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Stability behavior of differential equations

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STABILITY BEHAVIOR OF DIFFERENTIAL EQUATIONS

By

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B.A., University of Montana, 1965

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ABSTRACT

The purpose of this paper is to investigate the stability behavior of solutions of systems of differential equations. Conditions for stability, asymptotic stability, uniform stability and uniform asymptotic stability of linear systems are given in terms of fundamental matrices. The variation of constants formula is used to examine the stability behavior of certain nonlinear systems. Also the second method of Lyapunov and the concept of total stability are discussed. Several examples illustrating the theorems and applications are included.
ACKNOWLEDGEMENTS

Most of the material in this thesis was given in a course at the University of Montana in the summer of 1966. The author makes no claim of originality for the material used here.

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J.T.C.
A solution of a differential equation may be interpreted as describing the motion of a particle in a force field. The solutions may be determined by the initial position and the initial velocity of the particle. If one wanted an equation describing the motion of the particle, the initial values could be measured and then the equation of motion could be obtained by solving the differential equation. However, there may be errors in the initial values due to imprecise measurements and the solution obtained would not be a description of the actual motion. On the other hand, if the solution of the equation which describes the motion satisfies certain stability properties, a small error in initial values would not have much effect and the solution obtained would be a good approximation of the motion. As an example, consider the equation $y'' + k^2 y = 0$. The solution of this equation which satisfies the initial conditions $y(0) = y_0$, $y'(0) = v_0$ is given by $y(t) = (v_0/k) \sin(kt) + y_0 \cos(kt)$. Suppose the initial values obtained by measurement were $y_0 + \Delta y_0$ and $v_0 + \Delta v_0$. The solution of the above equation satisfying these initial conditions is $z(t) = \frac{(v_0 + \Delta v_0) \sin(kt)}{k} + (y_0 + \Delta y_0) \cos(kt)$. Since $|z(t) - y(t)| < |\frac{\Delta v_0}{k}| + |\Delta y_0|$, $z(t)$ is a good approximation of the solution $y(t)$ provided that the errors $\Delta y_0$ and $\Delta v_0$ are sufficiently small. Thus the solution
$y(t)$ exhibits a definite type of stability with respect to changes in initial values.

Having determined the stability behavior of the solutions of a certain differential equation, it might be suspected that the equation could be altered slightly, the solutions of the resulting equation exhibiting the same stability behavior as the original equation. For example, how small must $g(t)$ be in order that the solutions of $y'' + g(t)y' + k^2y = 0$ exhibit the same stability behavior as the solutions of $y'' + k^2y = 0$? These and similar types of questions will be answered in this paper. Chapter one gives the necessary properties of vectors and matrices and chapter two deals with differential systems. The basic properties of linear differential systems are given in chapter three. The definitions of the various types of stability to be considered and the main results of the thesis occur in chapters four and five.
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The set of all n-dimensional vectors $x = (x_1, \ldots, x_n)$ with real or complex entries will be denoted by $\mathbb{R}^n$. The norm of a vector $x$, is a real number, denoted $|x|$, satisfying:

1. $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$
2. $|ax| = |a| |x|$ for any scalar $a$
3. $|x + y| \leq |x| + |y|$\n
The set of all $n \times n$ matrices $A = (a_{ij})$ with real or complex entries will be denoted by $\mathbb{M}^n$. If $A$ is an $n \times n$ matrix, $|A|$ is defined by

$$|A| = \sup_{x \neq 0} \frac{|Ax|}{|x|}$$

(1.1), (1.2), and (1.3) are satisfied for matrices. Also,

1. $|Ax| \leq |A| |x|$ for $A \in \mathbb{M}^n$ and $x \in \mathbb{R}^n$
2. $|AB| \leq |A| |B|$ for $A, B \in \mathbb{M}^n$
3. If $|Ax| < \varepsilon$ for $0 < |x| < \delta$, then $|A| < \varepsilon / \delta$.

Proof of (1.7): $|A| = \sup_{x \neq 0} |Ax| = \sup_{x \neq 0} |Ax| = \sup_{|x| = 1} |Ax| = \sup_{|x|=1} |ax| = |a| \sup_{|x|=a} |Ax| = |a| \sup_{|x|=a} |Ax| \leq |a| \varepsilon / \delta$ . Letting $a \to \delta$, we obtain $|A| \leq \varepsilon / \delta$ .
With their respective norms, $\mathbb{R}^n$ and $\mathbb{M}^n$ are Banach spaces (complete normed vector spaces). For finite dimensional vector spaces it can be shown that all norms are equivalent. Consequently, a theorem proved using one norm is valid for the other norms and the choice of the norm is usually dictated by the ease of the proof. For definiteness, the sup norm will be employed. That is, for $x=(x_1,\ldots,x_n)$, $|x| = \max_{1 \leq i \leq n} |x_i|$.

(1.8) With the sup norm, the norm of a matrix $A=(a_{ij})$ is given by

$$|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Proof: Let $S = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$, say $S = \sum_{j=1}^{n} |a_{kj}|$, where $k$ is fixed. $|Ax| = \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| |x_i|$

$$= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| |x_i| = S |x|.$$ Thus $|Ax| \leq S$ for $x \neq 0$, so $|A| \leq S$. For the reverse inequality $|A| \geq S$, let $x=(x_1,\ldots,x_n)$ where $x_j = |a_{kj}| / a_{kj}$ if $a_{kj} \neq 0$, $x_j = 0$ otherwise. We may assume $a_{kj} \neq 0$ for some $j$ since otherwise $A=0$ and the theorem is obvious. Then $|x|=1$ and $|A| \geq |Ax| = \max_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$

$$= \left| \sum_{j=1}^{n} a_{kj} x_j \right| = S.$$ This completes the proof.

We will consider vector functions $x(t)=(x_1(t),\ldots,x_n(t))$ and also matrix functions $A(t)=(a_{ij}(t))$ of a real variable $t$. Integrals and derivatives of a vector or matrix...
function are defined by taking the integral or derivative of the components. For example, \( x'(t) = (x'_1(t), \ldots, x'_n(t)) \) and \( \int_a^b x(s) \, ds = (\int_a^b x_1(s) \, ds, \ldots, \int_a^b x_n(s) \, ds) \). It now follows that if \( x(t) \) is continuous on an interval \([a, b]\), then

\[
\left| \int_a^b x(s) \, ds \right| \leq \int_a^b |x(s)| \, ds.
\]

The algebra of derivatives of vector or matrix functions is analogous to that for real-valued functions. For example, if \( A \) and \( B \) are differentiable matrices, then \( AB \) is also differentiable and

\[ (AB)' = AB' + A'B. \]

If \( A \in M^n \), the exponential series \( e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} \) converges. Here \( I \) is the identity matrix. To see this, consider the partial sums \( S_m = \sum_{k=0}^{m} \frac{A^k}{k!} \). Since \( |A^k| \leq |A|^k \) for integers \( k \geq 0 \),

\[
|S_{m+p} - S_m| = \left| \sum_{k=m+1}^{m+p} \frac{A^k}{k!} \right| \leq \sum_{k=m+1}^{m+p} \frac{|A|^k}{k!} \leq \sum_{k=m+1}^{\infty} \frac{|A|^k}{k!}.
\]

The last sum tends to 0 as \( m \to \infty \) since it is the remainder after \( m \) terms of the series expansion for \( e^{\lambda A} \). Thus by the completeness of \( M^n \), \( e^A = \lim_{m \to \infty} S_m \) exists.

(1.9) If \( AB = BA \), then \( e^{A+B} = e^A e^B \).

Before proving this, we state the following formula which can be shown by induction. Let \( A_1, B_1 \in M^n, i=0,1,\ldots,m \).

Then

\[
\sum_{k=0}^{m} \sum_{j=0}^{m} A_k B_j - \sum_{k=0}^{m} \sum_{j=0}^{m} A_j B_{k-j} = \sum_{k=0}^{m} \sum_{j=0}^{m} A_k B_j.
\]

Proof of (1.9): \( e^{A+B} = \lim_{m \to \infty} T_m \) where

\[
T_m = \sum_{k=0}^{m} (A+B)^k/k!. \quad \text{Since } AB = BA, \text{ the binomial theorem holds and therefore } (A+B)^k = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j}. \quad \text{Thus } T_m = \sum_{k=0}^{m} \sum_{j=0}^{k} \frac{A^j B^{k-j}}{j!(k-j)!}.
\]

On the other hand, \( e^A e^B = \lim_{m \to \infty} S_m \) where \( S_m = \left( \sum_{k=0}^{m} \frac{A^k}{k!} \right) \left( \sum_{j=0}^{m} \frac{B^j}{j!} \right) = \).
Let $A_k = A^k/k!$ and $B_k = B^k/k!$. Then

$$T_m = \sum_{k=0}^{m} \sum_{j=0}^{m} A_j B_{k-j}$$

and $S_m = \sum_{k=0}^{m} \sum_{j=0}^{m} A_k B_j$. Using the formula stated, $S_m - T_m = \sum_{k=0}^{m} \sum_{j=0}^{m} A_k B_j$. Also

$$|e^{A+B} - e^A e^B| \leq |e^{A+B} - T_m| + |T_m - S_m| + |S_m - e^A e^B|$$

independent of $m$. Both $|e^{A+B} - T_m|$ and $|S_m - e^A e^B|$ tend to 0 as $m \to \infty$.

