambda}}{h}. \]

But \(|\tanh u| < 1\) for all \(u\), so \(\lambda > m_1 - h^2\).

Thus if \(\lambda < m_1 - h^2\), the solution to (ii) has no zeros on \([a, b]\).

When \(\lambda \leq m_1\), the solution to (i) is a linear or hyperbolic function, depending on whether we have \(\lambda = m_1\) or \(\lambda < m_1\), and hence the solution to (ii) has at most one zero on \([a, b]\) by theorem (2.6). On the other hand, if \(m_1 < \lambda < m_1 + \left(\frac{\pi}{b-a}\right)^2\), \(n \geq 1\), the solution

\[ y(x, \lambda) = \frac{h}{\sqrt{\lambda - m_1}} \sin \sqrt{\lambda - m_1} (x-a) + \cos \sqrt{\lambda - m_1} (x-a) \]

to (i) has at most
n zeros on \([a, b]\) and the solution to (ii) has also at most \(n\) zeros on \([a, b]\) by theorem (2.6). If \(\lambda > M_1 + \left(\frac{n\pi}{b-a}\right)^2\), \(n \geq 1\), the solution
\[ y(x, \lambda) = \frac{h}{\lambda - M_1} \sin \sqrt{\lambda - M_1} (x-a) + \cos \sqrt{\lambda - M_1} (x-a) \]
to (iii) has at least \(n\) zeros on \([a, b]\) and the solution to (ii) has at least \(n\) zeros on \([a, b]\) by theorem (2.6). Since \(m_1 < m\) and \(M_1 > M\), we have that if \(\lambda < m-h^2\), the solution to (ii) has no zeros on \([a, b]\). Also, if \(\lambda < m + \left(\frac{n\pi}{b-a}\right)^2\), then the solution to (ii) has at most \(n\) zeros on \([a, b]\).
Finally, if \(\lambda > M + \left(\frac{n\pi}{b-a}\right)^2\), then the solution to (ii) has at least \(n\) zeros on \([a, b]\).

Define \(N(\lambda)\) to be the number of zeros on \([a, b]\) possessed by the solution to the system
\[ y'' - (q(x) - \lambda)y = 0, \quad y(a) = 1, \quad y'(a) = h. \]

From the preceding discussion, we have that \(N(\lambda)\) is a monotone increasing function and, for each \(\lambda\), \(N(\lambda)\) is a non-negative integer.

**Theorem 2.9.** There exists a sequence \(\{\mu_i\}\) of real numbers such that \(N(\lambda) = 0\) for \(\lambda < \mu_0\), \(N(\lambda) = i + 1\) for \(\mu_i \leq \lambda < \mu_{i+1}\) and \(i = 0, 1, \ldots\), and \(y(b, \mu_i) = 0\) for \(i = 0, 1, 2, \ldots\). The number \(\mu_i\) is the least upper bound of the non-empty set \(A_i\), where \(A_i\) is the set of real numbers \(\lambda\) such that \(N(\lambda) = i\).

**Proof.** Let \(A_0\) be the set of real numbers \(\lambda\) such that \(N(\lambda) = 0\). \(A_0\) is non-empty and bounded above; hence there exists a real number \(\mu_0\) such that \(\mu_0 = \text{l.u.b.} A_0\). Consider \(y(x, \mu_0)\), where \(y(x, \lambda)\) is a solution of \(y'' - (q(x) - \lambda)y = 0, \quad y(a) = 1, \quad y'(a) = h. \)
Suppose there exists a constant \(c\), \(a < c < b\), such that \(y(c, \mu_0) = 0\). From earlier results, we know that \(y(x, \lambda)\) is continuous in both
variables together, so \( \lim_{\lambda \to \lambda_0} y(x, \lambda) = y(x, \lambda_0) \).

When \( \lambda < \lambda_0 \), \( y(x, \lambda) > 0 \) for \( a \leq x \leq b \), so \( y(x, \lambda_0) = 0 \).

Now \( y(c, \lambda_0) = 0 \) together with \( y(x, \lambda_0) \geq 0 \) for \( a \leq x \leq b \) and \( a < c < b \) implies \( y'(c, \lambda_0) = 0 \), while \( y(c, \lambda_0) = y'(c, \lambda_0) = 0 \) implies a trivial solution, contrary to the explicit assumption that \( y(x, \lambda) \) is non-trivial. Thus, \( c \) is not in \( (a, b) \) and \( N(\lambda_0) \leq 1 \). If \( \lambda > \lambda_0 \), then \( N(\lambda) \geq 1 \) and \( y(x, \lambda) \) has at least one zero in \( (a, b) \).

Let \( \sigma_n = \lambda_0 + \frac{1}{n} \), and let \( s_n \) be the first (i.e. least) zero of \( y(x, \sigma_n) \) in \( (a, b) \). From the first comparison theorem

\( a < s_1 < s_2 < \ldots \leq b \). The sequence \( \{s_n\} \) is a monotone, bounded sequence and has a limit, say \( \alpha \), \( \lim_{n \to \infty} s_n = \alpha \) and \( \alpha \) is in \( [a, b] \).

Thus \( 0 = y(\alpha, \lambda_0) = \lim_{n \to \infty} y(s_n, \sigma_n) \), by the continuity of \( y(x, \lambda) \).

Since \( \alpha \) cannot be in \( (a, b) \), we have \( \alpha = b \) and \( N(\lambda_0) = 1 \).

The set \( A_i \) is non-empty, bounded above and below, and hence there exists a real number \( \mu_i \) such that \( \mu_i = \text{l.u.b.} \ A_i \). Let

\( \tau_n = \mu_i + \frac{1}{n} \), so that \( \lim_{n \to \infty} \tau_n = \mu_i \) and \( y(x, \tau_n) \) has at least two zeros in \( (a, b) \). Let \( t_{1n} \) and \( t_{2n} \) be the first and second zeros of \( y(x, \tau_n) \) in \( (a, b) \). From the first comparison theorem

\( a < t_{11} < t_{12} < \ldots \leq b \), and \( a < t_{21} < t_{22} < \ldots \leq b \). Let

\( \lim_{n \to \infty} t_{1n} = \beta_1 \) and \( \lim_{n \to \infty} t_{2n} = \beta_2 \). The constants \( \beta_1 \) and \( \beta_2 \) differ, for suppose \( \beta_1 = \beta_2 \). Then there exists \( t_{1n} < t_{2n} \), such that

\( y'(t_{1n}, \tau_n) = 0 \); this follows from Rolle's theorem. Clearly

\( \lim_{n \to \infty} t_{1n} = \beta_1 \) and \( 0 = \lim_{n \to \infty} y'(t_{1n}, \tau_n) = y' \left( \beta_1, \mu_i \right) \). By the continuity of \( y'(x, \lambda) \). From the continuity of \( y(x, \lambda) \), we have

\( 0 = \lim_{n \to \infty} y(t_{1n}, \tau_n) = y(\beta_1, \mu_i) \). The statement \( y(\beta_1, \mu_i) = y'(\beta_1, \mu_i) = 0 \) implies a trivial solution, contrary to the assumption that the
solution is non-trivial. Hence, \( \beta_1 \neq \beta_2 \), \( N(\mu_1) = 2 \), and \( \mu_0 \neq \mu \).

Let \( \{\rho_n\} \) be an increasing sequence such that \( \mu_0 < \rho_n < \mu \), and 
\[
\lim_{n \to \infty} \rho_n = \mu_i. 
\]
Clearly, \( N(\rho_n) = 1 \) for all \( n \). If \( r_n \) is the zero of \( y(x, \rho_n) \) in \( (a, b) \), then, since \( y(a, \rho_n) > 0 \), we have 
\[
y(x, \rho_n) < 0 \quad \text{for} \quad r_n < x \leq b. 
\]
Also 
\[
0 \geq \lim_{n \to \infty} y(x, \rho_n) = y(x, \mu_i) = y(x, \mu_j) \quad \text{for} \quad r_n < x \leq b. 
\]
The sequence \( \{r_n\} \) is a monotone, decreasing, bounded sequence so there exists a point \( \gamma' \), \( a < \gamma' < b \), such that 
\[
\lim_{n \to \infty} r_n = \gamma'. 
\]
Then 
\[
0 = \lim_{n \to \infty} y(r_n, \rho_n) = y(\gamma', \mu_i). 
\]
There exists no zero of \( y(x, \mu_i) \) in \( (a, \gamma') \) nor \( (\gamma', b) \). For if \( \delta \) is in \( (a, \beta) \) and \( y(\delta, \mu_i) = 0 \), then 
\[
y(x, \mu_i) \geq 0 \quad \text{for} \quad a < x \leq \delta \quad \text{and} \quad y(\delta, \mu_i) = 0 \quad \text{imply} \quad y'(\delta, \mu_i) = 0, 
\]
contrary to the assumption of a non-trivial solution. Suppose there exists an \( \varepsilon \) in \( (\gamma', b) \) such that \( y(\varepsilon, \mu_i) = 0 \). Then there exists a \( \gamma' \) such that \( \gamma' \) is in \( (\gamma', \varepsilon) \). Further, there exists an \( M \) such that if \( n > M \), then 
\[
\begin{align*}
\gamma' < r_n < \gamma, \\
y(x, \rho_n) \text{ has no zeros in } (\gamma', b], \quad \text{and} \\
y(x, \rho_n) < 0 \quad \text{for} \quad r_n < x \leq b. 
\end{align*}
\]
Thus, if \( n > M \), \( y(x, \mu_i) \leq 0 \) for \( \gamma' \leq x \leq b \) and \( y(\varepsilon, \mu_i) = 0 \), contradicting the assumption of a non-trivial solution. Hence, 
\[
N(\mu_i) \leq 2. 
\]
Combining \( N(\mu_i) \leq 2 \) with \( N(\mu_j) \leq 2 \) and observing that 
\[
y(x, \mu_i) \neq 0 \quad \text{for} \quad a < x < \gamma \quad \text{or} \quad \gamma' < x < b, 
\]
we arrive at the conclusion that 
\[
N(\mu_i) = 2 \quad \text{and} \quad y(b, \mu_i) = 0. 
\]
Proceed by induction. Let \( A_1 \) be the set of all real numbers \( \lambda \) such that \( N(\lambda) = i \), \( i = 0, 1, 2, \ldots \). Suppose \( k \) is a non-negative integer. Suppose that \( A_i \) is non-empty and bounded for \( i = 1, 2, \ldots, k \). Suppose also that if \( \mu_1 = l.u.b.A_1 \), \( i = 0, 1, 2, \ldots, k \), then 
\[
\mu_0 < \mu_1 < \mu_2 < \cdots < \mu_k, \quad N(\mu_i) = i + 1, \quad y(b, \mu_i) = 0, 
\]
i = 0, 1, 2, \ldots, k. Then it remains to be shown that \( A_{k+i} \) is non-empty.
and bounded, and that if $\mu_{k+1} = \text{l.u.b.} A_{k+1}$ then

$\mu_k < \mu_{k+1}$, $N(\mu_{k+1}) = k + 2$, and $y(b, \mu_{k+1}) = 0$. This is established in a manner entirely similar to the above discussion.

Now it will be established that the system

$$y'' + (\lambda - q(x)) y = 0$$

$y(a) = 1$, $y'(a) = h$, $y'(b) - Hy(b) = 0$

has a solution $y(x, \lambda)$ for certain values for $\lambda$. The existence of a solution to

$$y'' + (\lambda - q(x)) y = 0, \quad y(a) = 1, \quad y'(a) = h,$$

for every $\lambda$, has been established in theorem (1.1).

Now $y(b, \lambda) = 0$ if and only if $\lambda = \mu_i$, $i = 0,1,2, \ldots$.

Define $F(\lambda) = \frac{y'(b, \lambda)}{y(b, \lambda)}$ for $\lambda \neq \mu_i$, $i = 0,1,2, \ldots$.

**Theorem 2.10.** $F(\lambda)$ is a continuous, strictly decreasing function in each of the intervals, $(-\infty, \mu_0)$, $(\mu_i, \mu_{i+1})$, $i = 0,1,2, \ldots$, such that

$$\lim_{\lambda \to \mu_1^+} F(\lambda) = +\infty, \quad \lim_{\lambda \to \mu_1^-} F(\lambda) = -\infty, \quad \lim_{\lambda \to -\infty} F(\lambda) = +\infty$$

$i = 0,1,2, \ldots$.

