1964

Introduction to Mathieu functions

Dennis Charles Pilling

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AN INTRODUCTION TO MATHIEU FUNCTIONS

by

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B.A. Western Washington State College, 1962
Presented in partial fulfillment of the requirements for
the degree of

Master of Arts

MONTANA STATE UNIVERSITY

1964

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ACKNOWLEDGEMENTS

I wish to express my appreciation to Professor Krishan K. Gorowara for his guidance and instruction throughout the preparation of this thesis. Also, I wish to thank Professor William R. Ballard, Professor William M. Myers and Randolph Jeppesen for their critical reading of the thesis.

D. C. P.
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INTRODUCTION

In a certain class of three-dimensional problems associated with sinusoidal wave motion, a displacement $u$, at any point $(x, y, z)$, must satisfy the wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

where $k$ is a constant.

The elliptic cylindrical coordinates $(\xi, \eta, z)$ can be related to the rectangular coordinates by the equations

$$x = c \cosh \xi \cos \eta$$
$$y = c \sinh \xi \sin \eta$$
$$z = z$$

where $c$ is an arbitrary real parameter, $0 \leq \xi < \infty$, $0 \leq \eta \leq 2\pi$ and $-\infty < z < \infty$. Thus the surface $\xi = \xi_0$ represents an elliptic cylinder

$$\frac{x^2}{c^2 \cosh^2 \xi_0} + \frac{y^2}{c^2 \sinh^2 \xi_0} = 1$$

and the surface $\eta = \eta_0$ is defined by the hyperbolic cylinder

$$\frac{x^2}{c^2 \cos^2 \eta_0} - \frac{y^2}{c^2 \sin^2 \eta_0} = 1$$

To transform the wave equation to elliptical coordinates, write

$$\mathcal{J} = x + iy = c \cosh \xi \cos \eta + ic \sinh \xi \sin \eta$$
$$= c \cosh (\xi + i \eta)$$
$$= c \cosh \omega.$$
\[ \vec{f} = x - iy = c \cosh \xi \cos \eta - ic \sinh \xi \sin \eta \]

\[ = c \cosh (\xi - i\eta) \]

\[ = c \cosh \vec{\omega} \]

Then \[ \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{1}{2} + \frac{\partial u}{\partial y} \cdot \frac{1}{21} = \]

\[ \frac{1}{2} \left[ \frac{\partial u}{\partial x} + \frac{1}{1} \frac{\partial u}{\partial y} \right]. \]

\[ \frac{\partial^2 u}{\partial \bar{r} \partial \bar{r}} = \frac{1}{2} \left[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{1}{1} \frac{\partial u}{\partial y} \right) \frac{\partial x}{\partial \bar{r}} + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{1}{1} \frac{\partial u}{\partial y} \right) \frac{\partial y}{\partial \bar{r}} \right] \]

\[ = \frac{1}{2} \left[ (-\frac{\partial^2 u}{\partial x^2} + \frac{1}{1} \frac{\partial^2 u}{\partial y \partial x}) \frac{1}{2} + (\frac{\partial^2 u}{\partial x \partial y} + \frac{1}{1} \frac{\partial^2 u}{\partial y^2}) (-\frac{1}{21}) \right] \]

\[ = \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

or \[ 4 \frac{\partial^2 u}{\partial \bar{r} \partial \bar{r}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \]

Similarly, \[ 4 \frac{\partial^2}{\partial \omega \partial \bar{\omega}} = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}. \]

Since \[ \bar{f} = c \cosh \omega \quad \text{and} \quad \bar{\omega} = c \cosh \bar{\omega} ; \quad \frac{\partial \omega}{\partial \bar{r}} = \frac{1}{c \sinh \omega} \]

\[ \frac{\partial \bar{\omega}}{\partial \bar{r}} = \frac{1}{c \sinh \bar{\omega}}, \quad \frac{\partial \omega}{\partial \bar{\omega}} = 0, \quad \frac{\partial \bar{\omega}}{\partial \bar{\omega}} = 0. \]

Therefore \[ \frac{\partial u}{\partial \bar{r}} = \frac{1}{c \sinh \omega} \frac{\partial u}{\partial \omega} \quad \text{and} \quad \frac{\partial u}{\partial \bar{\omega}} = \frac{1}{c \sinh \bar{\omega}} \frac{\partial u}{\partial \bar{\omega}}. \]

\[ \frac{\partial^2 u}{\partial \bar{\omega} \partial \bar{\omega}} = \frac{1}{c^2 \sinh \omega \sinh \bar{\omega}} \frac{\partial^2 u}{\partial \omega \partial \bar{\omega}}. \]

\[ 4 \frac{\partial^2}{\partial \bar{r} \partial \bar{r}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{4}{c^2 \sinh \omega \sinh \bar{\omega}} \frac{\partial^2}{\partial \omega \partial \bar{\omega}} \]

\[ = \frac{4}{c^2 \sinh \omega \sinh \bar{\omega}} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \]

\[ = \frac{2}{c^2 (\cosh 2\xi - \cos 2\eta)} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right). \]

Therefore \[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{2}{c^2 (\cosh 2\xi - \cos 2\eta)} \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right). \]

Thus the wave equation in elliptic cylindrical co-ordinates is
\[
\frac{d^2 u}{d\xi^2} + \frac{d^2 u}{d\eta^2} + \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta) \frac{d^2 u}{dz^2} + k^2 \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta) u = 0.
\]

We now assume a solution \( u \) of the form \( u = M(\xi)m(\eta)Z(z) \) can be found, and we insert this expression into the equation, obtaining
\[
mZ \frac{d^2 M(\xi)}{d\xi^2} + MZ \frac{d^2 m(\eta)}{d\eta^2} + \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta) Mm \frac{d^2 Z(z)}{dz^2} + k^2 \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta) mMZ = 0.
\]

By the method of separation of variables, we divide through by \( u \) and replace \( \frac{d^2 Z(z)}{dz^2} \) by \( (-\alpha^2) \) getting
\[
\frac{1}{M} \frac{d^2 M(\xi)}{d\xi^2} + \frac{1}{m} \frac{d^2 m(\eta)}{d\eta^2} + \frac{c^2}{2} (\cosh 2\xi - \cos 2\eta)(\alpha^2) + \frac{c^2k^2}{2} (\cosh 2\xi - \cos 2\eta) = 0.
\]

Hence we obtain the following three ordinary differential equations,
\[
\frac{d^2 M(\xi)}{d\xi^2} + [-\lambda + \frac{c^2}{2} (\kappa^2 - \alpha^2) \cosh 2\xi ] M(\xi) = 0
\]
\[
\frac{d^2 m(\eta)}{d\eta^2} + [ \lambda - \frac{c^2}{2} (\kappa^2 - \alpha^2) \cos 2\eta ] m(\eta) = 0
\]
\[
\frac{d^2 Z(z)}{dz^2} + \alpha^2 Z(z) = 0.
\]

The first and second of these equations are known as the modified Mathieu equation and the Mathieu equation, respectively. The last is the simple harmonic equation.

We write the Mathieu equation in the form
\[
\frac{d^2 y}{dz^2} + [\alpha - 2q \cos 2z] y = 0.
\]
In this thesis we propose to discuss solutions of the Mathieu equation only.
CHAPTER I

FUNCTIONS OF INTEGRAL ORDER

\[
\frac{d^2y}{dz^2} + (a-2q \cos 2z)y = 0
\]

is the standard form of the (1.1) Mathieu equation with \( a, q = k^2 \) real parameters, whose coefficients are periodic functions. For the present we shall confine our attention to solutions having period \( \pi \) or \( 2\pi \) in \( z \).

When \( q = 0 \), the equation becomes \( \frac{d^2y}{dz^2} + ay = 0 \). Putting \( a = m^2 \), \( m = 1, 2, 3, \ldots \) we obtain solutions \( \cos mz, \sin mz \).

We shall adopt the convention that the coefficients \( A \) of \( A \cos mz \) and \( B \) of \( B \sin mz \) are unity unless stated otherwise.

When \( q \neq 0 \), \( a \) must be a function of \( q \) for the solution of the Mathieu equation to have period \( \pi \) or \( 2\pi \). We write

\[
a = m^2 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 + \ldots \,
\]

then when \( q = 0 \) the equation reduces to

\[
\frac{d^2y}{dz^2} + m^2 y = 0
\]

whose solutions are \( \cos mz, \sin mz \) adopting the positive sign for convention.

To illustrate one method of finding a particular periodic solution of the Mathieu equation, consider the case where \( a = m^2 = 1 \) when \( q = 0 \). Then \( a = 1 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 + \ldots \). Since the solution is to reduce to, say, \( \cos z \) when \( q \) vanishes, we assume that

\[
y = \cos z + q c_1(z) + q^2 c_2(z) + \ldots \]

with the \( c_i \)'s to be determined. Substituting into the Mathieu equation,

\[
y'' + ay - 2q \cos 2zy = - \cos z + qc_1'' + q^2 c_2'' + q^3 c_3'' + \ldots,
\]

\[
+ \cos z + q(c_1'' + \alpha_1 \cos z) + q^2(c_2'' + \alpha_1 c_1' + \alpha_2 \cos z) + \ldots
\]

\[
- q(\cos z + \cos 3z) - 2q^2 c_1 \cos 2z - 2q^3 c_2 \cos 2z - \ldots = 0,
\]

-5-
Equating coefficients of like powers of \( q \) to zero gives for

\[ q^0: \cos z - \cos z = 0, \]

and for

\[ q: \quad c''_1 + c_1 - \cos 3z + (a_1 - 1) \cos z = 0. \]

The particular integral corresponding to \( (a_1 - 1) \cos z \) is the non-periodic function \( \frac{1}{2}(1-a_1) z \sin z \). Since \( y \) is to be periodic, this term must vanish, so \( a_1 = 1 \), while \( c''_1 + c_1 = (1,0) \cos 3z \). Now the particular integral of \( v'' + v = A \cos mz \) is \( -A \frac{\cos \frac{m \pi z}{m^2-1}}{m^2-1} \), so with \( A = 1 \) and \( m = 3 \), \( c_1 = -\frac{1}{8} \cos 3z \). (1.0)

Inclusion of the complementary function of (1,0) in (1.0a) would involve a term in \( q \sin z \) and \( q \cos z \) in the solution. Also the \( \sin z \) is odd. Hence the C.F. of (1,0) and subsequent equations is omitted. Hence to show,

\[ q^2: \quad c''_2 + c_2 + a_1 c_1 - 2c_1 \cos 2z + 2c_2 \cos z = 0. \]

Substituting for \( a_1, c_1 \) we get,

\[ c''_2 + c_2 - \frac{1}{8} \cos 3z + \frac{1}{8} \cos 5z + (\frac{1}{8} + \alpha_2) \cos z = 0. \]

Again \( \alpha_2 = -\frac{1}{8} \) for the particular integral of \( (\frac{1}{8} + \alpha_2) \cos z \) is non-periodic.

\[ c''_2 + c_2 = \frac{1}{8} \cos 3z - \frac{1}{8} \cos 5z, \]

\[ c_2 = -\frac{1}{64} \cos 3z + \frac{1}{192} \cos 5z. \]

Continuing in this way and substituting for \( c_1, c_2 \ldots \) into

\[ y = \cos z + q c_1(z) + q^2 c_2(z) + q^3 c_3(z) + \ldots \]

gives a solution of Mathieu's equation, periodic in \( z \), with period \( 2\pi \) denoted by \( ce_1(z,q) \) and represented by the series

\[ ce_1(z,q) = \cos z - \frac{1}{8} q \cos 3z + \frac{1}{64} q^2 (-\cos 3z + \frac{1}{3} \cos 5z) \]
\[ -\frac{1}{512} q^3 \left( \frac{1}{3} \cos 3z - \frac{4}{9} \cos 5z + \frac{1}{18} \cos 7z \right) \]
\[ + \frac{1}{4096} q^4 \left( \frac{11}{9} \cos 3z + \frac{1}{6} \cos 5z - \frac{1}{12} \cos 7z + \frac{1}{180} \cos 9z \right) \]
\[ + O(q^5). \]

For a given \( q \), the value of \( a \) is called the characteristic number of the Mathieu function \( ce_1(z,q) \).

The notation \( ce_m(z,q) \) signifies a cosine type of Mathieu function of order \( m \). Since \( m \) may be any positive integer, there is an infinite number of solutions of type \( ce_m(z,q) \); each is an even function of \( z \).

For a second solution of the Mathieu equation we assume that \( y = \sin z + qs_1(z) + q^2s_2(z) + q^3s_3(z) + \ldots \) where \( a = 1 \) when \( q = 0 \) and proceed as above, obtaining a sine type of Mathieu function designated \( se_1(z,q) \). We find

\[ se_1(z,q) = \sin z - \frac{1}{8} q \sin 3z + \frac{1}{64} q^2(\sin 3z + \frac{1}{3}\sin 5z) \]
\[ - \frac{1}{512} q^3(\frac{1}{3} \sin 3z + \frac{4}{9} \sin 5z + \frac{1}{18} \sin 7z) \]
\[ + \frac{1}{4096} q^4(-\frac{11}{9} \sin 3z + \frac{1}{6} \sin 5z + \frac{1}{12} \sin 7z + \frac{1}{180} \sin 9z) + O(q^5) \]

where \( a = 1 - q - \frac{1}{8} q^2 + \frac{1}{64} q^3 - \frac{1}{1536} q^4 - \frac{11}{36864} q^5 + O(q^5) \) is the characteristic number for the Mathieu function \( se_1(z,q) \), periodic in \( z \) with period \( 2\pi \).

