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**Tensors as algebraic systems**

John A. Peterson

*The University of Montana*

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TENSORS
AS
ALGEBRAIC SYSTEMS

by

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B.A., Montana State University, 1949

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The aim of this investigation is to examine Tensor Theory from an abstract point of view and develop a simple representation for what is commonly referred to as "Tensors". The representation shall present the Tensor Theory in general in such a way as to be readily understandable to one possessing no previous knowledge of the subject.

The definition of Tensor Fields and Tensor Field Systems lends itself well to expression in terms of matrices and Jacobians. This simple representation enables the mathematician, with the elementary operations of matrices and Algebra in mind, to grasp the basic ideas of Tensor Theory very rapidly, avoiding the tedious, time consuming procedure of deciphering bulky, complicated notation.

It is the opinion of the author that this representation will remove the veil of mystery presently shrouding the "Tensor" for many students.
PRELIMINARY CONCEPTS

In associating Tensor Theory and Matrix Algebra we shall be concerned with Euclidean n-space, \( \mathbb{R}^n \), and groups, \( \mathcal{G} \), of automorphisms of \( \mathbb{R}^n \). The word "group" is used in the abstract sense.

In applied mathematics it is convenient when considering transformations to regard the image point as a new representation of the old point. Abstractly we say simply that the transformation is a mapping of one point onto another, i.e., given \( g \in \mathcal{G} \) and \( r, \) a point in \( \mathbb{R}^n \), \( g(r) = \mathbf{f} \), or \( r \rightarrow \mathbf{f} \) by the transformation \( g \). The abstract view lends itself more readily to developing the theory, whereas the other is quite useful in setting up a mathematical model for reality.
Consider a transformation, \( g \), and a point \( r = (x^1, x^2, \ldots, x^n) \) of \( \mathbb{R}^n \) which under the transformation \( g \) maps into \( \tilde{r} = (\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n) \), i.e., \( g(r) = \tilde{r} \). \( \tilde{r}^i = \tilde{r}^i(x^1, x^2, \ldots, x^n) \) under the transformation \( g \), and \( r^j = r^j(x^1, x^2, \ldots, x^n) \) under the transformation \( g^{-1} \). Associated with this transformation is a Jacobian, \( J_g \), defined as:

\[
J_g = \begin{bmatrix}
\frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} & \cdots & \frac{\partial \tilde{x}^1}{\partial x^n} \\
\frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} & \cdots & \frac{\partial \tilde{x}^2}{\partial x^n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \tilde{x}^n}{\partial x^1} & \frac{\partial \tilde{x}^n}{\partial x^2} & \cdots & \frac{\partial \tilde{x}^n}{\partial x^n}
\end{bmatrix}
\]

(1) \( J_g = \)

\[
J_{g^{-1}} = \begin{bmatrix}
\frac{\partial x^1}{\partial \tilde{x}^1} & \frac{\partial x^1}{\partial \tilde{x}^2} & \cdots & \frac{\partial x^1}{\partial \tilde{x}^n} \\
\frac{\partial x^2}{\partial \tilde{x}^1} & \frac{\partial x^2}{\partial \tilde{x}^2} & \cdots & \frac{\partial x^2}{\partial \tilde{x}^n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x^n}{\partial \tilde{x}^1} & \frac{\partial x^n}{\partial \tilde{x}^2} & \cdots & \frac{\partial x^n}{\partial \tilde{x}^n}
\end{bmatrix}
\]

(2) \( J_{g^{-1}} = \)
The subscripts, $g$, are to identify the transformation.

In the sequel we shall consider only those transformations, $g$ and $g^{-1}$, whose components are independent and possess continuous first partial derivatives. This implies the Jacobian of the transformation exists and is non-singular. Groups, $\mathcal{G}$, which consist entirely of such transformations will be called admissible.

$J^t_g$ shall denote the transpose of $J_g$.

