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Tensors as algebraic systems

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TENSORS

AS

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INTRODUCTION

The aim of this investigation is to examine Tensor Theory from an abstract point of view and develop a simple representation for what is commonly referred to as "Tensors". The representation shall present the Tensor Theory in general in such a way as to be readily understandable to one possessing no previous knowledge of the subject.

The definition of Tensor Fields and Tensor Field Systems lends itself well to expression in terms of matrices and Jacobians. This simple representation enables the mathematician, with the elementary operations of matrices and Algebra in mind, to grasp the basic ideas of Tensor Theory very rapidly, avoiding the tedious, time consuming procedure of deciphering bulky, complicated notation.

It is the opinion of the author that this representation will remove the veil of mystery presently shrouding the "Tensor" for many students.

PRELIMINARY CONCEPTS

In associating Tensor Theory and Matrix Algebra we shall be concerned with Euclidean n-space, $\bar{R}_n$, and groups, $\mathcal{G}$, of automorphisms of $\bar{R}_n$. The word "group" is used in the abstract sense.

In applied mathematics it is convenient when considering transformations to regard the image point as a new representation of the old point. Abstractly we say simply that the transformation is a mapping of one point onto another, i.e., given $g \in \mathcal{G}$ and r, a point in $\bar{R}_n$, $g(r) = \bar{r}$, or $r \rightarrow \bar{r}$ by the transformation g. The abstract view lends itself more readily to developing the theory, whereas the other is quite useful in setting up a mathematical model for reality.

JACOBIANS

Consider a transformation, g, and a point $r:(x^1, x^2, \ldots, x^n)$ of R_n which under the transformation g maps into $\bar{r}:(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$, i.e., $g(r) = \bar{r}$. $\bar{x}^i = \bar{x}^i(x^1, x^2, \ldots, x^n)$ under the transformation g, and $x^j = x^j(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ under the transformation g^{-1}. Associated with this transformation is a Jacobian, J_g, defined as:

$$(1) \quad J_g = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \cdot & & \cdot \\ \vdots & \cdot\cdot & \cdot & \cdot \\ & & & \cdot \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{\partial x^2} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}$$

$$(2) \quad J_{g^{-1}} = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^1}{\partial \bar{x}^2} & \cdots & \frac{\partial x^1}{\partial \bar{x}^n} \\ \frac{\partial x^2}{\partial \bar{x}^1} & \cdot & & \cdot \\ \vdots & & \cdots & \cdot \\ \frac{\partial x^n}{\partial \bar{x}^1} & \frac{\partial x^n}{\partial \bar{x}^2} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix}$$

The subscripts, g, are to identify the transformation.

In the sequel we shall consider only those transformations, g and g^{-1}, whose components are independent and possess continuous first partial derivatives. This implies the Jacobian of the transformation exists and is non-singular.[1] Groups, $\mathcal{G}$, which consist entirely of such transformations will be called <u>admissable</u>.

J'_g shall denote the transpose of J_g.

$$(3) \quad J'_g = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^1} & \cdots & \frac{\partial \bar{x}^n}{\partial x^1} \\ \frac{\partial \bar{x}^1}{\partial x^2} & \cdot & & \cdot \\ \vdots & \cdot & \cdots & \cdot \\ \frac{\partial \bar{x}^1}{\partial x^n} & \frac{\partial \bar{x}^2}{\partial x^n} & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}$$

<u>Theorem 1</u>: If g is an element of the admissable group, $\mathcal{G}$, then

$$(4) \qquad J_{g^{-1}} = J_g^{-1}$$

Proof:

$$J_g \, J_{g^{-1}} = \sum \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} = \delta_{ij}$$

1. Sokolnikoff, <u>Advanced Calculus</u>, pp 433-434

Hence $$J_g \, J_{g^{-1}} = I$$

$$J_g^{-1} \, J_g \, J_{g^{-1}} = J_g^{-1} \, I$$

$$J_{g^{-1}} = J_g^{-1}$$

<u>Theorem 2</u>: If g and h are elements of an admissable group, $\mathcal{G}$, then $J_{hg} = J_h \, J_g$.

Proof:

Let $g(r) = \bar{r}$ and $h(\bar{r}) = \bar{\bar{r}}$.