**Periodic Solution of the Mathieu Equation of order \( m \).**

Assume the cosine type of function,

\[ y = \cos mz + q\alpha_1(z) + q^2\alpha_2(z) + q^3\alpha_3(z) + \ldots \]

where \( a = m^2 + a_1q + a_2q^2 + a_3q^3 + \ldots \).
In general, the solution obtained is either of the type $ce_{2n}(z,q)$ or $ce_{2n+1}(z,q)$ according as $m$ is even or odd, thereby obtaining a solution of period $\pi$ or $2\pi$ respectively. In the case $m = 2$, we obtain the solution

$$ce_2(z,q) = \cos 2z - \frac{1}{8} q(\frac{2}{3} \cos 4z) + \frac{1}{384} q^2 \cos 6z - \frac{1}{512} q^3(\frac{1}{45} \cos 8z + \frac{43}{27} \cos 4z + \frac{44}{3}) + \ldots.$$  

with $a = 4 + \frac{5}{12} q^2 - \frac{763}{13824} q^4 + \frac{1002401}{79626240} q^6 + O(q^8)$.

When $q = 0$, $a = 4$ and $ce_2 = \cos 2z$.

When $q \neq 0$, $ce_2(z,q)$ has a constant term;

$$\frac{1}{4} q - \frac{5}{192} q^3 + \frac{1363}{221184} q^5 + O(q^7).$$

This function is periodic in $z$ with period $\pi$.

In the case $m = 0$, we obtain the solution

$$ce_0(z,q) = 1 - \frac{1}{2} q \cos 2z + \frac{1}{32} q^2 \cos 4z - \frac{1}{128} q^3(\frac{1}{9} \cos 6z - 7z \cos 2z) + \frac{1}{73728} q^4(\cos 8z - 320 \cos 4z) + O(q^5)$$  

with $a = -\frac{1}{2} q^2 + \frac{7}{128} q^4 - \frac{29}{2304} q^6 + O(q^8)$.

when $q = 0$, $a = 0$ and $ce_0 = 1$.

Similarly for the sine type of function

$$y = \sin mz + qs_1(z) + q^2s_2(z) + q^3s_3(z) + \ldots$$

$$a = m^2 + a_1q + a_2q^2 + a_3q^3 + \ldots.$$  

In general, the solution obtained is either $se_{2n+1}$ or $se_{2n+2}$ according as $m$ is odd or even.

In the case $m = 2$, we obtain the solution

$$se_2(z,q) = \sin 2z - \frac{1}{12} q \sin 4z + \frac{1}{384} q^2 \sin 6z - \frac{1}{512} q^3(\frac{1}{45} \sin 8z - \frac{5}{27} \sin 4z) +$$  

with $a = 4 - \frac{1}{12} q^2 + \frac{5}{13824} q^4 + O(q^6)$.  

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When \( q = 0 \), \( a = 4 \) and \( \text{se}_2(z,0) = \sin 2z \).

The function \( \text{se}_2(z,q) \) is periodic in \( z \) with period \( \pi \). The characteristic numbers for \( \text{ce}_m(z,q) \) and \( \text{se}_m(z,q) \) are designated by \( a_m, b_m \) respectively.

The preferred forms of the series \( \text{ce}_m(z,q) \) and \( \text{se}_m(z,q) \) are given below to facilitate the application of standard convergence tests.

\[
\begin{align*}
\text{ce}_{2n}(z,q) &= \sum_{r=0}^{\infty} A_{2r} (2n)^{r} \cos 2rz \quad (a_{2n}) \quad (1.2) \\
\text{ce}_{2n+1}(z,q) &= \sum_{r=0}^{\infty} A_{2r+1} (2n+1)^{r} \cos(2r+1)z \quad (a_{2n+1}) \quad (1.3) \\
\text{se}_{2n+1}(z,q) &= \sum_{r=0}^{\infty} B_{2r+1} (2n+1)^{r} \sin(2r+1)z \quad (b_{2n+1}) \quad (1.4) \\
\text{se}_{2n+2}(z,q) &= \sum_{r=0}^{\infty} B_{2r+2} (2n+2)^{r} \sin(2r+2)z \quad (b_{2r+2}) \quad (1.5)
\end{align*}
\]

where \( A, B \) are functions of \( q \).

If in the Mathieu equation (1.1) we replace \( z \) by \( \frac{1}{2}z \), it becomes

\[y'' + (a + 2q \cos 2z)y = 0.\]

Then the solutions of (1.1) with period \( \pi \), \( 2\pi \), \( q \) being negative, are obtained if the above substitution is made in (1.2) to (1.5). Hence we define,
\[
\text{ce}_{2n}(z,-q) = (-1)^n \text{ce}_{2n}\left(\frac{1}{2\pi} - z, q\right) \\
= (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cos 2rz \quad (a_{2n}) (1.6)
\]

\[
\text{ce}_{2n+1}(z,-q) = (-1)^n \text{se}_{2n+1}\left(\frac{1}{2\pi} - z, q\right) \\
= (-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \cos (2r+1)z \quad (b_{2n+1}) (1.7)
\]

\[
\text{se}_{2n+1}(z,-q) = (-1)^n \text{ce}_{2n+1}\left(\frac{1}{2\pi} - z, q\right) \\
= (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \sin (2r+1)z \quad (a_{2n+1}) (1.8)
\]

\[
\text{se}_{2n+2}(z,-q) = (-1)^n \text{se}_{2n+2}\left(\frac{1}{2\pi} - z, q\right) \\
= (-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} \sin (2r+2)z \quad (b_{2n+2}) (1.9)
\]

The multiplier \((-1)^n\) insures that when \(q = 0\), the functions reduce to \(\cos mz\) in \(\sin mz\) as the case may be.

Orthogonality of \(\text{ce}_m\), \(\text{se}_m\): Let \(y_1\), \(y_2\) be solutions of 
\[
y'' + (a - 2q \cos 2z)y = 0 \quad \text{for the same value of } q, \text{ but usually different values of } a.
\]

Then 
\[
y_1'' + (a_1 - 2q \cos 2z)y_1 = 0 \quad \text{and} \quad y_2'' + (a_2 - 2q \cos 2z)y_2 = 0.
\]

Multiplying the first equation by \(y_2\), the second by \(y_1\) and subtracting gives 
\[
y_1''y_2 - y_2''y_1 = (a_2 - a_1)y_1y_2.
\]

Integrating both sides of this equation between the limits \(z_1, z_2\) we get 
\[
[y_1'y_2 - y_2'y_1]_{z_1}^{z_2} = (a_2 - a_1) \int_{z_1}^{z_2} y_1y_2 \, dz.
\] (1.10)

For a given \(q\), the \(a\)'s corresponding to \(y_1 = \text{ce}_m(z, q)\),
\[ y_2 = ce_p(z,q) \text{ where } m \neq p, \text{ are different. These functions have period } \pi \text{ or } 2\pi \text{ so with } z_1 = 0, z_2 = 2\pi \text{ the left hand side of } (1.10) \text{ vanishes.} \]

Hence
\[
\int_0^{2\pi} ce_m(z,q) \; ce_p(z,q) dz = 0 \quad m \neq p
\]

\[
\int_0^{2\pi} ce^2_{2n}(z,q) dz = 2\pi \left[ A_0^{(2n)} \right]^2 + \pi \sum_{r=1}^{\infty} \left[ A_{2r}^{(2n)} \right]^2 \quad p = m = 2n \quad (1.11)
\]
since we shall prove later on in the next chapter that the series for \( ce_{2n}(z,q), se_{2n}(z,q) \) are uniformly and absolutely convergent, then we can integrate term-by-term.

Similarly,
\[
\int_0^{2\pi} ce_{2n+1}(z,q) dz = \pi \sum_{r=0}^{\infty} \left[ B_{2r+1}^{(2n+1)} \right]^2
\]

\[
\int_0^{2\pi} se_m(z,q) \; se_p(z,q) dz = 0 \quad m \neq p
\]

\[
\int_0^{2\pi} se^2_{2n+1}(z,q) dz = \pi \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+1)}]^2
\]

\[
\int_0^{2\pi} ce_m(z,q)se_p(z,q) dz = 0 \quad m,p \text{ positive integers}
\]

\[
\int_0^{2\pi} se^2_{2n+2}(z,q) dz = \pi \sum_{r=0}^{\infty} [B_{2r+2}^{(2n+2)}]^2
\]

Therefore \( ce_m(z,q), se_m(z,q) \) are orthogonal.

**Normalization of \( ce_m(z,q), se_m(z,q) \):** Since our convention was that the coefficients A and B of \( \cos mz \) and \( \sin mz \) be unity for all values of q, we have \( A_m^{(m)} = B_m^{(m)} = 1 \) for all q. We will normalize the functions \( ce_m(z,q), se_m(z,q) \) by the stipulation that
\[
\frac{1}{\pi} \int_0^{2\pi} ce^2_m(z,q) dz = \frac{1}{\pi} \int_0^{2\pi} se^2_m(z,q) dz = 1
\]
for all real values of $q$. Then by (1.11) we have

$$2[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty} [A_{2r}^{(2n)}]^2 = 1. \quad (1.12)$$

Now $c_0 e_0(z,0) = 1$ by our convention, so if it is normalized, its constant term must be $A_0^{(0)} = 1/\sqrt{2}$ for $q = 0$.

Also

$$\frac{1}{\pi} \int_0^{2\pi} c_0 e_{2n+1}^2(z,q)dz = \sum_{r=0}^{\infty} [A_{2r+1}^{(2n+1)}]^2 = 1$$

$$\frac{1}{\pi} \int_0^{\infty} s e_{2n+1}^2(z,q)dz = \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+1)}]^2 = 1$$

$$\frac{1}{\pi} \int_0^{2\pi} s e_{2n+2}^2(z,q)dz = \sum_{r=0}^{\infty} [B_{2r+2}^{(2n+2)}]^2 = 1.$$
CHAPTER II
CHARACTERISTIC NUMBERS AND COEFFICIENTS

Recurrence Relations for the Coefficients.

If the series \( c_\ell(z, q) = \sum_{\ell=0}^{\infty} A_{2\ell}^n \cos 2\ell z \) is substituted into the Mathieu equation \( y'' + (a-2q \cos 2z)y = 0 \)
we get \([aA_o-qA_2] + [(a-4)A_2-q(A_4+2A_0)] \cos 2z
+ \sum_{r=0}^{\infty} [(a-4r^2)A_{2r}-q(A_{2r+2} + A_{2r-2})] \cos 2rz = 0.\)

Equating the coefficients of \( \cos 2\ell z \) to zero for \( \ell = 0, 1, 2, \ldots \) the following recurrence relations are obtained:

\[
\begin{align*}
\text{for } c_\ell(z, q) & : \\
\ell = 0, & aA_o-qA_2 = 0 \\
\ell \geq 1, & (a-4)A_2-q(A_4+2A_0) = 0 \\
\end{align*}
\]

Similarly, if the series \( c_{\ell+1}(z, q), s_{\ell+1}(z, q) \) and \( s_{\ell+2}(z, q) \) are substituted into the Mathieu equation and the coefficients of \( \cos(2\ell+1)z, \sin(2\ell+1)z, \sin(2\ell+2)z \) equated to zero we obtain

\[
\begin{align*}
\text{for } c_{\ell+1}(z, q) & : \\
\ell \geq 1, & (a-4\ell^2)A_{2\ell+1}-q(A_{2\ell+3}+A_{2\ell-1}) = 0 \\
\text{for } s_{\ell+1}(z, q) & : \\
\ell \geq 1, & (a-4\ell^2)B_{2\ell+1}-q(B_{2\ell+3}+B_{2\ell-1}) = 0 \\
\text{for } s_{\ell+2}(z, q) & : \\
\ell \geq 2, & (a-4\ell^2)B_{2\ell}-q(B_{2\ell+2} + B_{2\ell-2}) = 0 \\
\end{align*}
\]

If \( q \) is small enough, the formulae for a given in (1.0b)
may be used, since the term $O(q^6)$ is very small and we can approximate $a$ rather closely. But as $q$ becomes larger this term will affect the value of $a$ considerably so in general, $a$ must be calculated using one of the methods below.

From the second recurrence relation for $c_{2n}(z,q)$ we have

$$(4-a)A_2 + q(A_4 + 2A_o) = 0.$$  

Writing $v_o = A_2/A_o$, $v_2 = A_4/A_2$, then $v_ov_2 = A_4/A_o$  

Therefore $(4-a)A_2/A_o + q(A_4/A_o + 2A_o/A_o) = 0$

or $(4-a)v_o + q(v_ov_2 + 2) = 0,$

$$-v_o(a-4qv_2) = -2q,$$

$$-v_o = \frac{1}{2}q[1- \frac{1}{4}(a-qv_2)].$$

In the same way, using the third recurrence relation for $c_{2n}(z,q)$ with $v_{2r-2} = A_{2r}/A_{2r-2}$, $v_{2r} = A_{2r+2}/A_{2r}$,

we get $(4r^2-a)v_{2r-2} + q(v_{2r}v_{2r-2} + 1) = 0$

and $-v_{2r-2} = (q/4r^2)[1-(\frac{1}{4r^2})(a-qv_{2r})]$ $r \geq 2$  

Substituting $r = 2$ in $-v_{2r-2}$ yields

$$-v_2 = \frac{q}{16}[1 - \frac{1}{16}(a-qv_4)]$$

and upon replacing this into the equation for $-v_o$ we get

$$-v_o = \frac{1}{2}q \left\{1 - \frac{1}{4}a - \frac{a}{64} / [1 - \frac{1}{16}(a-qv_4)]\right\}$$

Continuing on in this way we ultimately get the infinite continued fraction:

$$-v_o = \frac{\frac{1}{2}q}{1 - \frac{1}{4}a} - \frac{\frac{1}{64}q^2}{1 - \frac{1}{16}a} - \frac{\frac{1}{276}q^2}{1 - \frac{1}{36}a} - \cdots \frac{q^2/16r^2(r-1)^2}{1 - a/4r^2}$$

From $aA_o - qA_2 = 0$, $v_o = A_2/A_o = a/q$, $-v_o = -a/q$ so the formula for $-v_o$ becomes

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Since $a$, $q$ are finite the denominator of the general term $\to$ unity and the numerator tends to zero as $r \to \infty$. Hence (2.7) is convergent.