**Proof.** Let $\lambda_1$ and $\lambda_2$ be two real numbers such that $\mu_1 < \lambda_1 < \lambda_2 < \mu_{i+1}$, $i = 0,1,2, \ldots$. From the Second Comparison Theorem, we have $F(\lambda_1) > F(\lambda_2)$ and, further, because of the continuity of $y(x, \lambda)$ and $y'(x, \lambda)$ in $(\mu_i, \mu_{i+1})$, $i = 0,1,2, \ldots$, $F(\lambda)$ is continuous in the same intervals. Let

$$\sigma_{in}$$

be a decreasing sequence of real numbers such that

$$\mu_1 < \sigma_{in} < \mu_{i+1}, \quad i = 0,1,2, \ldots, \quad n = 1,2,3, \ldots,$$
\[
\lim_{n \to \infty} \sigma_{in} = \mu_i, \ i = 0,1,2, \ldots \quad \text{Now if } i \text{ is even,}
\]
\[
y'(b, \sigma_{in}) < 0 \text{ and } y(b, \sigma_{in}) < 0 \text{ for large } n; \text{ if } i \text{ is odd, the}
\]
inequalities are reversed. But in either case \(F(\sigma_{in}) > 0\) and
\[
\lim_{n \to \infty} F(\sigma_{in}) = +\infty, \ i = 0,1,2, \ldots \quad \text{Hence } \lim_{\lambda \to \mu_i} F(\lambda) = +\infty
\]
for \(i = 0,1,2, \ldots \). Similarly, if we define an increasing sequence
\[
\{\tau_{in}\}
\]
of real numbers such that
\[
\mu_i < \tau_{in} < \mu_{i+1}, \ i = 0,1,2, \ldots, \ n = 1,2,3, \ldots, \text{ and}
\]
\[
\lim_{n \to \infty} \tau_{in} = \mu_{i+1}, \ i = 0,1,2, \ldots, \text{ then } \lim_{n \to \infty} F(\tau_{in}) = -\infty,
\]
\(i = 0,1,2, \ldots \). Hence \(\lim_{\lambda \to \mu_i} F(\lambda) = -\infty\), for \(i = 0,1,2, \ldots \).

Also, in a similar manner it can be shown that \(\lim_{\lambda \to \mu_0} F(\lambda) = -\infty\).

Consider the case where \(\lambda < \mu_0\). Let \(m_1 < m = \min q(x)\) for
\(a \leq x \leq b\), and compare the system
\[
y'' + (\lambda - q(x)) y = 0, \ y(a) = 1, \ y'(a) = h
\]
with
\[
y'' + (\lambda - m_1) y = 0, \ y(a) = 1, \ y'(a) = h.
\]
We have shown already that if \(\lambda < m_1 - h^2\), then the solution corresponding to the last system has no zeros anywhere and the same is true for the solution corresponding to the first system. Thus, from
the Second Comparison Theorem,
\[
F(\lambda) = \frac{y'(b, \lambda)}{y(b, \lambda)} > \frac{y'(b)}{y(b)} ,
\]
where \(y(x, \lambda)\) and \(y(x)\) are the solutions to the first and second
systems, respectively, defined at the beginning of this paragraph.

But
\[
y(x) = \cosh \sqrt{m_1 - \lambda} (x-a) + \frac{h}{\sqrt{m_1 - \lambda}} \sinh \sqrt{m_1 - \lambda} (x-a), \quad \text{so}
\]
\[
\frac{y'(b)}{y(b)} = \frac{\sqrt{m, -\lambda} \sinh \sqrt{m, -\lambda} (b-a) + h \cosh \sqrt{m, -\lambda} (b-a)}{\cosh \sqrt{m, -\lambda} (b-a) + h \sinh \sqrt{m, -\lambda} (b-a)}
\]
\[
= \frac{\sqrt{m, -\lambda} \tanh \sqrt{m, -\lambda} (b-a) + h}{1 + h \tanh \sqrt{m, -\lambda} (b-a)}
\]

From the above, it is immediately apparent that \(\lim_{\lambda \to -\infty} \frac{y'(b)}{y(b)} = +\infty\).

Thus, since \(F(\lambda) = \frac{y'(b, \lambda)}{y(b, \lambda)} > \frac{y'(b)}{y(b)}\), \(\lim_{\lambda \to -\infty} F(\lambda) = +\infty\).

This completes the proof of the theorem.

Observe that since \(F(\lambda)\) is a continuous, strictly decreasing function on \((-\infty, M_0)\) and \((M_i, M_{i+1})\), \(i = 0, 1, 2, \ldots\), there exists a sequence \(\{\lambda_n\}\) of real numbers such that

\(\lambda_0 < M_0 < M_1 < \lambda_1 < M_{i+1}, \quad i = 0, 1, 2, \ldots\), and

\(F(\lambda_i) = H,\) for \(i = 0, 1, 2, \ldots\). Also, if \(\lambda \neq \lambda_i, \quad i = 0, 1, 2, \ldots\), then \(F(\lambda) \neq H\). But this is just the second boundary condition

\(y'(b) - H y(b) = 0;\) hence, the existence of a solution to

\(y'' + (\lambda - q(x)) y = 0,\)
\(y(a) = 1, \quad y'(a) = h, \quad y'(b) - H y(b) = 0,\)

and also to the system

\(y'' + (\lambda - q(x)) y = 0,\)
\(y'(a) - h y(a) = 0, \quad y'(b) - H y(b) = 0,\) for exactly one \(\lambda\),

\(\lambda < M_0\) or \(M_i < \lambda < M_{i+1},\) \(i = 0, 1, 2, \ldots\), is established. The values of \(\lambda\) for which the above system has a solution, that is, \(\lambda_0, \lambda, \lambda_2, \ldots\), are called eigenvalues, and the corresponding solutions, \(y(x, \lambda_0), y(x, \lambda_1), \ldots,\) are the eigenfunctions of the system.
CHAPTER III

ESTIMATES FOR $\lambda_n$ AND $u_n(x)$.

Section I

Consider the systems

(i) $y'' + (\lambda - q(x)) y = 0$

$y(a) = 1, y'(a) = h, y'(b) - H y(b) = 0,$

and

(ii) $y'' + (\lambda - q(x)) y = 0$

$y'(a) - h y(a) = 0, y'(b) - H y(b) = 0.$

Any solution to (i) will certainly be a solution to (ii). Suppose $u(x)$ is a non-trivial solution of (ii). Then it necessarily follows that $u(a) \neq 0$, and it is apparent that $\frac{u(x)}{u(a)}$ will be a solution to (i). Hence, any non-trivial solution to (ii) is a constant multiple of a solution to (i). Precisely the same remark apply to the systems

$y'' + (\lambda - q(x)) y = 0, y(a) = 1, y'(a) = h$

and

$y'' + (\lambda - q(x)) y = 0, y'(a) - h y(a) = 0.$

Definition 3.1. Two functions $f(x)$ and $g(x)$ are orthogonal on $[a,b]$ if $\int_a^b f(x)g(x)dx = 0$. They are normal or normalized on $[a,b]$ if $\int_a^b [f(x)]^2 dx = 1$ and $\int_a^b [g(x)]^2 dx = 1$.

Two eigenfunctions corresponding to the same eigenvalue are linearly dependent. For, suppose $u$ and $v$ are two eigenfunctions corresponding to the same eigenvalue $\lambda$. Then $u$ and $v$ must satisfy
the following conditions:
\[ u'' + (\lambda - q(x)) u = 0, \quad u'(a) - h u(a) = 0, \quad u'(b) - H u(b) = 0. \]
and
\[ v'' + (\lambda - q(x)) v = 0, \quad v'(a) - h v(a) = 0, \quad v'(b) - H v(b) = 0. \]

Consider the Wronskian of \( u \) and \( v \) evaluated at \( x = a \).
Then \( W(u,v) = u'(a) v(a) - v'(a) u(a) = 0 \), after applying the boundary conditions. Hence, \( u \) and \( v \) are linearly dependent. Thus, there is essentially only one eigenfunction associated with an eigenvalue.

**Theorem 3.1.** Two eigenfunctions corresponding to distinct eigenvalues of the system
\[ y'' + (\lambda - q(x)) y = 0 \]
\[ y'(a) - h y(a) = 0, \quad y'(b) - H y(b) = 0 \]
are orthogonal on \([a,b]\).

**Proof.** Suppose \( m \neq n \). Multiplying the equation
\[ y''(x, \lambda_n) + (\lambda_n - q(x)) y(x, \lambda_n) = 0 \]
by \( y(x, \lambda_n) \) and
\[ y''(x, \lambda_m) + (\lambda_m - q(x, \lambda_m) = 0 \]
substituting and integrating, we have
\[
(\lambda_m - \lambda_n) \int_a^b y(x, \lambda_n) y(x, \lambda_m) \, dx
= \int_a^b \left[ y''(x, \lambda_n) y(x, \lambda_m) - y''(x, \lambda_m) y(x, \lambda_n) \right] \, dx
= \int_a^b \frac{d}{dx} \left[ y'(x, \lambda_n) y(x, \lambda_m) - y'(x, \lambda_m) y(x, \lambda_n) \right] \, dx
= \left[ y'(x, \lambda_n) y(x, \lambda_m) - y'(x, \lambda_m) y(x, \lambda_n) \right] \bigg|_a^b
= 0,
\]
after applying the boundary conditions.

Theorem 3.2. Every eigenvalue of the system
\[ y'' + (\lambda - q(x))y = 0, \]
y(a) = 1, y'(a) = h, y'(b) - Hy(b) = 0
is real.

Proof. Suppose there exists a \( \lambda_j \) such that
\[ \lambda_j = \sigma + i\tau, \neq 0. \]
Then, it necessarily follows that the corresponding eigenfunction has the form \( y(x, \lambda_j) = s + it, t \neq 0 \), where \( s \) and \( t \) are real-valued, continuous functions. If this were not the case, we observe upon writing the equation
\[ y'' + (\lambda_j - q(x))y = 0 \]
in the form \( \lambda_j y = -y'' + q(x)y \)
that the left side is non-real for at least one value of \( x \), while the right side is real for all values of \( x \). This is a contradiction; hence, \( y(x, \lambda_j) \) cannot be real for all \( x \). It must also satisfy the system under consideration so that
\[ \frac{d^2 s}{dx^2} + \frac{d^2 t}{dx^2} + \int [\sigma + i\tau - q(x)] [s + it] = 0. \]
Equating real and complex parts to zero yields
\[ \frac{d^2 s}{dx^2} + [\sigma - q(x)] s - \tau t = 0 \]
and
\[ \frac{d^2 t}{dx^2} + [\sigma - q(x)] t + \tau s = 0. \]
Applying the boundary conditions gives
\[ s(a) = 1, s'(a) = h, t(a) = t'(a) = 0, \text{ and } \]
\[ s'(b) - Hs(b) = t'(b) - Ht(b) = 0. \]
Also, there will exists a \( \lambda_k \) such that \( \lambda_k = \sigma - i\tau, \tau \neq 0 \),
and the corresponding eigenfunction has the form

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y(x, \lambda_K) = s - t, t \neq 0. This is readily verified by utilizing the immediately preceding results. Observe that

\[ \lambda_k = \lambda_j, \quad y(x, \lambda_j) = y(x, \Lambda_j). \]

Consider

\[ y''(x, \lambda_j) + (\lambda_j - q(x)) y(x, \lambda_j) = 0 \]

and

\[ y''(x, \Lambda_j) + (\Lambda_j - q(x)) y(x, \Lambda_j) = 0. \]

Applying a familiar technique, multiplying both equation by suitable terms, subtracting, and integrating, we have

\[ \int_a^b y'(x, \lambda_j) y(x, \Lambda_j) - y'(x, \Lambda_j) y(x, \lambda_j) \ dx = (\Lambda_j - \lambda_j) \int_a^b y(x, \lambda_j) y(x, \lambda_j) \ dx. \]

The left side is zero upon applying the boundary conditions, and thus

\[ 0 = \int_a^b y(x, \Lambda_j) y(x, \lambda_j) \ dx = \int_a^b (s^2 + t^2) \ dx, \]

since \( \lambda_j \neq \Lambda_j \). But \( s \) and \( t \) are continuous, real-valued functions, therefore, we must have \( s \equiv t \equiv 0 \) on \([a, b]\).

This is a contradiction, since every eigenfunction is non-trivial.

**Theorem 3.3.** Suppose \( u \) and \( v \) are real-valued and non-trivial solutions of the systems

\[ u'' - G_1 u = 0, \quad u'(b) - H u(b) = 0 \]

and

\[ v'' - G_2 v = 0, \quad v'(b) - H v(b) = 0, \]

where \( G_1 \) and \( G_2 \) are continuous, real-valued functions, \( G_1 \geq G_2 \), and \( G_1 \neq G_2 \) on any subinterval of \([a, b]\). If \( p \) is the last
zero of $u$ in $[a, b]$, and if $q$ is the last zero of $v$ in $[a, b]$, then $p < q$.