If $|T_m - S_m| \to 0$ as $m \to \infty$, the proof will be complete. $|S_m - T_m|$

$$= \left| \sum_{k=0}^{m} \sum_{j=0}^{m} A_k B_{k-j} \right| \leq \sum_{k=0}^{m} \sum_{j=0}^{m} |A_k| |B_{k-j}| \leq \sum_{k=0}^{m} \sum_{j=0}^{m} \frac{|A_k|}{k!} \frac{|B_j|}{j!}$$

$$= \left( \sum_{k=0}^{m} \frac{|A_k|}{k!} \right) \left( \sum_{j=0}^{m} \frac{|B_j|}{j!} \right) - \sum_{k=0}^{m} \frac{(|A_k| + |B_j|)^k}{k!}$$

and this difference tends to $e^{|A|} e^{|B|} - e^{|A|+|B|} = 0$ as $m \to \infty$. This completes the proof.

Since $A$ and $-A$ commute, $e^A e^{-A} = e^{-A} e^A = I$. Thus for any $A \in M_n$, $e^A$ is nonsingular and $(e^A)^{-1} = e^{-A}$. If $A$ is a diagonal matrix, the notation $A = \text{diag}[a_{11}, \ldots, a_{nn}]$ will be used. In this case, $A^k = \text{diag}[a_{11}^k, \ldots, a_{nn}^k]$ for integers $k \geq 0$ and it follows that $e^A = \text{diag}[e^{a_{11}}, \ldots, e^{a_{nn}}]$. More generally, $A$ may consist of submatrices $B_i$ which have their diagonals coincident with the diagonal of $A$. That is, $A$ may consist of blocks along the diagonal and zeroes elsewhere.

$$A = \begin{bmatrix}
B_1 & 0 & \cdots & 0 \\
0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & B_r
\end{bmatrix}$$
If $A$ is of this form we write $A = \text{diag}[B_1, \ldots, B_r]$. In this case $A^k = \text{diag}[B_1^k, \ldots, B_r^k]$ and it follows that $e^A = \text{diag}[e^{B_1}, \ldots, e^{B_r}]$.

If $P$ is a nonsingular matrix, $(P^{-1}AP)^k = P^{-1}A^kP$ for all integers $k \geq 0$. It follows that $e^{P^{-1}AP} = P^{-1}e^AP$.

In the following, $I_s$ will denote the identity matrix of order $s$ and $N_s$ will denote the matrix of order $s$ with ones just below the diagonal and zeroes elsewhere. For example

$$N_4 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}$$

(1.10) Let $A \in \mathbb{M}^n$, and let $\lambda_1, \ldots, \lambda_j$ ($1 \leq j \leq n$) be the distinct characteristic roots of $A$. Then there are nonsingular matrices $P$ such that $B = P^{-1}AP$ has the form $B = \text{diag}[B_1, \ldots, B_r]$ where each $B_i$ is of the form $B_i = \lambda_i I_{s_i} + N_{s_i}$ for some root $\lambda \in \{\lambda_1, \ldots, \lambda_j\}$. Further, for each root $\lambda$, there is an $i$ such that $B_i = \lambda_i I_{s_1} + N_{s_1}$ and $B_i$ is called a companion matrix of the root $\lambda$. The matrix $B$ is uniquely determined apart from a permutation of the blocks $B_i$ and is said to be in Jordan canonical form.

For a proof of this theorem, see reference [5] page 106 where it is stated in a slightly different form but gives 1.10 for $A^T$ (A transposed).

The following is an example of a matrix in Jordan form.
Here, \( B = \text{diag}[B_1, B_2, B_3, B_4] \) where \( B_1 = 3I_3 + N_3 \), \( B_2 = [6] \), \( B_3 = 2I_2 + N_2 \), and \( B_4 = [6] \).

If all the companion matrices of a root \( \lambda \) are of order one, \( \lambda \) is called a root of simple type. In the above example, 6 is a root of simple type. If all the roots are distinct, each block is of order one and the Jordan form is a diagonal matrix with the characteristic roots on the diagonal.

Suppose that \( B = P^{-1}AP = \text{diag}[B_1, \ldots, B_r] \) is in Jordan form. Then \( A = PBP^{-1} \) so that \( e^A = Pe^B P^{-1} \). Also, \( e^B = \text{diag}[e^{B_1}, \ldots, e^{B_r}] \). The matrices \( e^{Bi} \) may be calculated directly. To see this, write \( B_i = \lambda I_s + N_s \). Since \( \lambda I_s \) and \( N_s \) commute, and since \( e^{\lambda I_s} = \text{diag}[e^\lambda, \ldots, e^\lambda] \), \( e^{Bi} = e^{\lambda I_s} e^{N_s} = e^\lambda e^{N_s} \). To calculate \( e^{N_s} \), note that \( N_s^2 \) is obtained from \( N_s \) by shifting the 1's down one row.

For example,

\[
N_s^2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

Higher powers of \( N_s \) are obtained by repeatedly shifting the
1's down one row. $N_{s}^{S-1}$ is a matrix with a 1 in the lower left corner and $N_{s}^{S} = 0$. Thus $e^{Ns} = \sum_{k=0}^{S-1} (N_{s})^{k}/k!$ and

$$e^{B_{i}} = e^{\lambda} e^{Ns} = e^{\lambda}$$

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CHAPTER 2

Differential Systems

We will be considering systems of differential equations of the form

\[ x'_1 = f_1(t, x_1, \ldots, x_n) \]
\[ x'_2 = f_2(t, x_1, \ldots, x_n) \]
\[ \vdots \]
\[ x'_n = f_n(t, x_1, \ldots, x_n) \]

where \( t \) is a real variable referred to as "time" and each \( f_i \) is a real or complex valued function defined on a subset of \( \mathbb{R}^{n+1} \). Unless stated otherwise, differentiation will always be with respect to the variable \( t \).

In vector form, the above system may be written

(1) \[ x' = f(t, x) \]

where \( x = (x_1, \ldots, x_n) \) and \( f(t, x) = (f_1(t, x), \ldots, f_n(t, x)) \).

This form encompasses a wide variety of differential equations. For example, a scalar \( n \)-th order equation

\[ y^{(n)} = F(t, y, \ldots, y^{(n-1)}) \]

where the superscript indicates the order of the derivative may be put in vector form by making the transform-
We obtain the system of equations

\[
\begin{align*}
x'_1 &= x_2 \\
x'_2 &= x_3 \\
&\quad \vdots \\
x'_n &= F(t, x_1, \ldots, x_n).
\end{align*}
\]

A vector function \( x(t) \) is said to be a solution of (1) on an interval provided it is differentiable and satisfies \( x'(t) = f(t, x(t)) \) at each point \( t \) in the interval. Let \( (t_0, x_0) \) be a point in \( \mathbb{R}^{n+1} \) (\( x_0 \in \mathbb{R}^n \)). The problem of finding a solution \( x(t) \) of (1) such that

\[
x(t_0) = x_0
\]

is called an initial value problem (IVP).

**Theorem (2.1)** If \( f(t, x) \) is continuous, then the initial value problem (1)-(2) is equivalent to the integral equation

\[
x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds.
\]

**Proof:** If \( x(t) \) satisfies (1)-(2), then integrating (1) from \( t_0 \) to \( t \) gives (3). Conversely suppose (3) is satisfied. Then \( x(t_0) = x_0 \) so that (2) is satisfied. \( x(t) \) is continuous since the integral is a continuous function of its upper limit. Therefore, \( f(t, x(t)) \) is continuous. Then the integral \( \int_{t_0}^{t} f(s, x(s)) \, ds \) is differentiable and has the derivative \( f(t, x(t)) \). From (3), \( x(t) \) is differentiable and \( x'(t) = f(t, x(t)) \).
Theorem (2.2) Let \( f(t,x) \) be continuous for \( t_0 \leq t \leq t_0 + h \), \(|x|<\infty\). Suppose also that \( f \) satisfies the Lipschitz condition \(|f(t,x) - f(t,y)| \leq L|x - y|\) where \( L \) is some positive constant. Then the initial value problem (1)-(2) has a unique solution on the interval \([t_0, t_0 + h]\).