The i^{th} row of J_h is

$$\frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^1}, \; \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^2}, \; \ldots\ldots, \; \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^n}$$

The j^{th} column of J_g is

$$\frac{\partial \bar{x}^1}{\partial x^j}, \; \frac{\partial \bar{x}^2}{\partial x^j}, \; \ldots\ldots, \; \frac{\partial \bar{x}^n}{\partial x^j}$$

Hence the element of the i^{th} row and j^{th} column of $J_h \, J_g$ is

$$\frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^1} \frac{\partial \bar{x}^1}{\partial x^j} + \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^2} \frac{\partial \bar{x}^2}{\partial x^j} + \ldots\ldots + \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^n} \frac{\partial \bar{x}^n}{\partial x^j}.$$

which is recognizable as $\dfrac{\partial \bar{\bar{x}}^i}{\partial x^j}$ and is the element of the i^{th} row and j^{th} column of J_{hg}.

Hence

(5) $$J_{hg} = J_h \, J_g.$$

From Theorem 2 and the properties of the transpose of matrix products, it follows that

(6) $$J'_{hg} = (J_h\, J_g)' = J'_g\, J'_h \,.$$

Similarly and by (4)

(7) $$J^{-1}_{hg} = (J_h\, J_g)^{-1} = J^{-1}_g\, J^{-1}_h = J_{g^{-1}}\, J_{h^{-1}} \,.$$

<u>Definition 1</u>: The transformation g will be called orthogonal if $g(r) = A\, r$, where A is a constant orthogonal matrix, i.e., $A\, A' = I$.

Vector Fields and Systems of Vector Fields

Before defining Tensor Fields and Tensor Field Systems in general, let us examine vector fields in matrix notation.

Definition 2: Suppose that for every $r \in R_n$ we have associated a vector (an $n \times 1$ matrix), $V(r)$, whose components are functions over R_m, for some m. We shall say such a collection of vectors, $V(r)$, $r \in R_n$, forms a Vector Field over R_n.

Definition 3: Let $\mathcal{G}$ be an admissable group of automorphisms of R_n and suppose that for every $g \in \mathcal{G}$ we have a vector field, $V_g(r)$, and that these vector fields be related in such a manner that for every $r \in R_n$

(8) $$V_g(r) = J_g(r)\, V_e(r)$$

where e is the identity of $\mathcal{G}$. We shall say that the collection of vector fields, $V_g(r)$, $g \in \mathcal{G}$, forms a Vector Field System with respect to the group $\mathcal{G}$.

Lemma: If every element of $\mathcal{G}$ is an orthogonal transformation, then condition (8) of Definition 3 is equivalent to

(9) $$V_g(r) = J'_{g^{-1}}(r)\, V_e(r)$$

This readily follows from Definition 1. That is

$$J_g J'_g = I$$

Hence $$J_g = (J'_g)^{-1} = J'_{g^{-1}}$$

Note that if $V_g(r) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$ for some $g \in \mathcal{G}$ it is zero for all $g \in \mathcal{G}$. Since for $g, h, k \in \mathcal{G}$,

$$\begin{aligned} V_k = V_{hg} &= J_{hg} V_e \\ &= J_h J_g V_e \\ &= J_h V_g \end{aligned}$$

Note: It is to be emphasized that expressions such as (8) and (9) are identities in r, however, the argument r shall be omitted whenever the meaning is clear without it.

Theorem 3: If $\mathcal{G}$ is an admissable group of orthogonal transformations of R_n, ($J_g J'_g = 1$, $|J_g| = 1$) and if V_g and U_g are two vector field systems with respect to the group $\mathcal{G}$, then the collection of vector field systems $V_g \times U_g$ forms a vector field system with respect to $\mathcal{G}$.

Proof: We wish to show that the following relationship holds:

$$V_g \times U_g = J_g (V_e \times U_e)$$

From Definition 3 we have

$$V_g = J_g V_e \qquad \text{and} \qquad U_g = J_g U_e$$

Hence $$V_g \times U_g = J_g V_e \times J_g U_e$$

$$J_g = A = \left[a_{ij}\right]$$

Carrying out the cross product gives for the first component of

$J_g V_e \times J_g U_e$:

$$(\sum_i a_{2i}v_i)(\sum_j a_{3j}u_j) - (\sum_j a_{3j}v_j)(\sum_i a_{2i}u_i);$$

and for $J_g (V_e \times U_e)$:

$$a_{11}(v_2u_3 - v_3u_2) + a_{12}(v_3u_1 - v_1u_3) + a_{13}(v_1u_2 - v_2u_1) .$$

Expansion gives:

$$(a_{22}a_{33} - a_{23}a_{32})(v_2u_3 - v_3u_2) + (a_{31}a_{23} - a_{21}a_{33})(v_3u_1 - v_1u_3)$$
$$+ (a_{21}a_{32} - a_{31}a_{22})(v_1u_2 - v_2u_1)$$

Comparison of coefficients of like terms shows the coefficient in

$J_g V_e \times J_g U_e$ to be the cofactor of the respective coefficient

in $\breve{J}_g (\breve{V}_e \times \breve{U}_e)$. Further expansion shows this true for all

components and that the original cross product may be written:

$$J_gV_e \times J_gU_e = (\text{Adj } J_g)' \, (V_e \times U_e)$$

Since J_g is orthogonal, $(\text{Adj } J_g) = J_g' \left|J_g\right| = J_g'$, since $\left|J_g\right| = 1$

by assumption. Hence, $V_g \times U_g = J_g(V_e \times U_e)$.

