Postive and oscillatory radial solutions of semilinear elliptic equations

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Positive and Oscillatory Radial Solutions of Semilinear Elliptic Equations

by

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B.S., Hangzhou University, PRC, 1982

Presented in partial fulfillment of the requirements for the degree of Master of Science
The University of Montana
1994

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December 4, 1994
In recent years, numerous authors have given substantial attention to the existence of positive solutions of semilinear elliptic equations involving critical exponents. In this paper we consider the solutions of the nonlinear partial differential equation

\[ \Delta u + f(u) + g(|x|, u) = 0, \quad \text{in } \mathbb{R}^n, \]

where \( f \) and \( g \) are continuous functions, with \( f(u) > 0 \) and \( g(|x|, u) > 0 \), whenever \( u > 0 \). We first present existence and uniqueness theorems by using fixed point theory. Then we prove that, under the assumption that \( \lim_{u \to 0^+} f(u)/u^q = B > 0 \), if \( 1 < q < n/(n-2) \), the problem has no positive solution; but if \( n/(n-2) \leq q < (n+2)/(n-2) \) the problem may have a positive solution. If we add the additional condition \( 2nF(u) - (n-2)uf(u) > 0 \) for \( u > 0 \) and \( g = 0 \), then the problem again has no solution, even when \( n/(n-2) \leq q < (n+2)/(n-2) \). We also prove that if \( g = 0 \) then, for certain \( f(u) \), the solutions will oscillate to zero as \( r \to \infty \), while for other kinds of \( f(u) \) the solutions will oscillate to \( b \), where \( b \) is a root of \( f(u) = 0 \).
ACKNOWLEDGMENTS

I would like to extend my gratitude to Dr. William R. Derrick, my advisor, for his valuable advice and keen insights. Throughout the process of writing this thesis, Dr. Derrick assisted me with many discussions, which provided me much of the inspiration that I needed to complete this thesis. In brief, words are not enough to express my gratitude to him.

I would also like to give thanks to Dr. Joseph A. Cima, a professor at the University of North Carolina at Chapel Hill, for his many good suggestions and professional assistance.

I am very grateful to my other two committee members, Professor Richard J. Hayden and Professor Leonid Kalachev, for their constructive criticisms.

Finally, I wish to thank my friends and colleagues at the University of Montana for their help in these past years.
# TABLE OF CONTENTS

Abstract ................................................................. ii

Acknowledgement ......................................................... iii

Chapter One  Introduction ................................................. 1

Chapter Two  Elementary Results ........................................ 7

Chapter Three  Existence and Uniqueness Theorem .................. 14

Chapter Four  Nonexistence of Positive Solutions ................. 19

Chapter Five  Oscillatory Behaviors ................................. 26

Chapter Six  Conclusions ............................................... 35

Appendix 1 .............................................................. 38

Appendix 2 .............................................................. 43

Appendix 3 .............................................................. 45

Appendix 4 .............................................................. 46

References ............................................................... 47
Chapter One

Introduction

Semilinear elliptic equations involving critical Sobolev exponents (i.e. there is a $p^*$ such that, when the exponent $p$ changes its value from $p < p^*$ to $p > p^*$, the behavior of the corresponding solutions also will change) have been considered by various authors (see [BN], [J], [MTW], [N], [NY], [P], [Po], [Y]). These equations arise in various branches of mathematics as well as in physics. The following model is typical.

In order to describe the dynamics of a globular cluster of stars, astrophysicists have used the following scaled equation

\[(1.1) \quad \Delta u + c e^u = 0, \quad x \in \mathbb{R}^3,\]

where $u$ represents the gravitational potential (therefore $u > 0$), $\rho = -\Delta u = ce^u$ represents the density, $\int_{\mathbb{R}^3} \rho dx$ represents the total mass of the cluster and $c$ is a constant. (A brief introduction to globular dynamics is given in Appendix 1.) Another equation that has been used for the same purpose is

\[(1.2) \quad \Delta u + u^p = 0, \quad x \in \mathbb{R}^3,\]

where $p > 2/3$. However, astrophysicists have found that equation (1.1) leads to an infinite total mass for the cluster and equation (1.2) also has the same result for some values of $p$. In order to avoid this difficulty, Matukuma [Ma], used the following equation

\[(1.3) \quad \Delta u + \frac{1}{1 + |x|^2} u^p = 0, \quad x \in \mathbb{R}^3,\]

to describe this situation, where $p > 1$.

Since the globular cluster has radial symmetry, a positive radial solution
(i.e. a solution is radial if \( u(x) = u(|x|) > 0 \), and is positive if \( u(x) > 0 \) for all \( x \)) is of particular interest. For radial solutions, equations (1.2) and (1.3) reduce in spherical coordinates to the ordinary differential equations

\[
\begin{align*}
(1.4) & \quad u'' + \frac{2}{r} u' + u^p = 0, \quad \text{for } 0 < r < \infty, \quad u(0) = u_0, \quad u'(0) = 0 \\
(1.5) & \quad u'' + \frac{2}{r} u' + \frac{1}{1+r^2} u^p = 0, \quad \text{for } 0 < r < \infty, \quad u(0) = u_0, \quad u'(0) = 0
\end{align*}
\]

respectively, where \( u_0 > 0 \). Studying the structure of the solution \( u(r) \) of (1.5), Matukuma conjectured that

1. if \( p < 3 \), then \( u(r) \) has a finite zero (i.e. \( u(R) = 0 \) for some \( R > 0 \)) for every \( u_0 > 0 \).

2. if \( p = 3 \), then \( u(r) \) is a positive solution with finite total mass for every \( u_0 > 0 \).

3. if \( p > 3 \), then \( u(r) \) is a positive solution with infinite total mass for every \( u_0 > 0 \).

Matukuma [Ma] found an interesting exact solution of (1.5):

\[ u(r) = 3^{1/2}(1 + r^2)^{-1/2}, \text{ with } u_0 = 3^{1/2} \text{ and } p = 3. \]

Since the total mass is

\[
\int_{\mathbb{R}^3} \rho \, dx = \int_{\mathbb{R}^3} u^p(\mathbf{x})/(1 + |\mathbf{x}|^2) \, d\mathbf{x} = C \int_0^\infty (1 + r^2)^{-5/2} r^2 \, dr < \infty
\]

which is finite, this solution confirms part of his conjecture (i.e. the second conjecture is true with \( u_0 = 3^{1/2} \), but we shall see it is not true for smaller \( u_0 > 0 \) by the following results).

The topic was seldom touched upon until the early 1980's when the critical point theorem was developed. The critical point theorem addresses the following problem: Find an extreme point \( u_0(x) \) of the functional \( G(u) = \int_\Omega (|\nabla u(x)|^2 - F(u(x)) \, dx \) in a suitable space, where \( F(u) = \int f(u) \, du \). Under certain assumptions on \( f \), this extreme point \( u_0(x) \) is a solution of the equation.
\[ \Delta u + f(u) = 0. \] Several authors studied (1.4) and (1.5) and obtained important results (see [BN], [J], [MTW], [N], [NY], [P], [Po], [Y], [YY]) using a variety of elementary techniques of ordinary differential equations. Wei-Ming Ni and Shoji Yotsutani [NY] have proved for (1.5) that

1. if \( 1 < p < 5 \), then \( u(r) \) has a finite zero for every sufficiently large \( u_0 > 0 \).

2. if \( 1 < p < 5 \), then \( u(r) \) is a positive solution with infinite total mass for every sufficiently small \( u_0 > 0 \).

3. if \( p \geq 5 \), then \( u(r) \) is a positive solution with infinite total mass for every \( u_0 > 0 \).

It is interesting to note that not only the exponent \( p \) but also the initial value \( u_0 \) has a vital influence on the behavior of the solution \( u(r) \).

In studying the equations (1.2) and (1.3), many authors consider more general equations of the form

\[ (1.6) \quad \Delta u + f(u) + g(|x|,u) = 0, \text{ in } \mathbb{R}^n, \]

where \( f \) and \( g \) are continuous functions, with \( f(u) > 0 \) and \( g(|x|,u) > 0 \) whenever \( u > 0 \). Such equations arise in many areas of applied mathematics (see [L],[NY], [CC], [YY]); positive solutions that exist in \( \mathbb{R}^n \) and satisfy \( u(x) \to 0 \) as \( |x| \to \infty \) are called ground states.

By a positive solution in \( \mathbb{R}^n \) we mean a solution \( u \) satisfying \( u(x) > 0 \) for all \( x \) in \( \mathbb{R}^n \). Equation (1.6) is said to involve critical exponents if \( f(u) = u^p + f_0(u) \), where \( f_0(u) \) is an algebraic rational function in \( u \) with order of growth \( o(u^p) \) at infinity and \( p = (n+2)/(n-2) \) is the critical Sobolev exponent.

If \( g(|x|,u) \equiv 0 \) and \( f \in C^1 + \epsilon \) on an interval \( [0,\delta) \), it is known [GNN] that positive solutions of (1.6) are radially symmetric. For other \( g(|x|,u) \), examples of positive nonradial ground state solutions have been constructed [L].

In this paper we will deal only with radially symmetric solutions of (1.6).
Hence, we need only consider the singular initial value problem

\begin{equation}
(1.7) \quad u'' + \frac{n-1}{r} u' + f(u) + g(r,u) = 0, \text{ for } 0 < r < \infty,
\end{equation}

\[ u(0) = u_0, \; u'(0) = 0. \]

Ni and Yotsutani [NY] considered the equation

\begin{equation}
(1.8) \quad u'' + \frac{n-1}{r} u' + \sum_{i=1}^{k} c_i r^{l_i}(u^+)^{p_i} = 0, \; u(0) = u_0 > 0, \; u'(0) = 0,
\end{equation}

where \( p_i > 1, \; l_i > -2, \; c_i > 0 \) (1 \( \leq \) i \( \leq \) k) and

\[ u^+ = \begin{cases} 
    u & \text{if } u > 0 \\
    0 & \text{if } u \leq 0
\end{cases}. \]

They conclude that for (1.8):

1. if \( p_i \leq (n+2+2l_i)/(n-2) \) (1 \( \leq \) i \( \leq \) k), with at least one being a strict inequality, then \( u(r) \) has a finite zero for every \( \alpha > 0 \).

2. if \( p_i \geq (n+2+2l_i)/(n-2) \) (1 \( \leq \) i \( \leq \) k), then \( u(r) \) is a positive solution for every \( \alpha > 0 \).

From these conclusions, we have the following special case. When \( k = 2, \; l_1 = 0, \; p_1 = q, \; c_2 = 1 \) and \( p_2 = p = (n+2)/(n-2) \) is the critical Sobolev exponent, then equation (1.8) becomes (for positive \( u \))

\[ u'' + \frac{n-1}{r} u' + cu^q + u^p = 0, \; u(0) = u_0 > 0, \; u'(0) = 0. \]

If \( q < p \), then all solutions have a finite zero, while if \( q > p \), then all solutions are positive.

However, almost all the methods developed in [NY] and other papers do not observe that the coefficient \( q = n/(n-2) \) also behaves like a critical exponent. Moreover, few authors have discussed the oscillatory behavior of these solutions.
We will prove that, if

$$\lim_{u \to 0^+} f(u)/u^q = B > 0 \text{ for } 1 < q < n/(n-2),$$

then problem (1.7) has no positive solution, while for $n/(n-2) \leq q < (n + 2)/(n - 2)$, the problem may have positive solutions. We will also prove that if $g \equiv 0$ then, for certain $f(u)$, the solutions will oscillate to zero as $r \to \infty$, while for other $f(u)$ all solutions will oscillate to $b$, where $b$ is a root of $f(u) = 0$.

This thesis is organized as follows:

In Chapter 2 we present some elementary results that will be used in our proofs. The main tools consist of an "energy function" that was developed in McLeod, Troy and Weissler [MTW] and used in Maier [M], a modification of two identities due to Pokhozhaev [Po], and an a priori estimate of the solutions.

In Chapter 3 we prove an existence and uniqueness theorem for the initial value problem (1.7) by using fixed point theory.