From equation (2.7) we can evaluate $a$ to any desired degree of accuracy provided $q$ is known.

Using (2.5) and (2.6) we can also find values for the $A$'s.

Theorem I: There do not exist two linearly independent periodic solutions corresponding to the same pair $(a,q)$.

proof:

First we shall show that for no one value of $a$ can we obtain two solutions of the types $ce_{2n}(z)$ and $se_{2k+1}(z)$.

We know that the Wronskian of two independent solutions of the Mathieu equation is a nonzero constant; i.e.

$ce_{2n}(z) se'_{2k+1}(z) - se_{2k+1}(z) ce'_{2n}(z) = c \neq 0$.

Now using the fact that

$ce_{2n+1}(z+\pi) = -ce_{2n+1}(z)$

$se_{2n+1}(z+\pi) = -se_{2n+1}(z)$

consider the value of the above wronskian at $z + \pi$.

$ce_{2n}(z+\pi) se'_{2k+1}(z+\pi) - se_{2k+1}(z+\pi) ce'_{2n}(z+\pi)$

$= - ce_{2n}(z) se'_{2k+1}(z) + se_{2k+1}(z) ce'_{2n}(z)$

$= - [ce_{2n}(z) se'_{2k+1}(z) - se_{2k+1}(z) ce'_{2n}(z)]$

$= - c$
which is a contradiction.

Therefore we have shown that \( ce_{2n}(z) \) cannot coexist with \( se_{2n+1}(z) \). Similarly it can be shown no two solutions of the following types can coexist: \( ce_{2n}(z) \) and \( ce_{2n+1}(z) \); \( se_{2n}(z) \) and \( ce_{2n+1}(z) \); \( se_{2n}(z) \) and \( se_{2n+1}(z) \).

Now we shall show that no Mathieu equation can have two solutions of the types \( ce_{2n}(z) \) and \( se_{2k}(z) \). We saw that if we express \( ce_{2n}(z) \) as a Fourier series

\[
ce_{2n}(z) = \sum_{r=0}^{\infty} A_{2r} \cos 2rz
\]

then the Fourier coefficients must satisfy the equations

\[
aA_{2n,0} - qA_{2n,2} = 0
\]

\[
2qA_{2n,0} + (4-a)A_{2n,2} + qA_{2n,4} = 0
\]

\[
qA_{2n,2r-2} + (4r^2 - a)A_{2n,2r} + qA_{2n,2r+2} = 0 \quad r = 2, 3, \ldots
\]

Note: We write \( A_{2r}^{(2n)} \) as \( A_{2n,2r} \) for convenience of use since we may write them either way. Similarly, if we express \( se_{2k}(z) \) as a Fourier series we obtain

\[
se_{2k}(z) = \sum_{r=1}^{\infty} B_{2r}^{(2k)} \sin 2rz
\]

where

\[
(4-a)B_{2k,2} + qB_{2k,4} = 0
\]

\[
q B_{2k,2r-2} + (4r^2 - a)B_{2k,2r} + qB_{2k,2r+2} = 0 \quad r = 2, 3, \ldots
\]

Now consider the determinant

\[
u_r = \begin{vmatrix}
B_{2k,2r} & A_{2n,2r} \\
B_{2k,2r+2} & A_{2n,2r+2}
\end{vmatrix}, \quad r = 2, 3, \ldots
\]

\[
= \begin{vmatrix}
B_{2k,2r} & A_{2n,2r} \\
-B_{2k,2r-2} - \frac{(4r^2 - a)}{q}B_{2k,2r} & -A_{2n,2r-2} - \frac{(4r^2 - a)}{q}A_{2n,2r}
\end{vmatrix}
\]
Thus we see that \( u = u_{r-1} \) = ... = \( u_2 \), so that the determinants are equal. We can express \( u_2 \) very simply in terms of \( A_{2n,0} \) since

\[
\begin{vmatrix}
B_{2k,2} & A_{2n,2} \\
B_{2k,4} & A_{2n,4}
\end{vmatrix}
= \begin{vmatrix}
\frac{1}{q} & A_{2n,2} \\
0 & -2A_{2n,0}
\end{vmatrix}
\]

We see that \( A_{2n,0} \) cannot vanish, for otherwise all \( A_{2n,2} \) would vanish, as shown by the recurrence relationships. Similarly, \( B_{2k,2} \) cannot vanish. But to insure the convergence of the Fourier series we require that

\[
\lim_{r \to \infty} A_{2n,2r} = \lim_{r \to \infty} B_{2k,2r} = 0
\]

which implies that

\[
\lim_{r \to \infty} u_r = 0.
\]

But we have just shown that \( u_r \) is a nonzero constant. Therefore our original assumption that a Mathieu equation had two solutions of the types \( ce_{2n}(z) \) and \( se_{2k}(z) \) must be false. Similarly it can be shown that \( ce_{2n+1}(z) \) and \( se_{2n+1}(z) \) cannot co-exist.

**Theorem II:** If \( y(z) \) is a periodic solution of the Mathieu equation, then either \( y(z) \) is even or \( y(z) \) is odd.
proof:

Suppose \( y(z) \) is a solution, then so is \( y(-z) \) since the differential equation is unchanged if \( z \) is replaced by \(-z\). Therefore unless \( y(z) \) is even or odd, \( y(-z) \) will be independent of \( y(z) \). By choosing a suitable multiple of \( y(z) \), we can insure that \( y(0) = 1 \). Then

\[
y_1(z) = \frac{y(z) + y(-z)}{2}, \quad y_2(z) = \frac{y(z) - y(-z)}{2}
\]

since the functions on the right have the same initial conditions as \( y_1(z) \) and \( y_2(z) \). Now if \( y(z) \) has period \( \pi \) or \( 2\pi \), then so do \( y_1(z) \) and \( y_2(z) \). If \( y(z) \) is even, then \( y(z) = y_1(z) \) and

\[
y(z) - y(-z) = 0.
\]

If \( y(z) \) is odd, then \( y(z) = y_2(z) \) and

\[
y(z) + y(-z) = 0.
\]

It is therefore sufficient to confine our attention to periodic solutions of Mathieu's equation which are either even or odd. These solutions, and these only, will be called Mathieu functions.

Consider the system,

\[
y'' + (a-2q \cos 2z)y = 0
\]

\[
y(0) = y(\pi)
\]

\[
y'(0) = y'(\pi).
\]

It may be shown that there exists \( \{\lambda_n\} \), characteristic numbers of the system, where \( \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \) and

\[
\lim_{n \to \infty} \lambda_n = +\infty \quad \text{[Ince; pg. 247]}
\]

It may further be shown that if we let \( y_n \) be the characteristic function corresponding to \( \frac{\lambda_n}{n} \), then
\[ y_{2n}(z) = \sum_{r=0}^{\infty} A_{2r}^2 \cos \, 2rz \quad \text{and} \quad y_{2n+1}(z) = \sum_{r=0}^{\infty} B_{2r+2}^2 \sin \,(2r+2)z \]
since these functions satisfy the conditions for a Fourier series. \( y_{2n}(z) \) is an even periodic function of period \( \pi \) and \( y_{2n+1}(z) \) an odd periodic function of period \( \pi \), where the characteristic numbers \( \lambda_{2n} = a_{2n} \) and \( \lambda_{2n+1} = b_{2n+2} \). In fact
\[ y_{2n}(z) = c_{2n}(z,q) \quad \text{and} \quad y_{2n+1}(z) = s_{2n+2}(z,q). \]

Similarly given the system
\[ y^{\prime\prime} + (a-2q \cos \, z)y = 0 \]
\[ y(0) = y(2\pi) \]
\[ y^{\prime}(0) = y^{\prime}(2\pi). \]
It can be shown that the characteristic numbers of the system, where \( \mu_0 < \mu_1 < \mu_2 < \ldots \) and \( \lim_{n \to \infty} \mu_n = +\infty \). Also
\[ y_n(z) = \sum_{r=0}^{\infty} A_{2r+1}^2 \cos(2r+1)z \quad \text{where} \quad \mu_{2n} = b_{2n+1} \]
\[ y_{2n+1}(z) = \sum_{r=0}^{\infty} A_{2r+1}^2 \cos(2r+1)z \quad \text{where} \quad \mu_{2n} = a_{2n+1} \]
\[ y_{2n+1}(z) = a_{2n+1}. \]
Likewise \( y_{2n}(z) = s_{2n+2}^{2n+1}(z,q) \) and \( y_{2n+1}(z) = c_{2n+1}^{2n+1}(z,q) \) for \( q \) positive.

For \( q < 0 \) we have \( y_{2n} = \sum_{r=0}^{\infty} A_{2r+1}^2 \cos(2r+1)z \) and
\[ y_{2n+1} = \sum_{r=0}^{\infty} B_{2r+1}^2 \sin(2r+1)z \quad \text{where} \quad \mu_{2n} = a_{2n+1} \]
\[ \mu_{2n+1} = b_{2n+1}. \]
It also follows that \( y_{2n}(z) = c_{2n+1}^{2n+1}(z,q) \) and \( y_{2n+1}(z) = s_{2n+2}^{2n+1}(z,q) \) with \( q \) negative.

**Theorem III:** The series for \( c_{2n}(z,q) \) is absolutely and uniformly convergent.

**proof:**
\[ y(z) = \sum_{r=0}^{\infty} A_{2r} \cos \, 2rz \]
is the Fourier cosine series
solution of the Mathieu equation \( y'' + (a - 2q \cos 2rz)y(z) = 0 \)
with initial conditions \( y(0) = y(\pi) \) and \( y'(0) = y'((\pi)) \). The Fourier coefficients \( A_{2r} = \frac{1}{\pi} \int_{0}^{\pi} y(t) \cos 2rt \, dt \). Using the fact that \( y''(t) \) exists and the initial conditions, we can integrate \( A_{2r} \) by parts obtaining
\[
A_{2r} = -\frac{1}{4r^2 \pi} \int_{0}^{\pi} y''(t) \cos 2rt \, dt
\]
\[
= -\frac{1}{4r^2 \pi} \int_{0}^{\pi} (a_{2n} - 2q \cos 2rz)y(z) \cos 2rt \, dt.
\]
Now since \( y(t) \) is continuous it is bounded, i.e.; \( |y(t)| \leq M \) on \([0, \pi]\). Also \( |a_{2n} - 2q \cos 2rt| \leq |a_{2n}| + |2q| \).
Hence \( |(a_{2n} - 2q \cos 2rt)y(t)| \leq M(|a_{2n}| + |2q|) = Q \) and \( \left| \int_{0}^{\pi} (a_{2n} - 2q \cos 2rt)y(t) \, dt \right| \leq Q\pi \).
Then \( |A_{2r}| \leq \frac{1}{4r^2 \pi} \cdot Q\pi = \frac{Q}{4r^2} \)
and \( |A_{2r} \cos 2rz| \leq \frac{Q}{4r^2} = \frac{D}{r^2} \) which is convergent. Therefore the series \( ce_{2n}(z,q) \) is uniformly and absolutely convergent.
Hence \( A_{2r} \to 0 \) and so \( v_{2r} = \frac{A_{2r} + 2}{A_{2r}} \to 0 \).
Similarly the series for \( ce_{2n+1}(z,q) \), \( se_{m}(z,q) \) can be proved to have the same properties. Hence all ordinary Mathieu functions are continuous for \( z \) real, and the series may be differentiated or integrated term by term provided, in the case of differentiation, that the resulting series are uniformly convergent.

Convergence of Series \( ce_{m}^{(p)}(z,q) \), \( se_{m}^{(p)}(z,q) \)
The pth derivative of \( ce_{2n}(z,q) \) is
\[
\frac{d^p}{dz^p} \left[ \c_{2n}(z,q) \right] = \begin{cases} 
\sum_{r=1}^{\infty} (2r)^p A_{2r} (-1)^{p/2} \cos 2rz & \text{if } p \text{ even} \\
\sum_{r=1}^{p+1} (2r)^p A_{2r} (-1)^{2} \sin 2rz & \text{if } p \text{ odd.}
\end{cases}
\]

Thus the ratio of the coefficients of the \((r+1)\)th and \(r\)th terms is
\[
\left| \frac{u_{r+1}}{u_r} \right| = \left( \frac{r+1}{r} \right)^p \left| v_{2r} \right| = (1 + \frac{1}{r})^p \left| v_{2r} \right| \to 0 \text{ as } r \to +\infty,
\]

so all derivatives of \(\c_{2n}(z,q)\) are absolutely and uniformly convergent for \(z\) real. Similarly for \(\c_{2n+1}(z,q)\) and \(\s_{m}(z,q)\).

When the tabular values of \(a_0, a_1, \ldots, a_5\) and \(b_1, b_2, \ldots, b_6\) in the range \(q = 0 - 40\) are plotted using cartesian coordinates, the chart of Fig. 1 is obtained. It is symmetrical about the \(a\)-axis. \(a_{2n}, b_{2n+2}\), the characteristic curves for \(\c_{2n}\), \(\s_{2n+2}\) respectively, are symmetrical about this axis, but \(a_{2n+1}, b_{2n+1}\) corresponding to \(\c_{2n+1}, \s_{2n+1}\) are asymmetrical. Nevertheless the symmetry of the diagram is maintained, since the curves for the functions of odd order, \(a_{2n+1}, b_{2n+1}\) are mutually symmetrical.