Proof: Let \( X \) be the set of all \( n \)-dimensional vector functions which are continuous on the interval \([t_0, t_0 + h]\). Let \( K > L \) and for \( x \in X \), define \( \|x\| = \sup_{t \in [t_0, t_0 + h]} e^{-K(t-t_0)}|x(t)| \).

With this norm, \( X \) is a Banach space. Define \( T:X \to X \) by

\[
Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds.
\]

Let \( x, y \in X \). Then

\[
|Tx(t) - Ty(t)| \leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| \, ds.
\]

Using the Lipschitz condition:

\[
\leq L \int_{t_0}^{t} |x(s) - y(s)| \, ds.
\]

Thus,

\[
e^{-K(t-t_0)}|Tx(t) - Ty(t)| \leq L \int_{t_0}^{t} e^{-K(s-t_0)}|x(s) - y(s)| \, ds
\]

\[
= L \int_{t_0}^{t} e^{-K(t-s)} [e^{-K(s-t_0)}|x(s) - y(s)|] \, ds
\]

\[
\leq L \|x - y\| \int_{t_0}^{t} e^{-K(t-s)} \, ds \leq (L/K) \|x - y\|.
\]

Therefore, \( \|Tx - Ty\| \leq (L/K) \|x - y\| \) and since \( L/K < 1 \), \( T \) is a contraction mapping. Applying Banach's fixed point theorem, \( T \) has a unique fixed point \( x = Tx \) which by theorem (2.1) is the unique solution of the IVP (1)-(2). This completes the proof.
Theorem (2.3) If \( f(t, x) \) is continuous in a neighborhood of the point \((t_0, x_0)\), then the initial value problem (1)-(2) has a solution which is defined in a neighborhood of \( t_0 \).

This theorem is more difficult to prove and requires the use of the Schauder fixed point theorem. A proof is given in reference [1].

In stability theory we are interested in solutions which are defined for all \( t > t_0 \). Theorem (2.3) gives sufficient conditions for local existence, in particular on an interval \([t_0, t_0 + h)\) where \( h > 0 \). A solution \( x(t) \) of (1) on the interval \([t_0, t_0 + h)\) is said to be continuable provided there is a solution \( y(t) \) on an interval \([t_0, t_0 + r)\) where \( r > h \) and \( y(t) = x(t) \) for all \( t \in [t_0, t_0 + h) \). Otherwise \( x(t) \) is said to be noncontinuable. The next theorem will be very useful later on when we must show that certain solutions are defined for all \( t > t_0 \).

Theorem (2.4) Let \( f(t, x) \) be continuous on a compact set \( C \) and let \((t_0, x_0)\) be any point of \( C \). Then any noncontinuable solution \( x(t) \) of the initial value problem (1)-(2) is defined on an interval \([t_0, t_1]\) such that the graph of \( x(t) \) is contained in \( C \) for \( t_0 < t < t_1 \) and has the point \((t_1, x(t_1))\) on the boundary of \( C \).

Proof: Let \( x(t) \) be any solution of (1)-(2) and define \( t_1 = \sup \{ t \mid (s, x(s)) \in C, t_0 < s < t \} \). We show first that \( x(t) \)
can be defined at \( t_1 \) so that (1) is satisfied. We have
\[
x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds \quad \text{for } t_0 \leq t < t_1.
\]
Define
\[
x(t_1) = \lim_{h \to 0^+} x(t_1 - h) = x_0 + \lim_{t \to t_1} \int_{t_0}^{t} f(s, x(s)) \, ds.
\]
This limit certainly exists since \((s, x(s)) \in C\) for \( s \in [t_0, t_1] \) and \( f \) is bounded on \( C \). Then by this definition, \( x(t) \) is continuous at \( t_1 \). Therefore since \( C \) is compact, \((t_1, x(t_1))\) is also in \( C \). Since the integral representation is satisfied at \( t_1 \), \( x'(t_1) \) exists and equals \( f(t_1, x(t_1)) \). Finally, if \((t_1, x(t_1))\) were not on the boundary of \( C \), it would have to lie in the interior of \( C \). In this case, \( f \) would be continuous in a neighborhood of \((t_1, x(t_1))\). By theorem (2.3) there is a solution \( y(t) \) in a neighborhood of \( t_1 \) which satisfies the new initial condition \( y(t_1) = x(t_1) \). Thus the solution \( x(t) \) is continuabale. This completes the proof.

If \( x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds \), then
\[
|x(t)| \leq |x_0| + \int_{t_0}^{t} |f(s, x(s))| \, ds.
\]
Gronwall's inequality provides a method for obtaining an upper bound on \( x(t) \) which is independent of \( x \) if \( f \) satisfies a certain inequality.

Gronwall's Inequality: Let \( h(t) \) be a real valued continuous function and \( k(t) \) a nonnegative continuous function on the interval \([a, b]\). Let \( y(t) \) be a real valued continuous function on \([a, b]\) and suppose \( y(t) \leq h(t) + \int_{a}^{t} k(s)y(s) \, ds \) for \( a \leq t \leq b \). Then on the same interval,
\[
y(t) \leq h(t) + \int_{a}^{t} h(s)k(s) \exp\left(\int_{s}^{t} k(u) \, du\right) \, ds.
\]
In particular, if \( h(t) = h \) is constant on \([a,b]\), then
\[
y(t) \leq h \exp \left( \int_a^t k(s) \, ds \right)
\]
Proof: Let \( z(t) = \int_a^t k(s) y(s) \, ds \). Then \( z'(t) = k(t) y(t) \)
\[
\leq k(t) h(t) + k(t) \int_a^t k(s) y(s) \, ds = k(t) h(t) + k(t) z(t).
\]
Therefore, \( z'(t) - k(t) z(t) \leq k(t) h(t) \).
Let \( w(t) = z(t) \exp \left( - \int_a^t k(s) \, ds \right) \). Then
\[
w'(t) = -z(t) k(t) \exp \left( - \int_a^t k(s) \, ds \right) + z'(t) \exp \left( - \int_a^t k(s) \, ds \right)
= (z'(t) - z(t) k(t)) \exp \left( - \int_a^t k(s) \, ds \right) \leq k(t) h(t) \exp \left( - \int_a^t k(s) \, ds \right)
\]
Since \( w(a) = 0 \), \( w(t) = \int_a^t w'(s) \, ds \leq \int_a^t k(s) h(s) \exp \left( - \int_a^s k(u) \, du \right) \, ds \)
Therefore, \( z(t) = w(t) \exp \left( \int_a^t k(s) \, ds \right) \)
\[
\leq \int_a^t k(s) h(s) \exp \left( - \int_a^s k(u) \, du + \int_s^t k(u) \, du \right) \, ds \\
= \int_a^t k(s) h(s) \exp \left( \int_s^t k(u) \, du \right) \, ds.
\]
Using this last inequality and the assumption that
\( y(t) \leq h(t) + z(t) \) we obtain the inequality
\[
y(t) \leq h(t) + \int_a^t k(s) h(s) \exp \left( \int_s^t k(u) \, du \right) \, ds,
\]
which was to be shown.

In particular, if \( h(t) = h \) is constant, then
\[
y(t) \leq h \left[ 1 + \int_a^t k(s) \exp \left( \int_s^t k(u) \, du \right) \, ds \right]. \tag{1}
\]
The integrand \( k(s) \exp \left( \int_s^t k(u) \, du \right) \) is of the form \( -g'(s) \exp(g(s)) \)
where \( g(s) = \int_s^t k(u) \, du \). Since \( \int_a^t -g'(s) \exp(g(s)) \, ds = -\exp(g(s)) \)
we obtain \( \int_a^t k(s) \exp \left( \int_s^t k(u) \, du \right) \, ds = \exp \left( \int_a^t k(u) \, du \right) - 1 \).
Substituting this value in (1), \( y(t) \leq h \exp \left( \int_a^t k(u) \, du \right) \)
and the proof is complete.
CHAPTER 3

Properties of Linear Differential Systems

A linear system of differential equations has the form

\[
\begin{align*}
x'_1 &= a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_1(t) \\
x'_2 &= a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + b_2(t) \\
&\quad \vdots \\
x'_n &= a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_n(t)
\end{align*}
\]

Letting \( x = [x_1, \ldots, x_n]^T \), \( A(t) = (a_{ij}(t)) \), and \( b(t) = [b_1(t), \ldots, b_n(t)]^T \), the above system may be written

\[
(1) \quad x' = A(t)x + b(t)
\]

The homogeneous equation associated with (1) is

\[
(2) \quad x' = A(t)x
\]

Several properties concerning linear systems will be needed. Unless stated otherwise, we always assume \( A(t) \) and \( b(t) \) are continuous for \( t \geq t_0 \).