In Chapter 4 we prove a nonexistence theorem for positive solutions of (1.7) by using bootstrapping methods and the second Pokhozhaev's identity.

In Chapter 5 we discuss the initial-value problem

$$u'' + \frac{n-1}{r} u' + f(u) = 0, \text{ for } 0 < r < \infty,$$

$$u(0) = u_0, \ u'(0) = 0.$$

For certain $f(u)$, we can prove that all solutions of (1.9) will oscillate to 0. We also discuss the particular case: $f(u) = (a + u)|u|^{p-1}$ with $a > 0$ and $p = (n + 2)/(n - 2)$. We will prove that all solutions to this problem oscillate to $-a$. 

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In Chapter 6 we summarize our results and discuss other interesting phenomena. We conjecture some results which we believe are true based on numerical calculations.

Finally, in the Appendices, we give a brief introduction to the globular star cluster model in which the equations (1.1) and (1.2) arise and verify the solutions of certain particular equations.
In what follows, we shall need certain elementary facts concerning solutions of the initial value problems (1.6) and (1.7). Suppose $f(u)$ and $g(r,u)$ are continuous functions. If:

(a) $g(r,u) \neq 0$. Multiply (1.7) by $r^{n-1}$, rewrite as

$$ (r^{n-1} u')' = -r^{n-1}[f(u) + g(r,u)], $$

and integrate from $r_0$ to $r$:

$$ (2.1) \quad r^{n-1} u'(r) = r_0^{n-1} u'(r_0) - \int_{r_0}^{r} [f(u(s)) + g(s, u(s))] s^{n-1} \, ds. $$

Dividing both sides by $r^{n-1}$, we get

$$ (2.2) \quad u'(r) = r_0^{n-1} u'(r_0) - \int_{r_0}^{r} [f(u(s)) + g(s, u(s))] (\frac{s}{r})^{n-1} \, ds. $$

Integrating from $r_0$ to $x$, and reversing the order of integration, yields

$$ u(x) - u(r_0) = \int_{r_0}^{x} (\frac{r_0}{r})^{n-1} u'(r_0) \, dr - \int_{r_0}^{x} \int_{r_0}^{r} [f(u(s)) + g(s, u(s))] \frac{s^{n-1}}{r^{n-1}} \, ds \, dr $$

$$ = \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{x} \right)^{n-2} \right] u'(r_0) - \int_{r_0}^{x} \int_{r_0}^{r} [f(u(s)) + g(s, u(s))] s^{n-1} \left( \int_{s}^{x} \frac{1}{r^{n-1}} \, dr \right) \, ds $$

$$ = \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{x} \right)^{n-2} \right] u'(r_0) + \frac{1}{n-2} \int_{r_0}^{x} [f(u(s)) + g(s, u(s))] s \left( \frac{s^{n-2}}{x^{n-2}} - 1 \right) \, ds, $$

or, substituting $r$ for $x$,

$$ (2.3) \quad u(r) = u(r_0) + \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{r} \right)^{n-2} \right] u'(r_0) $$

$$ + \frac{1}{n-2} \int_{r_0}^{r} [f(u(s)) + g(s, u(s))] s \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) \, ds. $$
If \( r_0 = 0 \), (2.3) becomes

\[
(2.4) \quad u(r) = u(0) + \frac{1}{n-2} \int_0^r [f(u(s)) + g(s, u(s))] s^{\frac{n-2}{n-2}} - 1 ds.
\]

Setting \( r_0 \equiv 0 \) and assuming that \( f(u) > 0 \) and \( g(r, u) > 0 \) when \( u > 0 \), from (2.2) and (2.4), it is easy to see that if \( u \) is a positive solution of equation (1.7), then \( u \) is strictly decreasing since \( u' < 0 \), so \( u(r) \rightarrow c \geq 0 \) and \( u'(r) \rightarrow 0 \) as \( r \rightarrow \infty \). Suppose \( c > 0 \), then since \( f \) is continuous on the interval \([c, u_0]\) it has a minimum \( f_{\text{min}} > 0 \) on this interval. Hence, from (2.1) with \( r_0 = 0 \),

\[
- r^{n-1} u'(r) = \int_0^r [f(u(s)) + g(s, u(s))] s^{n-1} ds \geq f_{\text{min}} r^n,
\]

implying that \( u'(r) \leq -f_{\text{min}} r^n / n \rightarrow -\infty \), a contradiction. Hence \( u \rightarrow 0 \) as \( r \rightarrow \infty \).

(b) \( g(r, u) \equiv 0 \). By the uniqueness theorem for solutions of initial value problems, a solution of (1.9) cannot satisfy both \( u'(r) = 0 \) and \( f(u(r)) = 0 \), unless \( u \) is constant. Thus, except for such cases, the critical points (values \( r \) where \( u'(r) = 0 \)) of any solution of (1.9) are isolated, and are minima whenever \( f(u(r)) < 0 \) and maxima whenever \( f(u(r)) > 0 \). Let \( u(r) \) be a solution of (1.9) and define the "energy function" of [MTW]:

\[
Q(u(r)) = \frac{(u')^2}{2} + F(u) \quad \text{where} \quad F(u) = \int_0^u f(u) du.
\]

If (1.9) is multiplied by \( u' \), we obtain

\[
(2.6) \quad \frac{dQ}{dr} = \left( \frac{(u')^2}{2} + F(u) \right)' = - \frac{n-1}{r} (u')^2 \leq 0.
\]

This implies that the "energy" function \( Q \) is strictly decreasing because the critical points of \( u \) are isolated. From (2.6), we have

\[
(2.7) \quad \frac{dQ}{dr} = - \frac{n-1}{r} (u')^2 = - \frac{2(n-1)}{r} (Q - F(u)).
\]
Multiply (2.7) by \(r^{2(n-1)}\), rewrite as

\[
\frac{d}{dr} (r^{2(n-1)}Q) = 2(n-1) r^{2n-3} F(u)
\]

and integrate from any number \(r_0 \geq 0\) to \(r\) obtaining

\[
r^{2(n-1)}Q(u(r)) - r_0^{2(n-1)}Q(u(r_0)) = 2(n-1) \int_{r_0}^{r} F(u(s)) s^{2n-3} ds.
\]

**Lemma 2.1** Suppose that \(u\) has a critical point at \(r_0\). If \(u(r_0)\) is a local maximum, then \(u(r) < u(r_0)\) for all \(r > r_0\), and if \(u(r_0)\) is a local minimum, then \(u(r) > u(r_0)\) for all \(r > r_0\).

**Proof.** Suppose \(u(r_1) = u(r_0)\) for \(r_1 > r_0\). Then, since \(F(u(r_1)) = F(u(r_0))\),

\[
Q(u(r_1)) = \frac{u'(r_1)^2}{2} + F(u(r_1)) \geq F(u(r_0)) = Q(u(r_0)),
\]

contradicting the fact that \(Q\) is strictly decreasing.

We also will use the following versions of two of Pokhozhaev's identities (see [Po]): **First Identity:**

\[
\int_0^r \left[ \Delta u(s) + f(u(s)) \right] \left[ s u'(s) + \alpha u(s) \right] s ds =
\]

\[
r^2 Q(u(r)) + \alpha r u(r) u'(r) + \frac{\alpha}{2} (n-2) (u^2(r) - u^2(0)) +
\]

\[
(n-2-\alpha) \int_0^r u^2(s) s ds + \int_0^r \left[ \frac{\alpha}{2} u(s) f(u(s)) - 2 F(u(s)) \right] s ds.
\]

**Second Identity:**

\[
\int_0^r \left[ \Delta u(s) + f(u(s)) \right] \left[ s u'(s) + \alpha u(s) \right] s^k ds =
\]

\[
r^{k+1} Q(u(r)) + \alpha r^k u(r) u'(r) + \frac{\alpha}{2} (n-1-k) r^{k-1} u^2(r) +
\]
\[
(n - 1 - \frac{k+1}{2} - \alpha) \int_0^r u'^2(s) s^k \, ds - \alpha \frac{(n - 1 - k)(k - 1)}{2} \int_0^r u^2(s) s^{k-2} \, ds + \int_0^r [\alpha u(s) f(u(s)) - (k+1) F(u(s))] s^k \, ds.
\]

Since \(\Delta u + f(u) = 0\) for any solution \(u(r)\) to (1.9), the left side of each identity is zero. Verification of each of these identities is given in Appendix 2.

**Lemma 2.2** Let \(u\) be a solution of (1.9) and let

\[(2.12) \quad J(r, u(r)) = r^n u'^2(r) + (n - 2) r^{n-1} u(r) u'(r) + 2 r^n F(u(r)).\]

If \(u'(r_0) = 0\) for \(r_0 \geq 0\), then for all \(r \geq r_0\),

\[(2.13) \quad J(r, u(r)) = \int_{r_0}^r [2 n F(u(s)) - (n - 2) u(s) f(u(s))] s^{n-1} \, ds + 2 r_0^n F(u(r_0)).\]

**Proof.** Differentiating (2.12) with respect to \(r\), we get

\[(2.14) \quad \frac{dJ(r,u)}{dr} = nr^{n-1} u'^2(r) + 2 r^n u'(r) u''(r) + (n - 2)(n - 1)r^{n-2} u'(r) u(r) + (n - 2)r^{n-1} u'(r) u''(r) + (n - 2)r^n F(u(r)) + 2nr^{n-1} u'(r) f(u(r)) = 2r^n u'(r)[u''(r) + \frac{n-1}{r} u'(r) + f(u(r))] + (n - 2)r^{n-1} u(r)[u'(r) + \frac{n-1}{r} u'(r) + f(u(r)) + 2nr^{n-1} F(u(r)) - (n - 2)r^n f(u(r)).\]

The terms in brackets are zero, so an integration yields the desired result.

**Lemma 2.3** Let \(f(u)\) be continuous and let \(u\) be a solution of the initial value
problem

\begin{equation}
(2.15) 
\frac{d^2 u}{dr^2} + \frac{n-1}{r} \frac{du}{dr} + f(u) = 0, \text{ for } 0 < r < \infty, \\
u(0) = u_0 \neq 0, \quad u'(0) = 0,
\end{equation}

and suppose that

\begin{equation}
(2.16) 
\lim_{u \to 0^+} \frac{f(u)}{u^q} = B > 0,
\end{equation}

for \(1 < q < p = (n + 2)/(n - 2)\). If \(f(u) > 0\) for all \(u > 0\), and there exists an \(r_0 \geq 0\) such that \(u(r) > 0\) for all \(r \geq r_0\), then for some \(c > 0\),

\begin{equation}
(2.17) 
u(r) \leq c \ r^{-2/(q-1)}, \text{ for all } \ r \geq r_3 > r_0.
\end{equation}

**Proof.** First we suppose that there exists an \(r_0 \geq 0\) such that \(u(r) > 0\) for all \(r > r_0\). Let \(r_1\) be the first maximum such that \(u(r) > 0\) for all \(r \geq r_1\). Then \(u'(r_1) = 0\) and by (2.1) with \(g(r, u(r)) = 0\),

\begin{equation}
(2.18) 
u'(r) = -\frac{1}{r^{n-1}} \int_0^r f(u(s)) s^{n-1} \, ds \\
= -\frac{1}{r^{n-1}} \left\{ \int_0^{r_1} f(u(s)) s^{n-1} \, ds + \int_{r_1}^r f(u(s)) s^{n-1} \, ds \right\} \\
= \frac{r_1^{n-1}}{r^{n-1}} u'(r_1) - \frac{1}{r^{n-1}} \int_{r_1}^r f(u(s)) s^{n-1} \, ds \\
= -\frac{1}{r^{n-1}} \int_{r_1}^r f(u(s)) s^{n-1} \, ds,
\end{equation}

which implies that \(u'(r) < 0\) for all \(r \geq r_1\). Since \(u\) decreases to 0, by (2.16) some \(r_2 \geq r_1\) exists for which \(f(u)/u^q > B/2\) for \(r \geq r_2\). Then from equation (2.18)

\begin{align*}
u'(r) & \leq -\frac{1}{r^{n-1}} \int_{r_1}^{r_2} f(u(s)) s^{n-1} \, ds - \frac{B}{2r^{n-1}} \int_{r_2}^r u^q(s) s^{n-1} \, ds \\
& \leq -\frac{1}{r^{n-1}} \int_{r_1}^{r_2} f(u(s)) s^{n-1} \, ds - \frac{1}{r^{n-1}} cu^q(r) \int_{r_2}^r s^{n-1} \, ds \\
& \leq -\frac{c}{n} u^q(r) [r - \frac{r_2^n}{r^{n-1}}],
\end{align*}
where \( c = B/2 \). Divide by \( u^q(r) \),

\[
\frac{du}{u^{q}(r)} \leq - \frac{c}{n} \left[ r - \frac{r^n}{r^{n-1}} \right] dr.
\]