**Proof.** Suppose $v$ has no zeros in $(p, b)$. Observe that the non-triviality of $u$ and $v$ implies $u(b) \neq 0$, $v(b) \neq 0$. Multiplying the first differential equation by $v$ and the second by $u$, subtracting the resulting equations, and integrating from $p$ to $b$ yields

$$\int_{p}^{b} (u'v - v'u)dx = (u'v - v'u)_{|p}^{b} = \int_{p}^{b} (G_1 - G_2) uv dx.$$ 

Consider four cases for $p < x < b$:

1. $u > 0$, $v > 0$,
2. $u > 0$, $v < 0$,
3. $u < 0$, $v > 0$,
4. $u < 0$, $v < 0$.

For cases (1) and (4), $(u'v - v'u)_{|p}^{b} \neq 0$, while

$$\int_{p}^{b} (G_1 - G_2) uv dx > 0;$$

for cases (2) and (3), $(u'v - v'u)_{|p}^{b} \geq 0$, while

$$\int_{p}^{b} (G_1 - G_2) uv dx < 0.$$ 

In all four cases a contradiction arises; hence, the theorem is established.

**Section II**

In the sequel the interval $[a, b]$ will be replaced by $[q, \pi]$. We suppose $m_1 < m = \min. q(x)$, and $M_1 > M = \max. q(x)$ for $0 \leq x \leq \pi$. An estimate for $\lambda_n$ will now be given.

Assume that $n = 2$, $n > 1 + 3|H|$, $n > 1 + 3|H|$, and $n > \frac{3}{8} - 2m_1$. From theorem (2.8), we have, for $n \geq 1$,

$$m + (n-1)^2 \leq \lambda_n \leq M + (n+1)^2.$$ 

Consider the system
(i) \( y'' + (\lambda_n - q(x))y = 0 \), \( y(0) = 1 \), \( y'(0) = h \), \( y'(\pi) - H y(\pi) = 0 \) 
whose solution is designated by \( y(x, \lambda_n) \), and the system 

(ii) \( y'' + (\lambda_n - m_1) y = 0 \), \( y(0) = 1 \), \( y'(0) = h \). 

Since \( \lambda_n > m_1 \), the solution to (ii) is 

\[ y(x) = \cos \sqrt{\lambda_n - m_1} x + \frac{h}{\sqrt{\lambda_n - m_1}} \sin \sqrt{\lambda_n - m_1} x. \]

Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( \beta_1, \beta_2, \ldots, \beta_n \) be the zeros of \( y(x, \lambda_n) \) and \( y(x) \), respectively, on \([0, \pi]\). From the First Comparison Theorem, \( \alpha_1 > \beta_1 \). Since \( \beta_1 \) is a zero of \( y(x) \) on \([0, \pi]\), 

\[ 0 = y(\beta_1) = \cos \sqrt{\lambda_n - m_1} \beta_1 + \frac{h}{\sqrt{\lambda_n - m_1}} \sin \sqrt{\lambda_n - m_1} \beta_1, \]

and 

\[ \left| \tan \sqrt{\lambda_n - m_1} \beta_1 \right| = \frac{\sqrt{\lambda_n - m_1}}{|h|}. \]

The conditions \( n > 1 + 3/|h| \) and \( \lambda_n \geq m + (n-1)^2 \) yield \( \left| \tan \sqrt{\lambda_n - m_1} \beta_1 \right| > 3 \). From considerations of the graph of 

\[ \tan \sqrt{\lambda_n - m_1} x, \]

we arrive at \( \beta_1 > \frac{3}{8} \sqrt{\lambda_n - m_1} \); from 

\[ \beta_n = \beta_1 + \frac{(n-1)\pi}{\sqrt{\lambda_n - m_1}} \]

and \( \alpha_n > \beta_n \), we conclude that \( \alpha_n' > (n-\frac{5}{8}) \sqrt{\lambda_n - m_1} \).

Now consider the system 

\( y'' + (\lambda_n - m_1) y = 0, y(\pi) = 1, y'(\pi) = H \) 

whose solution is 

\[ y(x) = \cos \sqrt{\lambda_n - m_1} (\pi - x) - \frac{H}{\sqrt{\lambda_n - m_1}} \sin \sqrt{\lambda_n - m_1} (\pi - x), \]

for \( \lambda_n > m_1 \), and has zeros \( \gamma_1, \gamma_2', \gamma_3', \ldots, \gamma_n' \) in \([0, \pi]\). 

Suppose the last zero is at \( \gamma_j' \). From theorem (3.3) \( \alpha_n < \gamma_j' < \pi \), and since \( \gamma_j' \) is a zero,
0 = y(\frac{\pi}{2}) = \cos \sqrt{\lambda_n - m} \left( \pi - \frac{\pi}{2} \right) - \frac{H}{\sqrt{\lambda_n - m}} \sin \sqrt{\lambda_n - m} \left( \pi - \frac{\pi}{2} \right).

Hence,
\[
\tan \sqrt{\lambda_n - m} \left( \pi - \frac{\pi}{2} \right) = \frac{\sqrt{\lambda_n - m}}{H} \quad \text{and} \quad \left( \pi - \frac{\pi}{2} \right) > \frac{3}{8} \sqrt{\lambda_n - m}
\]
after applying the argument used in showing that \( \beta > \frac{3}{8} \sqrt{\lambda_n - m} \).

From \( n > \frac{3}{8} - 2m \), and the estimates for \( \alpha_n \) and \( \pi - \frac{\pi}{2} \), it follows that \( \pi > (\pi - \frac{\pi}{2}) + \alpha_n > (n^{-\frac{1}{4}}) \frac{\pi}{\sqrt{\lambda_n - m}} \)
and consequently \( \lambda_n > (n^{-\frac{1}{2}})^2 \).

Now assume that \( n > 2M - \frac{3}{8}, n > 1 + \sqrt{M - m} + 9h^2 \), and \( n > 1 + \sqrt{M - m} + 9H^2 \). Consider the system
\[
y'' + (\lambda_n - q(x))y = 0, y(0) = 1, y'(0) = h, y'(\pi) - Hy(\pi) = 0
\]
whose solution, \( y(x, \lambda_n) \), has zeros \( \sigma_1, \sigma_2, \ldots, \sigma_n \), in \([0, \pi] \), and the system
\[
y'' + (\lambda_n - M) y = 0, y(0) = 1, y'(0) = h
\]
whose solution is
\[
y(x) = \cos \sqrt{\lambda_n - M} x + \frac{h}{\sqrt{\lambda_n - M}} \sin \sqrt{\lambda_n - M} x,
\]
and has zeros \( \tau_1, \tau_2, \ldots, \tau_k \), \( k \leq n \), on \([0, \pi] \). Two cases are considered. If \( n > k \), then \( \tau_{k+1} < \pi \);
\[
\tau_{k+1} = \tau_k + \frac{k \pi}{\sqrt{\lambda_n - M}} < \frac{(k+1) \pi}{\sqrt{\lambda_n - M}} \leq \frac{n \pi}{\sqrt{\lambda_n - M}} \quad \text{since}
\]
\[
\tau_k < \frac{\pi}{\sqrt{\lambda_n - M}}.
\]
Thus, \( \lambda_n < M + n^2 < (n + \frac{1}{2})^2 \) for
\[
n > 2M - \frac{3}{8}. \quad \text{If} \quad k = n, \text{then, from} \quad n > 1 + \sqrt{M - m} + 9h^2,
\]
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\[ \lambda_n > m_1 + (n - 1)^2, \] and
\[ 0 = y'(\tau) = \cos \sqrt{\lambda_n - M_1} \tau + \frac{h}{\sqrt{\lambda_n - M_1}} \sin \sqrt{\lambda_n - M_1} \tau, \]
we obtain \[ \frac{3}{8} \sqrt{\frac{\pi}{\lambda_n - M_1}} \leq \tau < \frac{5}{8} \sqrt{\frac{\pi}{\lambda_n - M_1}}. \]

Finally, consider the system
\[ y'' + (\lambda_n - M_1) y = 0, \quad y(\pi) = 1, \quad y'(\pi) = H \]
whose solution is
\[ y(x) = \cos \sqrt{\lambda_n - M_1} (\pi - x) - \frac{H}{\sqrt{\lambda_n - M_1}} \sin \sqrt{\lambda_n - M_1} (\pi - x), \]
for \( \lambda_n > M_1 \), and has zeros \( \rho_1, \rho_2, \rho_3, \ldots, \rho_n \) in \([0, \pi]\).
Suppose \( \rho_n \) is the last zero in \([0, \pi]\). Applying theorem (3.3) and using the inequalities \( \lambda_n > m_1 + (n-1)^2 \) and \( n > 1 + \sqrt{M_1 + m_1 + 9H^2} \),
we obtain \( \rho_n < x_n < \pi \).
\[ 0 = y(\rho_n) = \cos \sqrt{\lambda_n - M_1} (\pi - \rho_n) - \frac{H}{\sqrt{\lambda_n - M_1}} \sin \sqrt{\lambda_n - M_1} (\pi - \rho_n), \]
\[ \tan \sqrt{\lambda_n - M_1} (\pi - \rho_n) = \frac{\sqrt{\lambda_n - M_1}}{H}, \]
and
\[ \frac{\sqrt{\lambda_n - M_1}}{H} > 3. \] Since \( \tan \sqrt{\lambda_n - M_1} (\pi - \rho_n) > 3 \), then
\[ \frac{3}{8} \sqrt{\frac{\pi}{\lambda_n - M_1}} < \pi - \rho_n < \frac{5}{8} \sqrt{\frac{\pi}{\lambda_n - M_1}}. \] From the inequalities
\[ \pi - \sigma_n < \pi - \rho_n, \quad \sigma_n \leq \tau_n = \tau + n - 1, \]
\[ \tau < \frac{5}{8} \sqrt{\frac{\pi}{\lambda_n - M_1}}, \]
it follows that
\[ \pi - (\frac{5}{8} + n - 1) \sqrt{\frac{\pi}{\lambda_n - M_1}} < \pi - \sigma_n < \frac{5}{8} \sqrt{\frac{\pi}{\lambda_n - M_1}}, \] solving.
this last inequality for $\lambda_n$, we arrive successively at

$$\pi < \frac{(n + \frac{1}{2})\pi}{\sqrt{\lambda_n - M_1}}, \quad \lambda_n < M_1 + (n + \frac{1}{2})^2,$$

and finally,

$$\lambda_n < (n + \frac{1}{2})^2,$$

after observing that $n > 2M_1 - \frac{3}{8}$.

Section III

Definition 3.2. The notation $a_n = (b_n)$ means there exists a positive integer $N$ and a positive number $K$ such that $|a_n| \leq K |b_n|$ whenever $n > N$. Similarly $f_n(x) = O(g_n(x))$ means there exists a positive integer $N$ and a positive integer $K$ such that $|f_n(x)| \leq K |g_n(x)|$ for all $x$ in the domain of definition of $f_n(x)$ and $g_n(x)$ whenever $n > N$.

Definition 3.3. A function $f(x)$ defined on $[a, b]$ is of bounded variation on $[a, b]$ if there exists a positive number $K$ such that if $a = x_0 < x_1 < \ldots < x_n = b$ is an arbitrary subdivision of $[a, b]$, then

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < K.$$

A simple and useful criterion for bounded variation is the following: A function $f(x)$ defined on $[a, b]$ is of bounded variation there if and only if it is representable as the difference of two monotone increasing functions on $[a, b]$.