Theorem (3.1) Let \( x_0 \in \mathbb{R}^n \). Then (1) has a unique solution \( x(t) \) on the interval \([t_0, +\infty)\) such that \( x(t_0) = x_0 \).

Proof: Let \( f(t, x) = A(t)x + b(t) \). Then

\[
|f(t, x) - f(t, y)| = |A(t)(x - y)| \leq |A(t)||x - y|.
\]

Since \( A(t) \) is continuous for \( t \geq t_0 \), on any finite interval \([t_0, t_1]\), there is a constant \( L \) such that \( |A(t)| \leq L \) for
By theorem (2.2), the initial value problem
\[ x' = f(t,x), \quad x(t_0) = x_0, \]
has a unique solution on \([t_0, t_1]\). Since \(t_1\) is arbitrary, the result follows.

Consider the matrix differential system

\[ Y' = A(t)Y \]

That is,
\[ y_{ij}' = \sum_{k=1}^{n} a_{ik}(t)y_{kj} \quad (i,j=1,\ldots,n). \]

This is actually a linear system of order \(n^2\) so that (3.1) applies to (3) as well, i.e., given a matrix \(C \in \mathbb{M}^n\), there is one and only one solution of (3) with \(Y(t_0) = C\). If \(Y(t)\) is a solution of (3) and \(C \in \mathbb{R}^n\), then \(x(t) = Y(t)c\) is a solution of the homogeneous equation (2) since
\[ x'(t) = Y'(t)c = A(t)Y(t)c = A(t)x(t). \]

Definition (3.2) Let \(Y(t)\) be a solution of (3). If for each solution \(x(t)\) of the homogeneous equation (2) there is a unique constant vector \(c\) such that \(x(t) = Y(t)c\), then \(Y(t)\) is called a fundamental matrix for (2).

Theorem (3.3) Let \(Y(t)\) be a solution of (3). Then \(Y(t)\) is a fundamental matrix for (2) if and only if \(Y(t)\) is nonsingular for all \(t_0 < t < t_1\).

Proof: Suppose first that \(Y(t)\) is a fundamental matrix for (2). If for some \(t_1 < t_0\), \(Y(t_1)\) is singular, then there is a non-zero vector \(c\) such that \(Y(t_1)c = 0\). Now \(x(t) = Y(t)c\) is a solution of (2) and \(x(t_1) = 0\) but the function identically zero on \([t_0, +\infty)\) is a solution of (2) and by the
uniqueness of solutions \( x(t) = 0 \) for all \( t \geq t_0 \). This contradicts the definition of a fundamental matrix since \( Y(t)c \) and \( Y(t)0 \) are distinct representations of the solution identically zero. Conversely suppose \( Y(t) \) is nonsingular for \( t \geq t_0 \). If \( x(t) \) is any solution of (2), \( z(t) = Y(t)Y^{-1}(t_0)x(t_0) \) is a solution such that \( z(t_0) = x(t_0) \). By uniqueness, \( z(t) = x(t) \) for all \( t \geq t_0 \). That is, \( x(t) = Y(t)Y^{-1}(t_0)x(t_0) \). Thus \( x(t) = Y(t)c \) where \( c = Y^{-1}(t_0)x(t_0) \). Further \( c \) is uniquely determined since \( Y(t) \) is nonsingular. This completes the proof.

A fundamental matrix \( Y(t) \) for (2) is extremely important. First of all, the solution of (2) which takes the value \( x_1 \) at the time \( t_1 \geq t_0 \) is given by \( x(t) = Y(t)Y^{-1}(t_1)x_1 \). Second, we can obtain the solutions of the inhomogeneous equation (1) in terms of \( Y(t) \). Suppose we wished to find the solution \( x(t) \) of (1) which satisfies \( x(t_0) = x_0 \). We assume a solution of the form \( x(t) = Y(t)c(t) \) where \( c(t) \) is a vector function to be determined. Differentiating,
\[
x'(t) = Y'(t)c(t) + Y(t)c'(t) = A(t)Y(t)c(t) + Y(t)c'(t)
\]
\[
= A(t)x(t) + Y(t)c'(t).
\]
We now determine \( c(t) \) so that \( Y(t)c'(t) = b(t) \), i.e., so that \( c'(t) = Y^{-1}(t_0)b(t) \).

Integrating from \( t_0 \) to \( t \), we obtain
\[
c(t) = c(t_0) + \int_{t_0}^{t} Y^{-1}(s)b(s)ds.
\]
Thus \( x(t) = Y(t)c(t) = Y(t)c(t_0) + Y(t) \int_{t_0}^{t} Y^{-1}(s)b(s)ds \).

Letting \( c(t_0) = Y^{-1}(t_0)x_0 \) we have \( x(t) = Y(t)Y^{-1}(t_0)x_0 + Y(t) \int_{t_0}^{t} Y^{-1}(s)b(s)ds \).

Clearly \( x(t_0) = x_0 \). Thus if \( Y(t) \) is a fundamental matrix
for the homogeneous equation (2), the solution \( x(t) \) of the inhomogeneous equation (1) which takes the value \( x_0 \) at the time \( t_0 \) is given by

\[
(3.4) \quad x(t) = Y(t)Y^{-1}(t_0)x_0 + Y(t) \int_{t_0}^{t} Y^{-1}(s)b(s)ds.
\]

This is called the variation of constants formula.

**Theorem (3.5)** Let \( Y(t) \) be a fundamental matrix for (2). Then a matrix \( X(t) \) is a fundamental matrix for (2) if and only if there is a constant nonsingular matrix \( C \) such that \( X(t) = Y(t)C \) for \( t \geq t_0 \).

Proof: If \( X(t) = Y(t)C \) as described, \( X'(t) = Y'(t)C = A(t)Y(t)C = A(t)X(t) \). Further \( X(t) \) is nonsingular for \( t \geq t_0 \) so by (3.3) \( X(t) \) is a fundamental matrix for (2). Conversely suppose that \( X(t) \) is a fundamental matrix for (2). Then for any constant vector \( c \), \( X(t)X^{-1}(t_0)c \) and \( Y(t)Y^{-1}(t_0)c \) are solutions of (2) which take the value \( c \) at the time \( t_0 \). By uniqueness, \( X(t)X^{-1}(t_0)c = Y(t)Y^{-1}(t_0)c \) for \( t \geq t_0 \). Since \( c \) was arbitrary, \( X(t)X^{-1}(t_0) = Y(t)Y^{-1}(t_0) \) or \( X(t) = Y(t)C \) where \( C = Y^{-1}(t_0)X(t_0) \). This completes the proof.

**Theorem (3.6)** If \( A(t) = A \) is a constant matrix, then \( Y(t) = e^{tA} \) is a fundamental matrix for (2).

Proof: From chapter 1 page 4, \( Y(t) \) is nonsingular for each \( t \) and \( Y^{-1}(t) = e^{-tA} \). It only remains to show that \( Y' = AY \).

\[
Y'(t) = \lim_{h \to 0} \frac{Y(t+h) - Y(t)}{h} = \lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h}
\]
\[
\lim_{h \to 0} \frac{e^{hA}e^{tA} - e^{tA}}{h} = \lim_{h \to 0} \left( \frac{e^{hA} - I}{h} \right) e^{tA}.
\]
Now
\[
e^{hA} - I = A + \frac{hA^2}{2!} + \frac{h^2A^3}{3!} + \ldots
\]
\[
= A + h \left( \sum_{k=2}^{\infty} \frac{h^{k-2}A^k}{k!} \right) \to A \text{ as } h \to 0.
\]
Hence \( Y'(t) = Ae^{tA} = AY(t) \) so that \( e^{tA} \) is a fundamental matrix for (2).

As an example, consider the equation

(i) \( y'' + 2uy' + k^2y = 0 \) \( (u > 0, \ k \neq 0) \)

Letting \( x_1 = y, \ x_2 = y' \), we obtain the equivalent system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k^2x_1 - 2ux_2
\end{align*}
\]
or in matrix form \( \mathbf{x}' = \mathbf{A}\mathbf{x} \) where

\[
\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -k^2 & -2u \end{bmatrix}
\]

The purpose of this example is to calculate the fundamental matrix \( e^{tA} \). Assume that \( k^2 > u^2 \) and let \( w = \sqrt{k^2 - u^2} \). In this case the equation (i) has the linearly independent solutions

\( y_1(t) = e^{-ut} \cos(wt) \) and \( y_2(t) = e^{-ut} \sin(wt) \).