Integrating from \( r_2 \) to \( r \), yields

\[
\frac{1}{1-q} \left[ u^{1-q}(r_2) - u^{1-q}(r) \right] \leq - \frac{c}{n} \left[ \frac{r^2}{2} + \frac{r^n}{(n-2)r^{n-2}} - \frac{r^n}{2} - \frac{r^n}{2(n-2)} \right]
\]

\[
= - \frac{c}{n} \left[ \frac{r^2}{2} + \frac{r^n}{(n-2)r^{n-2}} - \frac{nr^2}{2(n-2)} \right],
\]

which implies that

\[
u^{1-q}(r_2) + \frac{(q-1)c}{n} \left[ \frac{r^2}{2} + \frac{r^n}{(n-2)r^{n-2}} - \frac{nr^2}{2(n-2)} \right] \leq \frac{1}{u^{q-1}(r)}
\]

or

\[
u^{q-1}(r) \leq \frac{1}{u^{1-q}(r_2) + \frac{c(q-1)}{n} \left[ \frac{r^2}{2} + \frac{r^n}{(n-2)r^{n-2}} - \frac{nr^2}{2(n-2)} \right]} \leq c_1r^{-2}.
\]

Thus, we have proved that for \( r \) sufficiently large,

\[
u(r) \leq c \, r^{-2/(q-1)}, \quad \text{for all } r \geq r_3.
\]

**Corollary 2.4** Let \( f(u) \) be continuous and let \( u \) be a solution of (2.15), and assume that \( f(u) < 0 \) for all \( u < 0 \), and

\[
\lim_{u \to 0^-} \frac{f(u)}{|u|^{q-1}} = B > 0.
\]

If there is an \( r_0 \geq 0 \) such that \( u(r) < 0 \) for all \( r \geq r_0 \), then for some \( c > 0 \),

\[
|u(r)| \leq c \, r^{-2/(q-1)}, \quad \text{for all } r \geq r_3 > r_0.
\]
**Proof.** The proof follows by an argument almost identical to that of Lemma 2.3.

**Lemma 2.5** If $u(r)$ is a solution of (1.9) and $\beta = \lim_{r \to \infty} Q(u(r))$, then for any $r_0 \geq 0$, 

$$\lim_{r \to \infty} \left( \frac{1}{r} \int_{r_0}^{r} [F(u(s)) - \beta] \, ds \right) = 0.$$ 

**Proof.** Multiply (1.9) by $ru'(r)$ and integrate by parts from $r_0$ to $r$, getting 

$$\frac{1}{2} s u'^2(s) \bigg|_{r_0}^{r} + \left( n - \frac{3}{2} \right) \int_{r_0}^{r} u'^2(s) \, ds + sF(u(s)) \bigg|_{r_0}^{r} - \int_{r_0}^{r} F(u(s)) \, ds = 0,$$

or 

$$Q(u(r)) - \frac{r_0}{r} Q(u(r_0)) + \frac{(2n-3)}{r} \int_{r_0}^{r} Q(u(s)) \, ds = 2(n-1) \int_{r_0}^{r} F(u(s)) \, ds.$$

Letting $r \to \infty$, we get by L’Hospital’s rule, 

$$2(n-1)\beta = 2(n-1) \lim_{r \to \infty} \left( \frac{1}{r} \int_{r_0}^{r} F(u(s)) \, ds \right).$$

Thus our conclusion is true.
Chapter 3

Existence and Uniqueness Theorem

In this chapter, we use the fixed point theorem to prove the existence and uniqueness of the initial value problem (1.7).

Consider the initial value problem

\begin{align}
\tag{3.1}
    u'' + \frac{n-1}{r} u' + g(r, u) &= 0, \quad \text{for } 0 \leq r_0 < r < \infty, \\
\tag{3.2}
    u(r_0) &= a, \quad u'(r_0) = b,
\end{align}

where a and b are any real number. We suppose that g satisfies a local Lipschitz condition with respect to the second variable, i.e.

\begin{equation}
    \tag{3.3}
    |g(r,u) - g(r,v)| \leq h(r,u,v) |u - v| \quad \text{with } \lim_{r \to \infty} r^\alpha h(r,u,v) < \infty,
\end{equation}

and that g(r,u) is continuous in r and

\begin{equation}
    \tag{3.4}
    \lim_{r \to \infty} r^\alpha |g(r,u)| < \infty \quad \text{for all } 0 \leq \alpha < 1.
\end{equation}

By (2.2) and (2.3) with f(u) = 0 and r_0 > 0 we have

\begin{equation}
    \tag{3.5}
    u'(r) = \left( \frac{r_0}{r} \right)^{n-1} b - \int_{r_0}^{r} g(s,u(s)) \left( \frac{s}{r} \right)^{n-1} ds.
\end{equation}

and

\begin{equation}
    \tag{3.6}
    u(r) = a + \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{r} \right)^{n-2} \right] b \\
    + \frac{1}{n-2} \int_{r_0}^{r} g(s,u(s)) \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) ds.
\end{equation}

Then letting r_0 \to 0 and using the condition (3.4), we can see that both
\[
\int_0^r g(s,u(s)) s \left( s^{n-2} - 1 \right) ds \quad \text{and} \quad \int_0^r g(s,u(s)) (\frac{s}{r})^{n-1} ds
\]
are convergent. Thus (3.6) holds for all \( r_0 \geq 0 \). Now we verify that if \( u(r) \) is a solution of (3.6), then \( u(r) \) also is a solution of (3.1) and (3.2). It is very clear from (3.5) that \( u(r) \) represented by (3.6) satisfies the initial value conditions (3.2) and has a first order derivative. Differentiating (3.5), we have

\[
(3.7) \quad u''(r) = -(n-1) \left( \frac{r_0}{r} \right)^{n-1} b - g(r,u(r)) + (n-1) \int_{r_0}^r g(s,u(s)) (\frac{s}{r})^{n-1} \frac{ds}{r}
\]

Substituting (3.5) and (3.7) into (3.1), we get

\[
u'' + \frac{n-1}{r} u' + g(r,u) = -(n-1) \left( \frac{r_0}{r} \right)^{n-1} b - g(r,u(r)) + (n-1) \int_{r_0}^r g(s,u(s)) (\frac{s}{r})^{n-1} \frac{ds}{r} + \frac{n-1}{r} \left[ (\frac{r_0}{r})^{n-1} b - \int_{r_0}^r g(s,u(s)) (\frac{s}{r})^{n-1} ds \right] + g(r,u) = 0.
\]

Thus \( u(r) \) is also a solution of (3.1) and the integral equation (3.6) is equivalent to (3.1) and (3.2). Now let us prove the local existence and uniqueness theorem.

**Theorem 3.1** Suppose (3.3) and (3.4) hold. Then there exists an \( r_1 > r_0 \) such that the integral equation (3.6) has a unique solution \( u(r) \in C^2([r_0, r_1]) \cap C^1([r_0, r_1]) \).

**Proof.** Let

\[
S = \{ u(r) \in C([r_0, r_1] \mid u(r_0) = a \text{ and } |u(r) - a| \leq 1 \}
\]

with sup norm in \([r_0, r_1]\) where \( r_1 > r_0 \) will be determined later. It is clear that \( S \) is a closed subspace of \( C([r_0, r_1]) \). Define a mapping \( v = \Gamma u \) from \( S \) to \( S \) as follows:

\[
v(r) = \Gamma u(r) = a + \frac{1}{n-2} r_0 [1 - (\frac{r_0}{r})^{n-2}] b
\]
where \( u(r) \in S \). It is clear that \( v(r) \in C([r_0, r_1]) \) and \( v(r_0) = a \).

Suppose \( r_0 \neq 0 \). Since \( g \) is bounded on \([r_0, r]\), we obtain using (3.4)

\[
|v(r) - a| = \left| \frac{1}{n-2} r_0 [1 - (\frac{r_0}{r})^{n-2}] g(s, u(s)) \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) \right| ds
\]

\[
+ \left| \frac{1}{n-2} \int_{r_0}^{r} g(s, u(s)) \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) ds \right| \leq c (r - r_0) + c \int_{r_0}^{r} \left| \frac{s^{n-2}}{r^{n-2}} - 1 \right| ds
\]

Select a value \( r_1 \) such that

\[
c_1(r_1 - r_0) \leq 1,
\]

which implies that \( v(r) \in S \). Now we prove that the mapping \( \Gamma \) is a contraction if \( r_1 - r_0 \) is sufficiently small. Let \( u_1(r), u_2(r) \in S \), then using (3.3),

\[
|v_1(r) - v_2(r)| = |\Gamma u_1(r) - \Gamma u_2(r)|
\]

\[
= \left| \frac{1}{n-2} \int_{r_0}^{r} g(s, u_1(s)) \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) ds \right|
\]

\[
- \left| \frac{1}{n-2} \int_{r_0}^{r} g(s, u_2(s)) \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) ds \right|
\]

\[
\leq \frac{1}{n-2} \int_{r_0}^{r} \left| g(s, u_1(s)) - g(s, u_2(s)) \right| s \left| \frac{s^{n-2}}{r^{n-2}} - 1 \right| ds
\]

\[
\leq c_2(r_1 - r_0) \max_{r \in [r_0, r_1]} |u_1(r) - u_2(r)|.
\]

Again choose \( r_1 - r_0 \) sufficiently small such that \( c_2(r_1 - r_0) < 1 \). Then \( \Gamma \) maps \( S \) into \( S \) and is a contraction. By the Banach Contraction Principle, \( \Gamma \) has a unique fixed
point \( u(r) \in S \) such that \( u(r) = \Gamma u(r) \). This fixed point is the unique solution of (3.6).

If \( r_0 = 0 \), then from (3.10), (3.12), (3.4) and (3.3) we have

\[
|v(r) - a| = \left| \frac{1}{n-2} \int_0^r g(s, u(s)) s \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) ds \right| 
\leq c \int_0^r s^{1-\alpha} \left| \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) \right| ds \leq c_1 r_1^{2-\alpha}.
\]

\[
|v(r_1) - v_2(r)| \leq \frac{1}{n-2} \int_0^r s^{1-\alpha} \left| g(s, u_1(s)) - g(s, u_2(s)) \right| s \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) ds
\leq c \max_{r \in [0, r_1]} |u_1(r) - u_2(r)| \cdot \int_0^r s^{1-\alpha} \left| \left( \frac{s^{n-2}}{r^{n-2}} - 1 \right) \right| ds
\leq c_2 r_1^{2-\alpha} \max_{r \in [0, r_1]} |u_1(r) - u_2(r)|.
\]

Thus, if we choose \( r_1 \) sufficiently small, again \( \Gamma \) maps \( S \) into \( S \) and is a compression. Finally, from (3.5) and (3.7), we can easily see that \( u(r) \in C^2((r_0, r_1]) \cap C^1([r_0, r_1]) \). The proof is finished.

Now let us extend the local solution \( u(r) \) of (3.1) and (3.2) to whole interval \((0, \infty)\) if an a priori estimate is known.

**Theorem 3.2** Under the assumptions of Theorem 1, if we know the solution \( u(r) \) of (3.1) and (3.2) is bounded in \((0, \infty)\), then \( u(r) \) can be extended to the whole interval \((0, \infty)\).