Section IV

In the subsequent material a better estimate for $\lambda_n$ is given, where $\lambda_n$ is the $n+1^{st}$ eigenvalue of the system given below. The system
\[ y'' + (\lambda - q(x))y = 0 \]
\[ y(0) = 1, \quad y'(0) = h, \quad y'(\pi) - H y(\pi) = 0, \]

where \( q(x) \) is a real-valued, continuous function of bounded variation on \([0, \pi]\), has a solution only for certain values of \( \lambda \). The solution of the above differential equation subject to the initial conditions \( y(0) = 1 \) and \( y'(0) = h \) is, for \( \lambda > 0 \),

\[ y(x, \lambda) = \cos \sqrt{\lambda} x + \frac{h}{\sqrt{\lambda}} \sin \sqrt{\lambda} x + \frac{1}{\sqrt{\lambda}} \int_{0}^{x} \sin \sqrt{\lambda}(x-t)q(t)y(t, \lambda)dt. \]

Imposing the second boundary condition \( y'(\pi) - H y(\pi) = 0 \) and applying Leibniz's Rule, gives

\[ 0 = y'(\pi) - H y(\pi) = \left[-\sqrt{\lambda} + \frac{h}{\sqrt{\lambda}} \right] - \int_{0}^{\pi} \left(\sin \sqrt{\lambda}t - \frac{H}{\sqrt{\lambda}} \cos \sqrt{\lambda}t\right)q(t)y(t, \lambda)dt \sin \sqrt{\lambda}t \]
\[ + \left[h - H + \int_{0}^{\pi} \left(\cos \sqrt{\lambda}t + \frac{H}{\sqrt{\lambda}} \sin \sqrt{\lambda}t\right)q(t)y(t, \lambda)\right] \cos \sqrt{\lambda}t. \]

Hence

\[ \tan \sqrt{\lambda} \pi = \frac{h - H + \int_{0}^{\pi} \left(\cos \sqrt{\lambda}t + \frac{H}{\sqrt{\lambda}} \sin \sqrt{\lambda}t\right)q(t)y(t, \lambda)dt}{\sqrt{\lambda} - \frac{H h}{\sqrt{\lambda}} + \int_{0}^{\pi} \left(\sin \sqrt{\lambda}t - \frac{H}{\sqrt{\lambda}} \cos \sqrt{\lambda}t\right)q(t)y(t, \lambda)dt}. \]

Since the system under consideration has a solution if and only if \( \lambda \) is an eigenvalue, the last equation is satisfied if and only if \( \lambda \) is an eigenvalue. The boundedness of \( y(t, \lambda_n) \) on \([0, \pi]\) can be inferred from its continuity on \([0, \pi]\). Suppose \( M_n \) is the least upper bound of \( y(x, \lambda_n) \) on \([0, \pi]\); then

\[ |y(x, \lambda_n)| \leq \left(1 + \frac{h^2}{\lambda_n}\right)^{\frac{1}{2}} + \frac{M_n}{\sqrt{\lambda_n}} \int_{0}^{\pi} |q(t)| dt, \]

this follows from the expression for \( y(x, \lambda_n) \), and finally,
\[ M_n \leq \left( 1 + \frac{h^2}{\lambda_n} \right)^{\frac{1}{2}} \left( 1 - \frac{1}{\sqrt{\lambda_n}} \int_0^\pi q(t) \, dt \right)^{-1} \]

for \( \lambda_n > \left[ \int_0^\pi |q(t)| \, dt \right]^2 \). Thus, \( M_n = 1 + 0 \left( \frac{1}{n} \right) \), and there exists a constant \( K \) such that for \( n > K \), \( |y(x, \lambda_n)| \leq 2 \) for all \( x \) in \([0, \pi]\).

In view of these last results, the expression for \( \tan(\sqrt{\lambda_n} \pi) \) is such that its numerator is of the order of \( \sqrt{\lambda_n} \). So, for sufficiently large \( n \), \( |\tan(\sqrt{\lambda_n} \pi)| < \frac{c}{n} \) for a suitable constant \( c \). Let

\[ \alpha_n = \sqrt{\lambda_n} - n. \]

Then, since \( |\lambda_n - n| < \frac{1}{2} \) for large \( n \), it follows that

\[ |\alpha_n| < \frac{1}{2} \text{ for large } n. \]

Hence,

\[ |\alpha_n \pi| \leq |\tan(\alpha_n \pi)| = |\tan((n+\alpha_n)\pi)| = |\tan(\sqrt{\lambda_n} \pi)| < \frac{c}{n} \]

for large \( n \). Thus, \( \alpha_n = O\left(\frac{1}{n}\right) \) and \( \sqrt{\lambda_n} = n + O\left(\frac{1}{n}\right) \).

With the aid of two easily established results,

\[ \cos(\sqrt{\lambda_n} \pi) x = \cos nx + O\left(\frac{1}{n}\right) \text{ and } \sin(\sqrt{\lambda_n} \pi) x = \sin nx + O\left(\frac{1}{n}\right), \]

we shall conclude that \( y(x, \lambda_n) = \cos nx + O\left(\frac{1}{n}\right) \).

Substituting the expressions for \( \lambda_n \), \( y(x, \lambda_n) \), \( \cos(\sqrt{\lambda_n} \pi) x \), and \( \sin(\sqrt{\lambda_n} \pi) x \) into the equation for \( \tan(\sqrt{\lambda_n} \pi) \) yields

\[ \tan(\sqrt{\lambda_n} \pi) = \frac{h - H + \int_0^\pi q(t) \cos^2 n t \, dt + O\left(\frac{1}{n}\right)}{n - \frac{1}{2} \int_0^\pi q(t) \sin 2n t \, dt + O\left(\frac{1}{n}\right)} = \frac{h - H + \frac{1}{2} \int_0^\pi q(t) dt + \frac{1}{2} \int_0^\pi q(t) \cos 2n t \, dt + O\left(\frac{1}{n}\right)}{n - \frac{1}{2} \int_0^\pi q(t) \sin 2n t \, dt + O\left(\frac{1}{n}\right)}. \]

Since \( q(t) \) is a real-valued, continuous function of bounded variation on \([0, \pi]\), we may apply the Second Mean Value Theorem for Integrals.
to \( \int_0^\pi q(t) \cos 2\pi t \, dt \) and \( \int_0^\pi q(t) \sin 2\pi t \, dt \)

to establish that these integrals are equal to \( O\left(\frac{1}{n}\right) \).

So finally, we conclude that

\[
\tan \frac{\sqrt{\lambda_n}}{\pi} = \frac{d}{n} + O\left(\frac{1}{n^2}\right), \quad \text{where} \quad d = h + \frac{1}{2} \int_0^\pi q(t) \, dt.
\]

By a process similar to the one employed in establishing that

\( a_n = O\left(\frac{1}{n}\right) \), we arrive at

\( a_n = \frac{d}{n\pi} + O\left(\frac{1}{n^2}\right) \),

and hence

\[
\sqrt{\lambda_n} = n + \frac{d}{n\pi} + O\left(\frac{1}{n^2}\right)
\]

and

\[
y(x, \lambda_n) = \cos nx + \frac{1}{n} \left[ -\frac{dx}{\pi} + h + \frac{1}{2} \int_0^x q(t) \, dt \right] \sin nx + O\left(\frac{1}{n^3}\right) \, .
\]

This last formula follows from

\[
y(x, \lambda_n) = \cos \frac{\sqrt{\lambda_n}}{\pi} x + \frac{h}{\sqrt{\lambda_n}} \sin \frac{\sqrt{\lambda_n}}{\pi} x + \frac{1}{\sqrt{\lambda_n}} \int_0^x \sin \frac{\sqrt{\lambda_n}}{\pi} (x-t)q(t)y(t, \lambda_n) \, dt,
\]

after \( \sqrt{\lambda_n} \) is replaced by its new value.

Now consider the normalization of the above orthogonal

eigenfunctions. The problem is to determine a constant \( c_n \), for each \( n \)
such that

\[
u(x, \lambda_n) = c_n^2 y(x, \lambda_n), \quad \int_0^\pi u(x, \lambda_n) \, dx = 1 \quad \text{and} \quad u(0, \lambda_n) > 0.
\]

\[
1 = \int_0^\pi [u(x, \lambda_n)]^2 \, dx \\
= c_n^2 \int_0^\pi [y(x, \lambda_n)]^2 \, dx \\
= c_n^2 \left[ \frac{\pi}{2} + O\left(\frac{1}{n^2}\right) \right]
\]

for \( n \neq 1 \), using
\begin{align*}
y(x, \lambda_n) &= \cos nx + \frac{1}{n} \left[ \frac{dx}{\pi} + h + \frac{1}{2} \int_0^x q(t) dt \right] \sin nx + O\left(\frac{1}{n^2}\right). \\
\end{align*}

It follows quite readily that
\begin{align*}
c_n &= \sqrt{\frac{2}{\pi}} + O\left(\frac{1}{n^2}\right) \\
u(x, \lambda_n) &= \sqrt{\frac{2}{\pi}} \left\{ \cos nx + \frac{1}{n} \left[ -\frac{dx}{\pi} + h + \frac{1}{2} \int_0^x q(t) dt \right] \sin nx \right\} + O\left(\frac{1}{n^2}\right)
\end{align*}
for \( n \geq 1 \). Also, observe that \( u(0, \lambda_n) > 0 \). The functions \( u(x, \lambda_n) \) form an orthonormal system on \([0, \pi]\).

Section V

The problem might arise as to whether a given function \( f(x) \) defined on \([0, \pi]\) is capable of being represented by a series of the form \( \sum_{n=0}^{\infty} c_n u(x, \lambda_n) \). If so, how must the constants \( c_n \) be defined, and what convergence properties does the series have? This problem is similar to the problem of expanding an arbitrary function into a Fourier series.

Suppose \( f(x) \) has a continuous first derivative of bounded variation on \([0, \pi]\). Consider \( \sum_{n=1}^{\infty} a_n u(x, \lambda_n) \),

where \( a_n = \int_0^\pi f(x) u(x, \lambda_n) dx \) and \( u(x, \lambda_n) \) is the \( n + 1 \)st normalized eigenfunction. For convenience, let
\[ \phi(x) = \frac{dx}{\pi} + h + \frac{1}{2} \int_0^x q(t) dt. \]

Now \( a_n = \int_0^\pi f(x) u(x, \lambda_n) dx \)
\begin{align*}
&= \sqrt{\frac{2}{\pi}} \left\{ \int_0^\pi \left[ f(x) \cos nx + \frac{f(x)\phi(x)}{n} \sin nx \right] dx \right\} + O\left(\frac{1}{n^2}\right), \\
\end{align*}
and
\[
\int_0^\pi f(x) \cos nx \, dx = -\frac{1}{n} \int_0^\pi f'(x) \sin nx \, dx
\]
for \( n \geq 1 \), after integrating by parts. Using the bounded variation property of \( f'(x) \), we may write \( f'(x) = \alpha(x) - \beta(x) \), where \( \alpha(x) \) and \( \beta(x) \) are positive monotone functions. Applying the Second Mean Value Theorem for Integrals yields
\[
\int_0^\pi f'(x) \sin nx \, dx = \alpha(0) \int_0^\pi \sin nx \, dx + \alpha(\pi) \int_0^\pi \sin nx \, dx
\]
- \( \beta(0) \int_0^\pi \sin nx \, dx - \beta(\pi) \int_0^\pi \sin nx \, dx = 0(\frac{1}{n}). \]
where \( \xi \) and \( \xi_2 \) are in \([0, \pi]\). Also
\[
\int_0^\pi f(x) \phi(x) \sin nx \, dx = -f(x)\phi(x) \cos nx \int_0^\pi \frac{\sin nx}{n} \, dx + \int_0^\pi f(x)\phi(x) \cos nx \, dx \]
= \( 0(\frac{1}{n}) \).

Thus, we conclude that \( a_n = 0(\frac{1}{n^2}) \). There exist constants \( K \) and \( M \) such that \( |a_n| \leq \frac{K}{n^2} \) for \( n \geq 1 \), and \( |u(x, \lambda_n)| \leq M \). The last inequality is valid, since \( u(x, \lambda_n) = c_n y(x, \lambda_n) \), where
\[
c_n = \sqrt{\frac{2}{\pi}} + O(\frac{1}{n^2}) \text{ and } y(x, \lambda_n) \text{ is uniformly bounded on } [0, \pi].
\]
\[
\sum_{n=1}^{\infty} \frac{KM}{n^2} \text{ is convergent, and, according to the Weierstrass Comparison Theorem, } \sum_{n=0}^{\infty} a_n u(x, \lambda_n) \text{ converges uniformly to a continuous function, say } g(x), \text{ on } [0, \pi].
\]
Thus
\[
a_n = \int_0^\pi f(x) u(x, \lambda_n) \, dx,
\]
and
\[ a_n = \int_0^\pi \sum_{i=0}^{\infty} a_i u(x, \lambda_i) \ u(x, \lambda_n) \, dx \]

\[ = \int_0^\pi g(x) \ u(x, \lambda_n) \, dx. \]

Hence, \( \int_0^\pi [f(x) - g(x)] \ u(x, \lambda_n) \, dx = 0 \) for all \( n \).

From the completeness property of the eigenfunctions with respect to the class of continuous functions on \([0, \pi]\), which will be established shortly, it will be possible to conclude

\( f(x) = g(x) \) for \( 0 \leq x \leq \pi \), and hence \( f(x) = \sum_{n=0}^{\infty} a_n u(x, \lambda_n) \), for \( 0 \leq x \leq \pi \).
CHAPTER IV

MINIMAL PROPERTY OF THE EIGENFUNCTIONS.

Let $C$ be the class of functions which have continuous second derivatives on $[0, \pi]$ and satisfy the conditions

$$y'(0) - hy(0) = 0, \quad y'(\pi) - hy(\pi) = 0,$$

and

$$\int_0^\pi [y'(x)]^2 \, dx = 1.$$

Every function of the class $C$ is called an admissible function.

Define $J(y)$ as follows:

$$J(y) = \int_0^\pi \left\{ [y'(x)]^2 + q(x) [y(x)]^2 \right\} \, dx - y'(x)y(x) \bigg|_0^\pi;$$

observe that this is also equivalent to

$$J(y) = \int_0^\pi y(x) \left[ -y''(x) + q(x) y(x) \right] \, dx.$$

One refers to $J(y)$ as a functional; by this, one means a function whose domain is the set of admissible functions and whose range is a set of real numbers. An important property is given by the following theorem.