Excepting the curve \(a_0\), which is tangential to the \(q\)-axis at the origin in Fig. 1, all curves intersect this axis twice, one value of \(q\) being positive, the other negative. Thus each characteristic has two real zeros in \(q\).
Fig. 1. - Stability chart for Mathieu functions of integral order. The characteristic curves $a_0$, $b_1$, $a_1$, $b_2$, ... divide the plane into regions of stability and instability.
Comments on Normalization: Under the convention that the coefficients $A$ of $A \cos m z$ in $ce_m(z,q)$ and $B$ of $B \sin m z$ in $se_m(z,q)$ are unity for all values of $q$, the remaining coefficients are infinite for certain values of $q \neq 0$.

Consider $ce_2(z,q)$ whose characteristic number is $a_2$. Then the recurrence relation (2.1) gives $q/a = A_2/A_2$. Now when $q = \pm 21.28$, $a_2 = 0$ (see Fig. 1) which implies $A_2/A_2$ is infinite, and since $A_2 = 1$, $A_2 = \infty$. Also with $a_2 = 0$ in the second recurrence formula in (2.1) we have $2qA_2 + q A_4 = 0$ and since $A_2 = \infty$, $A_2 = 1$ $q \neq 0$, we get $A_4 = -\infty$, and so on. Therefore when $q = \pm 21.28$, $a_2 = 0$, and all the coefficients except $A_2$ become infinite.

A similar conclusion may be reached for the coefficients of any Mathieu function of integral order, except $ce_0$, $ce_1$, $se_1$, $se_2$.

For $ce_0$, the recurrence formula (2.1) gives $A_0/A_2 = q/a$. Since $A_0 = 1$ for all $q$, and $a_0$ never vanishes for $q > 0$ (see Fig. 1), $A_2$ is always finite. By use of the second recurrence formula for $ce_0$, it can be shown that $A_4$ is finite, and so on.

For $ce_1$, the recurrence formula (2.2) gives $A_1/A_3 = \frac{q}{a_1-1-q}$ and since $A_1 = 1$ for all $q$ and $a_1-1-q \neq 0$ for $q > 0$, $A_3$ cannot become infinite. The functions $se_1$, $se_2$ are located similarly.

It appears, that for functions of order $2n$, $2n + 1$ there are $n$ values of $q \neq 0$ for which all coefficients, except that of the same order as the function, become infinite. Therefore,
the convention mentioned above is inadmissible for general purposes, although it is sometimes useful when q is small.

It may be shown that none of the coefficients in the series ce₀, ce₁, se₁, se₂ has a zero in q > 0. For ce₀, the first recurrence formula (2.1) gives \( A_2/A_0 = a_0/q \). From (Fig. 1), \( a_0 \) is negative if q > 0 so \( a_0/q \) is negative. Since \( A_0 = 1 \), \( A_2 \) was finite, \( A_2/A_0 < 0 \) so neither can have a zero for q > 0. Using the first two recurrence relations (2.1) we get \( A_4/A_2 = \frac{(a_0-4)}{q} - 2q/a_0 \). For q > 0, \( a_0(a_0-4) > 2q^2 \). Thus \( A_4/A_2 \) will not vanish in q > 0. Hence \( A_4 \) has no zero in q > 0. Similarly it can be shown that the remaining \( A_{2r} \) have no zeros in q > 0. This conclusion is valid for the coefficients in the series for ce₁, se₁, se₂.

Under the normalization of ceₘ(z,q), seₘ(z,q), it appears that if \( a > 0 \), \( A_{2n}, A_{2r+1}, B_{2r+1}, B_{2r+2} \) each has r real zeros in q > 0. For ce₂₂ₙ, the first recurrence relation (2.1) gives \( A_2/A_0 = a_{2n}/q \) when \( n > 0 \), \( a_{2n} \) vanishes once q > 0 (see Fig. 1) and since \( A_0 \) is finite and has no zero in q > 0, \( A_2 = 0 \) when \( a_{2n} = 0 \). With q large enough it can be shown that \( a_{2n} \leq -2q + (8n + 2)q^{1/2} \) and as q → ∞, \( A_2/A_0 \rightarrow -2 \) and \( A_0 \) never vanishes in q > 0. Also from the recurrence relations (2.1) we obtain \( A_4/A_2 = \frac{q a_{2n}}{4a_{2n}} \). Then at the two intersections of \( 2q^2 + 4a_{2n} - a_{2n}^2 \) and the curve \( a_{2n} \) (q > 0), we get \( A_4/A_2 = 0 \). Hence \( A_4 \) vanishes twice for q > 0. Continuing thus we find that \( A_6 \) vanishes thrice for q > 0, and so on. The coefficients for other functions may be treated in
a similar way.

Behavior of Coefficients as $q \rightarrow 0$: From $aA_o - qA_2 = 0$ for $ce_2(z,q)$, $A_o \sim A_2 q/4$ and since $A_2$ is finite, $A_o \rightarrow 0$ as $q \rightarrow 0$. Also from (2.1) $(a_2 - 4) A_2 = q (2A_o + A_4)$. Substituting $a_2 = 4 + \frac{5}{12} q^2 + O(q^4)$ and for $A_o$ from above into 

$$(a_2 - 4) A_2 = q (2A_o + A_4)$$

gives $[\frac{5}{12} q^2 + O(q^4)] A_2 = \frac{1}{2} q^2 A_2 + qA_4$. When $q$ is small $A_4 \sim -\frac{1}{12} qA_2 \rightarrow 0$ as $q \rightarrow 0$.

Again by (2.1) with $r = 2$, $(a_2 - 16) A_4 = q(A_2 + A_6)$ and upon substituting for $a_2$ and $A_4$ above

$$-\frac{1}{12} A_2 [-12 + O(q^2)] = A_2 + A_6.$$

Thus when $q \rightarrow 0$, $A_6 \rightarrow 0$ and so on for the other coefficients.

Now using (1.12), it follows from above that $A_2^{(2)} \rightarrow 1$ as $q \rightarrow 0$. Similarly it can be shown that in (1.2) - (1.5), $A_m^{(m)} \rightarrow 1$, $B_m^{(m)} \rightarrow 1$ as $q \rightarrow 0$ while all the other $A, B$ tend to zero.

Behavior of Coefficients as $q \rightarrow + \infty$: We use the recurrence relations (2.1) - (2.4) and $a \sim -2q$. Then for $ce_{2n'} aA_o - qA_2 = 0$ so $A_2 A_o \rightarrow -2$ as $q \rightarrow + \infty$. Also

$$(a - 4) A_2 - q(A_4 + 2A_o) = 0$$

and substituting for $a$, $A_2 - 4A_o + A_4 + 2A_o = 0$ giving $A_4/A_o \rightarrow 2$ as $q \rightarrow + \infty$. Continuing in this way we find that $A_2^{(2n')}/A_o^{(2n')} \rightarrow (-1)^r 2$ as $q \rightarrow + \infty$. By (1.12) it follows that for $n \geq 0$, all the $A^{(2n)} \rightarrow 0$ as $q \rightarrow + \infty$. Similar conclusions are reached for $A^{(2n+1)}$, $B^{(m)}$. Similarly

$$A_2^{2r+1}/A_1 \rightarrow (-1)^r (2r+1) B_2^{2r+1}/B_1 \rightarrow (-1)^r (r+1).$$

Behavior of coefficients as $n \rightarrow + \infty$: For a large, positive, and greater than $q$, $a_{2n} \sim 4n^2$. From (2.1), $A_o/A_2 = q/a \sim q/4n^2 \rightarrow 0$ as $n \rightarrow + \infty$. Hence $A_o \rightarrow 0$. Also from
\(2.1\) \(A_2/A_4 = q/a - 4 \approx q/4n^2 - 4 \to 0\) as \(n \to + \infty\). Hence \(A_2 \to 0\).

Therefore it can be shown that all the \(A_{2r} \to 0\) except \(A_{2(2n)}\) which tends to unity, and similarly for the coefficients of the other functions of integral order.
A complete solution of a linear differential equation of the second order is \( y = Ay_1(z) + By_2(z) \) where \( A, B \) are (3.11) arbitrary constants and \( y_1(z), y_2(z) \) are any two solutions which constitute a fundamental system. For the Mathieu function we desire the solution to be single-valued and periodic with period \( \pi \) to start with.

Writing \((z+\pi)\) for \( z \), then \( y_1(z+\pi), y_2(z+\pi) \) are also solutions. Therefore

\[
\begin{align*}
  y_1(z+\pi) &= \alpha_1 y_1(z) + \alpha_2 y_2(z) \quad (3.2) \\
  y_2(z+\pi) &= \beta_1 y_1(z) + \beta_2 y_2(z) \quad (3.3)
\end{align*}
\]

where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are determinable constants. Differentiating we get

\[
\begin{align*}
  y_1'(z+\pi) &= \alpha_1 y_1'(z) + \alpha_2 y_2'(z) \quad (3.4) \\
  y_2'(z+\pi) &= \beta_1 y_1'(z) + \beta_2 y_2'(z) \quad (3.5)
\end{align*}
\]

Choosing the initial conditions \( y_1(0) = 1, y_1'(0) = 0, y_2(0) = 0, y_2'(0) = 1 \) and substituting in above we find \( \alpha_1 = y_1(\pi) \), \( \beta_1 = y_2(\pi) \), \( \alpha_2 = y_1'(\pi) \), \( \beta_2 = y_2'(\pi) \)

\[
y(z+\pi) = Ay_1(z+\pi) + By_2(z+\pi)
\]

\[
\begin{align*}
  &= A[\alpha_1 y_1(z) + \alpha_2 y_2(z)] + B[\beta_1 y_1(z) + \beta_2 y_2(z)] \\
  &= (A\alpha_1 + B\beta_1)y_1(z) + (A\alpha_2 + B\beta_2)y_2(z) \\
  &= \varphi y(z)
\end{align*}
\]

if \( \varphi \), a constant, can be found \( (A\alpha_1 + B\beta_1) = \varphi A \) and \( (A\alpha_2 + B\beta_2) = \varphi B \).
\[ (A_1 + B_2) = \varphi_A \Rightarrow - \frac{A}{B} = \frac{\beta_1}{\alpha_1 - \varphi} \]
\[ (A_2 + B_3) = \varphi_B \Rightarrow - \frac{A}{B} = \frac{\beta_2 - \varphi}{\alpha_2} . \]

Hence \((a_1 - \varphi)(\beta_2 - \varphi) = a_2 \beta_1\) if \(A, B \neq 0\)
or \(\varphi^2 = (a_1 + \beta_2) \varphi + (a_1 \beta_2 - a_2 \beta_1) = 0. \quad (3.6)\)

Therefore \(\varphi\) can be expressed in terms of \(y_1(\pi), y_1'(\pi), y_2(\pi), y_2'(\pi)\). Later on, it will be shown that one value of \(\varphi\) can be taken as \(e^{\mu \pi}\). Therefore let \(\varphi = e^{\mu \pi}\) and define \(\varphi(z) = e^{-\mu \pi} y(z)\).

Substituting \((z + \pi)\) for \(z\) yields
\[
\varphi(z + \pi) = e^{-\mu(z + \pi)} y(z + \pi) = e^{-\mu(z + \pi)} \varphi y(z) = e^{-\mu z} y(z) = \varphi(z) .
\]

Hence \(\varphi(z)\) is periodic in \(z\) with period \(\pi\). Since \(y(z)\) is a solution, then \(e^{\mu z} \varphi(z)\) is a solution.

Consider \(\frac{d^2 y}{dz^2} + (a - 2q \cos 2z) y = 0\) and \(\frac{d^2 y}{dz^2} + [a - 2q \psi(z)] y = 0\) where \(\psi(z)\) is an even, differentiable function of \(z\), with period \(\pi\). For example, \(\psi(z) = \sum_{r=1}^{\infty} \theta_{2r} \cos 2rz\). By the periodicity of \(\psi(z)\) with period \(\pi\) let,
\[
\varphi(z) = \sum_{r=-\infty}^{\infty} c_{2r} e^{2rzi} \\
\varphi(z) = \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rzi} .
\]

If \(\mu = \alpha + i\beta, \alpha, \beta\) real, then \(e^{\mu z} \varphi(z)\) is a formal solution of either \((3.7)\) or \((3.8)\). Since both of these equations are unchanged if \(-z\) be written for \(z\), \(e^{-\mu z} \varphi(-z)\) is an independent...
solution, provided either $\alpha \neq 0$ or, when $\alpha = 0$, $\beta$ is non-integral. Hence the complete solution is

$$y(z) = Ae^{\mu z} \phi(z) + Be^{-\mu z} \phi(-z)$$

$$= Ae^{\mu z} \sum_{-\infty}^{\infty} c_{2r} e^{2\mu z} + Be^{-\mu z} \sum_{-\infty}^{\infty} c_{2r} e^{-2\mu z}. \quad (3.9)$$

As $z \to \pm \infty$, a solution of (3.8) is defined to be unstable if it tends to $\pm \infty$ and stable if it tends to zero or remains bounded. The solution is unstable if $\mu$ is any real number $\neq 0$. The solution is unstable if $\mu = \alpha + i\beta$, $|\alpha| > 0$, $|\beta| > 0$. The solution is stable if $\mu = i\beta$; $\beta$ real.