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_n} \\
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial x_1} & \frac{\partial x_n}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_n}
\end{bmatrix}
\]

(3) $J^t_g = 
\begin{bmatrix}
\frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_n} \\
\frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial x_1} & \frac{\partial x_n}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_n}
\end{bmatrix}$

**Theorem 1:** If $g$ is an element of the admissible group, $\mathcal{G}$, then

(4) $J_{g^{-1}} = J^{-1}_g$

Proof:

$J_g J_{g^{-1}} = \sum \frac{\partial x_i}{\partial x_k} \frac{\partial x_k}{\partial x_j} = \delta_{ij}$

Hence \[ J_g J_{g^{-1}} = I \]
\[ J_{g^{-1}} J_g J_{g^{-1}} = J_{g^{-1}} I \]
\[ J_{g^{-1}} = J_{g^{-1}} \]

**Theorem 2.** If \( g \) and \( h \) are elements of an admissible group, \( G \), then \[ J_{hg} = J_h J_g . \]

**Proof:**

Let \( g(r) = F \) and \( h(F) = \bar{F} \).

The \( i^{th} \) row of \( J_h \) is
\[ \begin{array}{c}
\frac{\partial F_i}{\partial x^1} \\
\frac{\partial F_i}{\partial x^2} \\
\vdots \\
\frac{\partial F_i}{\partial x^n}
\end{array} \]

The \( j^{th} \) column of \( J_g \) is
\[ \begin{array}{c}
\frac{\partial x_j}{\partial x^1} \\
\frac{\partial x_j}{\partial x^2} \\
\vdots \\
\frac{\partial x_j}{\partial x^n}
\end{array} \]

Hence the element of the \( i^{th} \) row and \( j^{th} \) column of \( J_h J_g \) is
\[ \frac{\partial F_i}{\partial x^1} \frac{\partial x_j}{\partial x^1} + \frac{\partial F_i}{\partial x^2} \frac{\partial x_j}{\partial x^2} + \cdots + \frac{\partial F_i}{\partial x^n} \frac{\partial x_j}{\partial x^n} \]
which is recognizable as \( \frac{\partial x_j}{\partial x^i} \) and is the element of the \( i^{th} \) row and \( j^{th} \) column of \( J_{hg} \).

Hence

(5) \[ J_{hg} = J_h J_g . \]
From Theorem 2 and the properties of the transpose of matrix products, it follows that

(6) \[ J_{hg}^t = (J_h J_g)^t = J_g^t J_h^t. \]

Similarly and by (4)

(7) \[ J_{hg}^{-1} = (J_h J_g)^{-1} = J_g^{-1} J_h^{-1} = J_g^{-1} J_h^{-1}. \]

Definition 1: The transformation \( g \) will be called orthogonal if \( g(r) = A r \), where \( A \) is a constant orthogonal matrix, i.e.,

\[ A A' = I. \]
Before defining Tensor Fields and Tensor Field Systems in general, let us examine vector fields in matrix notation.

**Definition 2:** Suppose that for every \( r \in \mathbb{R}_n \) we have associated a vector (an \( n \times 1 \) matrix), \( V(r) \), whose components are functions over \( \mathbb{R}_m \), for some \( m \). We shall say such a collection of vectors, \( V(r) \), \( r \in \mathbb{R}_n \), forms a **Vector Field** over \( \mathbb{R}_n \).

**Definition 3:** Let \( \mathcal{G} \) be an admissible group of automorphisms of \( \mathbb{R}_n \) and suppose that for every \( g \in \mathcal{G} \) we have a vector field, \( V_g(r) \), and that these vector fields be related in such a manner that for every \( r \in \mathbb{R}_n \)

\[ V_g(r) = J_g(r) V_e(r) \]

where \( e \) is the identity of \( \mathcal{G} \). We shall say that the collection of vector fields, \( V_g(r) \), \( g \in \mathcal{G} \), forms a **Vector Field System** with respect to the group \( \mathcal{G} \).

**Lemma:** If every element of \( \mathcal{G} \) is an orthogonal transformation, then condition (3) of Definition 3 is equivalent to

\[ V_g(r) = J_g^{-1}(r) V_e(r) \]
This readily follows from Definition 1. That is

\[ J_g J'_g = I \]

Hence

\[ J_g = (J'_g)^{-1} = J'_g^{-1} \]

Note that if \( \mathbf{V}_g(r) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \) for some \( g \in \mathcal{G} \) it is zero for all \( g \in \mathcal{G} \). Since for \( g, h, k \in \mathcal{G} \),

\[
\mathbf{V}_k = \mathbf{V}_{hg} = J_{hg} \mathbf{V}_e \\
= J_h J_g \mathbf{V}_e \\
= J_h \mathbf{V}_g
\]

Note: It is to be emphasized that expressions such as (3) and (9) are identities in \( r \), however, the argument \( r \) shall be omitted whenever the meaning is clear without it.