**Theorem 4.1.** There exists a $K$ such that if $y$ is in $C$, then $J(y) \leq K$.

**Proof.** To establish a lower bound for $J(y)$, estimates for the terms

$$\int_0^\pi [y'(x)]^2 \, dx, \quad \int_0^\pi q(x) [y(x)]^2 \, dx, \text{ and } y'(x)y(x) \bigg|_0^\pi$$

will be given. Now $\int_0^\pi [y'(x)]^2 \, dx \geq 0$, and, from the First Mean Value Theorem for Integrals,

$$\left| \int_0^\pi q(x) [y(x)]^2 \, dx \right| = \left| q(\xi) \right| \int_0^\pi [y(x)]^2 \, dx \leq L,$$

where $0 \leq \xi \leq \pi$ and $L = \max \, |q(x)|$ for $0 \leq x \leq \pi$. Clearly

$$\left| \int_0^{\xi} y'(x) \, dx \right| = \left| y(\xi) - y(0) \right|.$$
Using the Schwarz Inequality, we have
\[ |y(0) - y(\xi)| \leq \left[ \xi \int_0^\xi (y'(x))^2 \, dx \right]^{\frac{1}{2}} \leq \left[ t \int_0^t (y'(x))^2 \, dx \right]^{\frac{1}{2}} \]
for \( 0 \leq \xi < t \leq \pi \), and
\[ |y(0)| \leq |y(\xi)| + \left[ t \int_0^t (y'(x))^2 \, dx \right]^{\frac{1}{2}}. \]

It is possible to choose \( \xi \) in \([0,t]\) such that \( |y(\xi)| \leq t^{-\frac{1}{2}} \). If this were not the case, then it would follow that \( y(x) > t^{-\frac{1}{2}} \) for \( 0 \leq x \leq t \) and
\[ \int_0^\pi (y(x))^2 \, dx > \int_0^t (y(x))^2 \, dx > 1, \]
but this is a contradiction. Thus,
\[ |y(0)| \leq t^{-\frac{1}{2}} + \left[ t \int_0^\pi (y'(x))^2 \, dx \right]^{\frac{1}{2}}. \]

If the integral in the brackets exceeds \( \pi^{-2} \), let \( t^{-1} = \left[ \int_0^\pi (y'(x))^2 \, dx \right]^{\frac{1}{2}} \); otherwise let \( t = \pi \). In the first case
\[ |y(0)| \leq 2 \left[ \int_0^\pi (y'(x))^2 \, dx \right]^{\frac{1}{4}}, \]
in the second case \( |y(0)| \leq 2 \pi^{-\frac{1}{2}} \). Considering the integral
\[ \int_\xi^\pi (y'(x))^2 \, dx \]
and applying an argument similar to the one above, we arrive at precisely the same results for \( y(\pi) \) instead of \( y(0) \).

Combining the various results obtained so far, one concludes that
\[ |H y^2(\pi) - h y^2(0)| \leq 4 \left[ |H| + |h| \right] \left[ \int_0^\pi (y'(x))^2 \, dx \right]^{\frac{1}{2}} \]
for \( \int_0^\pi (y'(x))^2 \, dx > \pi^{-2} \), and
\[ |H y^2(\pi) - h y(0)| \leq \frac{4}{\pi} \left[ |H| + |h| \right] \]
for \( \int_0^\pi (y'(x))^2 \, dx \leq \pi^{-2} \), where \( y'(x)y(x) \bigg|_0^\pi = H y^2(\pi) - h y^2(0) \).
since $y$ is in $C$. Thus

$$J(y) \geq -L - \frac{2}{r} \left[ |H| + |h| \right] \text{ for } \int_0^\pi \left[ y'(x) \right]^2 \, dx \leq \pi^2 - 2,$$

and

$$J(y) \geq \int_0^\pi \left[ y'(x) \right]^2 \, dx - L - 4 \left[ H + h \right] \left[ \int_0^\pi \left[ y'(x) \right]^2 \, dx \right]^{1/2}$$

for $\int_0^\pi \left[ y'(x) \right]^2 \, dx > \pi^2 - 2$. If $u = \int_0^\pi \left[ y'(x) \right]^2 \, dx$ in the above inequalities, then $J(y)$ becomes

$$J(y) \geq u^2 - L - 4 \left[ |H| + |h| \right] u$$

$$\geq - \left[ L + 4 \left( |H| + |h| \right) \right],$$

for $\int_0^\pi \left[ y'(x) \right]^2 \, dx > \pi^2 - 2$, after completing the square and discarding the non-negative part. Hence, in either case, $J(y)$ is bounded below.

**Theorem 4.2.** Consider the system

$$y'' + (\lambda - q(x))y = 0$$

$$y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0.$$

If $R = \text{g.l.b.} J(y)$, where the greatest lower bound is taken over all admissible functions, then $R = \lambda_0$, $\lambda_0$ being the smallest eigenvalue of the above system.

**Proof.** Note that $R \leq \lambda_0$ since $J(y_0) = \lambda_0$, where $y_0(x)$ is the normalized eigenfunction corresponding to $\lambda_0$. Suppose $R < \lambda_0$, then $R \neq \lambda_k$, $k = 0, 1, 2, \ldots$, where $\lambda_k$ is the $k + 1^{st}$ eigenvalue of the above system. Let $u_1(x), u_2(x), \ldots$ be admissible functions such that $J(u_n) < R + \frac{1}{n}$ and $\lim_{n \to \infty} J(u_n) = R$.

The results of theorem (4.1) are applicable to the functions $u_n(x)$, $n = 1, 2, 3, \ldots$, so that

$$\int_0^\pi \left[ u_n'(x) \right]^2 \, dx \leq (R + 1) + L + 4 \left[ |H| + |h| \right] \left[ \int_0^\pi \left[ u_n'(x) \right]^2 \, dx \right]^{1/2}$$
for \( \int_0^\pi [u_n'(x)]^2 \, dx > \pi^{-2} \); otherwise \( \int_0^\pi [u'(x)]^2 \, dx \leq \pi^{-2} \).

It is readily established that
\[
\left[ \int_0^\pi [u_n'(x)]^2 \, dx \right]^{\frac{1}{2}} \leq 4 \left[ |H| + |h| \right] + \left[ |R| + L + 1 \right]^{\frac{1}{2}}
\]
for \( \int_0^\pi [u_n'(x)]^2 \, dx > \pi^{-2} \).

The sequence of functions \( \{u_n(x)\} \) is uniformly bounded on \([0, \pi]\), for clearly, \([u_n'(x)]^2 + 1 \geq u_n'(x)\). From
\[
\left| u_n(x) - u_n(0) \right| = \left| \int_0^x u_n'(t) \, dt \right|
\]
it then follows that
\[
\left| u_n(x) - u_n(0) \right| \leq \int_0^\pi \left\{ [u_n'(x)]^2 + 1 \right\} \, dx
\]
\[
\leq \left[ 4 \left( |H| + |h| \right) + \left( |R| + L + 1 \right)^{\frac{1}{2}} \right]^2 + \pi
\]
for \( \int_0^\pi [u_n'(x)]^2 \, dx > \pi^{-2} \); otherwise \( \left| u_n(x) - u_n(0) \right| \leq \pi^{-2} + \pi \)
for \( 0 \leq x \leq \pi \). Let \( u_n(x) = c_n + \vartheta_n(x) \), where \( c_n = u_n(0) \)
and \( \vartheta_n(x) = u_n(x) - u_n(0) \). There exists a constant \( K \) such that
\[
\left| \vartheta_n(x) \right| \leq K \text{ for } 0 \leq x \leq \pi \text{ and for all } n.
\]
The sequence of constants \( \{c_n\} \) is also uniformly bounded. For, suppose this
were not so. Then there exists an \( m \) such that \( |c_m| \geq K + 2 \).
Since \( u_m(x) = c_m + \vartheta_m(x) \), \( |u_m(x)| \leq |c_m| - |\vartheta_m(x)| \), and hence
\[
|u_m(x)| > 2.
\]
This together with the normality of \( u_m(x) \) gives
\[
1 = \int_0^\pi [u_m(x)]^2 \, dx \geq 4 \pi,
\]
which is a contradiction. Hence, the sequence \( \{c_n\} \) is uniformly bounded. Consequently, the sequence
of functions \( \{u_n(x)\} \) is uniformly bounded for \( 0 \leq x \leq \pi \).

Define a sequence of functions \( \{f_n(x)\} \) by the equations
\[
u_n'' + (R - q(x)) u_n = f_n, \quad n = 1, 2, 3, \ldots
\]
Multiplying these equations by \( u_n \) and integrating from 0 to \( \pi \) yields
\[
\int_0^\pi u_n(x) \left[u_n''(x) + (R - q(x))u_n(x)\right] \, dx = -\int_0^\pi u_n(x)\left[-u_n''(x) + q(x)u_n(x)\right] \, dx + R
\]
\[ \int_0^\pi f_n(x)u_n(x)dx, \]

or equivalently,

\[ R - J(u_n) = \int_0^\pi f_n(x)u_n(x)dx. \]

Since \( \lim_{n \to \infty} J(u_n) = R, \lim_{n \to \infty} \int_0^\pi f_n(x)u_n(x)dx = 0. \)

Under the assumption that \( R \) is not an eigenvalue, there exists

for each \( n \) a solution to the system

\[ y'' + (R - q(x)) y = u_n, \]
\[ y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0. \]

This is established as follows: Suppose \( \alpha_1(x) \) and \( \alpha_2(x) \) are

linearly independent solutions of \( y'' + (R - q(x)) y = 0 \), then the

linear combination \( c_1 \alpha_1(x) + c_2 \alpha_2(x) \), for arbitrary constants \( c_1 \)

and \( c_2 \), is also a solution. Applying the boundary conditions

\[ y'(0) - h y(0) = 0 \text{ and } y'(\pi) - H y(\pi) = 0 \]

to

\[ c_1 \alpha_1(x) + c_2 \alpha_2(x), \]

we obtain a system of two homogeneous equations

in \( c_1 \) and \( c_2 \),

\[ c_1 \begin{bmatrix} \alpha_1'(0) - h \alpha_1(0) \\ \alpha_1'(\pi) - H \alpha_1(\pi) \end{bmatrix} + c_2 \begin{bmatrix} \alpha_2'(0) - h \alpha_2(0) \\ \alpha_2'(\pi) - H \alpha_2(\pi) \end{bmatrix} = 0, \]

which has a non-trivial solution if and only if the determinant

of the coefficients of \( c_1 \) and \( c_2 \) is zero; i.e.,

\[ \begin{vmatrix} \alpha_1'(0) - h \alpha_1(0) & \alpha_2'(0) - h \alpha_2(0) \\ \alpha_1'(\pi) - H \alpha_1(\pi) & \alpha_2'(\pi) - H \alpha_2(\pi) \end{vmatrix} = 0. \]

But the system

\[ y'' + (\lambda - q(x)) y = 0, \]
\[ y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0 \]

has a solution only for the eigenvalues; hence, if \( R \) is not an
eigenvalue, then \( c_1 = c_2 = 0 \) and the determinant must differ from zero.

The differential equation

\[
y'' + (R - q(x)) y = u_n
\]

has a solution given by

\[
U_n(x) = c_{n3} \phi_1(x) + c_{n4} \phi_2(x) + \int_0^x \left[ \frac{\phi_1(x) \phi_2(t) - \phi_2(x) \phi_1(t)}{W[\phi_2, \phi_1]} \right] u_n(t) \, dt,
\]

where \( c_{n3} \) and \( c_{n4} \) are arbitrary constants, \( \phi_1(x) \) and \( \phi_2(x) \) are linearly independent solutions to \( y'' + (R - q(x)) y = 0 \), and \( W[\phi_2, \phi_1] \) is the Wronskian of \( \phi_2(x) \) and \( \phi_1(x) \).