Let

\[
\mathbf{X}(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{bmatrix}
\]

Since \( y_1 \) and \( y_2 \) are solutions of (i) it follows that

\( \mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t) \). Also \( \mathbf{X}(t) \) is nonsingular since the solutions...
are linearly independent. Thus $X(t)$ is a fundamental matrix for the system $x' = Ax$. Since $e^{tA}$ is also a fundamental matrix, by (3.5), $e^{tA} = X(t)C$. Setting $t=0$, we have $I = X(0)C$ or $C = X^{-1}(0)$. Thus $e^{tA} = X(t)X^{-1}(0)$. Explicitly

$$X(t) = e^{-ut} \begin{bmatrix} \cos(wt) & \sin(wt) \\ -w\sin(wt)-uc\cos(wt) & w\cos(wt)-us\sin(wt) \end{bmatrix}$$

and it follows that $X^{-1}(0) = \begin{bmatrix} w & 0 \\ u & 1 \end{bmatrix}$. Hence

$$e^{tA} = \frac{e^{-ut}}{w} \begin{bmatrix} w\cos(wt) + us\sin(wt) & \sin(wt) \\ (-w^2-u^2)\sin(wt) & w\cos(wt)-us\sin(wt) \end{bmatrix}$$

Finally, we will need Jacobi's formula.

**Theorem (3.7)** Let $Y(t)$ be a solution of the matrix equation (3). Then

$$\det Y(t) = \det Y(t_0) \exp \left( \int_{t_0}^{t} \text{Tr}[A(s)]ds \right)$$

where $\text{Tr}[A(s)] = \sum_{i=1}^{n} a_{ii}(s)$ is the sum of the diagonal elements of $A(s)$, called the trace of $A(s)$.

**Proof:** Using the product rule for differentiating a determinant, $(\det Y)' = \sum_{i=1}^{n} \det D_i$, where $D_i$ is the matrix obtained from $Y$ by differentiating the $i$-th row. Since

$$y'_{ij} = \sum_{k=1}^{n} a_{ik}y_{kj} \quad \text{for } j=1, \ldots, n$$
Thus \( \det D_i = \sum_{k=1}^{n} a_{ik} \det E_k \) where

\[
\begin{bmatrix}
y_{11} & \cdots & y_{1n} \\
\vdots & \ddots & \vdots \\
y_{n1} & \cdots & y_{nn}
\end{bmatrix}
\]

\( \leftarrow i\text{-th row} \)

Now \( \det E_k = 0 \) if \( k \neq i \) since the \( i \)th row is then the same as the \( k \)th row. Thus \( \det D_i = a_{ii} \det E_i = a_{ii} \det Y \). Hence

\[
(\det Y)' = \sum_{i=1}^{n} \det D_i = \left( \sum_{i=1}^{n} a_{ii} \right) \det Y = \text{Tr}[A] \det Y.
\]

Solving this scalar differential equation for \( \det Y \),

\[
\det Y(t) = \det Y(t_0) \exp \left( \int_{t_0}^{t} \text{Tr}[A(s)] ds \right).
\]
Stability Behavior of Linear Differential Systems

We first define the various types of stability to be considered. Let $x(t)$ be a solution of the vector differential equation

(i) $x' = f(t,x)$

which is defined for all $t > t_0$.

The solution $x(t)$ is said to be stable if

(4.1) for each $\epsilon > 0$ there is a corresponding $\delta = \delta(\epsilon) > 0$ such that any solution $y(t)$ of (i) which satisfies $|y(t_0) - x(t_0)| < \delta$ is defined and satisfies $|y(t) - x(t)| < \epsilon$ for all $t > t_0$.

The solution $x(t)$ is said to be asymptotically stable (a.s.) if it is stable and in addition

(4.2) there is a $\Delta > 0$ such that any solution $y(t)$ of (i) which satisfies $|y(t_0) - x(t_0)| < \Delta$ has $\lim_{t \to \infty} |y(t) - x(t)| = 0$.

The solution $x(t)$ is said to be uniformly stable (u.s.) if

(4.3) for each $\epsilon > 0$ there is a corresponding $\delta = \delta(\epsilon) > 0$ such that any solution $y(t)$ of (i) which satisfies $|y(t_1) - x(t_1)| < \delta$ for some $t_1 > t_0$, is defined and satisfies $|y(t) - x(t)| < \epsilon$ for all $t > t_1$. 

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In particular, uniform stability implies stability since with \( t_1 = t_0 \), (4.3) is precisely the same as (4.1).

The solution \( x(t) \) is said to be **uniformly asymptotically stable** (u.a.s.) if it is uniformly stable and in addition there is a fixed positive real number \( \Delta \) which satisfies the following conditions:

\[
(4.4) \quad \text{for each } \epsilon > 0 \text{ there is a corresponding } T = T(\epsilon) > 0 \text{ such that any solution } y(t) \text{ of } (i) \text{ with } |y(t_1) - x(t_1)| < \Delta \text{ for some } t_1 \geq t_0, \text{ is defined and satisfies } |y(t) - x(t)| < \epsilon \text{ for all } t \geq t_1 + T.
\]

Uniform asymptotic stability implies asymptotic stability since \( \Delta \) in (4.4) is independent of \( \epsilon \) and with \( t_1 = t_0 \), we have \( |y(t) - x(t)| < \epsilon \text{ for all } t > t_0 + T(\epsilon) \) provided only \( |y(t_0) - x(t_0)| < \Delta \).

As a simple example illustrating these definitions, consider the equation \( x' = kx \) (\( k \) a constant). All solutions of this equation are of the form \( x(t) = x_0 e^{kt} \). For the case \( k = 0 \), the solutions are constant. Thus the zero solution is stable, in fact uniformly stable, but it is not asymptotically stable. For \( k < 0 \), the solutions are either strictly increasing or strictly decreasing and approach zero as \( t \) becomes infinite. The zero solution is uniformly asymptotically stable. For \( k > 0 \), the solutions (not identically zero) are all unbounded so that the zero solution is not stable.

An autonomous equation is one that is independent of the variable \( t \). That is, an equation of the form \( x' = F(x) \).
Autonomous equations have the property that if \( x(t) \) is any solution, then \( x(t+h) \) is also a solution for any constant \( h \). We use this fact to prove the following theorem.

**Theorem (4.5)** Suppose that \( x(t) = c \) is a constant solution of an autonomous equation \( x' = F(x) \). Then this solution is uniformly stable if and only if it is stable. It is uniformly asymptotically stable if and only if it is asymptotically stable.

**Proof:** Suppose that the constant solution \( x(t) \) is stable. Let \( \varepsilon > 0 \) and \( \delta = \delta(\varepsilon) \) in the definition of stability. If \( y(t) \) is any solution of \( x' = F(x) \) with \( |y(t_1) - c| < \delta \) for some \( t_1 > t_0 \), \( z(t) = y(t + [t_1 - t_0]) \) is a solution with \( |z(t_0) - c| < \delta \). Thus by the stability of \( x(t) \), \( z(t) \) is defined and satisfies \( |z(t) - c| < \varepsilon \) for all \( t > t_0 \). That is, \( y(t) \) is defined and satisfies \( |y(t) - c| < \varepsilon \) for \( t > t_1 \). Thus the constant solution is uniformly stable. In an analogous manner, the asymptotic stability of the constant solution implies that it is also uniformly asymptotically stable.

In particular, this result applies to the zero solution of the equation \( x' = Ax \) where \( A \) is a constant matrix.

Now consider the linear equations

\[
\begin{align*}
(1) & \quad x' = A(t)x + b(t) \\
(2) & \quad x' = A(t)x
\end{align*}
\]

We shall assume that \( A(t) \) and \( b(t) \) are continuous for \( t > t_0 \). By the previous results, every solution of (1) or (2) is defined for all \( t > t_0 \).
Theorem (4.6) Any solution $x(t)$ of (1) is stable (a.s., u.s., u.a.s.) if and only if the zero solution of (2) is stable (a.s., u.s., u.a.s.).

Proof: Let $x(t)$ be any solution of (1). If $y(t)$ is any other solution of (1), $z(t) = y(t) - x(t)$ is a solution of (2). Thus if the zero solution of (2) is stable, $|y(t) - x(t)| = |z(t)| < \varepsilon$ provided $|z(t_0)| < \delta(\varepsilon)$. Conversely, if $x(t)$ is stable and $z(t)$ is any solution of (2), $y(t) = z(t) + x(t)$ is a solution of (1) so that $|z(t)| < \varepsilon$ provided that $|z(t_0)| = |x(t_0) - y(t_0)| < \delta(\varepsilon)$. Similar arguments apply for a.s., u.s., and u.a.s..

Thus it is legitimate to speak of the stability of the system (1) rather than the stability of a particular solution. The next theorem gives necessary and sufficient conditions for the stability of the linear system (1).