Theorem 3: If \( \mathcal{G} \) is an admissible group of orthogonal transformations of \( R_n \), \( (J_g J'_g = 1, |J_g| = 1) \) and if \( \mathbf{V}_g \) and \( \mathbf{U}_g \) are two vector field systems with respect to the group \( \mathcal{G} \), then the collection of vector field systems \( \mathbf{V}_g \times \mathbf{U}_g \) forms a vector field system with respect to \( \mathcal{G} \).

Proof: We wish to show that the following relationship holds:

\[
\mathbf{V}_g \times \mathbf{U}_g = J_g (\mathbf{V}_e \times \mathbf{U}_e)
\]

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From Definition 3 we have

\[ \mathbf{v}_g = J_g \mathbf{v}_e \quad \text{and} \quad \mathbf{u}_g = J_g \mathbf{u}_e \]

Hence

\[ \mathbf{v}_g \times \mathbf{u}_g = J_g \mathbf{v}_e \times J_g \mathbf{u}_e \]

\[ J_g = A = [a_{ij}] \]

Carrying out the cross product gives for the first component of

\[ J_g \mathbf{v}_e \times J_g \mathbf{u}_e \]

\[ \left( \sum_j a_{2j} \mathbf{v}_j \right) \left( \sum_j a_{3j} \mathbf{u}_j \right) - \left( \sum_j a_{3j} \mathbf{v}_j \right) \left( \sum_i a_{i2} \mathbf{u}_i \right) \]

and for \( J_g \) \((\mathbf{v}_e \times \mathbf{u}_e)\):

\[ a_{11}(v_2u_3 - v_3u_2) + a_{12}(v_3u_1 - v_1u_3) + a_{13}(v_1u_2 - v_2u_1) \]

Expansion gives:

\[ \left( a_{22}a_{33} - a_{23}a_{32} \right) (v_2u_3 - v_3u_2) + \left( a_{31}a_{23} - a_{21}a_{33} \right) (v_3u_1 - v_1u_3) + \left( a_{21}a_{32} - a_{31}a_{22} \right) (v_1u_2 - v_2u_1) \]

Comparison of coefficients of like terms shows the coefficient in \( J_g \mathbf{v}_e \times J_g \mathbf{u}_e \) to be the cofactor of the respective coefficient in \( J_g(\mathbf{v}_e \times \mathbf{u}_e) \). Further expansion shows this true for all components and that the original cross product may be written:

\[ J_g \mathbf{v}_e \times J_g \mathbf{u}_e = (\text{Adj } J_g)' (\mathbf{v}_e \times \mathbf{u}_e) \]

Since \( J_g \) is orthogonal, \( (\text{Adj } J_g) = J_g^T J_g = J_g^T \), since \( |J_g| = 1 \)

by assumption. Hence,

\[ V_g \times U_g = J_g(\mathbf{v}_e \times \mathbf{u}_e) \]
Definition 4: A Tensor Field of rank two is an \( n \times n \) matrix, \( M(r) \), associated with each point \( r \in \mathbb{R}^n \), whose elements are functions over \( \mathbb{R}^n \), for some \( n \).

Many examples of tensor fields will be recognized as we proceed.

To stay with the familiar, however, let us consider the Jacobian of the transformation \( g \) which maps \( r: (x^1, x^2, \ldots, x^n) \) onto \( r^*: (x^1, x^2, \ldots, x^n) \). For fixed \( g \in \mathcal{G} \), \( J_g \) is a tensor field, since for each point, \( r: (x^1, x^2, \ldots, x^n) \), of \( \mathbb{R}^n \) there is associated a constant matrix, \( J_g \).

It is customary to call a vector field a tensor field of rank one and these terms will be used interchangeably in the sequel.

Thus the following definition is merely a repetition of Definition 3.