TENSOR FIELD SYSTEMS

Definition 4: A Tensor Field of rank two is an $n \times n$ matrix, $M(r)$, associated with each point $r \in R_n$, whose elements are functions over R_m, for some m.

Many examples of tensor fields will be recognized as we proceed. To stay with the familiar, however, let us consider the Jacobian of the transformation g which maps $r:(x^1, x^2, \ldots, x^n)$ onto $\bar{r}:(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$. For fixed $g \in \mathcal{G}$, J_g is a tensor field, since for each point, $r:(x^1, x^2, \ldots, x^n)$, of R_n there is associated a constant matrix, J_g.

It is customary to call a vector field a tensor field of rank one and these terms will be used interchangeably in the sequel. Thus the following definition is merely a repetition of Definition 3.

Definition 5: Let $\mathcal{G}$ be an admissable group of automorphisms of R_n and suppose that for every $g \in \mathcal{G}$ we have a tensor field, M_g, of rank one, and that these tensor fields be related in such a manner that for every $r \in R_n$

(9) $$M_g = J_g M_e$$

where e is the identity of $\mathcal{G}$. Such a collection of tensor fields shall be called a Contravariant Tensor Field System of rank one with respect to the group $\mathcal{G}$.

Example:- Differentials.

Let $f(x^1, x^2, \ldots, x^n)$ be a function over R_n. Then we define df to be a function over R_{2n} as follows:

$$df(x^1, x^2, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) = \frac{\partial f}{\partial x^1} x^{n+1} + \frac{\partial f}{\partial x^2} x^{n+2} + \ldots + \frac{\partial f}{\partial x^n} x^{2n}.$$

In applications $x^{n+1}, x^{n+2}, \ldots, x^{2n}$ are interpreted as increments of the independent variables $x^1, x^2, \ldots, x^n$ respectively.

Accordingly, for $g \in \mathcal{G}$, $d\bar{x}^1, d\bar{x}^2, \ldots, d\bar{x}^n$ are defined as above, to be functions over R_{2n}, i.e.:

$$(10) \qquad d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1} x^{n+1} + \frac{\partial \bar{x}^i}{\partial x^2} x^{n+2} + \ldots + \frac{\partial \bar{x}^i}{\partial x^n} x^{2n}.$$

Then $D_g = \begin{bmatrix} d\bar{x}^1 \\ d\bar{x}^2 \\ \cdot \\ \cdot \\ \cdot \\ d\bar{x}^n \end{bmatrix}$ is a tensor field.

Now for $g = e$, $\bar{x}^i = x^i$ so that $d\bar{x}^i = x^{n+i}$, and

$$D_\bullet = \begin{bmatrix} x^{n+1} \\ x^{n+2} \\ \cdot \\ \cdot \\ \cdot \\ x^{2n} \end{bmatrix}$$ is also a tensor field.

It follows that $D_g = J_g D_\bullet$ is another way of writing (10)

Hence the vector fields $\breve{D}_g$ form a tensor field system of rank one.

<u>Definition 6:</u> Let $\mathcal{G}$ be an admissable group of automorphisms of R_n and suppose that for every $g \in \mathcal{G}$ we have a tensor field, M_g, of rank one and that these tensor fields be related in such a manner that for every $r \in R_n$

(11) $$M_g = J'_{g^{-1}} M_\bullet.$$

Such a collection of tensor fields shall be called a Covariant Tensor Field System of rank one with respect to the group $\mathcal{G}$.

Example:- Gradient.

Let $s(r)$ be a real-valued function over R_n possessing continuous first partial derivatives. In applied fields such functions arise and are required to remain invariant under a transformation.