**Proof.** Denote by \( u(r; a, b, r_0) \) the unique solution of (3.1) and (3.2) guaranteed by Theorem 3.1 in some interval \([r_0, r_1]\). We shall continue this solution to the right by setting \( u(r_1; a, b, r_0) = a_1 \) and \( u'(r_1; a, b, r_0) = b_1 \) and again solving (3.1). We can get a solution \( u(r; a_1, b_1, r_1) \) defined in \([r_1, r_2]\). Now we need to prove that \( u(r; a, b, r_0) = u(r; a_1, b_1, r_1) \) for \( r \in [r_1, r_2] \). Using (3.6) and (3.5), we have

\[
u(r; a_1, b_1, r_1) = u(r_1; a, b, r_0) + \frac{1}{n-2} r_1[1 - \left( \frac{r_1}{r} \right)^{n-2}] u'(r_1; a, b, r_0)
\]
\[
+ \frac{1}{n-2} \int_{r_1}^{r} g(s,u(s)) s \left( \frac{s^{n-2}}{s^{n-2}} - 1 \right) ds 
\]
\[
= a + \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{r_1} \right)^{n-2} \right] b 
\]
\[
+ \frac{1}{n-2} \int_{r_0}^{r_1} g(s,u(s)) s \left( \frac{s^{n-2}}{r_1^{n-2}} - 1 \right) ds 
\]
\[
+ \frac{1}{n-2} \int_{r_1}^{r} g(s,u(s)) s \left( \frac{s^{n-2}}{r_1^{n-2}} - 1 \right) ds 
\]
\[
= a + \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{r_1} \right)^{n-2} + \left( \frac{r_0}{r_1} \right)^{n-2} - \left( \frac{r_0}{r_1} \right)^{n-2} \right] b 
\]
\[
+ \frac{1}{n-2} \int_{r_0}^{r_1} g(s,u(s)) s \left( \frac{s^{n-2}}{r_1^{n-2}} - 1 - \frac{s^{n-2}}{r_1^{n-2}} + \frac{s^{n-2}}{r_1^{n-2}} \right) ds 
\]
\[
+ \frac{1}{n-2} \int_{r_1}^{r} g(s,u(s)) s \left( \frac{s^{n-2}}{r_1^{n-2}} - 1 \right) ds 
\]
\[
= a + \frac{1}{n-2} r_0 \left[ 1 - \left( \frac{r_0}{r_1} \right)^{n-2} \right] b 
\]
\[
+ \frac{1}{n-2} \int_{r_0}^{r_1} g(s,u(s)) s \left( \frac{s^{n-2}}{r_1^{n-2}} - 1 \right) ds. 
\]

Thus, according to this process, we can extend \( u(r) \) continuously to the right as long as \( u(r) \) is bounded. Finally, we prove \( u(r) \) can be extended to \( r = \infty \). Suppose that on the \( i^{th} \) step \( u(r) \) can be extended to \([r_0, r_i] \). If \( r_i \to \alpha < \infty \) as \( i \to \infty \) and \([r_0, \alpha) \) is the maximum interval of \( u(r) \), let \( r \to \alpha \) in (3.5) and (3.6). We get \( u(\alpha) \) and \( u'(\alpha) \) are finite since \( u(r) \) is bounded. Then, using the numbers \( u(\alpha) \) and \( u'(\alpha) \) as the initial values and solving (3.1) again, we can extend \( u(r) \) to \([r_0, \alpha + h] \) for some small \( h > 0 \), which is a contradiction. This proves the theorem.
Chapter 4

Nonexistence of Positive Solutions

We now show that problem (1.7) does not have positive radial solutions if the order $q$ of the zero of $f(u)$ at the origin is in the interval $1 < q < n/(n - 2)$. For $q$ outside this interval, positive solutions may exist: for example, if $n = 3$ and $q \geq n/(n - 2) = 3$, then

$$u'' + \frac{2}{r} u' + \frac{4q u_0}{(q - 1)^2} \left( \frac{u}{u_0} \right)^{2q - 1} + \frac{2(q - 3)u_0}{(q - 1)^2} \left( \frac{u}{u_0} \right)^q = 0,$$

has the positive solution $u(r) = u_0(1 + r^2)^{-1/(q - 1)}$. (In Appendix 3, we verify this fact.)

Suppose that the solution $u$ of (1.7) oscillates about zero a finite number of times and has a local maximum at $r_0$ after which $u(r) > 0$ for all $r \geq r_0$. We call such solutions eventually positive solutions.

**Theorem 4.1** If $f$ and $g$ are continuous, $f(u) > 0$ and $g(r,u) > 0$ for $u > 0$, and

$$\lim_{u \to +0} \frac{f(u)}{u^q} = B > 0, \quad \text{for } 1 < q < n/(n - 2),$$

then the initial-value problem

$$u'' + \frac{n-1}{r} u' + f(u) + g(r,u) = 0, \quad \text{for } 0 < r < \infty,$$

$$u(0) = u_0 > 0, \quad u'(0) = 0,$$

has no positive solutions. Moreover, if $u_0 < 0$ or $u(r)$ becomes negative, there is no point $r_* \geq 0$ such that $u(r) > 0$ for all $r > r_* \geq 0$, that is, (4.2) has no eventually positive solutions.

**Proof.** Suppose that such a solution exists. Then there is a point $r_* \geq 0$ such that
$u(r) > 0$ for all $r_* \geq 0$. Let $\bar{r}$ be the first maximum such that $u(r) > 0$ for all $r \geq \bar{r}$.

By a calculation similar to (2.18), we have

\begin{equation}
(4.3) \quad u'(r) = -\frac{1}{r^{n-1}} \int_0^r [f(u(s)) + g(s,u(s))] s^{n-1} \, ds
\end{equation}

\begin{equation*}
= -\frac{1}{r^{n-1}} \int_{\bar{r}}^r [f(u(s)) + g(s,u(s))] s^{n-1} \, ds.
\end{equation*}

It follows that $u'(r) < 0$ for all $r > \bar{r}$ and $u \to 0$ as $r \to \infty$ by the proof after (2.5). Since $n > (n-2)q$, select $0 < \epsilon < 1/q$ and an integer $k > 0$ such that both of the following inequalities hold:

$$
\eta = n - (n-2)q - \frac{(n-2-\epsilon)(q-1)}{q^k - 1} > 0,
$$

and

\begin{equation*}
\frac{n-2-(n-2-\epsilon)(q-1)}{q^k - 1} > \frac{1}{q} > \epsilon = \frac{2 - \eta - [n-(n-2)q-\eta]q^k}{q-1}.
\end{equation*}

By L'Hopital's rule and (4.3), we have

\begin{equation}
(4.4) \quad \lim_{r \to \infty} \frac{u(r)}{r^{-\beta}} = \lim_{r \to \infty} \frac{u'(r)}{-\beta r^{-\beta - 1}} = \lim_{r \to \infty} \frac{\int_{\bar{r}}^r [f(u(s)) + g(s,u(s))] s^{n-1} \, ds}{\beta r^{n-2-\beta}}.
\end{equation}

If $\beta = n-2$, then since the integral in (4.4) is positive, some constant exists so that

\begin{equation}
(4.5) \quad u(r) \geq c_0 r^{-(n-2)}, \quad \text{for all } r \geq r_0 \geq \bar{r}.
\end{equation}

Define $\beta_{j+1} = \beta_j q - 2 + \eta$ with $\beta_0 = n-2$. Now we prove by induction that

\begin{equation}
(4.6) \quad \beta_j = (n-2-\epsilon) \left(1 - \frac{q^{j-1}}{q^k - 1}\right) + \epsilon.
\end{equation}

For $j = 0$, it is obvious that $\beta_0 = n-2$. Suppose (4.6) is true for $\beta_j$, then

\begin{equation*}
\beta_{j+1} = \beta_j q - 2 + \eta = q[(n-2-\epsilon) \left(1 - \frac{q^{j-1}}{q^k - 1}\right) + \epsilon] - 2 + \eta
\end{equation*}
\[= q(n-2-\epsilon) \left(1-\frac{q^j-1}{q^k-1}\right) + \epsilon q + n-(n-2)q - \frac{(n-2-\epsilon)(q-1)}{q^k-1} - 2\]

\[= (n-2-\epsilon) \left\{ \left(q-\frac{q^{j+1}-q}{q^k-1}\right) - q - \frac{q-1}{q^k-1} + 1 \right\} + \epsilon\]

\[= (n-2-\epsilon) \left(1-\frac{q^{j+1}-1}{q^k-1}\right) + \epsilon.\]

Thus (4.6) holds for all \(j < k\) by induction. From hypothesis, \(f(u)/u^q \geq \frac{1}{2} \beta\) for all \(u \leq u_*\), and since \(u \to 0\) as \(r \to \infty\), we can assume \(u(r) \leq u_*\) for all \(r \geq r_0\) (otherwise we can take \(r_0\) larger). Using (4.4) with \(\beta = \beta_1\) and (4.5), observe that

\[
\int_{r_0}^{r} \frac{[f(u(s)) + g(s,u(s))] s^{n-1} \, ds}{\beta_1 r^{n-2-\beta_1}} \geq \frac{1}{2} \beta \int_{r_0}^{r} u^q(s) s^{n-1} \, ds
\]

\[
\geq \frac{1}{2} B c_0 q \int_{r_0}^{r} s^{n-1-2(n-2)q} \, ds
\]

\[
= \frac{B c_0 q}{2 \beta_1 r^{n-2-\beta_1}} \left( \frac{r^{n-2(n-2)q} - r_0^{n-2(n-2)q}}{n-2(n-2)q} \right)
\]

\[
= C r^n \left(1 - \left(\frac{r_0}{r}\right)^{n-2(n-2)q}\right) \to \infty, \text{ as } r \to \infty.
\]

Thus, by (4.4) with \(\beta = \beta_1\), there exist constants \(c_1 > 0\) and \(r_1 > r_0\) such that

\[
(4.8) \quad u(r) \geq c_1 r^{-\beta_1}, \text{ for all } r \geq r_1.
\]

We can repeat the process in (4.7) with \(\beta = \beta_2\), obtaining \(u(r) \geq c_2 r^{-\beta_2}\), for \(r \geq r_2 > r_1\), and in general

\[
(4.9) \quad u(r) \geq c_j r^{-\beta_j}, \text{ for all } r \geq r_j > r_{j-1}, \text{ } j \leq k.
\]

Since \(\beta_k = \epsilon\), we have proved that \(u(r) \geq c_k r^{-\epsilon}\) for all \(r \geq r_k\). However, by (4.3) and (4.7),

\[
|u'(r)| = \frac{1}{r^{n-1}} \int_{r}^{R} [f(u(s)) + g(s,u(s))] s^{n-1} \, ds \geq \frac{1}{r^{n-1}} \frac{1}{2} \beta \frac{c_k q}{r^{n-1}} \int_{r_k}^{R} s^{n-1-\epsilon q} \, ds
\]
\[ C r^{l-\epsilon q} \left( 1 - \left( \frac{r_{\star}}{r} \right)^{n-\epsilon q} \right) \to \infty, \text{ as } r \to \infty, \]

which is a contradiction of the statements following (2.5). Thus, no positive solution of (4.2) exists and there is no point \( r_{\star} \geq 0 \) such that \( u(r) > 0 \) for all \( r > r_{\star} \geq 0 \). The theorem is proved.

**Corollary 4.2** Let \( f \) and \( g \) be continuous, \( uf(u) > 0 \) and \( ug(r,u) > 0 \) for all \( u \neq 0 \) and all \( r \), and

\[ B > 0, \quad 1 < q < n/(n-2). \]

Then the initial value problem

\[ u'' + \frac{n-1}{r} u' + f(u) + g(r,u) = 0, \text{ for } 0 < r < \infty, \]

\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \]

has no eventually positive or eventually negative solutions.