Definition 5: Let \( \mathcal{G} \) be an admissible group of automorphisms of \( \mathbb{R}^n \) and suppose that for every \( g \in \mathcal{G} \) we have a tensor field, \( M_g \), of rank one, and that these tensor fields be related in such a manner that for every \( r \in \mathbb{R}^n \)

\[
M_g = J_g M_e
\]
where \( e \) is the identity of \( G \). Such a collection of tensor fields shall be called a Contravariant Tensor Field System of rank one with respect to the group \( G \).

Example: Differentials.

Let \( f(x^1, x^2, \ldots, x^n) \) be a function over \( \mathbb{R}^n \). Then we define \( df \) to be a function over \( \mathbb{R}^{2n} \) as follows:

\[
\begin{align*}
df(x^1, x^2, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) &= \frac{\partial f}{\partial x^1} x^{n+1} + \frac{\partial f}{\partial x^2} x^{n+2} + \\
&\quad \ldots + \frac{\partial f}{\partial x^n} x^{2n}.
\end{align*}
\]

In applications, \( x^{n+1}, x^{n+2}, \ldots, x^{2n} \) are interpreted as increments of the independent variables \( x^1, x^2, \ldots, x^n \) respectively.

Accordingly, for \( g \in G \), \( dx^1, dx^2, \ldots, dx^n \) are defined as above, to be functions over \( \mathbb{R}^{2n} \), i.e.:

\[
(10) \quad dx^i = \frac{\partial x^i}{\partial x^1} x^{n+1} + \frac{\partial x^i}{\partial x^2} x^{n+2} + \ldots + \frac{\partial x^i}{\partial x^n} x^{2n}
\]

Then \( D_g = \begin{bmatrix} dx^1 \\ dx^2 \\ \vdots \\ dx^n \end{bmatrix} \) is a tensor field.

Now for \( g = e \), \( x^i = x^i \) so that \( dx^i = x^{n+1} \), and
\[
\mathbf{D}_0 = \begin{bmatrix}
\mathbb{R}^{n+1} \\
\mathbb{R}^{n+2} \\
\vdots \\
\vdots \\
\mathbb{R}^{2n}
\end{bmatrix}
\]

is also a tensor field.

It follows that \( \mathbf{D}_g = J_g \mathbf{D}_0 \) is another way of writing (10).

Hence the vector fields \( \mathbf{D}_g \) form a tensor field system of rank one.

**Definition 6:** Let \( \mathcal{L} \) be an admissible group of automorphisms of \( \mathbb{R}^n \) and suppose that for every \( g \in \mathcal{L} \) we have a tensor field, \( M_g \), of rank one and that these tensor fields be related in such a manner that for every \( r \in \mathbb{R}^n \)

\[
M_g = J_{g^{-1}} M_0.
\]

Such a collection of tensor fields shall be called a Covariant Tensor Field System of rank one with respect to the group \( \mathcal{L} \).

**Example:** Gradient.

Let \( s(r) \) be a real-valued function over \( \mathbb{R}^n \) possessing continuous first partial derivatives. In applied fields such functions arise and are required to remain invariant under a
transformation.

Define \( \nabla s(r) = \left[ \frac{\partial s}{\partial x^1} \right. \)
\[ \cdots \]
\[ \left. \frac{\partial s}{\partial x^n} \right] \]

Accordingly, we define \( \mathcal{E}(r) \equiv s(r^*) \), where \( g(r^*) = r \).

That is, we have a new function, \( \mathcal{E}(r) \), over \( \mathbb{R}^n \) such that its value for the point \( r \) equals \( s(r^*) \). The question arises, what is the relationship between \( \nabla s(r) \) and \( \nabla \mathcal{E}(r) \). Certainly, each of these is a tensor field of rank one. In addition it is well known from analysis that the following holds:

\[
\frac{\partial \mathcal{E}}{\partial x^i} = \sum_j \frac{\partial s}{\partial x^j} \frac{\partial x^j}{\partial x^i}
\]

Hence

\[
\nabla \mathcal{E}(r) = \sum_{g \in G_{\text{ad}}} \nabla s(r)
\]

We may summarize in the following theorem:

**Theorem 4**: Let \( s_e(r) \) be a real-valued function over \( \mathbb{R}^n \) possessing continuous first partial derivatives, and for each element \( g \) of the admissible group \( G \), let \( s_g(r) = s_e(r^*) \), where \( g(r^*) = r \).