Applying the boundary conditions to the function \( U_n(x) \) gives

\[
c_{n3} \left[ \phi_1'(0) - h \phi_1(0) \right] + c_{n4} \left[ \phi_2'(0) - h \phi_2(0) \right] = 0,
\]

\[
c_{n3} \left[ \phi_1'(\pi) - H \phi_1(\pi) \right] + c_{n4} \left[ \phi_2'(\pi) - H \phi_2(\pi) \right] = H \int_0^\pi \left[ \frac{\phi_1(x) \phi_2(t) - \phi_2(x) \phi_1(t)}{W[\phi_2, \phi_1]} \right] u_n(t) \, dt - \int_0^\pi \left[ \frac{\phi_1(x) \phi_2(t) - \phi_2(x) \phi_1(t)}{W[\phi_2, \phi_1]} \right] u_n(t) \, dt.
\]

But the determinant of the coefficients of \( c_1 \) and \( c_2 \) is not zero, which is also true of the determinant of the coefficients of \( c_{n3} \) and \( c_{n4} \). Thus, it is possible to find values for \( c_{n3} \) and \( c_{n4} \), such that the function \( U_n(x) \) is a solution to the above system. The sequence of functions \( \{U_n(x)\} \) is uniformly bounded on \( [0, \pi] \), since the sequence \( \{u_n(x)\} \) is uniformly bounded and \( \{c_{n3}\}, \{c_{n4}\} \) are also uniformly bounded on \( [0, \pi] \). Define \( v_n \) by the equation

\[
v_n(x) = u_n(x) + c U_n(x),
\]

where \( c \) is a constant to be chosen shortly in an appropriate manner. The function \( v_n \) satisfies all defining conditions of the admissible functions except that of normality, which may or may not be satisfied.
That the function \( v_n \) is a solution of the system
\[
\begin{align*}
v''_n + (R - q(x)) v_n &= f_n + c u^n, \\
v'(0) - h v_n(0) &= 0, \quad v'(\pi) - H v_n(\pi) = 0,
\end{align*}
\]
follows from the differential equations
\[
\begin{align*}
u''_n + (R - q(x)) u_n &= f_n \quad \text{and} \quad U''_n + (R - q(x)) U_n = u_n.
\end{align*}
\]
Multiplying the above differential equation by \( v_n \) and integrating from 0 to \( \pi \) yields
\[
\int_0^\pi \left[ v_n v''_n + (R-q(x))v_n^2 \right] dx = \int_0^\pi \left[ v_n f_n + c v_n u_n \right] dx.
\]
Since
\[
\int_0^\pi \left[ v_n v''_n + (R-q(x))v_n^2 \right] dx = R \int_0^\pi v_n^2 dx - J(v_n),
\]
\[
R \int_0^\pi v_n^2 dx = \int_0^\pi \left[ v_n f_n + c v_n u_n \right] dx + J(v_n).
\]
Multiplying the equation
\[
u''_n + (R-q(x)) u_n = f_n \quad \text{by} \quad U_n
\]
and the equation
\[
U''_n + (R-q(x)) U_n = u_n \quad \text{by} \quad u_n,
\]
subtracting, and integrating from 0 to \( \pi \) yields
\[
\int_0^\pi \left[ U^n_0 f_n - u_n^2 \right] dx = 0
\]
Thus,
\[
R \int_0^\pi v_n^2 dx - J(v_n) = \int_0^\pi \left[ v_n f_n + c v_n u_n \right] dx
\]
\[
= \int_0^\pi \left[ u_n + c U_n \right] f_n dx + c \int_0^\pi \left[ u_n + c U_n \right] u_n dx
\]
\[
= \int_0^\pi f_n u_n dx + c \int_0^\pi \left[ U_n f_n - u_n^2 \right] dx + 2c \int_0^\pi u_n^2 dx + c^2 \int_0^\pi u_n U_n dx
\]
\[
= \int_0^\pi f_n u_n dx + 2c + c^2 \int_0^\pi u_n U_n dx.
\]
The sequence \( \left\{ \int_0^\pi u_n U_n dx \right\} \) is uniformly bounded, since \( \{ u_n \} \).
and \( \{ U_n \} \) are uniformly bounded sequences on \([0, \pi]\). Let \( \left\| \int_0^\pi u_n u'_n \, dx \right\| \leq B \), where B is a positive constant. Choose
\[
c = \frac{1}{B} \quad \text{and} \quad \delta = \frac{1}{B}.
\]
Since \( \lim_{n \to \infty} \int_0^\pi u_n f \, dx = 0 \), by definition there exists an \( N \) such that if \( n > N \), then 
\[
-\delta < \int_0^\pi u_n f \, dx < \delta.
\]
Thus,
\[
R \int_0^\pi v_n^2 \, dx - J(v_n) > 0 \quad \text{for} \quad n > N.
\]
Now \( \int_0^\pi v_n^2 \, dx > 0 \). For, if \( v_n \equiv 0 \) on \([0, \pi]\), then
\[
R \int_0^\pi v_n^2 \, dx - J(v_n) = 0,
\]
contrary to the inequality
\[
R \int_0^\pi v_n^2 \, dx - J(v_n) > 0.
\]
Let \( \beta_n = \left[ \int_0^\pi v_n^2 \, dx \right]^{\frac{1}{2}} \) and
\[
\mathcal{z}_n = \frac{v_n}{\beta_n}.
\]
The function \( \mathcal{z}_n \) is an admissible function and imparts to \( J \) a value less than or equal to \( R \), i.e.,
\[
R - \frac{J(v_n)}{\beta_n^2} = R - J(\mathcal{z}_n) \leq 0.
\]
This contradicts the statement that \( R = \text{g.l.b. } J(y) \) for all admissible \( y \); hence, it follows that \( R = \lambda_\circ \). This completes the proof of the theorem. For convenience let \( \mathcal{z}_i \) be the normal solution to the system under consideration, when \( \lambda = \lambda_i \), \( i = 0, 1, 2, \ldots \).

**Theorem 4.3.** Suppose \( n \) is a non-negative integer. Suppose \( v \) has a continuous second derivative on \([0, \pi]\),
\[
v'(0) = h v(0), \quad v'(\pi) - H v(\pi) = 0, \quad \int_0^\pi v^2 \, dx = 1, \quad \text{and} \quad \int_0^\pi v \mathcal{z}_i \, dx = 0, \quad i = 0, 1, 2, \ldots, n - 1,
\]
where \( \mathcal{z}_i \) is the \( i + 1 \)st normalized eigenfunction of the system
\[
y'' + (\lambda - q(x))y = 0, \quad y'(0) = h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0.
\]
Then $J(v) \geq J(z_n) = \lambda_n$.

**Proof.** The proof is by induction. It is true for $n = 0$, the orthogonality being vacuously satisfied. Hence, suppose $n$ is a positive integer. Suppose that if $v$ has a continuous second derivative on $[0, \pi]$, 

$v'(0) - h v(0) = 0, v'(\pi) - H v(\pi) = 0,$

$\int_0^\pi v^2 dx = 1, \text{ and } \int_0^\pi v z_i dx = 0, i = 0, 1, 2, \ldots, n - 1,$

then $J(v) \geq J(z_n) = \lambda_n$. Now suppose $u$ has a continuous second derivative on $[0, \pi]$, 

$u'(0) - h u(0) = 0, u'(\pi) - H u(\pi) = 0,$

$\int_0^\pi u^2 dx = 1, \text{ and } \int_0^\pi u z_i dx = 0, i = 0, 1, 2, \ldots, n.$

It must be shown that $J(u) \leq J(z_{n+1}) = \lambda_{n+1}$.

Let $R_{n+1} = \text{g.l.b } J(u)$, where the greatest lower bound is taken over all functions $u$ satisfying the above conditions; then

$\lambda_n \leq R_{n+1} \leq \lambda_{n+1}$. Suppose $R_{n+1} < \lambda_{n+1}$; then either

$R_{n+1} = \lambda_n$ or $R_{n+1}$ is not an eigenvalue. As in theorem (4,2), define a sequence of functions $\{u_m\}$ satisfying the above conditions for $u$ such that $J(u_m) < R_{n+1} + \frac{1}{m}$ and $\lim_{m \to \infty} J(u_m) = R_{n+1}$. 

Now define another sequence of functions $\{f_m\}$ by the equation $u_m'' + (R_{n+1} - q(x)) u_m = f_m$. It is readily established that

$R_{n+1} - J(u_m) = \int_0^\pi f_m u_m dx \text{ and } \lim_{m \to \infty} \int_0^\pi f_m u_m dx = 0.$

Consider the system

$y'' + (R_{n+1} - q(x)) y = u_m,$

$y'(0) - h y(0) = 0, y'(\pi) - H y(\pi) = 0.$

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If $R_{n+1}$ is not an eigenvalue, then the system has a solution which will be denoted by $U_m$, as was established in the last theorem.

If $R_{n+1} = \lambda_n$, this system will still have a solution. For, suppose $\alpha_1$ and $\alpha_2$ are two linearly independent solutions of

$$y'' + (R_{n+1} - q(x)) y = 0, \quad R_{n+1} = \lambda_n.$$ 

The general solution to $y'' + (R_{n+1} - q(x)) y = 0$ is

$$U_m = c_1 \alpha_1(x) + c_2 \alpha_2(x) + \int_0^x \left[ \frac{\alpha_1(x)\alpha_2(t) - \alpha_2(x)\alpha_1(t)}{W[\alpha_2, \alpha_1]} \right] u_m(t) \, dt,$$

where $W[\alpha_2, \alpha_1]$, the Wronskian of $\alpha_2$ and $\alpha_1$, is a constant here.

It will be shown that the constants $c_1$ and $c_2$ may be chosen such that the above function $U_m(x)$ satisfies the boundary conditions.

Let $c_1 = \alpha_2'(0) - h \alpha_1(0)$, and let $c_2 = -[\alpha_2'(0) - h \alpha_1(0)]$. Then

$$U_m'(0) - h U_m(0) = c_1 \left[ \alpha_1'(0) - h \alpha_1(0) \right] + c_2 \left[ \alpha_2'(0) - h \alpha_2(0) \right]$$

$$= 0,$$

and

$$U_m'(\pi) - H U_m(\pi) = c_1 \left[ \alpha_1'(\pi) - H \alpha_1(\pi) \right] + c_2 \left[ \alpha_2'(\pi) - H \alpha_2(\pi) \right]$$

$$+ \int_0^\pi \left[ \frac{\alpha_1'(\pi)\alpha_2(t) - \alpha_2'(\pi)\alpha_1(t)}{W[\alpha_2, \alpha_1]} \right] u_m(t) \, dt - H \int_0^\pi \left[ \frac{\alpha_1(\pi)\alpha_2(t) - \alpha_2(\pi)\alpha_1(t)}{W[\alpha_2, \alpha_1]} \right] u_m(t) \, dt.$$

To show that $U_m'(\pi) - H U_m(\pi) = 0$, we proceed as follows:

Let $y_n(x) = k_1 \alpha_1(x) + k_2 \alpha_2(x)$, where $k_1$ and $k_2$ are appropriate constants such that $y_n(x)$ is an $n+1^{st}$ eigenfunction of the system corresponding to $\lambda_n$.

$$y'' + (R_{n+1} - q(x)) y = 0,$$

$$y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0.$$
Clearly
\[ y'_n(0) - h y_n(0) = 0 \]
\[ = k_1 \left[ \alpha'_1(0) - h \alpha_1(0) \right] + k_2 \left[ \alpha'_2(0) - h \alpha_2(0) \right], \]
and
\[ y'_n(\pi) - H y_n(\pi) = 0 \]
\[ = k_1 \left[ \alpha'_1(\pi) - H \alpha_1(\pi) \right] + k_2 \left[ \alpha'_2(\pi) - H \alpha_2(\pi) \right]. \]

Observe that since \( k_1 \) and \( k_2 \) cannot be zero simultaneously,
the determinant of the coefficients of \( k_1 \) and \( k_2 \) is zero; i.e.,
\[
\begin{vmatrix}
\alpha'_1(0) - h \alpha_1(0) & \alpha'_2(0) - h \alpha_2(0) \\
\alpha'_1(\pi) - H \alpha_1(\pi) & \alpha'_2(\pi) - H \alpha_2(\pi)
\end{vmatrix} = 0.
\]

Thus,
\[ c_1 \left[ \alpha'_1(\pi) - H \alpha_1(\pi) \right] + c_2 \left[ \alpha'_2(\pi) - H \alpha_2(\pi) \right] = 0, \]
and
\[ U'_m(\pi) - H U_m(\pi) = \left[ \frac{\alpha'_1(\pi) - H \alpha_1(\pi)}{\alpha'_2, \alpha_1} \right] \int_0^\pi \alpha_2(t) u_m(t) dt - \left[ \frac{\alpha'_2(\pi) - H \alpha_2(\pi)}{\alpha'_2, \alpha_2} \right] \int_0^\pi \alpha_1(t) u_m(t) dt. \]
\[ \int_0^\pi \left[ k_1 \alpha'_1(t) + k_2 \alpha'_2(t) \right] u_m(t) dt = 0, \]
because \( y_n(x) \) is orthogonal to \( u_m(x) \) on \([0, \pi]\). But \( y_n(x) \) is non-trivial, so we may suppose \( k_1 \neq 0 \). Then the above integral can be rewritten as
\[ \int_0^\pi \alpha_1(t) u_m(t) dt = - \frac{k_2}{k_1} \int_0^\pi \alpha_2(t) u_m(t) dt. \]

Substituting this into the expression for \( U'_m(\pi) - H U_m(\pi) \) yields
\[ U'_m(\pi) - H U_m(\pi) \]
\[ = \left[ \frac{k_1 \left[ \alpha'_1(\pi) - H \alpha_1(\pi) \right]}{\alpha'_2, \alpha_1} + k_2 \left[ \alpha'_2(\pi) - H \alpha_2(\pi) \right]\right] \int_0^\pi \alpha_2(t) u_m(t) dt \]
\[ = 0, \]
since
\[ k_1 \left[ q_1'(\pi) - H q_1(\pi) \right] + k_2 \left[ q_2'(\pi) - H q_2(\pi) \right] = 0. \]
The same result is obtained if \( k_2 \neq 0 \). Hence, we conclude that the system
\[ y'' + (R_{n+1}-q(x)) y = u_m \]
\[ y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0 \]
has a solution for \( R_{n+1} = \lambda_n \), namely \( U_m(x) \).