**Proof.** Theorem 4.1 gives the eventually positive case. The eventually negative case follows by letting \( v = -u \).

**Theorem 4.3** Let \( f \) be continuous, \( f(u) > 0 \) for \( u > 0 \),

(4.10) \[ \lim_{u \to 0^+} \frac{f(u)}{u^q} = B > 0, \quad \text{for } 1 < q < (n+2)/(n-2) \]

and suppose that

(4.11) \[ 2 F(u) \geq \left( 1 - \frac{2}{n} \right) u f(u) > 0 \text{ for all } u > 0. \]

Then the initial-value problem

(4.12) \[ u'' + \frac{n-1}{r} u' + f(u) = 0, \text{ for } 0 < r < \infty, \]
\[ u(0) = u_0 > 0, \quad u'(0) = 0, \]

has no positive solutions. Moreover, if \( u_0 < 0 \) or \( u(r) \) becomes negative, then, there is no point \( r_* \geq 0 \) such that \( u(r) > 0 \) for all \( r \geq r_* \), that is, (4.12) has no eventually positive solutions.

**Proof.** If \( 1 < q < n/(n-2) \), the conclusions are a special case of Theorem 4.1. So we only prove the conclusions for \( n/(n-2) \leq q < (n+2)/(n-2) \) or \( 2q/(q-1) \leq n < 2(q+1)/(q-1) \). Suppose the conclusions are not true. Then there exists a point \( r_* \geq 0 \) such that \( u(r) > 0 \) for all \( r \geq r_* \). By Lemma 2.3, we obtain that

\[
0 < u(r) \leq c r^{-2/(q-1)}, \quad \text{for all } r \geq r_0 \geq r_*.
\]

Using (4.3) with \( g(r,u) = 0 \) and (4.13), we get

\[
|u'(r)| = \left| \int_0^r f(u(s)) \left( \frac{s}{r}\right)^{n-1} ds \right|
\]

\[
\leq \frac{C}{r^{n-1}} + \frac{1}{r^{n-1}} \int_0^r \frac{f(u(s))}{u^q(s)} u^q(s) s^{n-1} ds
\]

\[
\leq \frac{C}{r^{n-1}} + \frac{2Bc^q}{r^{n-1}} \int_0^r s^{n-1-2q/(q-1)} ds
\]

\[
\leq \frac{C}{r^{n-1}} + c_1 r^{-(q+1)/(q-1)},
\]

where \( r_0 \geq r_* \) is such that

\[
0 < \frac{f(u(r))}{u^q(r)} \leq 2B \quad \text{for } r \geq r_0.
\]

Since \( q \geq n/(n-2) \), it is easy to show that \( n-1 \geq (q+1)/(q-1) \). Thus from (4.14),

\[
|u'(r)| \leq c_2 r^{-(q+1)/(q-1)} \quad \text{for large } r.
\]

By L'Hospital's rule, we have from (4.10)

\[
\lim_{r \to \infty} \frac{F(u(r))}{u^q+1(r)} = \lim_{r \to \infty} \frac{\int_0^r f(u)du}{u^q+1(r)} = \lim_{r \to \infty} \frac{f(u(r))}{(q+1)u^q(r)} = \frac{B}{q+1}.
\]
Now using (4.13), (4.16) and (4.15), we get

\begin{align*}
(4.17) \quad r^n F(u(r)) &\leq r^n \frac{F(u(r))}{u^{q+1}(r)} \left[ c r^{-2/(q-1)} \right]^{q+1} \\
&\leq c r^{n-2(q+1)/(q-1)} = c r^{-\alpha},
\end{align*}

\begin{align*}
(4.18) \quad r^{n-1} u(r)|u'(r)| &\leq r^{n-1}(c r^{-2/(q-1)}) (c_2 r^{-(q+1)/(q-1)}) \\
&\leq c r^{n-2(q+1)/(q-1)} = c r^{-\alpha},
\end{align*}

\begin{align*}
(4.19) \quad r^n |u'(r)|^2 &\leq r^n (c_2 r^{-(q+1)/(q-1)})^2 = c r^{-\alpha},
\end{align*}

for large \( r \), where \( \alpha = 2(q+1)/(q-1) - n > 0 \). Let \( r_1 \) be the first maximum point such that \( u(r) > 0 \) for all \( r > r_1 \). Then \( u'(r_1) = 0 \). Using Lemma 2.2 and (4.11), we get

\begin{align*}
(4.20) \quad J(r, u) = r^n u'^2(r) + (n-2) r^{n-1} u(r) u'(r) + 2 r^n F(u(r)) \\
&= \int_{r_1}^r \left[ 2 n F(u(s)) - (n-2) u(s) f(u(s)) \right] s^{n-1} ds + 2 r_1^n F(u(r_1)) \\
&\geq 2r_1 F(u(r_1)) > 0.
\end{align*}

But on the other hand, using (4.17) - (4.19), \( J(r, u(r)) \leq (n+1)c r^{-\alpha} \) and letting \( r \to \infty \), we get a contradiction of (4.20). The proof is finished.

**Corollary 4.4** Let \( f \) be continuous, \( u f(u) > 0 \) for \( u \neq 0 \), and assume that

\[ \lim_{u \to 0} \frac{f(u)}{u|u|^{q-1}} = B > 0, \quad 1 < q < (n+2)/(n-2), \]

and

\[ 2 F(u) \geq (1 - \frac{2}{B^q}) u f(u), \quad \text{for all } u. \]

Then

\[ u'' + \frac{n-1}{r} u' + f(u) = 0, \quad \text{for } 0 < r < \infty, \]
\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \]

has no eventually positive or eventually negative solutions.

**Proof.** The proof is almost identical with that of Theorem 4.3.

**Example 1.** In \( \mathbb{R}^3 \), both \( f(u) = u^3(1 + u^2)^{1/2} + u^5 \) and \( f(u) = u^4/(1 + u) + u^5 \) satisfy the conditions of Corollary 4.4 (the proof will be given in Appendix 4). Thus, the initial-value problem

\[ u'' + \frac{n-1}{r} u' + f(u) = 0, \quad \text{for } 0 < r < \infty, \]

\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \]

has no eventually positive or eventually negative solutions in either case.
Chapter 5

Oscillatory Behaviors

In this chapter we shall discuss some oscillatory behaviors for two kinds of equations. Consider the following initial-value problem

\[ u'' + \frac{n-1}{r} u' + f(u) = 0, \text{ for } 0 < r < \infty, \]

\[ u(0) = u_0 \neq 0, \quad u'(0) = 0, \]

**Lemma 5.1** Let \( f(u) \) be continuous. Suppose \( u \) is a solution of (5.1) that oscillates infinitely about the value \( b \) and converges to \( b \) as \( r \to \infty \). Then \( f(b) = 0 \).

**Proof.** Assume \( f(b) \neq 0 \). By continuity it follows that \( f(u(r)) \) does not change sign for \( r \) sufficiently large. Without loss of generality assume \( f(u(r)) < 0 \) for \( r > r_* \). Let \( r_0 > r_* \) be the value of the first local maximum of \( u(r) \) after \( r_* \). Then by (2.3), for all \( r > r_0 \),

\[ u'(r) = -\frac{1}{r^{n-1}} \int_{r_0}^{r} f(u(s)) s^{n-1} ds > 0, \]

which is impossible, since \( u(r_0) \) is a local maximum. Thus \( f(b) = 0 \).

Next we show that we need not assume convergence to \( b \) as \( r \to \infty \) for certain functions \( f(u) \).

**Lemma 5.2** let \( f(u) \) be continuous and suppose \( F(b) = \inf \{F(u) : u \in \mathbb{R} \} \). Suppose the solution of (5.1) oscillates about \( b \) as \( r \to \infty \), and assumes no other root of \( f(u(r)) \) for large \( r \). Then \( \lim_{r \to \infty} u(r) = b \).

**Proof.** Since \( f \) is continuous, \( f(b) = F'(b) = 0 \), so \( b \) is a root of \( f \). Suppose

\[ \lim_{r \to \infty} \sup u(r) \neq \lim_{j \to \infty} \inf u(r). \]
Let the local minimums of \( u(r) \) occur at \( r_{2j-1} \) and the local maximums at \( r_{2j} \). Then, by Lemma 2.1,

\[ u(r_1) < u(r_3) < u(r_5) < \ldots < u(r_6) < u(r_4) < u(r_2). \]

Thus, the maximums are bounded below, while the minimums are bounded above, so both series converge, say to \( u_+ \) and \( u_- \) respectively, with \( u_+ > u_- \). From (2.6), the energy function \( Q(u(r)) \) decreases and is bounded below since

\[ (5.2) \quad F(u_+) = \lim_{j \to \infty} F(u(r_{2j})) = \lim_{j \to \infty} Q(u(r_{2j})) = \lim_{r \to \infty} Q(u(r)) = \lim_{j \to \infty} Q(u(r_{2j-1})) = F(u_-). \]

By Lemma 2.5, with \( \beta = F(u_+) \),

\[ (5.3) \quad \lim_{j \to \infty} \frac{1}{r} \int_{r_0}^{r} [F(u_+) - F(u(s))] \, ds = 0. \]

Let \( g(s) = F(u_+) - F(u(s)) \). Then \( g(s) \) oscillates between the maximums \( g(t_i) = \beta - F(b) \) at \( u(t_i) = b \), and local minimums \( g(r_i) = \beta - F(u(r_i)) \to 0 \) as \( i \to \infty \) with \( r_i > t_i \). Select \( i_0 \) large enough that \( g(r_i) > -\delta \) for all \( i \geq i_0 \), where \( \delta = [\beta - F(b)]/5 \) and the only root of \( f(u) = 0 \) is \( u > b \) for \( r \geq r_0 \). Then, by the mean value theorem

\[ \frac{1}{t_j} \int_{t_0}^{t_j} g(s) \, ds = \frac{1}{t_j} \sum_{i=i_0}^{j-1} \int_{t_i}^{t_{i+1}} g(s) \, ds \]

\[ = \frac{1}{t_j} \sum_{i=i_0}^{j-1} g(s_i^*) (t_{i+1} - t_i) \geq \left[ \inf_{i} g(s_i^*) \right] \left( 1 - \frac{t_0}{t_j} \right). \]

If we can show that \( \inf_{i} g(s_i^*) > 0 \), then

\[ \lim_{j \to \infty} \frac{1}{t_j} \int_{t_0}^{t_j} g(s) \, ds > 0, \]

which will be a contradiction of (5.3).
Set \( A_i = \{ s \in (t_i, t_{i+1}) : |g(s)| < \delta \} \) and \( B_i = (t_i, t_{i+1}) \backslash A_i \). Denote the Lebesgue measure of a set \( A \) by \( \text{meas}(A) \). Then

\[
g(s_i^*) = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} g(s) \, ds = \frac{1}{\text{meas}(A_i) + \text{meas}(B_i)} \left( \int_{A_i} g(s) \, ds + \int_{B_i} g(s) \, ds \right)
\]

\[
\geq \frac{\delta \text{meas}(B_i)}{\text{meas}(A_i) + \text{meas}(B_i)} - \frac{\delta \text{meas}(A_i)}{\text{meas}(A_i) + \text{meas}(B_i)}.
\]

If we can prove that \( \text{meas}(A_i) \leq \frac{1}{4}(t_{i+1} - t_i) \), then it will follow that \( g(s_i^*) \geq \frac{3}{4}\delta - \frac{1}{4}\delta = \frac{1}{2}\delta > 0 \). From (2.4), since \( Q \) and \( F \) are bounded, it follows that \( |u'| \leq M < \infty \). By the mean value theorem

\[
0 < g(u_+) < g(u(t_{i+1})) - g(u(r_i)) = |f(u(r^*))| |u'(r^*)| (t_{i+1} - r_i) \leq c(t_{i+1} - r_i).
\]

Similarly, \( g(u_+) < c(r_i - t_i) \), so it follows that \( (t_{i+1} - t_i) \geq a > 0 \) for all \( i \).