Theorem (4.7) Let $Y(t)$ be a fundamental matrix for the system (2). Then the system (1) is

a) stable if and only if there is a positive constant $K$ such that
   
   i) $|Y(t)| < K$ for all $t \geq t_0$

b) uniformly stable if and only if there is a positive constant $K$ such that
   
   ii) $|Y(t)Y^{-1}(s)| < K$ for $t_0 < s < t < +\infty$

c) asymptotically stable if and only if
   
   iii) $|Y(t)| \to 0$ as $t \to \infty$

d) uniformly asymptotically stable if and only if there exist positive constants $K, \alpha$ such that
   
   iv) $|Y(t)Y^{-1}(s)| \leq Ke^{-\alpha(t-s)}$ for $t_0 < s < t < +\infty$
Proof: If \( Y(t_0) \neq I \), by theorem (3.5), \( X(t) = Y(t)Y^{-1}(t_0) \) is a fundamental matrix for (2) such that \( X(t_0) = I \). The conditions (i)-(iv) hold for \( X(t) \) if and only if they hold for \( Y(t) \). Thus there is no loss of generality in assuming that \( Y(t_0) = I \). Also, by theorem (4.6), only the zero solution of (2) needs investigation. The solution of (2) which takes the value \( x_0 \) at \( t_0 \) is given by \( x(t) = Y(t)x_0 \).

a) Suppose that (i) holds. If \( x(t) \) is a solution of (2) such that \( |x(t_0)| \leq \varepsilon/K \), then \( |x(t)| = |Y(t)x(t_0)| \leq |Y(t)||x(t_0)| \leq K|x(t_0)| \leq \varepsilon \). Thus the zero solution of (2) is stable. Conversely if (2) is stable, \( |x(t)| = |Y(t)x(t_0)| \leq \varepsilon \) provided \( |x(t_0)| \leq \delta \). By (1.7), \( |Y(t)| \leq \varepsilon/\delta \) for all \( t \geq t_0 \).

b) Suppose that (ii) holds. The solution of (2) which takes the value \( c \) at the time \( t_1 > t_0 \) is given by \( x(t) = Y(t)Y^{-1}(t_1)c \). Then \( |x(t)| = |Y(t)Y^{-1}(t_1)c| \leq |Y(t)||Y^{-1}(t_1)||c| \leq \varepsilon/|c| \) if \( t \geq t_1 \). Hence, \( |x(t)| \leq \varepsilon/|c| \) for all \( t \geq t_1 \) if \( |c| \leq \varepsilon/K \). Conversely suppose that (2) is u.s.. Then \( |Y(t)Y^{-1}(t_1)c| \leq \varepsilon/|c| \) provided \( |c| \leq \delta \). Again by (1.7), \( |Y(t)Y^{-1}(t_1)c| \leq \varepsilon/|c| \) for \( t \geq t_1 \). Hence the condition (ii) is satisfied.

c) Suppose (iii) holds. Since \( Y(t) \) is continuous, it is bounded and therefore by (a), the zero solution of (2) is stable. Furthermore, all solutions approach zero as \( t \) becomes infinite since \( |x(t)| = |Y(t)x(t_0)| \leq |Y(t)||x(t_0)| \rightarrow 0 \) as \( t \rightarrow \infty \). Therefore the zero solution is asymptotically stable. Conversely, if the zero solution is a.s., there is a \( \Delta > 0 \) such that \( |x(t)| = |Y(t)x(t_0)| \rightarrow 0 \) as \( t \rightarrow \infty \) provided \( |x(t_0)| \leq \Delta \).
In particular, there is a sequence \( \{ t_k \} \) such that \( |Y(t)x(t_0)| < 1/k \) for all \( t \geq t_k \), and therefore by (1.7), \( |Y(t)| \leq 1/k \Delta \) for \( t \geq t_k \) and hence \( |Y(t)| \to 0 \) as \( t \to \infty \).

d) Suppose that (iv) holds. Then \( |Y(t)Y^{-1}(s)| < K \) for \( t_0 < s < t \). Hence by (b), (2) is uniformly stable. Let \( \Delta = 1 \). We have

\[
|x(t)| = |Y(t)Y^{-1}(t_1)x(t_1)| \leq |Y(t)Y^{-1}(t_1)| |x(t_1)| \leq K |x(t_1)| \quad \text{for } t \geq t_1 > t_0.
\]

If \( |x(t_1)| < 1 \) and \( \epsilon < K \), then \( |x(t)| < \epsilon \) for all \( t \geq t_1 \).

If \( |x(t_1)| < 1 \) and \( \epsilon < K \), let \( T(\epsilon) = -(1/\alpha) \log(\epsilon/K) \). If \( t \geq t_1 + T(\epsilon) \),

\[
|x(t)| \leq |Y(t)Y^{-1}(t_1)| |x(t_1)| < |Y(t)Y^{-1}(t_1)| \leq K e^{-\alpha(t-t_1)} < \epsilon.
\]

Thus (2) is u.a.s.. Conversely suppose that the zero solution of (2) is u.a.s.. Let \( \Delta \) satisfy the conditions of u.a.s. and let \( 0 < \epsilon < \Delta \). There is a \( T = T(\epsilon) > 0 \) such that

\[
|x(t)| = |Y(t)Y^{-1}(t_1)c| < \epsilon \quad \text{for } t \geq t_1 + T \quad \text{if } |c| < \Delta.
\]

By (1.7), \( |Y(t)Y^{-1}(t_1)| \leq \epsilon / \Delta \) for \( t \geq t_1 + T \). Let \( \theta = \epsilon / \Delta < 1 \). Then \( |Y(t)Y^{-1}(t_1)| \leq \theta \) for all \( t \geq t_1 + T \). Since \( t_1 > t_0 \) is arbitrary, we have

\[
* |Y(t+T)Y^{-1}(t)| \leq \theta \quad \text{for all } t \geq t_0.
\]

Since (2) is u.s., there is a positive constant \( K \) such that

\[
** |Y(t)Y^{-1}(s)| \leq K \quad \text{for } t \geq s \geq t_0.
\]

We shall use (*) and (**) to show that \( |Y(t)Y^{-1}(s)| \leq (K/\theta)e^{-\alpha(t-s)} \) where \( \alpha = (-1/T) \log \theta > 0 \).

Let \( t \geq s \geq t_0 \). There is a non negative integer \( n \) such that \( s + nT < t < s + (n+1)T \).

\[
|Y(t)Y^{-1}(s)| = |Y(t)Y^{-1}(s+nT)Y(s+nT)Y^{-1}(s)| \leq |Y(t)Y^{-1}(s+nT)| |Y(s+nT)Y^{-1}(s)| \leq K |Y(s+nT)Y^{-1}(s)|
\]

\[
= K |Y(s+nT)Y^{-1}(s+[n-1]T)Y(s+[n-1]T)Y^{-1}(s)| \leq K |Y(s+nT)Y^{-1}(s+[n-1]T)| |Y(s+[n-1]T)Y^{-1}(s)|
\]

\[
\leq K^2 |Y(s+[n-1]T)Y^{-1}(s)| \leq \ldots \leq \theta^k \epsilon < \epsilon.
\]
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\[ K^N_{0,1} Y(s+T) Y^{-1}(s) \leq K^N = (K/\theta)^{n+1} = (K/\theta) e^{-\alpha(n+1)T} < (K/\theta) e^{-\alpha(t-s)}. \]

This completes the proof.

Theorem (4.8) If \( \lim_{t \to \infty} \sup \Re \left( \int_{t_0}^{+\infty} \text{Tr}[A(s)] ds \right) = +\infty, \)
then (1) is unstable. (\( \Re \) denotes the real part)

Proof: By Jacobi's formula (3.7),
\[ \det Y(t) = \det Y(t_0) \exp \left( \int_{t_0}^{+\infty} \text{Tr}[A(s)] ds \right). \]
Thus
\[ |\det Y(t)| = |\det Y(t_0)| \exp \left( \Re \int_{t_0}^{+\infty} \text{Tr}[A(s)] ds \right) \]
and therefore, \( \det Y(t) \) is unbounded. This implies that \( Y(t) \) is unbounded since a determinant is a continuous function of its argument. Thus by (4.7), the equation (1) is not stable.