Then for every \( g \in G \) the matrices \( \nabla s_g = \left[ \frac{\partial s_g}{\partial x^1} \right. \)
\[ \cdots \]
\[ \left. \frac{\partial s_g}{\partial x^n} \right] \] form
a tensor field of rank one and the collection of tensor fields $\nabla g \circ g$, $g \in \mathcal{G}$, form a covariant tensor field system of rank one.

From preceding remarks it is clear that $s_g(r) = \bar{s}(r)$ is such that

$$\nabla s_g = J g^{-1} \nabla s_0.$$

**Definition.** Let $\mathcal{G}$ be an admissible group of automorphisms of $R^n$ and suppose that for every $g \in \mathcal{G}$ we have a tensor field, $M_g$, of rank two and that these tensor fields be related in such a manner that for every $r \in R^n$

$$(12) \quad M_g = J g \circ g \circ J g^{-1}.$$

Such a collection of tensor fields shall be called a Contravariant Tensor Field System of rank two with respect to the group $\mathcal{G}$.

**Theorem 5.** Let $V_g$ and $U_g$ be vector field systems with respect to an admissible group $\mathcal{G}$. Then the collection of tensor fields $V_g, U_g$ form a contravariant tensor field system of rank two with respect to $\mathcal{G}$.

**Proof:**

$$V_g = J_g V_0 \quad \text{and} \quad U_g = J_g U_0$$

by (3)

$$V_g U_g = (J_g V_0)(J_g U_0)^* = J_g (V_0 U_0^*) J_g^*$$

In Theorem 3 we showed that the vector cross product formed a
vector field system with respect to a group of orthogonal transformations.

In the approach to tensors from the applied point of view, the components of the vector cross product are used to exemplify a contravariant tensor field system. We wish to point out that the components of the vector cross product are elements of \( V \times U = U \times V \).

**Definition:** Let \( \mathcal{G} \) be an admissible group of automorphisms of \( R_n \) and suppose that for every \( g \in \mathcal{G} \) we have a tensor field, \( M_g \), of rank two, and that these tensor fields be related in such a manner that for every \( r \in R_n \)

\[
M_g = J_{g-1} M_e J_{g-1}.
\]

Such a collection of tensor fields shall be called a Covariant Tensor Field System of rank two with respect to the group \( \mathcal{G} \).

**Example:** Euclidean Metric Tensor

Let \( \mathcal{G} \) be an admissible group of automorphisms of \( R_n \).

Define \( G_g \) as follows:

\[
G_g = J_{g-1} J_{g-1}
\]

Then

\[
G_g = J_{g-1} I J_{g-1}
\]

\[
= J_{g-1} G_e J_{g-1}
\]

And thus the tensor fields, \( G_g \), form a Covariant Tensor Field System.
of rank two.

For fixed \( g \in \mathcal{G} \), consider the differential vector field, \( D_g \).

(See example 5) and define the \( 1 \times 1 \) matrix \( ds^2 \) of functions over \( \mathbb{R}^{2n} \) as follows:

\[
ds^2 = D_g^T D_g = D_g^T J_g^{-1} J_g^{-1} D_g = D_g^T G_g D_g
\]

In applied fields the right member, expanded, is called the Fundamental Quadratic Form in space and the elements of \( G_g \) are the coefficients in the expansion.

**Definition 2**: Let \( \mathcal{G} \) be an admissible group of automorphisms of \( \mathbb{R}^n \) and suppose that for every \( g \in \mathcal{G} \) we have a tensor field, \( M_g \), of rank two, and that these tensor fields be related in such a manner that for every \( r \in \mathbb{R}^n \)

\[
(14) \quad M_g = J_g M_r J_g^{-1}
\]

Such a collection of tensor fields shall be called a Mixed Tensor Field System of rank two with respect to the group \( \mathcal{G} \).