Since \( f(u(r)) \neq 0 \) if \( u(r) \neq b \), and \( F'(u) = f(u) \), there are constants \( \alpha > 0 \) and \( \gamma > 0 \) independent of \( i \) such that

\[
(5.4) \quad |f(u(s))| \geq \alpha, \quad \text{if} \quad |b - u(s)| > \gamma \quad \text{and} \quad |u(s) - b| > \gamma \quad \text{if} \quad F(u(s)) \geq F(b) + \delta.
\]

Let \( p_i < p_i^* \) be two numbers such that \( g(p_i) = g(p_i^*) = \delta \) and \( t_i < p_i < r_i < p_i^* < t_{i+1} \) (See Figure 1). Since \( u''(r) = -(n-1)u'(r)/t - f(u(r)) \), and \( f(u(r)) \) doesn't change.
sign on the interval \( r_i < r < p_i^* \), we have

\[
|u'(r)| = |\int_{r_i}^{r} (f(u(s)) + \frac{n-1}{s} u'(s)) \, ds |
\]

\[
\geq \int_{r_i}^{r} [|f(u(s))| - \frac{n-1}{s} |u'(s)|] \, ds \geq \frac{\alpha}{2} (r - r_i), \quad \text{for } r \in (r_i, p_i^*),
\]

when \( i_0 \) is chosen large enough so that \( (n-1)|u'(r)|/r < \alpha/2 \) for \( r \geq r_i^* \).

But \( u'(r) \) doesn't change sign in \( r_i < r < p_i^* \), so integrating (5.5) we have

\[
|u(p_i^*) - u(r_i)| = |\int_{r_i}^{p_i^*} u'(s) \, ds| = |\int_{r_i}^{p_i^*} |u'(s)| \, ds| \geq \frac{\alpha}{4} (p_i^* - r_i)^2.
\]

Using (5.4) and (5.6) it follows that

\[
2\delta \geq |g(p_i^*) - g(r_i)| = |F(u(p_i^*)) - F(u(r_i))| = |f(u^*)| |u(p_i^*) - u(r_i)| \geq \frac{\alpha^2}{4} (p_i^* - r_i)^2.
\]

Similarly, \( 2\delta \geq (\alpha^2/4) (r_i - p_i)^2 \). Thus,

\[
\text{meas}(A_i) = p_i^* - p_i \leq \frac{4\sqrt{2}\delta}{a} \leq \frac{1}{4} a \leq \frac{1}{4} (t_{i+1} - t_i),
\]

if \( \delta > 0 \) is small enough. Hence, the proof is complete.

**Theorem 5.3** Let \( f(u) \) satisfy local Lipschitz condition and assume that

\[
\lim_{u \to 0^+} \frac{|f(u)|}{|u|^q} = B > 0, \quad \text{for } 1 < q < (n+2)/(n-2).
\]

Further, assume that

\[
2 F(u) \geq (1 - \frac{2}{n}) u f(u) > 0 \text{ for } u \neq 0.
\]

Then the solution of (5.1) will oscillate infinitely and tend to 0 as \( r \to \infty \).

**Proof.** From (5.7) and (5.8) it follows that the hypotheses of Corollary 4.4 are satisfied, so the solution \( u \) of (5.1) cannot be eventually positive or negative.
Hence, the solution must oscillate infinitely about 0. By (5.8) and continuity, it follows that $u=0$ is the only root of $f(u)=0$ and $F(u) \geq F(0) = 0$. The conclusion then follows from Lemma 5.2.

**Remark 5.4** If we add a stricter condition:

$$
(n-2)u f(u) > 2 F(u) \geq (1 - \frac{2}{n}) u f(u) > 0 \quad \text{for all } u \neq 0
$$

instead of (5.8), then, we have the a-priori estimate:

$$
|u(r)| \leq c r^{-2/(q+1)}, \quad \text{for all } r > 0.
$$

To verify (5.10) use the first Pokhozhaev identity (2.10) with $\alpha = n - 2$:

$$
r^2 \left[ \frac{1}{2} u'^2(r) + F(u(r)) \right] + (n-2) u(r) u'(r) + \frac{1}{2} (n-2)^2 (u^2(r)-u^2(0))
$$

\[ + \int_0^r [(n-2) u(s) f(u(s)) - 2F(u(s))] s \, ds = 0. \]

or

$$
r^2 F(u(r)) + \frac{1}{2} [ru'(r) + (n-2) u(r)]^2 + \int_0^r [(n-2) u(s) f(u(s)) - 2F(u(s))] s \, ds
$$

\[ = \frac{1}{2} (n-2)^2 u^2(0). \]

From (5.9) and (5.11), we get

$$
0 \leq (1 - \frac{2}{n}) u f(u) \leq 2F(u) \leq (n-2)^2 u^2(0) \, r^{-2}.
$$

Finally, using (5.7) we obtain (5.10).

**Example 2.** If $f(u) = (a|u|^{q-1} + |u|^{p-1})u$ with $a > 0$, then it is easy to see that $f(u)$ satisfies (5.7) and (5.8) or even (5.9). Thus all solutions of (5.1) will oscillate to 0 as $r \to \infty$ and (5.10) holds.
Now let us consider the following particular initial-value problem

\[ u'' + \frac{n-1}{r} u' + (u + a)|u|^{p-1} = 0, \text{ for } 0 < r < \infty, \]

\[ u(0) = u_0 \neq 0, \quad u'(0) = 0. \]

**Theorem 5.5** If \( p = (n+2)/(n-2) \) and \( a > 0 \), then all solutions of (5.12) will oscillate infinitely and tend to \(-a\) as \( r \to \infty\).

**Proof.** First, we prove that \( u(r) \) does not approach \(-a\) monotonically after finitely many oscillations. Suppose otherwise, and let \( r_1 \) be the last extreme point of \( u \). Then \( u(r) + a \) does not change sign for \( r > r_1 \). Then, since

\[ u'(r_1) = -\frac{1}{r^{n-1}} \int_0^{r_1} |u(s)|^{p-1}(a + u(s)) s^{n-1} ds = 0 \]

by L' Hospital's rule with \( \beta \leq n-2 \),

\[ \lim_{r \to \infty} \frac{u(r) + a}{r^\beta} = \lim_{r \to \infty} \frac{u'(r)}{-\beta r^{-\beta - 1}} \]

\[ = \lim_{r \to \infty} \left[ \int_0^{r_1} |u(s)|^{p-1}(a + u(s)) s^{n-1} ds + \int_{r_1}^r |u(s)|^{p-1}(a + u(s)) s^{n-1} ds \right] \]

\[ = \lim_{r \to \infty} \frac{\int_{r_1}^r |u(s)|^{p-1}(a + u(s)) s^{n-1} ds}{\beta r^{-\beta - 2 + n}}. \]

First, take \( \beta = n - 2 \). Then we have

\[ \lim_{r \to \infty} \frac{u(r) + a}{r^{-(n-2)}} = \lim_{r \to \infty} \int_{r_1}^r |u(s)|^{p-1}(a + u(s)) s^{n-1} ds \neq 0, \]

and the limit exists or is infinite since \( u(r) + a \) does not changes sign. So there is a constant \( c_1 > 0 \), and an \( r_2 > r_1 \) such that

\[ |u(r) + a| \geq c_1 r^{-(n-2)} \quad \text{and} \quad |u(r)|^{p-1} \geq \frac{a}{2}, \]

for \( r \geq r_2 \). If \( n-2 > 1 \), let \( \beta = n - 3 \) in (5.13) to obtain

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\[ \lim_{r \to \infty} \frac{u(r) + a}{r^{-(n-3)}} = \lim_{r \to \infty} \int_{r_2}^{r} \frac{|u(s)|^{p-1}|a + u(s)| s^{n-1} \, ds}{(n-3)r} \]
\[ \geq \lim_{r \to \infty} \frac{ac_1}{2(n-3)r} \int_{r_2}^{r} s \, ds = \infty. \]

Hence there is a constant \( c_2 > 0 \) such that
\[ |u(r) + a| \geq c_2 r^{-(n-3)} \]

for \( r \geq r_2 \). If \( n-3 > 1 \), we can continue the same steps above until \( n-k \leq 1 \). Then we get
\[ |u(r) + a| \geq c_{k-1} r^{-1} \]

for \( r \geq r_2 \). On the other hand
\[ |u'(r)| = \left| - \frac{1}{r^{n-1}} \int_{r_1}^{r} |u(s)|^{p-1}(a + u(s)) s^{n-1} \, ds \right| \]
\[ = \frac{1}{r^{n-1}} \int_{r_1}^{r} |u(s)|^{p-1}|a + u(s)| s^{n-1} \, ds \]
\[ \geq \frac{1}{r^{n-1}} \int_{r_2}^{r} |u(s)|^{p-1}|a + u(s)| s^{n-1} \, ds \]
\[ \geq \frac{ac_{k-1}}{2r^{n-1}} \int_{r_2}^{r} s^{n-2} \, ds \geq c_0 > 0, \]

for \( r > r_2 \), which is a contradiction because \( u'(r) \to 0 \) as \( r \to \infty \).

Next, we prove that \( u(r) \) cannot increase to 0 if \( u(0) < 0 \). Suppose otherwise, then \( u'(r) > 0 \) for all \( r \), and \( u(r) \to 0^- \), \( u'(r) \to 0 \) as \( r \to \infty \). Since
\[ u'(r) = - \frac{1}{r^{n-1}} \int_{0}^{r} |u(s)|^{p-1}(a + u(s)) s^{n-1} \, ds > 0 \]
and \( u(s) + a > 0 \) for all \( s > R \) large enough, we have
\[ \int_{0}^{\infty} |u(s)|^{p-1}(a + u(s)) s^{n-1} \, ds < \infty. \]
In fact, if (5.14) is not true, we have

\[ \int_{R}^{\infty} |u(s)|^{p-1}(a+u(s)) s^{n-1} ds = \infty. \]

Then \( u'(r) < 0 \) for sufficiently large \( r \) which is a contradiction. Thus

\[
\lim_{r \to \infty} \frac{-u(r)}{r^{-(n-2)}} = \lim_{r \to \infty} \frac{u'(r)}{(n-2)r^{-(n-1)}} = c > 0
\]

or \( |u(r)| \leq c_1 r^{-(n-2)} \) and \( u'(r) \leq c_2 r^{-(n-1)} \) for all \( r > 0 \). Using Lemma 2.2 with \( r_0 = 0 \) we get

\[
2r^n \left( \frac{a|u(r)|^{p-1}u(r)}{p} + \frac{|u(r)|^{p+1}}{p+1} \right) + r^n u'^2(r)
\]

\[
= \frac{a(n-2)^2}{n+2} \int_{0}^{r} u(s) |u(s)|^{p-1} s^{n-1} ds < 0.
\]

Let \( r \to \infty \), then the left item of the equation (5.15) tends to 0, which is a contradiction.

Next, we prove that there exist \( c > 0 \) and \( r_0 > 0 \) such that \( u(r_0) < -c \) for all \( r \geq r_0 \). Since \( u(r) \) cannot be positive by Theorem 4.3 and cannot be always decreasing, there exists an \( r_1 > 0 \) such that \( u(r_1) < 0 \) and

\[
u'(r_1) = - \frac{1}{r_1^{n-1}} \int_{0}^{r_1} |u(s)|^{p-1}(a+u(s)) s^{n-1} ds = 0
\]

or

\[
a \int_{0}^{r_1} |u(s)|^{p-1}s^{n-1} ds = - \int_{0}^{r_1} |u(s)|^{p-1}u(s) s^{n-1} ds.
\]

On the other hand, applying Lemma 2.2 with \( r_0 = 0 \) and \( r = r_1 \), we obtain

\[
2r_1^n \left( \frac{a|u(r_1)|^{p-1}u(r_1)}{p} + \frac{|u(r_1)|^{p+1}}{p+1} \right)
\]

\[
= \frac{a(n-2)^2}{n+2} \int_{0}^{r_1} u(s) |u(s)|^{p-1} s^{n-1} ds.
\]
Using (5.16) and (5.17) we get

\[
2r_1^n Q(r_1) = 2r_1^n \left( \frac{a|u(r_1)|^{p-1}u(r_1)}{p} + \frac{|u(r_1)|^{p+1}}{p+1} \right)
= -\frac{a^2(n-2)^2}{n+2} \int_0^{r_1} |u(s)|^{p-1}s^{n-1} \, ds < 0.
\]

Thus \(0 > Q(u(r_1)) > Q(u(r))\) for all \(r > r_1\) which implies that \(u(r) < -c\) for some \(c > 0\) and all \(r \geq r_1\).