Choose as a first solution of Mathieu equation

$$y_1(z) = \frac{1}{2}A \left[ e^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2\mu r z} + e^{-\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2\mu r z} \right]$$

$$= A \sum_{r=-\infty}^{\infty} c_{2r} \cosh(\mu + 2\pi r)z + c_{-2r} \cosh(\mu + 2\pi r)z$$

$$= A \sum_{r=-\infty}^{\infty} c_{2r} \cosh(\mu + 2\pi r)z \quad (3.10)$$

$$y_1'(z) = A \sum_{r=-\infty}^{\infty} (\mu + 2\pi r) c_{2r} \sinh(\mu + 2\pi r)z$$

As a second solution we get

$$y_2(z) = B \sum_{r=-\infty}^{\infty} c_{2r} \sinh(\mu + 2\pi r)z. \quad (3.11)$$

If $y_1(0) = 1$, then $A = 1/\sum_{r=-\infty}^{\infty} c_{2r}^2$

$$\alpha_1 = y_1(\pi) = A \sum_{r=-\infty}^{\infty} c_{2r} \cosh \mu \pi = \cosh \mu \pi$$

and $y_1(n\pi) = \cosh \mu n\pi$.

If $y_1'(0) = 1$, then $B = 1/\sum_{r=-\infty}^{\infty} (\mu + 2\pi r) c_{2r}$

$$\beta_2 = y_2'(\pi) = \cosh \mu \pi$$

and $y_2(n\pi) = \cosh \mu n\pi$.

$$\alpha_2 = y_2'(\pi) = A \sum_{r=-\infty}^{\infty} (\mu + 2\pi r) c_{2r} = A/B \sinh \mu \pi$$

and $y_1(n\pi) = A/B \sinh \mu n\pi$.
\[\begin{align*}
\beta_1 &= y_2(\pi) = B/A \sinh \mu \pi \quad \text{and} \quad y_2'(\pi) = B/A \sinh \mu \pi \\
y_1(\pi) &= y_2'(\pi) \quad \text{or} \quad \alpha_1 = \beta_2 \\
\alpha_1 \beta_1 &= y_2'(\pi)y_2(\pi) = \sinh^2 \mu \pi \\
\alpha_1 \beta_2 &= y_2'(\pi)y_1(\pi) = \cosh^2 \mu \pi \\
\end{align*}\]

Therefore \[\alpha_1 \beta_2 - \alpha_2 \beta_1 = y_2'(\pi)y_1(\pi) - y_1'(\pi)y_2(\pi) = 1\]

Substituting into (3.6) we get \[\phi^2 - (2 \cosh \mu \pi) \phi + 1 = 0\]

\[\phi_1 = e^{\mu \pi} \quad \phi_2 = e^{-\mu \pi} \quad \phi_1 \phi_2 = 1 \quad \phi_1 = 1/\phi_2.\]

In general, we certainly cannot expect the solutions of the differential equation \(y'' + \phi(z)y = 0\) with \(\phi(z) = a-2q \cos 2z\), to be periodic. But we can find solutions such that \(y(z+\pi) = \phi y(z)\) for some constant \(\phi\).

Consider (3.6) the quadratic equation for \(\phi\):
\[\phi^2 - [y_1(\pi) + y_2'(\pi)] \phi + y_1'(\pi)y_2(\pi) - y_1'(\pi)y_2'(\pi) = 0\]

or \[\phi^2 - [y_1(\pi) + y_2'(\pi)] \phi + 1 = 0.\]

This equation, in general, has two different roots, \(\phi_1 \quad \phi_2\), corresponding to each of which we can find a set of coefficients \(c_1 \quad \text{and} \quad c_2\), which gives us two solutions of the differential equation, say \(u_1(z) \quad \text{and} \quad u_2(z)\) such that
\[u_1(z+\pi) = \phi_1 u_1(z)\]
\[u_2(z+\pi) = \phi_2 u_2(z)\]

Looking at the discriminant of the quadratic equation (3.11a) we must distinguish between three distinct cases.

In the first case, if \(|y_1(\pi) + y_2'(\pi)| < 2\) then the roots of (3.11a) are complex numbers of absolute value 1. Then \(u_1 \quad \text{and} \quad u_2\) are solutions whose absolute value is bounded and
therefore all solutions are bounded. Therefore in this case all solutions are stable.

In the second case, if \( |y_1(\pi) + y_2(\pi)| > 2 \), then the roots are real and since \( \Phi_1 \Phi_2 = 1 \) either \( |\Phi_1| < 1 \) and \( |\Phi_2| > 1 \) or \( |\Phi_1| > 1 \) and \( |\Phi_2| < 1 \). Thus one solution is bounded while the other is unbounded.

In the third case, consider the two subcases

\[
y_1(\pi) + y_2(\pi) = \pm 2.
\]

When \( y_1(\pi) + y_2(\pi) = 2 \) there is only one double root \( \Phi = 1 \) and when \( y_1(\pi) + y_2(\pi) = -2 \) there is also only one double root \( \Phi = -1 \).

In any one of these cases we can in general expect only one solution with the property \( y(z+\pi) = \Phi y(z) \) although in exceptional cases there may be two such solutions. For \( \Phi = 1 \), there is a solution of period \( \pi \); i.e. \( y(z+\pi) = y(z) \).

For \( \Phi = -1 \), the solution has period \( 2\pi \) since \( y(z+2\pi) = -y(z+\pi) = y(z) \). In Theorem II, it has been shown that we can restrict ourselves to purely even or odd solutions.

Now suppose that \( y_1(z) \) is an even periodic solution of period \( \pi \), such that \( y_1(z+\pi) = y_1(z) \) and also \( y_1'(z+\pi) = y_1'(z) \).

Since \( y_1(z) \) is even, \( y_1'(z) \) is odd. Then for \( z = -\frac{\pi}{2} \),

\[
y_1'\left(-\frac{\pi}{2}+\pi\right) = y_1'(\frac{\pi}{2}) = y_1'(\frac{\pi}{2}) = y_1'(\frac{\pi}{2}) = 0.
\]

Conversely, if \( y_1'(\frac{\pi}{2}) = 0 \), then \( y_1(z) \) is periodic of period \( \pi \). Since \( y_1(z+\pi) \) satisfies Mathieu's equation

\[
y_1(z+\pi) = c_1 y_1(z) + c_2 y_2(z)
\]

\[
y_1'(z+\pi) = c_1 y_1'(z) + c_2 y_2'(z).
\]
To find $c_1$ and $c_2$ we let $z = -\frac{\pi}{2}$ and observe that $y_1(z)$ and $y'_1(z)$ are even, and $y'_2(z)$ and $y_2(z)$ are odd. Then

\[ y_1\left(\frac{\pi}{2}\right) = c_1 y_1\left(\frac{\pi}{2}\right) - c_2 y_2\left(\frac{\pi}{2}\right) \]
\[ y'_1\left(\frac{\pi}{2}\right) = -c_1 y'_1\left(\frac{\pi}{2}\right) + c_2 y'_2\left(\frac{\pi}{2}\right). \]

When $y'_1(\frac{\pi}{2})$ vanishes, $y'_2(\frac{\pi}{2})$ cannot vanish. Therefore, if $y'_1(\frac{\pi}{2}) = 0$ the second of the above equation tells us that $c_2 = 0$, and the first equation shows that $c_1 = 1$. Similarly for the other three cases.

Thus there exists a nontrivial periodic solution that is

1. even and of period $\pi$ iff $y'_1\left(\frac{\pi}{2}\right) = 0$
2. odd and of period $\pi$ iff $y_2\left(\frac{\pi}{2}\right) = 0$
3. even and of period $2\pi$ iff $y_1\left(\frac{\pi}{2}\right) = 0$
4. odd and of period $2\pi$ iff $y'_2\left(\frac{\pi}{2}\right) = 0$.

Consider $y(z+\pi) = \varphi y(z)$, if $y_1(z)$ is a solution of either (3.7) or (3.8). Then $y_1(z+\pi) = e^{\mu\pi}u_1(z)$; also if $y_2(z)$ is a linearly independent solution $y_2(z+\pi) = e^{\mu\pi}y_2(z)$.

Hence, \[
\cosh \left\{ \begin{array}{l}
\mu\pi = \frac{u_1(z+\pi) + y_1(z-\pi)}{2y_1(z)} \\
\sinh \end{array} \right. \]
\[
= \frac{y_1(\pi) + y_1(-\pi)}{2y_1(0)} \quad \text{if } z = 0 \text{ and } y_1(\pi) \neq 0
\]

Similarly, \[
\cosh \left\{ \begin{array}{l}
\mu\pi = \frac{y_2(z+\pi) + y_2(z+\pi)}{2y_2(z)} \\
\sinh \end{array} \right. \]
\[
= \frac{y_2(0) + y_2(2\pi)}{2y_2(\pi)} \quad \text{if } z = \pi \text{ and } y_2(0) \neq 0.
\]

Consider a More general Case. From $e^{-\mu z}y(z) = \varphi(z)$ the solution of (3.7) or (3.8) has the form $y = e^{\mu z}y(z)$ where we
shall take 2\pi as the period of \( \phi(z) \). Then \( \phi(-\pi) = \phi(\pi) \) and

\[ y(-\pi) = e^{i\mu\pi}y(\pi), \quad y(\pi) = e^{i\mu\pi}y(-\pi) \]

so

\[ y(\pi) - e^{2i\mu\pi}y(-\pi) = 0 \quad (3.12) \]

similarly

\[ y'(\pi) - e^{2i\mu\pi}y'(-\pi) = 0 \quad (3.13) \]

Letting \( y_1(z) \) and \( y_2(z) \) be any two linearly independent solutions, and substituting \( y = Ay_1(z) + By_2(z) \) into (3.12) and (3.13) yields

\[
A[y_1(\pi) - e^{2i\mu\pi}y_1(-\pi)] + B[y_2(\pi) - e^{2i\mu\pi}y_2(-\pi)] = 0
\]

and

\[
A[y_1'(\pi) - e^{2i\mu\pi}y_1'(-\pi)] + B[y_2'(\pi) - e^{2i\mu\pi}y_2'(-\pi)] = 0.
\]

If \( A, B \neq 0 \), we get

\[
[y_1(\pi) - e^{2i\mu\pi}y_1(-\pi)][y_2(\pi) - e^{2i\mu\pi}y_2(-\pi)]
\]

\[
- [y_2(\pi) - e^{2i\mu\pi}y_2(-\pi)][y_1(\pi) - e^{2i\mu\pi}y_1(-\pi)] = 0
\]

or

\[
e^{4i\mu\pi} - D/c^2 e^{2i\mu\pi} + 1 = 0
\]

where

\[
D = y_1(-\pi)y_2'(\pi) + y_1'(\pi)y_2(-\pi)y_1'(\pi) - y_2(\pi)y_1'(\pi)
\]

\[
c^2 = y_1(\pi)y_2'(\pi) - y_2(\pi)y_1'(\pi) = y_1(\pi)y_2'(\pi) - y_2(\pi)y_1'(\pi)
\]

Therefore

\[
cosh 2\mu\pi = \frac{y_1'(\pi) + y_2'(\pi)}{2} \quad \text{if } y_1(\pi) = y_2(-\pi) = 1
\]

\[
y_2(-\pi) = y_1'(\pi) = 0.
\]

So far, the above work has been based upon a solution of the type \( e^{\mu z} \phi(z) \) where \( \phi(z) = \sum_{r=-\infty}^{\infty} c_{2r} e^{2rz} z \), which has period \( \pi \). We want \( \mu \) to be real or imaginary \( (\mu = i\beta, 0 < \beta < 1) \) so it is essential that \( \phi(z) \) have period \( 2\pi \). Without loss of generality then we take

\[
\phi(z) = \sum_{r=-\infty}^{\infty} c_{2r+1} e^{(2r+1)zi}
\]

and change \( 2r \) to \( (2r+1) \) in all infinite series.

Also \( y_1(n\pi) = (-1)^n \cosh \mu n \pi \).

Consider a form of the solution when \( \mu = i\beta, 0 < \beta < 1 \).

Writing \( \mu = i\beta \) in (3.10) we get

\[
y_1(z) = A \sum_{r=-\infty}^{\infty} c_{2r} \cos(2r+\beta)z. \quad (3.14)
\]
Similarly in (3.11) \( y_2(z) = B \sum_{-\infty}^{\infty} c_{2r} \sin(2r+\beta)z \) (3.15)

the \( i \) being dropped, since it is merely a constant multiplier. Hence the complete solution of the Mathieu equation takes the form

\[
y(z) = A \sum_{-\infty}^{\infty} c_{2r} \cos(2r+\beta)z + B \sum_{-\infty}^{\infty} c_{2r} \sin(2r+\beta)z
\]

with \( y_1(n\pi) = y_2'(n\pi) = \cos \beta n\pi \) and \( y_1'(n\pi)y_2(n\pi) = -\sin^2 \beta n\pi \)

which can be checked by using the initial conditions.

Also replacing \( 2r \) by \( (2r+1) \) in (3.14) - (3.16) leads to the independent solutions,

\[
y_1(z) = A \sum_{-\infty}^{\infty} c_{2r+1} \cos(2r+1+\beta)z
\]

\[
y_2(z) = B \sum_{-\infty}^{\infty} c_{2r+1} \sin(2r+1+\beta)z
\]

and the complete solution

\[
y(z) = A \sum_{-\infty}^{\infty} c_{2r+1} \cos(2r+1+\beta)z + B \sum_{-\infty}^{\infty} c_{2r+1} \sin(2r+1+\beta)z \] (3.18a)

with \( y_1(n\pi) = y_2'(n\pi) = (-1)^n \cos \beta n\pi \)

(3.18a) is another form of the solution of the Mathieu equation. It may be added that \( A, B \) are quite arbitrary and need not have the values which can be obtained from the initial conditions stated earlier. As a matter of fact any appropriate initial conditions may be chosen.

Consider the solution when \( \beta = 0 \) or 1. Take any point \( (a,q), q > 0 \) in a stable region of Fig. 1 near \( a_{2n} \) but not upon it. Then the Mathieu equation has two independent coexistent solutions, (3.14) and (3.15). On \( a_{2n} \), \( \beta = 0 \) and (3.15) is not a solution. Substituting (3.14) into the Mathieu equation and equating the coefficient of \( \cos 2rz \) to zero for \( r = -\infty \) to \( \infty \) we
get the recurrence relation

\[(a-4r^2) c_{2r-q} (c_{2r+2} + c_{2r-2}) = 0\] with \(\beta = 0\) \(r \neq 0\).