Define $$\nabla s(r) \equiv \begin{bmatrix} \frac{\partial s}{\partial x^1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial s}{\partial x^n} \end{bmatrix}$$

Accordingly, we define $\bar{s}(r) \equiv s(r^*)$, where $g(r^*) = r$. That is, we have a new function, $\bar{s}(r)$, over R_n such that its value for the point r equals $s(r^*)$. The question arises, what is the relationship between $\nabla s(r)$ and $\nabla \bar{s}(r)$. Certainly, each of these is a tensor field of rank one. In addition it is well known from analysis that the following holds:

$$\frac{\partial \bar{s}}{\partial \bar{x}^i} = \sum_j \frac{\partial s}{\partial x^j} \frac{\partial x^j}{\partial \bar{x}^i}$$

Hence $$\nabla \bar{s}(r) = J'_{g^{-1}} \nabla s(r)$$

We may summarize in the following theorem:

<u>Theorem 4</u>: Let $s_e(r)$ be a real-valued function over R_n possessing continuous first partial derivatives, and for each element g of the admissable group $\mathcal{G}$, let $s_g(r) = s_e(r^*)$, where $g(r^*) = r$. Then for every $g \in \mathcal{G}$ the matrices $\nabla s_g = \begin{bmatrix} \frac{\partial s_g}{\partial x^1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial s_g}{\partial x^n} \end{bmatrix}$ form

a tensor field of rank one and the collection of tensor fields ∇s_g, $g \in \mathcal{G}$, form a covariant tensor field system of rank one.

From preceding remarks it is clear that $s_g(r) = \bar{s}(r)$ is such that

$$\nabla s_g = J'_{g^{-1}} \nabla s.$$

<u>Definition 7</u>: Let $\mathcal{G}$ be an admissable group of automorphisms of R_n and suppose that for every $g \in \mathcal{G}$ we have a tensor field, M_g, of rank two and that these tensor fields be related in such a manner that for every $r \in R_n$

(12) $$M_g = J_g M_e J'_g .$$

Such a collection of tensor fields shall be called a Contravariant Tensor Field System of rank two with respect to the group $\mathcal{G}$.

<u>Theorem 5</u>: Let V_g and U_g be vector field systems with respect to an admissable group $\mathcal{G}$. Then the collection of tensor fields $V_g U'_g$ form a contravariant tensor field system of rank two with respect to $\mathcal{G}$.

Proof: $V_g = J_g V_e$ and $U_g = J_g U_e$ by (8)

$$V_g U'_g = (J_g V_e)(J_g U_e)' = J_g(V_e U'_e)J'_g$$

In Theorem 3 we showed that the vector cross product formed a vector field system with respect to a group of orthogonal transformations.

In the approach to tensors from the applied point of view, the components of the vector cross product are used to exemplify a contravariant tensor field system. We wish to point out that the components of the vector cross product are elements of $V_g U'_g - U_g V'_g$

<u>Definition 8</u>: Let $\mathcal{G}$ be an admissable group of automorphisms of R_n and suppose that for every $g \in \mathcal{G}$ we have a tensor field, M_g, of rank two, and that these tensor fields be related in such a manner that for every $r \in R_n$

$$(13) \qquad M_g = J'_{g^{-1}} M_e J_{g^{-1}} \quad .$$

Such a collection of tensor fields shall be called a Covariant Tensor Field System of rank two with respect to the group $\mathcal{G}$.

Example:- Euclidean Metric Tensor

Let $\mathcal{G}$ be an admissable group of automorphisms of R_n.

Define G_g as follows:

$$G_g = J'_{g^{-1}} J_{g^{-1}}$$

Then
$$G_g = J'_{g^{-1}} I J_{g^{-1}}$$
$$= J'_{g^{-1}} G_e J_{g^{-1}}$$

And thus the tensor fields, G_g, form a Covariant Tensor Field System of rank two.

For fixed $g \in \mathcal{G}$, consider the differential vector field, D_e, (See example 5) and define the 1×1 matrix ds^2 of functions over R_{2n} as follows:

$$ds_g^2 = D_e' D_e = D_g' J_{g^{-1}}' J_{g^{-1}} D_g = D_g' G_g D_g$$

In applied fields the right member, expanded, is called the Fundamental Quadratic Form in space and the elements of G_g are the coefficients in the expansion.

<u>Definition 9</u>: Let $\mathcal{G}$ be an admissable group of automorphisms of R_n and suppose that for every $g \in \mathcal{G}$ we have a tensor field, M_g, of rank two, and that these tensor fields be related in such a manner that for every $r \in R_n$

(14) $$M_g = J_g M_e J_{g^{-1}} .$$

Such a collection of tensor fields shall be called a Mixed Tensor Field System of rank two with respect to the group $\mathcal{G}$.