The only remaining possibility is that \(u(r)\) oscillates about some negative number as \(r \to \infty\). By Lemma 5.1 that number must be a zero of \(f(u) = 0\), hence \(u\) oscillates about \(-a\). Then, by Lemma 5.2 it follows that \(u \to -a\) as \(r \to \infty\). Hence the proof is complete.
Chapter 6

Conclusions

From the previous chapters we arrive at the following conclusions:

1. Consider the following initial-value problem

\begin{equation}
\tag{6.1}
\frac{2n}{r} u'' + u' + g(r, u) = 0, \quad \text{for } 0 \leq r_0 < r < \infty,
\end{equation}

\begin{equation}
\tag{6.2}
u(r_0) = a, \quad u'(r_0) = b,
\end{equation}

where \(a\) and \(b\) are any real number. We suppose that \(g\) satisfies a local Lipschitz condition with respect to the second variable, i.e.

\begin{equation}
\tag{6.3}
|g(r,u) - g(r,v)| \leq h(r,u,v) |u - v| \quad \text{with } \lim_{r \to 0^+} r^\alpha h(r,u,v) < \infty,
\end{equation}

and that \(g(r,u)\) is continuous in \(r\) and

\begin{equation}
\tag{6.4}
\lim_{r \to 0^+} r^\alpha |g(r,u)| < \infty \quad \text{for all } 0 \leq \alpha < 1.
\end{equation}

Then there exists an \(r_1 > r_0\) such that (6.1) and (6.2) have a unique solution \(u(r) \in C^2([r_0, r_1]) \cap C^1([r_0, r_1])\), which is a local solution because the interval \(r_1 - r_0\) may be very small. But if we know the solution \(u(r)\) of (6.1) and (6.2) is bounded in \((0, \infty)\), then \(u(r)\) can be extended to the whole interval \((0, \infty)\).

2. If \(f\) and \(g\) are continuous, \(f(u) > 0, \ g(r,u) > 0\) for \(u > 0\), and \(\lim_{u \to 0^+} f(u)/u^q = B > 0\) for \(1 < q < n/(n - 2)\), then the initial-value problem

\begin{equation}
\tag{6.5}
\frac{2n}{r} u'' + u' + f(u) + g(r,u) = 0, \quad \text{for } 0 < r < \infty,
\end{equation}

\[u(0) = u_0 > 0, \quad u'(0) = 0,\]
has no positive solutions. It is interesting that the nonexistence of positive solutions is a function of the order of the zero of \( f(u) \) at \( u=0 \), rather than the critical Sobolev exponent or some other leading term. This result is motivated by the following example.

The function \( u(r) = A/(B + r^2)^{q-1} \), \( A, B > 0 \), is a positive solution of the problem (see Appendix 3)

\[
(6.6) \quad u'' + \frac{n-1}{r} u' + f(u) = 0, \quad u(0) = A/B^{q-1}, \quad u'(0) = 0,
\]

with

\[
f(u) = \frac{4qB}{(q-1)^2A^{2(q-1)}} u^{q-1} + \frac{2(q(n-2)-n)}{(q-1)^2A^{q-1}} u^q.
\]

For both terms of \( f(u) \) to have nonnegative coefficients we must have \( q \geq n/(n-2) \). Observe that positive solutions exist for arbitrarily large exponents \( q > (n+2)/(n-2) = p \), the critical Sobolev exponent.

3. If \( \lim_{u \to 0^+} f(u)/u^q = B > 0 \) for \( 1 < q < (n+2)/(n-2) \) and \( 2nF(u) > (n-2)uf(u) > 0 \) for all \( u > 0 \), then the initial-value problem

\[
(6.7) \quad u'' + \frac{n-1}{r} u' + f(u) = 0, \quad 0 < r < \infty,
\]

\[
u(0) = u_0, \quad u'(0) = 0,
\]

with \( u_0 > 0 \) has no positive solution. The condition \( 2nF(u) > (n-2)uf(u) > 0 \) for all \( u > 0 \) is possibly too strong. We conjecture the condition can be weakened to \( f(u) > 0 \) for all \( u > 0 \) and

\( 2nF(u) > (n-2)uf(u) \) for sufficiently small \( u > 0 \).

Numerical calculations show that the conclusion is still true under this weakened condition.

4. If \( \lim_{u \to 0} uf(u)/|u|^{q+1} = B > 0 \) for \( 1 < q < (n+2)/(n-2) \) and \( 2nF(u) > 0 \) for all \( u > 0 \), then the initial-value problem

\[
\]

with \( u_0 > 0 \) has no positive solution. The condition \( 2nF(u) > (n-2)uf(u) > 0 \) for all \( u > 0 \) is possibly too strong. We conjecture the condition can be weakened to

\( f(u) > 0 \) for all \( u > 0 \) and

\( 2nF(u) > (n-2)uf(u) \) for sufficiently small \( u > 0 \).

Numerical calculations show that the conclusion is still true under this weakened condition.
(n - 2)uf(u) > 0 for all u ≠ 0, then all solutions of (5.1) will oscillate to 0 as r→∞.
However, the condition that 2nF(u) > (n - 2)uf(u) > 0 for all u ≠ 0 is possibly too strong. We conjecture the condition can be weakened to

2nF(u) > (n-2)uf(u) > 0 for sufficiently small u ≠ 0.
Numerical calculations show that the conclusion is still true.

5. If f(u) = (a + u)|u|^p in (6.7) with p = (n+2)/(n-2), then all solutions of (6.7) will oscillate to −a as r→∞. For the more general case f(u) = a|u|^q + |u|^{p-1}u with 1 < q < p = (n+2)/(n-2), I can prove that the solution will oscillate infinitely about b and tend to b, where b = −a^{1/(p-q)}, if q + 1 ≤ p. I plan to give that proof in my Ph. D thesis.

6. The theorems in this thesis apply to functions f(u) that satisfy uf(u) ≥ 0. The comments in 5. apply to f(u) ≥ 0 for u ≥ −a. However, it is of considerable interest to consider f(u) < 0 on several intervals. For example, f(u) = u^3(u+1)(u+2) is negative in −1 < u < 0 and u < −2. Hence it may be possible to find u_0 > 0 such that if u(0) < u_0 then the solution oscillates to zero, but if u(0) > u_0, the solution oscillates to −2. Such bifurcation phenomena have not been studied in the literature.

7. In Remark 5.4 I showed that |u| ≤ c r^{−2/(q+1)} implies that u ∈ L_m(R) for m > (q+1)/2. However, in arriving at that conclusion, I discarded several terms which indicates that much stronger growth conditions may be possible. It would be interesting to discover if the solutions were in L_2, as a Fourier analysis of u would then be possible.

8. Numerical experiments show that the interval between zeros of the solution grows as r → ∞. But does it stay bounded? In particular, are there constants m and M such that

m ≤ |u(r_i)|(t_{i+1} - t_i) ≤ M?

I plan to discuss these problems in my Ph. D thesis.
Appendix 1

In this appendix we give a brief explanation of a derivation of the equation (1.1), whose main idea comes from Kurth [K].

Consider a real stellar system with \( n \) gravitating particles, where \( n \gg 1 \). We assume that

1. four laws are valid: the law of conservation of mass, Newton's equations of motion, Newton's law of gravitation, and Schwarzschild's distribution law (the definition will be given later).

2. the real stellar system has a material continuum, streaming in a 6-dimensional position-velocity space.

3. the real system is considered by both a particle and continuum model, i.e., we define continuous probability distributions for the positions and states of motion of discrete particles, and then evaluate and interpret them. This applies statistical mechanics to the stellar system.

4. the system is stationary and has rotational symmetry.

Consider now a system of \( n \) particles. A particle \( j \) will be assumed to act on a particle \( i \) with a force \( k_{ij} \), whose magnitude is given by

\[
|k_{ij}| = G \frac{m_i m_j}{|r_i - r_j|^2},
\]

and whose force can be expressed by

\[
(A.1) \quad k_{ij} = G \frac{m_i m_j}{|r_i - r_j|^3} (r_i - r_j),
\]

where \( r_i = (x_i^1, x_i^2, x_i^3) \) and \( r_j \) are the position vectors of the two particles, \( m_i \) and \( m_j \) are their masses, and \( G \) is the constant of gravitation. By Newton's equations of motion, we have

\[
m_i \ddot{r}_i = G \sum_{j=1}^{n} \frac{m_i m_j}{|r_i - r_j|^3} (r_i - r_j), \quad \text{i = 1,2,\ldots,n.}
\]
For convenience, we consider the force exerted by $n - 1$ particles, of mass $m_1, \cdots, m_{n-1}$, with vector coordinates $r_1, \cdots, r_{n-1}$, on an $n^{th}$ particle of unit mass at the point $r$. We assume the $n$ positions are distinct. Then from (A.1) the force is given by

$$F = -G \sum_{i=1}^{n-1} \frac{m_i}{|r - r_i|^3} (r - r_i).$$

The right-hand side can be expressed as a gradient of (with respect to $r$)

$$V = G \sum_{i=1}^{n-1} \frac{m_i}{|r - r_i|},$$

which is called the potential. Thus the force $F = (F_1, F_2, F_3)$ can be expressed as the negative gradient of the scalar potential function $V$:

$$F_i = -\frac{\partial V}{\partial x_i}, \quad i = 1, 2, 3.$$

We also assume that distribution of mass in the system can be described by a mass density $\rho(r, V)$ having continuous derivatives with respect to the coordinates, and that this density and its derivatives decrease to zero as $|r| \to \infty$, such that the integrals what we deal with converge. We then represent the force at the point $r$ by

$$F_1(r) = -G \int \int \int_{R^3} \frac{\rho(\xi, \eta, \zeta, V)}{r^2} (x_1 - \xi) \, d\xi d\eta d\zeta,$$

with similar equations for $F_2$ and $F_3$, where $r = \sqrt{(x_1 - \xi)^2 + (x_2 - \eta)^2 + (x_3 - \zeta)^2}$, and the potential $V$ is given by

$$V(r) = G \int \int \int_{R^3} \frac{\rho(\xi, \eta, \zeta, V)}{r} \, d\xi d\eta d\zeta.$$

From this we can obtain that $V$ satisfies Poisson's equation

(A.2) $$\Delta V = \sum_{i=1}^{3} \frac{\partial^2 V}{\partial x_i^2} = 4\pi G \rho(r, V).$$

Now we discuss the relationship among $\rho$, $V$, and $r$. Let $x = (x_1, x_2, x_3)$ be

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orthogonal Cartesian coordinates in \( \mathbb{R}^3 \) and \( u = (u_1, u_2, u_3) \) be the corresponding velocity components. We first describe the distribution of material in \( \mathbb{R}^3 \) as a material density \( f(u,x) \geq 0 \), which is called the frequency function of the system. The integral

\[
\int \int _G f(u,x)dudx
\]

over any sub-region \( G \) of \( \mathbb{R}^3 \) shall equal the total mass of all the stars within \( G \), which implies that

\[
(A.3) \quad \rho = \int \int \int_{-\infty}^{+\infty} f(u,x)du_1du_2du_3.
\]

Schwarzschild's distribution law says that the frequency function has the form

\[
(A.4) \quad dn = n \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} \exp\left(-\frac{1}{2}\left( \frac{u_1^2}{\sigma_1^2} + \frac{u_2^2}{\sigma_2^2} + \frac{u_3^2}{\sigma_3^2} \right)\right)du_1du_2du_3,
\]

which is a good approximation to the number of stars whose random velocities lie in the interval

\[
u_1 \pm \frac{1}{2} du_1, \quad u_2 \pm \frac{1}{2} du_2, \quad u_3 \pm \frac{1}{2} du_3
\]

when suitable Cartesian axes are chosen. Here \( n \) is the number of stars considered, and the constants \( \sigma_1, \sigma_2, \sigma_3 \) are the root mean square dispersions of the velocity components in the three directions (see [K, page 7]). From (A.3) and (A.4), we get

\[
d\rho = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} g(x) \exp\left(-\frac{1}{2}\left( \frac{u_1^2}{\sigma_1^2} + \frac{u_2^2}{\sigma_2^2} + \frac{u_3^2}{\sigma_3^2} \right)\right)du_1du_2du_3dx_1dx_2dx_3
\]

where \( g(x_1,x_2,x_3) \) is a position function.