As an example, consider the scalar \( n \)-th order equation
\[ y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = 0. \]
Making the transformation \( x_1 = y, \ldots, x_n = y^{(n-1)} \), we obtain the equivalent system
\[ x_1' = x_2 \]
\[ x_2' = x_3 \]
\[ \vdots \]
\[ x_n' = -a_n(t)x_1 - \ldots - a_1(t)x_n \]
or in matrix notation, \( x' = A(t)x \) where

\[
A(t) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 1 & \vdots \\
-a_n(t) & -a_{n-1}(t) & \ldots & -a_1(t) & 0
\end{bmatrix}
\]

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Here $\text{Tr}[A(t)] = -a_1(t)$ so that the equation is unstable if

$$\lim_{t \to \infty} \sup \left( \int_{t_0}^{t} a_1(s) \, ds \right) = +\infty.$$ 

Theorem (4.7) will now be applied to the case where $A(t) = A$ is a constant matrix. Here, $Y(t) = e^{tA}$. Let $B = P^{-1}AP$ be in Jordan canonical form. Then $B = \text{diag}[B_1, \ldots, B_r]$ where $B_i = \lambda_i I_{s_i} + N_{s_i}$ and from the results of chapter (1), $e^{tB} = \text{diag}[e^{tB_1}, \ldots, e^{tB_r}]$. Also, $e^{tB_i} = e^{\lambda_i t} e^{tN_{s_i}}$. As in chapter (1), a direct calculation shows

$$e^{tB_i} = e^{\lambda_i t} \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
t & 1 & 0 & \cdots & 0 \\
t^2/2! & t & 1 & 0 & \cdots \\
t^3/3! & t^2/2! & t & 0 & \cdots \\
t^{(s_i-1)/s_i!} & 0 & \cdots & \cdots & \cdots & t & 1
\end{bmatrix}$$

Thus for $t > 0$, $|e^{tB_i}| = |e^{\lambda_i t}| [1 + t + \ldots + t^{(s_i-1)/s_i!}]$

$$= e^{R(\lambda_i)t} [1 + t + \ldots + t^{(s_i-1)/s_i!}],$$

and $|e^{tB}| = \max_{1 \leq i \leq r} |e^{tB_i}|$ for $t > 0$. If for any $i$, $R(\lambda_i) > 0$, then $|e^{tB}| \to 0$ as $t \to \infty$. If $|e^{tB}|$ is bounded if and only if $R(\lambda_i) \leq 0$ and for those $i$ with $R(\lambda_i) = 0$, $s_i = 1$. Also, $|e^{tB}| \to 0$ as $t \to \infty$ if and only if $R(\lambda_i) > 0, 1 < i < r$. Since $e^{tA} = Pe^{tB}P^{-1}$ (see page 5), we have the following theorem.
Theorem (4.9) If $A$ is a constant matrix, then the system \((1)\) is stable if and only if the characteristic roots of $A$ have non-positive real part and those roots with zero real part are of simple type. It is asymptotically stable if and only if each characteristic root has negative real part.

For example consider the second order system $x' = Ax$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{The characteristic equation is} \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

The characteristic roots are $\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$ and they have negative real part if and only if $a+d<0$ and $ad-bc>0$. 
Stability Behavior of Non-linear Differential Systems

When discussing the stability behavior of non-linear differential systems, the first questions concern the existence of solutions on an infinite interval. Then the behavior of these solutions, if they exist, must be determined. We first consider systems whose right side is the sum of a linear term and a "small" nonlinear term. Using the results for linear systems, it is sometimes possible to determine the stability behavior of the nonlinear system. For linear systems, either all solutions are stable or else they are all unstable. However, for nonlinear systems this is not the case and we will always consider the behavior of one particular solution. We first show that for theoretical purposes, the solution to be considered may be assumed to be identically zero.

Consider a particular solution $x(t)$ of a differential equation $x' = F(t,x)$. If $y(t)$ is any other solution, then the difference $z(t) = y(t) - x(t)$ satisfies the equation $z' = F(t,z+x(t)) - F(t,x(t)) = G(t,z)$. The zero solution of $z' = G(t,z)$ is stable (u.s.,a.s., u.a.s.) if and only if the solution $x(t)$ of the original equation is stable (u.s.,a.s., u.a.s.). Thus there is no loss of generality in assuming that $F(t,0) = 0$ and that the solution under consideration is the zero solution. Suppose further that $F$ has continuous partial derivatives $\frac{\partial F}{\partial x_j}$. Let $A(t) = [a_{ij}(t)]$ be the $n$ by $n$ matrix with $a_{ij}(t) = \frac{\partial F}{\partial x_j} \bigg|_{(t,0)}$. If we let $f(t,x) = F(t,x) - A(t)x$, the equation $x' = F(t,x)$ becomes $x' = A(t)x + f(t,x)$. 

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For each $t$, $f(t,x) = o(|x|)$, i.e., $|f(t,x)|/|x| \to 0$ as $|x| \to 0$.

Thus it might be expected that the zero solution of

$$x' = A(t)x + f(t,x)$$

has the same stability behavior as the linear system

$$x' = A(t)x$$

We assume that $A(t)$ is continuous for $t>t_0$ and that $f(t,x)$ is continuous for $t>t_0$ and $|x| \leq c$. Also $Y(t)$ will be the fundamental matrix for (2) such that $Y(t_0) = I$.

**Theorem (5.1)** Suppose there is a constant $K$ such that

$$|Y(t)Y^{-1}(s)| \leq K \quad \text{for } t > s > t_0$$

and suppose that $f$ satisfies the inequality

$$|f(t,x)| \leq r(t)|x|$$

where $r(t)$ is a continuous nonnegative function such that $\int_{t_0}^{+\infty} r(s)ds < +\infty$.

Then there is a positive constant $L$ such that if $t_1 > t_0$, any solution $x(t)$ of (1) for which $|x(t_1)| < c/L$, is defined and satisfies the inequality

$$|x(t)| \leq L|x(t_1)|$$

for all $t > t_1$. In particular, the zero solution of (1) is uniformly stable. If in addition $|Y(t)| \to 0$ as $t \to \infty$, the zero solution of (1) is asymptotically stable.

**Proof:** Let $L = 2K \exp\left( K \int_{t_0}^{+\infty} r(s)ds \right)$. Suppose that $x(t)$ is a solution of (1) such that $|x(t_1)| < c/L$ for some $t_1 \geq t_0$. Let $t_2$ be the right end point of the interval of definition of $x(t)$. We assume that $x(t)$ is not continu-able past $t_2$. Since $x(t)$ is a solution of the inhomogeneous equation

$$x' = A(t)x + f(t,x(t))$$

by the variation of constants formula (3.4).
x(t) = Y(t)Y^{-1}(t_1)x(t_1) + Y(t) \int_{t_1}^{t} Y^{-1}(s)f[s,x(s)]ds, \quad t_1 \leq t \leq t_2.

Therefore, under the assumptions in (5.1),
\[|x(t)| \leq |Y(t)Y^{-1}(t_1)||x(t_1)| + \int_{t_1}^{t} |Y(t)Y^{-1}(s)||f[s,x(s)]|ds\]
\[\leq K|x(t_1)| + \int_{t_1}^{t} Kr(s)|x(s)|ds.\]

By Gronwall's inequality, for \(t_1 \leq t < t_2\),
\[|x(t)| \leq K|x(t_1)|\exp\left(\int_{t_1}^{t} Kr(s)ds\right) \leq K|x(t_1)|\exp\left(K\int_{t_0}^{+\infty} r(s)ds\right)\]
\[= (L/2)|x(t_1)| < c/2.\]

If \(t_2 = +\infty\), by theorem (2.4), \(x(t)\) must be defined on \([t_1,t_2]\).

But then since \(|x(t_1)| < c/L\) and since \(x(t)\) is continuous,
\(|x(t_2)| \leq (L/2)|x(t_1)| < c/2.\) Hence by theorem (2.3), \(x(t)\)
could be continued past \(t_2\), a contradiction. Thus \(t_2 = +\infty\)
and we have \(|x(t)| \leq (L/2)|x(t_1)| \leq L|x(t_1)|\) for all \(t > t_1\).

Thus the zero solution of (1) is uniformly stable since any
solution with \(|x(t_1)| < \min(c/L, \varepsilon/L)\) is defined and satisfies
\(|x(t)| \leq \varepsilon/2 < \varepsilon\) for \(t > t_1\). Suppose in addition that \(Y(t) \to 0\)
as \(t \to \infty\), Let \(x(t)\) be a solution of (1) with \(|x(t_0)| < c/L\)
then \(x(t)\) is defined for all \(t > t_0\) and \(|x(t)| \leq L|x(t_0)|\). Let
\(\varepsilon > 0\) be given. Choose \(t_1\) so large that \(KL|x(t_0)|\int_{t_1}^{+\infty} r(s)ds < \varepsilon/2.\)

Let \(t > t_1\). Since \(Y(t_0) = I\), by the variation of constants
formula, \(x(t) = Y(t)x(t_0) + \int_{t_0}^{t} Y(t)Y^{-1}(s)f[s,x(s)]ds = Y(t)x(t_0) + Y(t)\int_{t_0}^{t} Y^{-1}(s)f[s,x(s)]ds + \int_{t_1}^{t} Y(t)Y^{-1}(s)f[s,x(s)]ds.\)

Now using the fact that \(|f[s,x(s)]| \leq r(s)|x(s)| \leq r(s)L|x(t_0)|\)
and the fact that \(|Y(t)Y^{-1}(s)| \leq K\) for \(t > s > t_0\),
\[|x(t)| \leq |Y(t)||x(t_0)| + |Y(t)||\int_{t_0}^{t} Y^{-1}(s)f[s,x(s)]ds| + KL|x(t_0)|\int_{t_1}^{+\infty} r(s)ds\]
\[< |Y(t)||x(t_0)| + |Y(t)||\int_{t_0}^{t} Y^{-1}(s)f[s,x(s)]ds| + \varepsilon/2.\]

Since \(Y(t) \to 0\) as \(t \to \infty\), there exists a \(t_2\) such that the first
two terms in the last sum are less than \(\varepsilon/2\) if \(t > t_2\). Thus
\[ |x(t)| < \varepsilon \text{ for all } t \geq \max(t_1, t_2). \] Since \( \varepsilon \) was arbitrary, \( |x(t)| \to 0 \text{ as } t \to \infty \) and the zero solution of (1) is asymptotically stable. This completes the proof.