**Example**: Kronecker Delta

The customary lengthy development of this tensor may be summarized by our methods as follows:

For each \( g \in \mathcal{G} \), let \( M_g = I \). Then the \( M_g \) form a mixed...
tensor field system of rank two, i.e.,

\[ I = J_g^{-1} J_g^{-1} \]

**Theorem 6:** Let \( M_g \) and \( M_g^* \) be two tensor field systems of the same rank and type. Then \( M_g + M_g^* \) is a tensor field system of that same rank and type.

**Proof:** This requires a trivial enumeration of the possibilities:

**Rank One:**

\[ M_g + M_g^* = J_g M_e + J_g M_g^* = J_g (M_e + M_g^*) \]

**Rank Two:**

\[ M_g + M_g^* = J_g^{-1} M_e + J_g^{-1} M_g^* = J_g^{-1} (M_e + M_g^*) \]

**Theorem 7:** Let \( M_g \) and \( M_g^* \) be two tensor field systems of rank one. Then the product, \( M_g^* M_g^{-1} \), is a tensor field system of rank two. If \( M_g \) and \( M_g^* \) are of the same type the product will be of that type; if they are of different types the product, or the transpose of the
product, will be of mixed type.

Proof: Again, an enumeration of the possibilities will suffice.

Contravariant:

\[
M^g_i^j = (J^g_e M^e_i)(J^e_o M^o_j)' = J^g_e (M^e_i M^o_j)^1 J^e_o
\]

Covariant:

\[
M^g_i^j = (J^g_{-1} M^e_i)(J^e_{-1} M^o_j)' = J^g_{-1} (M^e_i M^o_j)^1 J^e_{-1}
\]

Mixed:

\[
M^g_i^j = (J^g_e M^e_i)(J^e_{-1} M^o_j)^1 = J^g_e (M^e_i M^o_j)^1 J^e_{-1}
\]

\[
M^g_i^j = (J^g_{-1} M^e_i)(J^e_m M^o_j)^1 = J^g_{-1} (M^e_i M^o_j)^1 J^e_{-1}
\]

\[
(M^g_i^j)^1 = M^g_i^j = J^g_e (M^e_i M^o_j)^1 J^e_{-1}
\]

Theorem 8: Let \( M_i^g \) form a covariant (contravariant) tensor field system of rank two and \( M_i^g \) a contravariant (covariant) tensor field system of rank two with respect to a group \( \mathcal{G} \) of admissable automorphisms of \( R_n \). Then \( M_i^g M_i^g \) forms a mixed tensor field system of rank two.

Proof: By Definition 8:

\[
M_i^g = J^g_{-1} M^e_i J^e_{-1}
\]

By Definition 7:

\[
M_i^g = J^g M^e_i J^e
\]

Then

\[
M_i^g M_i^g = J^g M^e_i J^g_{-1} M^e_i J^e_{-1}
\]

\[
= J^g (M^e_i M^e_i)^1 J^e_{-1}
\]

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Theorem 9: Quotient Rule.

Let $V_g$ be a vector field for each $g \in \mathcal{J}$ and $U_g$ a covariant (contravariant), non-null vector field system. If the real-valued function over $R^n$, $V_g^t U_g$, is independent of $g$, i.e., $V_g^t U_g = V_h^t U_h$ for $g, h \in \mathcal{J}$, then $V_g$ is a contravariant (covariant) vector field system.

Proof:

By (9)

$U_g = J_g^{-1} U_e$

Then

$U_e = J_g^t U_g$

Since

$V_g^t U_g = V_e^t U_e$

$V_g^t U_g = V_e^t J_g U_e$

Multiply both sides on the right by $U_g^t$.

Then

$V_g^t U_g U_g^t = V_e^t J_g U_e U_g^t$

Now $U_g U_g^t \neq 0$ since $U_g \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by assumption. The cancellation law applies giving

$V_g^t = V_e^t J_g$

Or

$V_g = J_g V_e$. 


CONCLUSIONS

It should be evident from the foregoing that the basic ideas of Tensor Theory are much easier to grasp when presented from the abstract point of view. The applications can then be followed in any notation.

Those who must apply Tensor Theory will, of course, find it necessary to go through the details but will find the reading easier because of their understanding of the basic ideas.

The differentiation of Tensor Systems, the Euclidean Christoffel Symbols, and Covariant Derivatives have been omitted as they contribute nothing to the objective of this paper.
BIBLIOGRAPHY