If the distributions are spherical symmetry, i.e. if both \( \rho = \rho(r) \) and \( V = V(r) \) are functions only of the radial distance \( r \), then it takes the form

\[
\frac{d^2V}{dr^2} + \frac{2}{r} \frac{dV}{dr} = 4\pi G\rho(r,V).
\]
For convenience, we use the simplified law (see [K, page 125]):

\[(A.5)\quad f(u, r) = c_1 \exp(-E/\sigma^2),\]

where \(E = \frac{1}{2} (u_1^2 + u_2^2 + u_3^2) + V(r)\) is the energy function, \(c_1\) and \(\sigma\) are normalized constants. By (A.3), the corresponding density function is

\[\rho(r, V) = c_2 \exp[-V(r)/\sigma^2],\]

where \(c_2 = c_1 \int \int \int \exp[-\frac{1}{2}(u_1^2 + u_2^2 + u_3^2)] \, du_1 \, du_2 \, du_3\). Poisson's equation becomes

\[\frac{d^2V}{dr^2} + \frac{2}{r} \frac{dV}{dr} = c_2 \exp[-V(r)/\sigma^2].\]

Letting \(V = -\sigma^2v\), we obtain

\[\frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} + c \exp v = 0.\]

Thus we obtain the equation (1.1). Researchers later found that the exponential law (A.5) leads to an infinite total mass, so they tried to use other kinds of frequency functions \(f\). One of those functions is

\[f = \begin{cases} (A - E)^k, & \text{for } E \leq A, \\ 0, & \text{for } E > A, \end{cases}\]

where \(k > 0\) (see [K, page]). Substituting \(f\) into (A.3), we get

\[\rho = \int \int \int [A - V - \frac{1}{2}(u_1^2 + u_2^2 + u_3^2)]^k \, du_1 \, du_2 \, du_3\]

the integral being taken over the sphere \(u_1^2 + u_2^2 + u_3^2 \leq 2(A - V)\). Using spherical coordinates and then changing the integral variable \(s = w/\sqrt{2(A - V)}\), we have

\[\rho = \int_0^{\sqrt{2(A - V)}} (A - V - \frac{1}{2} r^2)^k 4\pi w^2 dw\]

\[= (A - V)^{3/2 + k} 2^{3/2} \cdot 4\pi \int_0^1 (1 - s^2)^k s^2 \, ds.\]

Finally, if we set
\[ u = 1 - \frac{V}{A}, \quad t = \frac{A}{2\pi GA^{3/2} + k} \cdot \frac{4\pi}{2^{3/2}} \int_0^1 (1 - s^2)^k s^2 \, ds \]

(A.2) becomes

\[ \frac{d^2 u}{dt^2} + \frac{2}{t} \frac{du}{dt} + u^{3/2} + k = 0. \]
Appendix 2

In this appendix we verify the two Pokhozhaev identities (2.10) and (2.11):

For the first identity, integrating by parts, we get

\[
\int_0^r [\Delta u(s) + f(u(s))] [su'(s) + \alpha u(s)] s \, ds
\]

\[
= \int_0^r [u''(s)u'(s) s^2 + (n - 1)u'^2(s) s + \alpha u''(s)u(s) s
\]

\[
+ \alpha(n - 1)u'(s)u(s) + f(u(s))u'(s) s^2 + \alpha f(u(s))u(s) s] \, ds
\]

\[
= \frac{1}{2} u'^2(s) s^2 \bigg|_0^r - \int_0^r u'^2(s) s \, ds + (n - 1) \int_0^r u'^2(s) s \, ds + \alpha u'(s)u(s) s \bigg|_0^r
\]

\[
- \alpha \int_0^r u'^2(s) s \, ds - \alpha \int_0^r u'(s)u(s) \, ds + \frac{\alpha(n - 1)}{2} u^2(s) \bigg|_0^r
\]

\[
+ F(u(s)) s^2 \bigg|_0^r - 2 \int_0^r F(u(s)) s \, ds + \alpha \int_0^r f(u(s))u(s) s \, ds
\]

\[
= \frac{1}{2} u'^2(r) r^2 + \alpha u'(r)u(r) r + \frac{\alpha(n - 1)}{2} [u'^2(r) - u'^2(0)] + F(u(r)) r^2
\]

\[
+ (n - 2 - \alpha) \int_0^r u'^2(s) s \, ds - \frac{\alpha}{2} u^2(s) \bigg|_0^r + \int_0^r [\alpha u(s)f(u(s)) - 2F(u(s))] s \, ds
\]

\[
= r^2 Q(u(r)) + \alpha r u(r) u'(r) + \frac{\alpha}{2} (n - 2) [u^2(r) - u^2(0)]
\]

\[
+ (n - 2 - \alpha) \int_0^r u'^2(s) s \, ds + \int_0^r [\alpha u(s)f(u(s)) - 2 F(u(s))] s \, ds.
\]

Thus the identity (2.10) holds. For the second identity, integrating by parts again, we obtain

\[
\int_0^r [\Delta u(s) + f(u(s))] [su'(s) + \alpha u(s)] s^k \, ds
\]

\[
= \int_0^r [u''(s)u'(s) s^{k+1} + (n - 1)u'^2(s) s^k + f(u(s))u'(s) s^{k+1}
\]

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\[ + \alpha u''(s)u(s) s^k + \alpha(n-1)u'(s)u(s) s^{k-1} + \alpha u(s)f(u(s)) s^k \] ds \\
\[ = \frac{1}{2} u'^2(s) s^{k+1} |_0^r - \frac{(k+1)}{2} \int_0^r u'^2(s) s^k ds + (n-1) \int_0^r u'^2(s) s^k ds \]
\[ + F(u(s)) s^{k+1} |_0^r - (k+1) \int_0^r F(u(s)) s^k ds + \alpha u'(s)u(s) s^k |_0^r \]
\[ - \alpha \int_0^r u'^2(s) s^k ds - \alpha k \int_0^r u'(s)u(s) s^{k-1} ds + \frac{\alpha(n-1)}{2} u^2(s) s^{k-1} |_0^r \]
\[ - \frac{\alpha(n-1)(k-1)}{2} \int_0^r u^2(s) s^{k-2} ds + \alpha \int_0^r u(s)f(u(s)) s^k ds \]
\[ = \frac{1}{2} u'^2(r) r^{k+1} + F(u(r)) r^{k+1} + \alpha u'(r)u(r) r^k + \frac{\alpha(n-1)}{2} u^2(r) r^{k-1} \]
\[ - \frac{\alpha k}{2} u^2(s) s^{k-1} |_0^r + (n-1 - \frac{k+1}{2} - \alpha) \int_0^r u'^2(s) s^k ds \]
\[ + \frac{\alpha k(k-1)}{2} \int_0^r u^2(s) s^{k-2} ds - \frac{\alpha(n-1)(k-1)}{2} \int_0^r u^2(s) s^{k-2} ds \]
\[ + \int_0^r [\alpha u(s)f(u(s)) - (k+1)F(u(s))] s^k ds \]
\[ = r^{k+1}Q(u(r)) + \alpha r^k u(r) u'(r) + \frac{\alpha}{2} (n-1-k) r^{k-1}u^2(r) \]
\[ + (n-1 - \frac{k+1}{2} - \alpha) \int_0^r u'^2(s) s^k ds - \frac{\alpha(n-1-k)(k-1)}{2} \int_0^r u^2(s) s^{k-2} ds \]
\[ + \int_0^r [\alpha u(s)f(u(s)) - (k+1) F(u(s))] s^k ds. \]

Thus the identity (2.11) holds.
Appendix 3

In this appendix we verify that

\[(A.6) \quad u(r) = \frac{u_0}{(1 + r^2)^{1/(q-1)}}\]

satisfies the equation

\[(A.7) \quad u'' + \frac{2}{q} u' + \frac{4qu_0}{(q-1)^2} \left( \frac{u}{u_0} \right)^{2q-1} + \frac{2(q-3)u_0}{(q-1)^2} \left( \frac{u}{u_0} \right)^q = 0,\]

\[u(0) = u_0 > 0, u'(0) = 0.\]

**Verification** Differentiating the function \(u(r)\) in (A.6), we have

\[(A.8) \quad u'(r) = -\frac{2ru_0}{(q-1)(1 + r^2)^{1/(q-1)} + 1} = -\frac{2ru_0}{(q-1)(1 + r^2)^{q/(q-1)}}.\]

It is clear that \(u(r)\) satisfies the initial conditions. Differentiate (A.8) again

\[(A.9) \quad u''(r) = -\frac{2u_0}{(q-1)(1 + r^2)^{q/(q-1)}} + \frac{4qr^2u_0}{(q-1)^2(1 + r^2)^{(2q-1)/(q-1)}}.\]

Substituting (A.6), (A.8) and (A.9) into (A.7), we get

\[-\frac{2u_0}{(q-1)(1 + r^2)^{q/(q-1)}} + \frac{4qr^2u_0}{(q-1)^2(1 + r^2)^{(2q-1)/(q-1)}} - \frac{4ru_0}{r(q-1)(1 + r^2)^{q/(q-1)}}

\[+ \frac{4qu_0}{(q-1)^2} \left( \frac{u_0}{u_0(1 + r^2)^{1/(q-1)}} \right)^{2q-1} + \frac{2(q-3)u_0}{(q-1)^2} \left( \frac{u_0}{u_0(1 + r^2)^{1/(q-1)}} \right)^q\]

\[= -\frac{6(q-1)u_0 + 2(q-3)u_0}{(q-1)^2(1 + r^2)^{q/(q-1)}} + \frac{4qr^2 + 1)u_0}{(q-1)^2(1 + r^2)^{(2q-1)/(q-1)}}\]

\[= \frac{4qu_0 - 6qu_0 + 6u_0 + 2qu_0 - 6u_0}{(q-1)^2(1 + r^2)^{q/(q-1)}} = 0.\]
Appendix 4

In this appendix we verify in $\mathbb{R}^3$ that

\[(A.10) \quad f(u) = u^3 \sqrt{1 + u^2} + u^5 \quad \text{and} \]
\[(A.11) \quad f(u) = \frac{u^4}{1 + u} + u^5 \]

satisfy the conditions

\[(A.12) \quad \lim_{u \to 0^+} \frac{f(u)}{u^q} = B > 0, \quad \text{for } 1 < q < \frac{(n+2)/(n-2)}{5}, \]
\[(A.13) \quad 2 F(u) \geq (1 - \frac{2}{3}) u f(u) > 0 \quad \text{for all } u > 0.\]

It is clear that both (A.10) and (A.11) satisfy (A.12) with $q = 3$ and $q = 4$, respectively. Set

$$g(u) = 2F(u) - \frac{1}{3} uf(u),$$

and note that $g(0) = 0$ and

$$g'(u) = 2f(u) - \frac{1}{3} f(u) - \frac{1}{3} uf'(u) = \frac{5f(u) - uf'(u)}{3}.$$

Thus for (A.10)

$$g'(u) = \frac{2u^3 + u^5}{3 \sqrt{1 + u^2}} > 0 \quad \text{for } u > 0$$

and for (A.11)

$$g'(u) = \frac{u^4}{3(1 + u)} + \frac{u^5}{3(1 + u)^2} > 0 \quad \text{for } u > 0.$$

So $g(u) > 0$ and (A.13) holds.
References


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