Then if \(c_0 \neq 0\) and \(c_{2r_{\infty}} = c_{-2r}\), \(r \geq 1\) (3.14) may be written

\[y_1(z) = A[c_0 + 2 \sum_{r=1}^\infty c_{2r} \cos 2rz]\]

provided \(\beta = 0\) and \(a = a_{2n}\). When \(\beta = 1\), \((a, q)\) is on \(b_{2n+1}\) and (3.14) is not a solution. Then the recurrence relation is

\[(a-(2r+1)^2) c_{2n-q} (c_{2r+2} + c_{2r-2}) = 0\]
\[(a-(2r+1)^2) c_{-2r-2-q} (c_{-2r} + c_{-2r-4}) = 0\] if \(r = -(r+1)\).

Then if \(c_{2r} = -c_{-2r-2}\), (3.15) may be written

\[y_2(z) = 2B \sum_{r=0}^\infty c_{2r+1} \sin(2r+1)z.\]

Similarly with \(\beta = 0\), (3.17) may be written

\[y_1(z) = 2A \sum_{r=0}^\infty c_{2r+1} \cos(2r+1)z\]

and with \(\beta = 1\), (3.18) may be written

\[y_2(z) = 2B \sum_{r=0}^\infty c_{2r+2} \sin(2r+2)z.\]

**Another form of general solution.** For a first solution let \(y_1(z) = e^{\mu z} \phi(z, \sigma)\), \(\phi\) being periodic in \(z\) with (3.19) period \(\pi\) or \(2\pi\), and \(\sigma\) a new parameter. When \((a, q)\) lies between \(a_0, b_2\) in Fig. 1, we get

\[\phi(z, \sigma) = \sin(z-\sigma) + s_3 \sin(3z-\sigma) + s_5 \sin(5z-\sigma) + \ldots + c_3 \cos(3z-\sigma) + c_5 \cos(5z-\sigma) + \ldots .\]  

(3.20)

Assume \(a = 1 + q f_1(\sigma) + q^2 f_2(\sigma) + q^3 f_3(\sigma) + \ldots\)

\[\mu = qg_1(\sigma) + q^2 g_2(\sigma) + q^3 g_3(\sigma) + \ldots,\]

and take \(\phi(z, \sigma) = \sin(z-\sigma) + qh_1(z, \sigma) + q^2 h_2(z, \sigma) + \ldots\).

Substituting \(y_1(z) = e^{\mu z} \phi(z, \sigma)\) into \(y'' + (a-2q \cos 2z)y = 0\) gives \(e^{\mu z}[\phi'' + 2\mu \phi' + (\mu^2 + a - 2q \cos 2z)\phi] = 0\), or
Equating the coefficients of $q^0, q, q^2,$ ... to zero we obtain

$q^0$: $-\sin(z-a) + \sin(z-a) = 0$

$q$: $h''_1 + h_1 + f_1 \sin(z-a) + 2g_1 \cos(z-a) - 2\sin(z-a) \cos 2z = 0$

and $h''_1 + h_1 + (2g_1 + \sin 2\sigma) \cos(z-a) + (f_1 + \cos 2\sigma) \sin(z-a) - \sin(3z-a) = 0$

since $2 \cos 2z \sin(z-a) = [\sin(3z-a) - \sin(z+a)] = [\sin(3z-a) - \sin(z-a) \cos 2\sigma - \cos(z-a) \sin 2\sigma].$

The particular integrals of $\cos(z-a)$ and $\sin(z-a)$ are

$1/2z \sin(z-a)$ and $-1/2z \cos(z-a)$ respectively, which are non-periodic. But $\sigma$ is to be periodic, hence $g_1 = -1/2 \sin 2\sigma$

$f_1 = -\cos 2\sigma$ leaving $h''_1 + h_1 = \sin(3z-a)$ so $h_1 = -1/8 \sin(3z-a).$

$q^2$: $h''_2 + h_2 + 2g_2 \cos(z-a) - 3/4g_1 \cos(3z-a) + (g_1^2 + f_2) \sin(z-a) +$

$1/8 \sin(3z-a) \cos 2\sigma + 1/4 \sin(3z-a) \cos 2z = 0,$

so $h''_2 + h_2 + 2g_2 \cos(z-a) + 1/8 \sin(3z-a) \cos 2\sigma + 1/8 \sin(5z-a) + \ldots = 0$

$g_2 = 0$, $f_2 = -1/4 + 1/8 \cos 4\sigma$ to avoid non-periodic terms, leaving

$h''_2 + h_2 + 3/8 \cos(3z-a) \sin 2\sigma + 1/8 \sin(3z-a) + 1/8 \sin(5z-a) = 0$

and $h_2 = 3 \sin 2\sigma \cos(3z-a) + \cos 2\sigma \sin(3z-a) + 1/3 \sin(5z-a).$

Continuing in this fashion we get,

$a = 1 - q \cos 2\sigma + 1/4 q^2 (-1 + 1/2 \cos 4\sigma) + \ldots$ \hspace{1cm} (3.21)

$= 1 - q \cos 2\sigma - \mu^2 + q S_3.$

$\mu = -1/2q \sin 2\sigma + 3/128 q^3 \sin 2\sigma - 3/1024 q^4 \sin 4\sigma - \ldots$ \hspace{1cm} (3.22)

$= 1/2q (-\sin 2\sigma + c_3).$

Since $a$ is an even function of $\sigma$ we can write $-\sigma$ for $\sigma.$
Writing \(-a\) for \(a\) in the formula for \(\mu\) alters its sign, since it is an odd function of \(a\).

Therefore the second independent solution of Mathieu's equation may take the form

\[
y_2 = e^{-\mu z} \phi(z, -a)
\]

and the complete solution becomes

\[
y = Ae^{\mu z} \phi(z, \sigma) + Be^{-\mu z} \phi(z, -\sigma)
\]

Making the proper substitution will reduce (3.19) and (3.20) to the series for \(ce_1(z, q)\), and similarly for \(se_1(z, q)\).

Consider the region in Fig. 2, lying between \(a_1\) and \(b_2\). On the curve \(a_1\), \(\sigma = -1/2\pi\) in (3.21). Now if \(\sigma = -1/2\pi + i\theta\), \(\cos 2\sigma = -\cosh 2\theta\), so if \(q > 0\) is fixed, (3.21) shows that \(a\) increases with increase in \(\theta\) until \(b_2\) is reached. On this curve \(a = 4 + 1/2q^2(1/3 - 1/2 \cos 2\sigma) + \ldots\) with \(\sigma = 0\).

If in \(a\), \(\sigma = i\theta\), \(a\) decreases with increase in \(\theta\) until \(a_1\) is regained. Thus between \(a_1\), \(b_2\) with \(q > 0\), \(\sigma\) is complex or imaginary. If \(\sigma = -1/2\pi + i\theta\), then

\[
\mu = -1/2q \sin 2\sigma + \frac{3}{128} q^3 \sin 2\sigma - \ldots
\]

\[
= 1/2i q \sinh 2\theta - \frac{31}{128} q^3 \sinh 2\theta + \ldots
\]

therefore \(\mu\) is imaginary, hence the solution is stable.

If \(\sigma\) is real, the series for \(\mu\) contains \(\sin 2\sigma\), \(\sin 4\sigma\), etc. Thus \(\mu\) is real, and the solution is unstable.

(1) When \((a, q), q > 0\) lies between \(a_m, b_{m+1}\) in Fig. 1, 2, 3, \(\mu\) is imaginary, and the two solutions of Mathieu's equation are stable.

(2) When \((a, q), q > 0\) lies between \(b_m, a_m\), \(\mu\) is real
and the complete solution of Mathieu's equation is constable.

When \((a,q)\) lies between \(2n\) and \(2n+1\), for the first solution we take
\[
y_1(z) = e^{i\beta z} \sum_{r=-\infty}^{\infty} c_{2r+1} e^{2rzi} = e^{i\beta z} \varphi(z)_{2r+1}
\]

or
\[
y_1(z) = \sum_{r=-\infty}^{\infty} c_{2r+1} \cos(2r+\beta)z
\]

or
\[
y_1(z) = e^{i\beta z} [\sin(2n z-\sigma) + s_2 \sin(2z-\sigma) + s_4 \sin(4z-\sigma) + \ldots +]
\]
\[
+ c_2 \cos(2z-\sigma) + c_4 \cos(4z-\sigma) + \ldots].
\]

If \(n \geq 1\), formulae for \(a\), \(i\beta = \mu\), and the coefficients \(c_{2r}, s_{2r}\)
can all be found as shown before.

When \((a,q)\) lies between \(2n+1\) and \(2n+2\), for the first solution we take
\[
y_1(z) = e^{i\beta z} \sum_{r=-\infty}^{\infty} c_{2r+1} e^{(2r+1)zi} = e^{i\beta z} \varphi(z)_{2r+1}
\]

or
\[
y_1(z) = \sum_{r=-\infty}^{\infty} c_{2r+1} \cos(2r+\beta)z
\]

or
\[
y_1(z) = e^{i\beta z} [\sin((2n+1)z-\sigma) + s_2 \sin((2z-\sigma) + s_3 \sin(3z-\sigma) + \ldots +]
\]
\[
+ c_1 \cos(z-\sigma) + c_3 \cos(3z-\sigma) + \ldots].
\]

If \(n \geq 1\), formulae for \(a\), \(i\beta = \mu\), and the coefficients \(c_{2r+1},
\]
\(s_{2r+1}\) may be found as shown above. The above results are for
\(q\) small and positive and give stable solutions.

When \((a,q)\) lies between \(2n+2\) and \(2n+3\), for the first solution we take
\[
y_1(z) = e^{i\beta z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rzi} = e^{i\beta z} \varphi(z)_{2r}
\]

or
\[
y_1(z) = e^{i\beta z} [\sin(2nz-\sigma) + s_2 \sin(2z-\sigma) + s_4 \sin(4z-\sigma) + \ldots +]
\]
\[
+ c_2 \cos(2z-\sigma) + c_4 \cos(4z-\sigma) + \ldots],
\]

where \(\mu\) is real and \(\mu > 0\).

When \((a,q)\) lies between \(2n+3\) and \(2n+4\), for the first solution we take
\[
y_1(z) = e^{i\beta z} \sum_{r=-\infty}^{\infty} c_{2r+1} e^{(2r+1)zi} = e^{i\beta z} \varphi(z)_{2r+1}
\]
Fig. 2. Variation in the parameter $\sigma$ in stable and unstable regions $a_m$, $b_m$ are mutually asymptotic and $\rightarrow - \infty$ as $q \rightarrow + \infty$ $a_m$, $a_{m+1}$; $b_m$, $b_{m+1}$ are mutually asymptotic and $\rightarrow - \infty$ as $q \rightarrow - \infty$. 

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or \( y_1(z) = e^{\mu z} \left[ \sin \left\{ (2n+1)z-\sigma \right\} + s_1 \sin(z-\sigma) + s_3 \sin(3z-\sigma) + \ldots + c_1 \cos(z-\sigma) + c_3 \cos(3z-\sigma) + \ldots \right] \).

The second solutions of the above four cases are derived by making the appropriate substitutions \(-z\) for \(z\), \(\sin\) for \(\cos\), and \(-\sigma\) for \(\sigma\). The two above cases are for \(q\) small and positive and give unstable solutions.

Consider the region in Fig. 3 between the curves \(a_1, b_2\). If for any assigned \(\beta\), the \(a\) are calculated for \(q\) increasing from zero in small steps, the characteristic curve \(\beta\) is obtained. Lying between those for \(ce_1, se_2\) we shall define it as that for the Mathieu function of real fractional order \((1+\beta)\).

In general, if \((a, q)\) lies between the curves \(a_m, b_{m+1}\), the order of the function will be \((m+\beta)\), and the value of \(a\) on any curve is that for \(ce_{m+\beta}(z, q)\) and \(se_{m+\beta}(z, q)\).

For \(q > 0, 0 < \beta < 1, \beta\) real, we adopt the definitions

\[
\begin{align*}
ce_{2n+\beta}(z, q) &= \sum_{-\infty}^{\infty} A_{2r}^{(2n+\beta)} \cos(2r+\beta)z \\
se_{2n+\beta}(z, q) &= \sum_{-\infty}^{\infty} A_{2r}^{(2n+\beta)} \sin(2r+\beta)z
\end{align*}
\]

coeexistnt solutions

(a, q) between \(a_{2n}\) and \(b_{2n+1}\)

\[
\begin{align*}
ce_{2n+1+\beta}(z, q) &= \sum_{-\infty}^{\infty} A_{2r+1}^{(2n+1+\beta)} \cos(2r+1+\beta)z \\
se_{2n+1+\beta}(z, q) &= \sum_{-\infty}^{\infty} A_{2r+1}^{(2n+1+\beta)} \sin(2r+1+\beta)z
\end{align*}
\]

coeexistnt solutions

(a, q) between \(a_{2n+1}\) and \(b_{2n+2}\).

The iso-\(\beta\) curves lie in the stable regions for Mathieu's equation, while the iso-\(\mu\) curves lie in the unstable regions. Each iso-\(\beta\) curve is single-valued in \(q\), but except for the region
below $a_o$, each iso-$\mu$ is double-valued in $q$. For constant
$\mu$, $q$ at the turning-point of the iso-$\mu$ curve increases with
increase in $a$.