Thus, so far, we have shown that the system
\[ y'' + (R_{n+1}-q(x)) y = u_m \]
\[ y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0 \]
has a solution \( U_m \) for \( R_{n+1} \neq \lambda_n \), and a solution \( U_m \) for \( R_{n+1} = \lambda_n \), under the assumption that \( u_m \) is orthogonal to \( y_0, y_1, y_2, \ldots, y_n \).

Now if \( R_{n+1} \neq \lambda_n \), then
\[ \int_0^\pi U_m y_i dx = 0, \quad i = 0, 1, 2, \ldots, n. \]
This is established as follows: Since
\[ \int_0^\pi u_m^2 dx = 0, \quad i = 0, 1, 2, \ldots, n, \quad m = 1, 2, 3, \ldots, \]
we have from the equations
\[ U_m'' + (R_{n+1}-q(x)) U_m = u_m, \]
\[ z_i'' + (\lambda_i - q(x)) z_i = 0, \]
that
\[ (R_{n+1} - \lambda_i) \int_0^\pi U_m z_i dx = (z_i', U_m - u_m z_i) \int_0^\pi \]
\[ = 0. \]

If \( R_{n+1} = \lambda_n \), the solution is not uniquely determined but has the form
\[ z_m = U_m + c_m z_n, \quad z_n \]
being the normalized eigenfunction corresponding to \( \lambda_n \), and \( c_m \) being an arbitrary real constant.
If \( c_m \) is defined by \( c_m = -\int_0^\pi U_m z_n dx \), then
\[
\int_0^\pi 2 U_m y_n dx = 0.
\]
Let \( U_m \) be equal to \( U_m \) for \( R_{n+1} \neq \lambda_n \),
and let \( U_m \) be equal to \( 2 U_m \) for \( R_{n+1} = \lambda_n \).

This function satisfies all conditions required except possibly normality. Define the function \( v_m \) by \( v_m = u_m + c U_m \), where \( c \) is
an arbitrary constant. This function also satisfies all conditions
required except possibly normality. From the equations
\[
\begin{align*}
&u''_m + (R_{n+1} - q(x)) u'_m = f_n, \\
&U''_m + (R_{n+1} - q(x)) U'_m = u'_m,
\end{align*}
\]
it follows that
\[
R_{n+1} \int_0^\pi v^2_m dx - J(v_m) = \int_0^\pi (v'_m + c v_m u_m) dx.
\]
Recalling that \( \int_0^\pi (U_m f_m - u_m^2) dx = 0 \) and replacing \( v_m \) by \( u_m + c U_m \) in
the right side of the above equation yields
\[
R_{n+1} \int_0^\pi v^2_m dx - J(v_m) = \int_0^\pi f_m u_m dx + 2c + c^2.
\]
By an argument employed at the end of theorem (4.2), we have
\[
R_{n+1} - J(v_m) > 0, \text{ where } w_m = v_m \left[ \int_0^\pi v^2_m dx \right]^{-\frac{1}{2}}.
\]
This is a contradiction, and hence \( R_{n+1} = \lambda_{n+1} \).
CHAPTER V

CONVERGENCE THEOREMS

Section I

Definition 5.1. Let $B$ be a set of real-valued functions on $[a,b]$, and let $C$ also be a set of real-valued functions on $[a,b]$. The set $B$ is closed with respect to $C$ on $[a,b]$, if whenever $f(x)$ is in $C$ and $\epsilon > 0$, there exist functions $f_1(x), f_2(x), \ldots, f_n(x)$ in $B$, and constants $c_1, c_2, \ldots, c_n$, such that

$$\left| \sum_{i=1}^{n} c_i f_i(x) - f(x) \right| < \epsilon$$

for all $x$ in $[a,b]$.

Definition 5.2. Suppose $B$ and $C$ are sets of real-valued, integrable functions on $[a,b]$. Then $B$ is closed in the mean with respect to $C$ on $[a,b]$, if for every $f(x)$ in $C$ and for every $\epsilon > 0$, there exist functions $f_1(x), f_2(x), \ldots, f_n(x)$ in $B$, and constants $c_1, c_2, \ldots, c_n$, such that

$$\int_{a}^{b} \left[ \sum_{i=1}^{n} c_i f_i(x) - f(x) \right]^2 dx < \epsilon.$$

Definition 5.3. Suppose $B$ and $C$ are sets of real-valued, integrable functions on $[a,b]$. Then $B$ is complete with respect to $C$ on $[a,b]$, if whenever $f(x)$ is in $C$ and $\int_{a}^{b} f(x)g(x)dx = 0$ for all $g(x)$ in $B$, it follows that $f(x) \equiv 0$ on $[a,b]$.
Lemma. If $v_1, v_2, \ldots, v_n$ are distinct eigenfunctions of the system

$$y'' + (\lambda - q(x)) y = 0, \quad y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0,$$

if $u$ has a continuous second derivative on $[0, \pi]$, if $u$ satisfies the conditions $u'(0) - h u(0) = 0$, $u'(\pi) - H u(\pi) = 0$, and if

$$\int_0^\pi u v_i dx = 0, \quad i = 1, 2, 3, \ldots, n,$$

then

$$J(u + \sum_{i=0}^{n} v_i) = J(u) + \sum_{i=0}^{n} J(v_i).$$

Proof. Now

$$\int_0^\pi (u''v - v''u) dx = \int_0^\pi (u'v - v'u)' dx = (u'v - v'u)' \bigg|_0^\pi = 0.$$

From the equation $v'' + (\alpha - q(x)) v = 0$ and the preceding result, it follows that

$$\int_0^\pi (u''v - v''u) dx = 2 \int_0^\pi q(x)v dx.$$

Hence,

$$J(u + v) = \int_0^\pi \left[ (u' + v')^2 + q(x)(u+v) \right] dx = (u+v)(u+v)' \bigg|_0^\pi$$

$$= J(u) + J(v).$$

The proof of the lemma is readily completed by induction, if we observe that

$$J(u + \sum_{i=0}^{n} v_i) = J(u + v_n + \sum_{i=0}^{n-1} v_i).$$

**Theorem 5.1.** The set of normalized eigenfunctions of the system

$$y'' + (\lambda - q(x)) y = 0,$$

$$y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0,$$

is closed in the mean with respect to the set $C$ of functions defined
on $[0, \pi]$, that have a continuous second derivative, and satisfy
the boundary conditions
\[ y'(0) = h y(0) = 0, \quad y' (\pi) = H y(\pi) = 0. \]

**Proof.** Consider
\[
\int_0^\pi \left[ f(x) - \sum_{i=0}^n c_i u_i(x) \right]^2 dx,
\]
where $f(x)$ is in $C$, $c_i = \int_0^\pi f(x)u_i(x)dx$, $i = 0,1,2, \ldots, n$,
and $u_i(x)$ is the $i + 1$st normalized eigenfunction of the above
system. Clearly
\[
0 \leq \int_0^\pi \left[ f(x) - \sum_{i=0}^n c_i u_i(x) \right]^2 dx
= \int_0^\pi [f(x)]^2 dx - \sum_{i=0}^n c_i^2.
\]
If $\rho_n(x) = f(x) - \sum_{i=0}^n c_i u_i(x)$, then
\[
\int_0^\pi \rho_n(x)^2 u_i(x) dx = 0,
\]
i = 0,1,2, \ldots, n. Suppose $\rho_n(x) \not= 0$ on $[0, \pi]$, for otherwise
the problem is trivial. Let $\sigma_n(x) = \rho_n(x) \left[ \int_0^\pi [\rho_n(x)]^2 dx \right]^{-\frac{1}{2}}$.

From theorem (4.3), it follows that
\[
\lambda_{n+1} \leq J(\sigma_n)
= J(\rho_n) \left[ \int_0^\pi [\rho_n(x)]^2 dx \right]^{-\frac{1}{2}},
\]
and
\[
\int_0^\pi [\rho_n(x)]^2 dx \leq \frac{J(\rho_n)}{\lambda_{n+1}}
\]
for $\lambda_{n+1} > 0$, which is certainly true for sufficiently large $n$.
Using the results of the lemma, we conclude that
\[
J(f) = J(\rho_n) + \sum_{i=0}^\infty c_i^2 \lambda_i.
\]
Now suppose

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\[ \lambda_0, \lambda_1, \ldots, \lambda_m \leq 0, \text{ while } 0 < \lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n, \text{ then} \]
\[ J(\rho_n) \leq J(f) - \sum_{i=0}^{m} c_i^2 \lambda_i, \]

for \( n > m \), after discarding the non-positive terms. The right side of this inequality is independent of \( n \). Consequently,

\[ |J(\rho_n)| \leq K \text{ for all } n \text{ and some constant } K, \]

and

\[ \lim_{n \to \infty} \int_0^\pi [\rho_n(x)]^2 \, dx \leq \lim_{n \to \infty} \frac{J(\rho_n)}{\lambda_{n+1}} = 0. \]

Theorem 5.2. (Weierstrass Approximation Theorem).

Any continuous function defined on \([a, b]\) may be approximated uniformly on this interval by polynomial functions (see Courant and Hilbert: Methods of Mathematical Physics, Vol. I).

Theorem 5.3. The set of normalized eigenfunctions of the system

\[ y'' + (\lambda - q(x))y = 0, \]

\[ y'(0) - h y(0) = 0, \ y'(\pi) - H y(\pi) = 0, \]

is closed in the mean with respect to the set of continuous functions on \([0, \pi]\).

Proof. Suppose \( f(x) \) is continuous on \([0, \pi]\). Suppose \( \varepsilon > 0 \). We want to determine a function \( g(x) \) such that it has a continuous second derivative, satisfies the boundary conditions, and

\[ |f(x) - g(x)| < \frac{\varepsilon}{2} (\frac{x}{\pi})^{-\frac{1}{2}} \text{ for } 0 \leq x \leq \pi. \]

According to the Weierstrass Approximation Theorem, there exists a polynomial \( p(x) \) such that

\[ |f(x) - p(x)| < \frac{\varepsilon}{4} (\frac{x}{\pi})^{-\frac{1}{2}} \text{ for } 0 \leq x \leq \pi. \]

A function \( r(x) \) will be determined such that if \( g(x) = p(x) + r(x) \), then \( g(x) \) has a continuous second derivative, satisfies the boundary conditions,
and \( r(x) < \frac{1}{4} \left( \frac{\epsilon}{\pi} \right)^{-\frac{1}{2}} \) for \( 0 \leq x \leq \pi \). If \( r(x) = \left[ h \left( p(0) - p'(0) \right) + \frac{\pi}{2} p(\pi) - p'(\pi) \right] \sin nx \)
\[
+ \left[ h \left( p(0) - p'(0) - \frac{\pi}{2} p(\pi) + p'(\pi) \right) \right] \frac{\sin(n+1)x}{n+1},
\]
where \( n \) is a non-zero, even, positive integer, then it follows readily that \( g(x) \) has the desired properties.

Finally,
\[
|f(x) - g(x)| = |f(x) - p(x)| + |p(x) - g(x)|
\]
\[
= |f(x) - p(x)| + |r(x)|
\]
\[
\leq \frac{1}{2} \left( \frac{\epsilon}{\pi} \right)^{-\frac{1}{2}}
\]
for \( 0 \leq x \leq \pi \) and for \( n > 8 \left( \frac{\pi}{\epsilon} \right)^{\frac{3}{2}} \), where
\[
K = \max \left\{ \left| h \left( p(0) - p'(0) + \frac{\pi}{2} p(\pi) - p'(\pi) \right) \right|, \left| h \left( p(0) - p'(0) - \frac{\pi}{2} p(\pi) + p'(\pi) \right) \right| \right\}.
\]

But for such a function \( g(x) \), there exists an \( N \) such that if \( n > N \),
then
\[
\int_{0}^{\pi} \left| g(x) - \sum_{i=0}^{n} c_{i} u_{i}(x) \right|^{2} \, dx < \frac{\epsilon}{4} \quad \text{for} \quad c_{i} = \int_{0}^{\pi} g(x) u_{i}(x) \, dx,
\]
this is so because of theorem (5.1). Combining all results obtained so far, it follows that
\[
\int_{0}^{\pi} \left| f(x) - \sum_{i=0}^{n} c_{i} u_{i}(x) \right|^{2} \, dx \leq \int_{0}^{\pi} \left| f(x) - g(x) \right|^{2} \, dx
\]
\[
+ 2 \int_{0}^{\pi} \left| f(x) - g(x) \right| \left| g(x) - \sum_{i=0}^{n} c_{i} u_{i}(x) \right| \, dx
\]
\[
+ \int_{0}^{\pi} \left( g(x) - \sum_{i=0}^{n} c_{i} u_{i}(x) \right)^{2} \, dx
\]
\[
< \epsilon.
\]

Theorem 5.4. The set of normalized eigenfunctions of the system
\[ y'' + (\lambda - q(x))y = 0, \]
\[ y'(0) - h y(0) = 0, y'(\pi) - H y(\pi) = 0, \]
is complete with respect to the set of continuous, real-valued functions on \([0, \pi]\).