Example:- Kronecker Delta

The customary lengthy development of this tensor may be summarized by our methods as follows:

For each $g \in \mathcal{G}$, let $M_g = I$. Then the M_g form a mixed tensor field system of rank two, i.e.,

$$I = J_g \, I \, J_{g-1}$$

<u>Theorem 6</u>: Let M_g and M_g^* be two tensor field systems of the same rank and type. Then $M_g + M_g^*$ is a tensor field system of that same rank and type.

Proof: This requires a trivial enumeration of the possibilities:

Rank One:

$$M_g + M_g^* = J_g M_e + J_g M_e^* = J_g (M_e + M_e^*)$$

$$M_g + M_g^* = J'_{g-1} M_e + J'_{g-1} M_e^* = J'_{g-1} (M_e + M_e^*)$$

Rank Two:

$$M_g + M_g^* = J_g M_e J'_g + J_g M_e^* J'_g = J_g (M_e + M_e^*) J'_g$$

$$M_g + M_g^* = J'_{g-1} M_e J_{g-1} + J'_{g-1} M_e^* J_{g-1} = J'_{g-1} (M_e + M_e^*) J_{g-1}$$

$$M_g + M_g^* = J_g M_e J_{g-1} + J_g M_e^* J_{g-1} = J_g (M_e + M_e^*) J_{g-1}$$

<u>Theorem 7</u>: Let M_g and M_g^* be two tensor field systems of rank one. Then the product, $M_g^* M'_g$, is a tensor field system of rank two. If M_g and M_g^* are of the same type the product will be of that type; if they are of different types the product, or the transpose of the

product, will be of mixed type.

Proof: Again, an enumeration of the possibilities will suffice.

Contravariant:

$$M_g^* M_g' = (J_g M_e^*)(J_g M_e)' = J_g (M_e^* M_e') J_g'$$

Covariant:

$$M_g^* M_g' = (J_{g-1}' M_e^*)(J_{g-1}' M_e)' = J_{g-1}' (M_e^* M_e') J_{g-1}$$

Mixed:

$$M_g^* M_g' = (J_g M_e^*)(J_{g-1}' M_e)' = J_g (M_e^* M_e') J_{g-1}$$

$$M_g^* M_g' = (J_{g-1}' M_e^*)(J_g M_e)' = J_{g-1}' (M_e^* M_e') J_g'$$

$$(M_g^* M_g')' = M_g M_g^{*\prime} = J_g (M_e M_e^{*\prime}) J_{g-1}$$

<u>Theorem 8</u>: Let M_g form a covariant (contravariant) tensor field system of rank two and M_g^* a contravariant (covariant) tensor field system of rank two with respect to a group $\mathcal{G}$ of admissable automorphisms of R_n. Then $M_g^* M_g'$ forms a mixed tensor field system of rank two.

Proof: By Definition 8: $\quad M_g = J_{g-1}' M_e J_{g-1}$

By Definition 7: $\quad M_g^* = J_g M_e J_g'$

Then

$$M_g^* M_g' = J_g M_e J_g' J_{g-1}' M_e' J_{g-1}$$

$$= J_g (M_e M_e') J_{g-1}$$

Theorem 9: Quotient Rule.

Let V_g be a vector field for each $g \in \mathcal{G}$ and U_g a covariant (contravariant), non-null vector field system. If the real-valued function over R_m, $V_g' U_g$, is independent of g, i.e., $V_g' U_g = V_h' U_h$ for $g, h \in \mathcal{G}$, then V_g is a contravariant (covariant) vector field system.

Proof: $\quad U_g = J'_{g^{-1}} U_e \quad$ by (9)

Then $\quad U_e = J_g' U_g$

Since $\quad V_g' U_g = V_e' U_e$

$$V_g' U_g = V_e' J_g' U_g$$

Multiply both sides on the right by U_g'.

Then $\quad V_g' U_g U_g' = V_e' J_g' U_g U_g'$

Now $U_g U_g' \neq 0$ since $U_g \neq \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{bmatrix}$ by assumption. The cancellation law applies giving

$$V_g' = V_e' J_g'$$

Or $\quad V_g = J_g V_e$.

CONCLUSIONS

It should be evident from the foregoing that the basic ideas of Tensor Theory are much easier to grasp when presented from the abstract point of view. The applications can then be followed in any notation.

Those who must apply Tensor Theory will, of course, find it necessary to go through the details but will find the reading easier because of their understanding of the basic ideas.

The differentiation of Tensor Systems, the Euclidean Christoffel Symbols, and Covariant Derivatives have been omitted as they contribute nothing to the objective of this paper.

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