For $q < 0$, write $\left(\frac{1}{2}\pi - z\right)$ for $z$, obtaining

\[
\ce_{2n+\beta}(\frac{1}{2}\pi - z, q) = \cos \frac{1}{2} \beta \pi \sum_{r=0}^{\infty} (-1)^r A_{2r}(2n+\beta) \cos(2r+\beta)z + \\
+ \sin \frac{1}{2} \beta \pi \sum_{r=0}^{\infty} (-1)^r A_{2r}(2n+\beta) \sin(2r+\beta)z \quad (3.25)
\]

\[
\se_{2n+\beta}(\frac{1}{2}\pi - z, q) = \sin \frac{1}{2} \beta \pi \sum_{r=0}^{\infty} (-1)^r A_{2r}(2n+\beta) \cos(2r+\beta)z - \\
- \cos \frac{1}{2} \beta \pi \sum_{r=0}^{\infty} (-1)^r A_{2r}(2n+\beta) \sin(2r+\beta)z. \quad (3.26)
\]

If we adopt the definitions

\[
\ce_{2n+\beta}(z, q) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}(2n+\beta) \cos(2r+\beta)z \\
\se_{2n+\beta}(z, -q) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}(2n+\beta) \sin(2r+\beta)z
\]

each function is a linearly independent solution of Mathieu's
equation. If the tabular values of (3.23) are known, those of
(3.27) may be calculated by the aid of the following relation­
ships derived from (3.25)-(3.27)

\[
\ce_{2n+\beta}(z, -q) = (-1)^n \left[ \cos \frac{1}{2} \beta \pi \ce_{2n+\beta}(\frac{1}{2}\pi - z, q) + \sin \frac{1}{2} \beta \pi \se_{2n+\beta}(\frac{1}{2}\pi - z, q) \right]
\se_{2n+\beta}(z, -q) = (-1)^n \left[ \sin \frac{1}{2} \beta \pi \ce_{2n+\beta}(\frac{1}{2}\pi - z, q) - \cos \frac{1}{2} \beta \pi \se_{2n+\beta}(\frac{1}{2}\pi - z, q) \right]
\]

Similar remarks apply to (3.24).

Consider a point $(a, q)$ in a stable region of Figures 1 or
3 between $a_{2n}$ and $b_{2n+1}$, then $0 < \beta < 1$. With $q > 0$ fixed,
let $a_{2n}$ be approached $\beta \to 0$. Then

\[
\ce_{2n+\beta}(z, q) \to \ce_{2n}(z, q) \quad \se_{2n+\beta}(z, q) \to 0
\]

so $A_{-2r} \to A_{2r} \to \frac{1}{2} A_{2r}, \quad A_{o} \to A_{o}, \quad a_{2n+\beta} \to a_{2n}.$
Fig. 3. Iso-$\beta\mu$ stability chart for Mathieu functions of fractional order. The iso-$\beta$ and iso-$\mu$ curves are symmetrical about the $a$-axis.
Similarly, as $\beta \to 1$

$$s e_{2n+\beta}(z,q) \to s e_{2n+1}(z,q), \quad c e_{2n+\beta}(z,q) \to 0$$

$$-A_{-2r-2} \to A_{2r} \to \frac{1}{2} B_{2r+1},$$

For $(a,q)$ between $a_{2n+1}$ and $b_{2n+2}$ as $\beta \to 0$

$$c e_{2n+1+\beta}(z,q) \to c e_{2n+1}(z,q), \quad s e_{2n+1+\beta}(z,q) \to 0$$

$$A_{-2r-1} \to A_{2r+1} \to \frac{1}{2} A_{2r+1},$$

as $\beta \to 1$

$$s e_{2n+1+\beta}(z,q) \to s e_{2n+2}(z,q)$$

$$c e_{2n+1+\beta}(z,q) \to 0$$

$$A_{-2r-3} \to A_{2r+1} \to \frac{1}{2} B_{2r+2}$$

Consider the segment of any line between the curves $b_m, a_m$, parallel to the $a$-axis in an unstable region. On $b_m$ and $a_m$, $\mu = 0$, while at some point on the line, $\mu$ attains a maximum value. Thus if the turning-point (nearest the $a$-axis) on the iso-$\mu$ curve is $q = q_0$, there are two values of $a$ for any $q > q_0$ on the curve. No two iso-$\mu$ curves intersect, corresponding to the same property of iso-$\beta$ curves. The iso-$\mu$ curves are asymptotic to the characteristic curves $b_m, a_m$ which bound the unstable region where they lie.

Consider functions of order $m + \mu$, $q > 0$. When $(a,q)$ lies in an unstable region between $b_{2n}$ and $a_{2n}$, we define the functions of order $(2n + \mu)$ by

$$c e u_{2n+\mu}(z,q) = k e^{H_2z} \left\{ \sum_{r=1}^{\infty} c_{2r} \cos(2rz + \varphi_2r) \right\}. \quad (3.28)$$
For the region between \( b_{2n+1} \), \( a_{2n+1} \) we get

\[
\text{ceu}_{2n+1+\mu}(\pm z, q) = k_1 e^{\pm uz} \sum_{r=0}^{\infty} \langle 2r+1 \rangle \cos[(2r+1)z \pm \phi_{2r+1}] \quad (3.29)
\]

where \( k \) and \( k_1 \) are normalizing constants defined thus

\[
k^2[2\rho_0^2 + \sum_{r=1}^{\infty} \rho_r^2] = 1 \quad \text{or} \quad k = 1/[2\rho_0^2 + \sum_{r=1}^{\infty} \rho_r^2]^{1/2} \quad \text{and} \quad k_1^2 \sum_{r=0}^{\infty} \rho_{2r+1}^2 = 1
\]

or \( k_1 = 1/[\sum_{r=0}^{\infty} \rho_{2r+1}^2]^{1/2} \).

On the upper and lower parts, and at the turning point of an iso-\( \mu \) curve, we use the symbols \( \text{ceu}_{m+\mu}^- \), \( \text{ceu}_{m+\mu}^+ \), \( \text{ceu}_{m+\mu}^0 \) respectively.

If \( \mu \) remains constant as \((a, q)\) moves towards \( a_{2n} \) in Fig. 3, \( \mu \to 0 \) and in (3.28) \( k\rho_0 \to A_0^{(2n)} \), \( k\rho_{2r} \to A_2^{(2n)} \), and \( \phi_{2r} \to 0 \), \( r \geq 1 \). When \( \mu = 0 \), by choosing \( k \) accordingly, we obtain \( \text{ce}_{2n}(z, q) \). As \((a, q)\) moves towards \( b_{2n} \), \( \mu \to 0 \), and \( \rho_0 \to 0 \)

\[
k_{2r} \to B_{2r}^{(2n)} \phi_{2r} \to -\frac{1}{2\pi}, \quad r \geq 1 \quad \text{so with} \quad \mu = 0 \quad \text{we get} \quad \text{se}_{2n}(z, q).
\]

Similarly the degenerate forms of (3.29) are \( \text{ce}_{2n+1}(z, q) \) and \( \text{se}_{2n+1}(z, q) \).

Consider functions of order \( m + \mu, \ q < 0 \). These are defined as follows:

\[
\text{ceu}_{2n+1+\mu}(\pm z, -q) = (-1)^n e^{\pm \pi \mu/2} \text{ceu}_{2n+1+\mu}[\pm(\pi/2 - \pm z), q]
\]

\[
= (-1)^n k_1 e^{\pm uz} \sum_{r=1}^{\infty} \langle 2r+1 \rangle \cos[(2r+1)z \mp \phi_{2r+1}] \quad (3.30)
\]

\[
\text{ceu}_{2n+1+\mu}(\pm z, -q) = (-1)^n e^{\pm \pi \mu/2} \text{ceu}_{2n+1+\mu}[\pm(\pi/2 - \pm z), q]
\]

\[
= (-1)^n k_1 e^{\pm uz} \sum_{r=0}^{\infty} \langle 2r+1 \rangle \sin[(2r+1)z \mp \phi_{2r+1}] \quad (3.31)
\]

when \( \mu = 0 \) in (3.30) and (3.31) the functions degenerate to \( \text{ce}_{2n}(z, -q), \text{se}_{2n+1}(z, -q) \); when \( \mu = 0 \), we get \( \text{se}_{2n}(z, q) \).
ce_{2n+1}(z;q) as may be expected from Fig. 3, where q < 0.

Mathieu's equation transformed to a Riccati type:

Write \( y = e^{\int z \omega(z)dz} \), where \( v = \sqrt{a} \) and \( \omega(z) \) is a differentiable function of \( z \). Then

\[
\frac{dy}{dz} = v\omega y
\]

\[
\frac{d^2y}{dz^2} = vy(d\omega/dz + v\omega^2).
\]

Letting \( 1 - (2q/a)\cos 2z \) = \( \beta^2 \), Mathieu's equation becomes

\[
y'' + a \beta^2 y = 0.
\]

Substitution from above yields the Riccati type

\[
\frac{1}{v} \frac{d\omega}{dz} + \omega^2 + \beta^2 = 0.
\]

(3.32)

Now suppose \( a \geq q > 0 \), a very large, then \( \frac{1}{v} \frac{d\omega}{dz} \) can be neglected since it is small and we get,

\[
\omega^2 + \beta^2 = 0
\]

\[
\omega = \pm i\beta = \pm i(1 - \frac{2q}{a}\cos 2z)^{1/2}
\]

expanding by the binomial theorem we obtain

\[
\omega \sim \pm i(1 - \frac{d}{a}\cos 2z).
\]

Hence \( v \int_2^{z} \omega dz \sim \pm iv(z - \frac{q}{2a}\sin 2z) \).

Therefore \( y = \exp(v \int_2^{z} \omega dz) \sim \exp(\pm iv[z - (q/2a)\sin 2z]) \).

By the theory of linear differential equations,

\[
y_1 \sim \frac{1}{2} \left\{ e^{iv[z - (q/2a)\sin 2z]} + e^{-iv[z - (q/2a)\sin 2z]} \right\} = 
\cos[v(z - q/2a \sin 2z)]
\]

and

\[
y_2 \sim \frac{1}{2i} \left\{ e^{iv[z - (q/2a)\sin 2z]} - e^{-iv[z - (q/2a)\sin 2z]} \right\} = 
\sin[v(z - q/2a \sin 2z)]
\]

\[
y = Ay_1 + By_2 = (A^2 + B^2)^{1/2}\cos[v(z - q/2a \sin 2z) - \tan^{-1}(\frac{B}{A})] = C \cos[v(z - q/2a \sin 2z) - \alpha].
\]
Expansion gives

\[ y \sim C \{ \cos(\frac{q}{2v} \sin 2z) \cos(vz-\alpha) + \sin(\frac{q}{2v} \sin 2z) \sin(vz-\alpha) \} \]

with \( h = \frac{q}{2v} \) and using formulae (25.8) and (25.9) on pg. 105 of [2]

\[ y \sim C \left\{ \left[ J_0(h) + 2 \sum_{r=1}^{\infty} J_{2r}(h) \cos 4rz \right] \cos (vz-\alpha) \right. \\
\left. + \left[ 2 \sum_{r=0}^{\infty} J_{2r+1}(h) \sin (4r+2)z \right] \sin (vz-\alpha) \right\} \]

\[ = C \sum_{r=-\infty}^{\infty} J_r(h) \cos [(v-2r)z-\alpha]. \quad (3.33) \]

The approximate solution of \( y'' + (a+2q \cos 2z)y = 0 \)

is obtained from (3.33) by writing \(-h\) for \( h \). Therefore we get

\[ y \sim C \sum_{r=-\infty}^{\infty} (-1)^r J_r(h) \cos [(v-2r)z-\alpha]. \]
HILL'S EQUATION

Hill's equation may take the form
\[ \frac{d^2y}{dz^2} + [a-2q\Psi(\omega z)]y = 0 \]  \hspace{1cm} (4.1)

where \( \Psi(z) \) is a periodic differentiable function such that
\[ |\Psi(\omega z)|_{\text{max}} = 1. \]
Usually \( \Psi(\omega z) \) is even in \( z \) with \( \int_{0}^{\frac{2\pi}{\omega}} \Psi(\omega z)dz = 0 \).

When \( \Psi(\omega z) = \cos 2z \), (4.1) becomes the standard Mathieu equation. In certain applications it is convenient to write
(4.1) in the form
\[ \frac{d^2y}{dz^2} + \left( \Theta_0 + 2 \sum_{r=1}^{\infty} \Theta_{2r} \cos 2rz \right)y = 0 \]  \hspace{1cm} (4.2)

where \( \Theta_0, \Theta_2, \Theta_4, \ldots \) are assigned parameters and \( \sum_{r=1}^{\infty} |\Theta_{2r}| \) converges. The theory of Chapter III applies to (4.1), (4.2).