Proof. If \( \epsilon > 0 \), then there exists \( N \) such that if \( n > N \),
\[
\int_{0}^{\pi} \left[ f(x) - \sum_{i=0}^{n} a_{i} u_{i}(x) \right]^{2} dx < \epsilon \quad \text{for} \quad a_{i} = \int_{0}^{\pi} g(x) u_{i}(x) dx,
\]
where \( g(x) \) has a continuous second derivative and satisfies the boundary conditions. Now
\[
\int_{0}^{\pi} \left[ f(x) - \sum_{i=0}^{n} b_{i} u_{i}(x) \right]^{2} dx \leq \int_{0}^{\pi} \left[ f(x) - \sum_{i=0}^{n} a_{i} u_{i}(x) \right]^{2} dx
\]
where \( b_{i} = \int_{0}^{\pi} f(x) u_{i}(x) dx \) and \( a_{i} = \int_{0}^{\pi} g(x) u_{i}(x) dx \),
since
\[
\int_{0}^{\pi} \left[ f(x) - \sum_{i=0}^{n} a_{i} u_{i}(x) \right]^{2} dx = \int_{0}^{\pi} \left[ f(x) \right]^{2} dx + \sum_{i=0}^{n} (a_{i} b_{i})^{2} - \sum_{i=0}^{n} b_{i}^{2}
\]
\[
= \int_{0}^{\pi} \left[ f(x) \right]^{2} dx - \sum_{i=0}^{n} b_{i}^{2}
\]
\[
= \int_{0}^{\pi} \left[ f(x) - \sum_{i=0}^{n} b_{i} u_{i}(x) \right]^{2} dx.
\]
Because of the closed in the mean property,
\[
\int_{0}^{\pi} \left[ f(x) \right]^{2} dx - \sum_{i=0}^{n} b_{i}^{2} = \int_{0}^{\pi} \left[ f(x) - \sum_{i=0}^{n} b_{i} u_{i}(x) \right]^{2} dx
\]
\[
\leq \epsilon
\]
for large \( n \), or simply
\[
\lim_{n \to \infty} \left[ \int_{0}^{\pi} \left[ f(x) \right]^{2} dx - \sum_{i=0}^{n} b_{i}^{2} \right] = \int_{0}^{\pi} \left[ f(x) \right]^{2} dx - \sum_{i=0}^{\infty} b_{i}^{2} = 0.
\]

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Suppose that \( \int_0^\pi f(x)u_i(x)dx = 0, \) \( i = 0,1,2, \ldots, \) then 
\( b_i = 0,1,2, \ldots, \) and \( \int_0^\pi [f(x)]^2 dx = 0. \) From the continuity of 
f(x) and the non-negative property of \( [f(x)]^2, \) we have \( f(x) \equiv 0 \)
for \( 0 \leq x \leq \pi. \) Thus, the completeness is established for the
normalized eigenfunctions. The properties proved for the normalized
eigenfunctions in the last several theorems are also possessed by
the eigenfunctions.

We may now turn to the remarks at the end of theorem (3.3) and
observe the following theorem.

**Theorem 5.5.** If \( \phi(x) \) has a continuous derivative of bounded
variation on \([0,\pi],\) then \( \phi(x) = \sum_{n=0}^{\infty} c_n u_n(x) \) and the convergence
is uniform, where \( c_n = \int_0^\pi \phi(x)u_n(x)dx \) and \( u_n(x) \) is the \( n + 1^{st} \)
normal eigenfunction of the system

\[ y'' + (\lambda - q(x)) y = 0, \]
\[ y'(0) - h y(0) = 0, y' (\pi) - H y(\pi) = 0. \]

**Theorem 5.6.** If \( a_n \) is a positive, decreasing function of
\( n, \) and if \( n a_n \leq 1, \) then there exists a constant \( K \) such that

\[ \left| \sum_{n=1}^{m} a_n \sin nx \right| \leq K \text{ for all } m \text{ and for all } x \] (see Hardy and
Rogosinski: *Fourier Series*).

The next theorem will compare the Sturm-Liouville development
of any continuous function \( f(x) \) on \([0,\pi],\) with the Fourier cosine
development of the same function.

**Theorem 5.7.** The Sturm-Liouville development of any continuous
function \( f(x) \) on \([0, \pi]\) converges (diverges) at any point in \([0, \pi]\) whenever the cosine development converges (diverges) at that point. The convergence is uniform on any subinterval of \([0, \pi]\) if and only if the cosine development converges uniformly in the same interval.

Before we give the proof, several observations will be made and several lemmas will be established.

Suppose \( f(x) \) is a continuous function on \([0, \pi]\). Let \( s_n(x) \) be the sum of the first \((n+1)\) terms of the Sturm-Liouville development, thus

\[
s_n(x) = \sum_{k=0}^{n} u_k(x)u_k(t)dt,
\]

where \( u_k(x) \) is the \( k+1\)st normalized eigenfunction of the system

\[
y'' + (\lambda - q(x))y = 0,
\]

\[
y'(0) - h y(0) = 0, \quad y'(\pi) - H y(\pi) = 0.
\]

Recall that

\[
u_k(x) = \sqrt{\frac{2}{\pi}} \cos n x + \beta(x) \frac{\sin k x}{k} + O\left(\frac{1}{n^2}\right), \quad k = 1, 2, \ldots,
\]

where

\[
\beta(x) = -\frac{dx}{\pi} + h + \frac{1}{2} \int_0^\pi q(t)dt.
\]

The system

\[
y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0
\]

has normalized solutions given by

\[
v_k = \sqrt{\frac{2}{\pi}} \cos k x, \quad k = 1, 2, 3, \ldots, \quad \text{and} \quad v_0(x) = \pi^{-\frac{1}{2}}
\]

Let

\[
\sigma_n(x) = \int_0^\pi f(x) \left[ \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^n \cos kx \cos k t \right]dt.
\]

Then \( \sigma_n(x) \) is the sum of the first \((n+1)\) terms of the cosine development. Hence, if
\[ \Phi_n(x,t) = \sum_{k=0}^{n} u_k(x)u_k(t) - \left[ \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{n} \cos kx \cos kt \right], \]

it follows that
\[ s_n(x) - \sigma_n(x) = \int_0^\pi \Phi_n(x,t)f(t)dt. \]

By means of this relation, theorem (5.7) will be established. The proof is based on two lemmas.

**Lemma I.** There exists a constant \( M \) such that
\[ |\Phi_n(x,t)| < M \text{ for all } n, x, \text{ and } t. \]

**Proof.** From the asymptotic form of \( u_k(x) \), we have, for \( k > 0,\)
\[ u_k(x)u_k(t) = \frac{2}{\pi} \cos kx \cos kt = \left[ \sqrt{\frac{2}{\pi}} \cos kx + \frac{\beta(x) \sin kx}{k} + O\left(\frac{1}{k^2}\right) \right] \left[ \sqrt{\frac{2}{\pi}} \cos kt + \frac{\beta(t) \sin kt}{k} + O\left(\frac{1}{k^2}\right) \right] \]
\[ = \frac{2}{\pi} \cos kx \cos kt \]
\[ = \frac{1}{\sqrt{2\pi}} \left[ \beta(x) + \beta(t) \right] \frac{\sin k(x+t)}{k} + \frac{1}{\sqrt{2\pi}} \left[ \beta(x) - \beta(t) \right] \frac{\sin k(x-t)}{k} + O\left(\frac{1}{k^2}\right). \]

But the sums of the series
\[ \sum_{k=1}^{n} \frac{\sin k(x+t)}{k} \text{ and } \sum_{k=1}^{n} \frac{\sin k(x-t)}{k} \]
are bounded according to the theorem (5.6), and \( \beta(x) \) is also bounded on \([0,\pi]\); hence, there exists a constant \( M \) such that \( |\Phi_n(x,t)| \leq M \)
for all \( n, x, \text{ and } t. \)

**Lemma II.** If \( \Phi(x) \) has a continuous second derivative on \([0,\pi]\), then \( \int_0^\pi \Phi_n(x,t) \phi(t) \) converges uniformly to zero.

**Proof.** If \( g_n(x) \) and \( h_n(x) \) are the sums of the first \((n+1)\) terms of the Sturm-Liouville and the cosine development of \( \phi(x) \), respectively, then
\[ g_n(x) - h_n(x) = \int_0^\pi \bar{F}_n(x,t) \phi(t) \, dt. \]

The uniform convergence of the Sturm-Liouville development of a function with a continuous first derivative of bounded variation to the function itself was already established. But in this case, the cosine development is a special case of the Sturm-Liouville development. Therefore, a similar conclusion is valid for the cosine development. Also, since a function with a continuous second derivative has a continuous first derivative of bounded variation, we have, for \( \epsilon > 0 \),

\[
\left| \int_0^\pi \bar{F}_n(x,t)\phi(t)\,dt \right| \leq \left| g_n(x) - \phi(x) \right| + \left| \phi(x) - h_n(x) \right| < \frac{\epsilon}{2}
\]

for sufficiently large \( n \), and for all \( x \) in \([0, \pi]\).

Now the proof of theorem (5.7) will be given.

**Proof.** Since \( f(x) \) is continuous on \([0, \pi]\), a sequence of functions \( \{ \phi_n(x) \} \) with continuous second derivatives may be formed, according to theorem (5.2), which converges uniformly to \( f(x) \) on \([0, \pi]\).

Then
\[
s_n(x) - \sigma_n(x) = \int_0^\pi \bar{F}_n(x,t) [f(t) - \phi_m(t)] \, dt + \int_c^T \bar{F}_n(x,t) \phi_m(t) \, dt.
\]

But \( \{ \phi_m(x) \} \) converges uniformly to \( f(x) \) on \([0, \pi]\), so given \( \epsilon > 0 \), there exists an integer \( N \) such that \( |f(t) - \phi_m(t)| < \frac{\epsilon}{2\pi M} \) for all \( t \) in \([0, \pi]\), and for \( m > N \).

Finally, we have, from lemma II, that
\[
\left| s_n(x) - \sigma_n(x) \right| \leq \int_0^\pi \bar{F}_n(x,t) \left| f(t) - \phi_m(t) \right| \, dt + \left| \int_c^T \bar{F}_n(x,t) \phi_m(t) \, dt \right|.
\]
for $0 \leq x \leq \pi$, and for all large $n$.

This theorem is of considerable importance. An enormous volume of work has been done concerning convergence or divergence of the Fourier cosine development of an arbitrary function. By means of the last theorem, the results of such investigations are applicable to the Sturm-Liouville development of that function. For example, the following theorems are consequences of theorem (5.7) and well-known theorems concerning the Fourier development.

**Theorem 5.8.** Let $f(x)$ be a function of bounded variation on $[0, \pi]$. Then

$$
\sum_{n=0}^{\infty} c_n u_n(x) = \frac{f(x^+) + f(x^-)}{2}
$$

for $0 < x < \pi$ where $f(x^+)$ and $f(x^-)$ exist; and, when $f(0^+)$ and $f(\pi^-)$ exist, then

$$
\sum_{n=0}^{\infty} c_n u_n(0) = f(0^+) \quad \text{and} \quad \sum_{n=0}^{\infty} c_n u_n(\pi) = f(\pi^-).
$$

**Theorem 5.9.** Suppose $f(x)$ is a continuous function on $[0, \pi]$. Denote by $s_n(x)$ the sum of the first $(n+1)$ terms of the Sturm-Liouville development of $f(x)$. Also, let $\sigma_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(x)$.

Then

$$
\lim_{n \to \infty} \sigma_n(x) = f(x) \quad \text{uniformly on} \quad [0, \pi].
$$

**Theorem 5.10.** If $f(x)$ is continuous and of bounded variation on $[0, \pi]$, then

$$
\sum_{n=0}^{\infty} c_n u_n(x) = f(x) \quad \text{uniformly on} \quad [0, \pi].
$$
LIST OF REFERENCES