As a solution of (4.2) we take
\[ y_1(z) = e^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rz} \]  \hspace{1cm} (4.3)

where \( \mu \) may be real, imaginary, or complex. Substituting
(4.3) into (4.2) gives
\[ \sum_{r=-\infty}^{\infty} (\mu+2ri) c_{2r} e^{2rz} + \Theta_0 \sum_{r=0}^{\infty} c_{2r} e^{2rz} + \Theta_2 \sum_{r=0}^{\infty} c_{2r+2} e^{2rz} + \cdots = 0 \]

Equating the coefficient of \( e^{2rz} \) to zero, \( r=-\infty \) to \( \infty \), we get
\[ (\mu+2ri)^2 c_{2r} + \Theta_0 c_{2r} + \Theta_2 (c_{2r+2} c_{2r+2}) + \Theta_4 (c_{2r+4} c_{2r+4}) + \cdots = 0 \]  \hspace{1cm} (4.4)
or
\[ (\mu+2ri)^2 c_{2r} + \sum_{m=-\infty}^{\infty} \Theta_{2m} c_{2r+2m} = 0 \]
with the convention that $\theta_{2m} = \theta_{-2m}$. Dividing (4.4) throughout by $(\mu^2 + r_1^2)^2 \theta = \theta_0 - (2r - \mu)^2 = \phi_2 r$ we get

$$c_{2r} + \frac{\theta_2}{\phi_2} (c_{2r-2} + \phi_2) + \frac{\theta_4}{\phi_2} (c_{2r-4}) + ... = 0.$$

Giving $r$ the values $-2, -1, 0, 1, 2, ...$ in succession leads to the set of equations

\[
\begin{align*}
\ldots \ldots + \frac{\theta_2}{\phi_2} c_{-2} + \phi_2 c_{0} + \frac{\theta_4}{\phi_2} c_{2} + \ldots \ldots &= 0 \\
\ldots \ldots + \frac{\theta_2}{\phi_2} c_{-2} + \phi_2 c_{0} + \frac{\theta_4}{\phi_2} c_{2} + \ldots \ldots &= 0 \\
\ldots \ldots + \frac{\theta_2}{\phi_2} c_{-2} + \phi_2 c_{0} + \phi_2 c_{2} + \ldots \ldots &= 0
\end{align*}
\]

The eliminant for the $c$ is given below provided $\theta_0 \neq (2r - \mu)^2$

\[
\Delta(i\mu) = \begin{vmatrix}
\ldots \ldots & \theta_2/\phi_2 & \theta_4/\phi_2 & \ldots \ldots \\
\ldots \ldots & 1 & \theta_2/\phi_2 & \ldots \ldots \\
\ldots \ldots & \theta_4/\phi_2 & \theta_2/\phi_2 & 1 & \ldots \ldots \\
\end{vmatrix}
\]

\[(4.5)\]

It can be shown that the determinant is absolutely convergent, provided none of $\phi_{2r}$ vanishes, (see pg. 67, ref. [3] and pg. 36, ref. [5]) and that $\cos \mu = 1 - \Delta(0)[1 - \cos \theta_0^{1/2}]$ or $\sin^2 \frac{1}{2} \mu = \Delta(0) \sin^2 \frac{1}{2} \theta_0^{1/2}$ where $\theta_0 \neq 4r^2$.

(see pg. 69 - ref. [3])

Thus the determinant
\[ \Delta(0) = \begin{vmatrix} 1 & \frac{\theta_2}{(\theta_o-4)} & \frac{\theta_4}{(\theta_o-4)} \\ \frac{\theta_2}{\theta_o} & 1 & \frac{\theta_2}{\theta_o} \\ \frac{\theta_4}{(\theta_o-4)} & \frac{\theta_2}{(\theta_o-4)} & 1 \end{vmatrix} \]

When \( \theta_2, \theta_4, \theta_6 \ldots \) are small enough, it can be shown by the aid of the expansion for \( \cot x \) that

\[
\Delta(0) \sim 1 + \frac{\pi \cot \frac{1}{2}\pi \theta_o^{1/2}}{4\theta_o^{1/2}} \left[ \frac{\theta_2^2}{2^2-\theta_o} + \frac{\theta_4^2}{3^2-\theta_o} + \ldots \right].
\]

Hill's equation transformed to a Riccati type.

First let us derive a more accurate approximate solution of \( y'' + [(a=2q]\Psi(\omega z)]y = 0 \). Starting with

\[
\frac{1}{v} \frac{d\omega}{dz} + \omega^2 + \rho^2 = 0
\]

we assume that

\[
\omega = \omega_o + \frac{1}{v} \omega_1 + \frac{1}{v^2} \omega_2 + \ldots \quad (4.6)
\]

the \( \omega(z) \) being differentiable functions of \( z \). Then

\[
\omega^2 = \omega_o^2 + \frac{2}{v} \omega_o \omega_1 + \frac{1}{v^2} (\omega_1^2 + 2 \omega_o \omega_2) + \ldots
\]

\[
\frac{1}{v} \frac{d\omega}{dz} = \frac{1}{v} \omega_o' + \frac{1}{v^2} \omega_1' + \frac{1}{v^3} \omega_2' + \ldots.
\]

Substituting the two above equations into the Riccati type equation (3.32) gives

\[
\omega_o^2 + \rho^2 + \frac{1}{v} (\omega_o' + 2 \omega_o \omega_1) + \frac{1}{v^2} (\omega_1' + \omega_1^2 + 2 \omega_o \omega_2) + \ldots = 0.
\]

Equating the coefficients of \( v^0, v^{-1}, v^{-2}, \ldots \) to zero yields

\[
\omega_o^2 + \rho^2 = 0 \quad (4.7)
\]

\[
\omega_o' + 2 \omega_o \omega_1 = 0 \quad (4.8)
\]

\[
\omega_1'' + \omega_1^2 + 2 \omega_o \omega_2 = 0 \quad (4.9)
\]
From (4.7), \( \omega_0 = \pm i\rho \) so \( \pm \int^z \omega_0 \, dz = \pm i \int^z \rho \, dz. \)

From (4.8) \( \omega_1 = -\frac{i}{2} \omega_0 \), so \( \int^z \omega_1 \, dz = -\frac{1}{2} \int^z \frac{d\omega_0}{\omega_1} + \text{constant} \)

\[ = -\frac{1}{2} \log A \omega_0 \]

\[ = \log \left( \pm i \rho \right) - \frac{1}{2} + \log A \]

\[ = \frac{1}{2} \left( \log \rho + \frac{1}{\rho^2} \right) + \log A \]

\[ = \log \left( A \rho^2 - \frac{1}{2} \right) + \frac{1}{\rho^2}. \]

From (4.9), \( \omega_2 = -\left( \frac{\omega_1^2 + \omega_1}{2\omega_0} \right) = \frac{1}{4} \frac{\omega_0}{\omega^2} - \frac{3}{2} \frac{\omega_0}{\omega^3} \)

\[ = \frac{1}{8} \rho^3 \left( 2 \rho^2 \rho - (3 \rho')^2 \right), \]

so \( \frac{1}{\rho} \int^z \omega_2 \, dz = \frac{1}{8} \int^z \left( 2 \rho^2 \rho - (3 \rho')^2 \right) \, dz. \)

Then by substitution from (4.6) - (4.9), if \( |a| > 1, |a| > |2q| \), to a second approximation

\( \int^z \omega \, dz = \pm i \int^z \rho \, dz + \log(A \rho^2 - 1) + \frac{1}{4} \rho + \frac{1}{8} \int^z \left( 2 \rho^2 \rho - (3 \rho')^2 \right) \, dz. \)

Hence \( \exp \left( \int^z \omega \, dz \right) \simeq \)

\[ \simeq \text{(constant)} \rho^{-\frac{1}{2}} \exp \left( \pm i \sqrt{a} \int^z \left( \frac{2 \rho^2 \rho - (3 \rho')^2}{8 \rho^3} \right) \, dz \right). \]

Combining the two solutions we get

\[ y_1(z) \simeq \text{constant} \rho^{-\frac{1}{2}} \cos \left[ a^2 \int^z \varphi(z) \, dz \right], \]

\[ y_2(z) \simeq \text{constant} \rho^{-\frac{1}{2}} \sin \left[ a^2 \int^z \varphi(z) \, dz \right], \]

where \( \varphi(z) = \rho^{-\frac{2}{80 \rho^3}} \).

Then (4.10) are independent solutions of \( y'' + [a-2q \rho \varphi(z)]y = 0. \)

Thus the Riccati type of equation is \( \frac{1}{\rho} \frac{d\omega}{dz} + \omega^2 + \rho^2 = 0 \)

where \( \rho^2 = [1-2q/a \varphi(z)]; \) and if \( |a| > q > 0, |\varphi(z)|_{\text{max}} = 1 \),

its solution is (4.10). In terms of \( \varphi \) we have
The argument of the circular functions is periodic (see pg. 95 ref. [3]) with the frequency of repetition being
\[ f = \frac{1}{2\pi} \frac{d}{dz} \left[ a^{1/2} \int_0^z \left( 1 - \frac{2q}{a} \right)^{1/2} dz + \omega q^2 \int_0^z \left\{ \psi'' + \frac{5q}{2a} \left[ 1 - \left( \frac{2q}{a} \right)^2 \right] \right\} \right]. \]  
(4.12)

Under the conditions \(|a| > q > 0\), \(|\psi(\omega z)|_{\text{max}} = 1\) we derive from (4.12) the approximation
\[ f \approx \frac{1}{2\pi} \left[ a^{1/2} q a^{-1/2} + \frac{1}{4} \omega^2 q a^{-3/2} \left\{ \psi''(1 + 5qa^{-1}) \right\} + \frac{5q}{2a} a^{-1} \psi'^2 (1 + 5qa^{-1}) \right]. \]  
(4.13)

The solution of (4.2) is obtained if in (4.10) we write \(\Theta_0\) for \(a\), and take \(\rho^2 = [1 + 2\Theta_0] \sum_{r=1}^{\infty} \Theta_2 r \cos 2rz \).

As a first approximation, with \(\varphi(z) = \rho\) we get
\[ y_1(z) \left\{ \begin{array}{c} \text{constant} \\ [1 + 2/\Theta_0] \sum_{r=1}^{\infty} \Theta_2 r \cos 2rz \end{array} \right\} \quad \sin \left[ \Theta_2 r \cos 2rz \right] \]  
\[ y_2(z) \left\{ \begin{array}{c} \text{constant} \\ [1 + 2/\Theta_0] \sum_{r=1}^{\infty} \Theta_2 r \cos 2rz \end{array} \right\} \quad \cos \left[ \Theta_2 r \cos 2rz \right]. \]  
(4.14)

This solution rests on the assumption that \(|\Theta_0| > 2 \sum_{r=1}^{\infty} |\Theta_2 r|\), and in common with (4.11) its boundedness implies stability, i.e. (4.14) does not hold for an unstable region of the \(a, q\)-plane for Hill's equation.
Writing (4.1) in the form \( y'' + 2q\frac{a}{2q} - \Phi(\omega z) ) y = 0 \) (4.15) with \( 2q \geq a > 0 \), \((\Phi(\omega z))_{\text{max}} = 1 \) and substituting \( y = e^{\int z^2 \omega \, dz} \) with \( z^2 = 2q \) we get \( \frac{d^2y}{dz^2} = y\left(\frac{d\omega}{dz} + \omega^2\right) \) (4.16)

so from (4.15), (4.16), if \( \xi^2 = [(a/2q) - \Phi] \), (4.15) transforms to \( \frac{1}{\sqrt{\xi}} \frac{d\omega}{dz} + \omega^2 + \xi^2 = 0 \).

On comparison with (3.32), the approximate solution of (4.15) is obtained by writing \( y \) for \( v \), and \( \xi \) for \( \rho \) in (4.10). Thus

\[
\begin{align*}
    y_1(z) &= (\text{constant}) \left[ -\frac{1}{2} \cos \left( \frac{1}{2q} \int z^2 \xi - \frac{1}{16q} \xi^3 \right) \right] dz, \\
y_2(z) &= \left( \text{constant} \right)^{1/4} \left[ \frac{\cos \left( \frac{1}{2q} \int z^2 \left[ a-2q\Phi(\omega z) \right] \right)^{1/2}}{\sin} \right] dz. \tag{4.17}
\end{align*}
\]

The solution of Mathieu's equation under the above conditions is obtained from (4.17) by writing \( \cos 2z \) for \( \Phi(\omega z) \).

**The Nature of the Second Solution:**

Let \( ce_{2n}(z) \) be the first solution of Mathieu's equation, then the second solution can be obtained in this way. Let \( y = v ce_{2n} \), then

\[
y'' = v'' ce_{2n} + 2v' ce'_{2n} + v ce''_{2n}.
\]

Substituting these values into the Mathieu equation we get

\[
v'' ce_{2n} + 2v' ce'_{2n} + v ce''_{2n} + (a-2q \cos 2z) v ce_{2n} = 0.
\]

Since \( ce_{2n} \) is a solution, the coefficients of \( v \) vanish leaving \( v'' ce_{2n} + 2v' ce'_{2n} = 0 \).

The solution of this equation is (by separation of variables)
\[ v' = A/(ce_{2n})^2 \]
\[ v = A \int 1/(ce_{2n})^2 \, dz + B. \]

Therefore
\[ y = A \, ce_{2n} \, \int 1/(ce_{2n})^2 \, dz + B \, ce_{2n}. \]

Let \( \phi_1 = ce_{2n}, \phi_2 = ce_{2n} \int 1/(ce_{2n})^2 \, dz \)
then \( W(\phi_1, \phi_2)(x_0) = 1. \)

Hence we obtained two linear independent solutions \( \phi_1 \) and \( \phi_2 \).
Similarly we can obtain second solutions of Mathieu's equation for the other types of first solutions.
Consider the mass-link-spring mechanism above. The mass $m$ slides on a smooth plane along the straight line CA. The links are massless and the pin-points at 0, A, B are frictionless. Also the spring is massless and at D is in its natural position. The displacement $y$ is small. Apply a driving force $F_1 = (f_0 \cos 2\omega t)y$ at B. It can be resolved into two components, one along BA and the other along DA. The component along DA is $(f_0 \cos 2\omega t)y$ and let $s$ be the spring constant.

Then the equation of motion is (mass x acceleration = force)

$$m \frac{d^2y}{dt^2} = (lf_0 \cos 2\omega t)y - 5y$$

or

$$\frac{d^2y}{dt^2} + \frac{5}{m}y = \frac{lf_0 \cos 2\omega t}{m} y = 0.$$
Put $\frac{S}{m} = a$, $\frac{f_0}{m} = 2q$

\[ \therefore \frac{d^2y}{dt^2} + (a-2q \cos 2\omega t)y = 0. \]

Finally putting $z = \omega t$, the equation becomes

\[ \frac{d^2y}{dz^2} + (a-2q \cos 2z)y = 0 \]

which is the Mathieu Equation.
REFERENCES