The results of (5.1) are also useful in determining the stability behavior of linear equations. If (2) is uniformly stable, if \( B(t) \) is a matrix which is continuous for \( t \geq t_0 \) and if \( \int_{t_0}^{+\infty} |B(t)| dt < +\infty \), then the linear system \( x' = A(t)x + B(t)x \) is also uniformly stable. This follows immediately from (5.1) with \( f(t, x) = B(t)x \).

For example, the second order scalar equation \( y'' + g(t)y' + k^2 y = 0 \) where \( g \) is continuous for \( t \geq t_0 \), \( k \neq 0 \), is uniformly stable if
\[
\int_{t_0}^{+\infty} |g(s)| ds < +\infty.
\]
If in addition \( g(t) \leq 0 \) for \( t \geq t_0 \), the condition is also necessary. This is seen by making the transformation \( x_1 = y, \ x_2 = y' \) to obtain the system
\[
\begin{bmatrix} 0 & 1 \\ -k^2 & -g(t) \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -g(t) \end{bmatrix} x.
\]
The system \( x' = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} x \) is uniformly stable and
\[
f(t, x) = \begin{bmatrix} 0 & 1 \\ 0 & -g(t) \end{bmatrix} x \]
satisfies the conditions of (5.1) with \( r(t) = |g(t)| \). On the other hand, if \( g(t) \leq 0 \) for \( t \geq t_0 \), \( |g(t)| = -g(t) \) and by (4.8) the system is unstable if \( \int_{t_0}^{+\infty} -g(s) ds = +\infty \).

This example also points out the fact that the condition \( \int_{t_0}^{+\infty} r(s) ds < +\infty \) in (5.1) cannot be replaced by the weaker condition \( r(t) \to 0 \text{ as } t \to +\infty \). To see this let \( g(t) = 1/(t_0 - 1 - t) \).
\[ r(t) = |g(t)| \to 0 \text{ as } t \to \infty \text{ but } \int_{t_0}^{\infty} -g(s) \, ds = +\infty \text{ so that the zero solution is unstable.} \]

We now consider conditions which imply asymptotic stability. The following is an example where \( f \) satisfies the conditions of (5.1), the equation (2) is asymptotically stable, and yet the zero solution of (1) is not asymptotically stable. Let \( 1 < 2a < 1 + e^{-\pi} \) and let

\[
A(t) = \begin{bmatrix}
-a & 0 \\
0 & \sin(\log t) + \cos(\log t) - 2a
\end{bmatrix}, \quad t_0 = 1.
\]

The system \( x' = A(t)x \) has the solutions \( x_1(t) = c_1 e^{-a(t-1)} \), \( x_2(t) = c_2 \exp[tsin(\log t) - 2at + 2a] \), both of which tend to zero exponentially as \( t \to \infty \). Thus the system is asymptotically stable.

Let \( B(t) = \begin{bmatrix} 0 & 0 \\ e^{-at} & 0 \end{bmatrix} \). Then \( f(t,x) = B(t)x \) satisfies the conditions of (5.1) with \( r(t) = e^{-at} \). We now show that the zero solution of \( x' = A(t)x + B(t)x \) is not stable. For this system we still have \( x_1(t) = c_1 e^{-a(t-1)} \) and it follows from the variation of constants formula that

\[
x_2(t) = e^{2a} \exp[tsin(\log t) - 2at] \left[ c_2 + c_1 e^a \int_0^t \exp[-ssin(\log s)] \, ds \right].
\]

We take \( c_2 = 0 \) and show that if \( c_1 \neq 0 \), then \( x_2(t) \) is unbounded. Since \( 0 < (2a-1)e^{-\pi} < 1 \), we may choose \( \theta \), \( 0 < \theta < \pi/2 \) so that \( \cos \theta > (2a-1)e^{-\pi} \). Let \( t_n = e^{(2n-1/2)\pi} \). Then \( \log t_n = (2n-1/2)\pi \) and \( \log(\ln t_n e^\theta) = (2n-1/2)\pi + \theta \). Hence for \( t_n < s < t_n e^\theta \),

\[
(2n-1/2)\pi < \log s < (2n-1/2)\pi + \theta.
\]

Since the sine function is...
increasing on the interval \([2n-1/2)\pi, 2n\pi]\), we have 
\[
\sin(\log t) \leq \sin[(2n-1/2)\pi + \theta] = -\cos\theta .
\]
Thus 
\[
-\sin(\log t) > -\cos\theta \text{ for } t_n < t < e^\theta .
\]
Hence
\[
\int_{t_n}^{t} \exp[\sin(\log s)] ds > \int_{t_n}^{t} \exp[\cos s] ds > t_n (e^\theta - 1) \exp(t_n \cos \theta).
\]
Since \(\sin(\log t(t_n e^\pi)) = 1\), we have since \(c = 0\),
\[
|x_2(t_n e^\pi)| = e^{3a} |c_1| \int_{t_n}^{t} \exp[(1-2a) t_n e^\pi] \int_{t_n}^{t} \exp[-\sin(\log s)] ds
\]
\[
|c_1| \int_{t_n}^{t} \exp[(1-2a) t_n e^\pi] t_n (e^\theta - 1) \exp(t_n \cos \theta)
\]
\[
= |c_1| t_n (e^\theta - 1) \exp \left( (1-2a) e^\pi + \cos \theta \right) t_n .
\]
Thus \(|x_2(t_n e^\pi)| \to \infty\) as \(n \to \infty\) since \((1-2a) e^\pi + \cos \theta > 0\).

The following lemma will be needed.

Lemma: Suppose there is a positive constant \(K\) such that
\[
\int_{t_0}^{t} |Y(t)Y^{-1}(s)| ds \leq K \text{ for } t > t_0 .
\]
Then there is a positive constant \(N\) such that
\[
|Y(t)| \leq Ne^{-t/K} \text{ for } t > t_0 .
\]
Proof: Let \(\psi(t) = 1/|Y(t)|\), \(\psi(t) = \int_{t_0}^{t} \phi(s) ds\).
Then \(\psi(t)Y(t) = \int_{t_0}^{t} Y(t)Y^{-1}(s)Y(s) \phi(s) ds\) so that
\[
\psi(t)Y(t) \leq \int_{t_0}^{t} |Y(t)Y^{-1}(s)||Y(s)||\phi(s) ds = \int_{t_0}^{t} |Y(t)Y^{-1}(s)| ds \leq K .
\]
Therefore \(\psi'(t) = \phi(t) = 1/|Y(t)| \geq (1/K) \psi(t)\). Let \(b(t) = \psi'(t) - (1/K) \psi(t)\) for \(t > t_0\). Then \(b(t)\) is continuous and nonnegative for \(t > t_0\). Let \(t_1 > t_0\) and consider the initial value problem \(x' = (1/K) x + b(t)\), \(x(t_1) = \psi(t_1)\). The solution of this problem is of course \(\psi(t)\) and by the variation of constants formula
\[
\psi(t) = \psi(t_1) e^{(1/K) (t-t_1)} + \int_{t_1}^{t} b(s) e^{(1/K) (t-s)} ds .
\]
Since \(b(s) \geq 0\) for all \(s\), we must have \(\psi(t) > \psi(t_1) e^{(1/K) (t-t_1)}\).
for all $t > t_1$. Thus
\[ |Y(t)| = \frac{1}{\psi(t)} < \frac{K}{\psi(t)} < (K/\psi(t_1)) e^{-(1/K)(t-t_1)} = \left( \frac{Ke^{t_1/K}}{\psi(t_1)} \right) e^{-t/K} \]
for $t > t_1$. Choosing $N > \frac{Ke^{t_1/K}}{\psi(t_1)}$ so large that $|Y(t)| < Ne^{-t/K}$
for $t_0 < t < t_1$, we get $|Y(t)| < Ne^{-t/K}$ for all $t > t_0$. This completes the proof.

**Theorem (5.2)** Suppose there is a positive constant $K$ such that
\[ \int_{t_0}^{t} |Y(t)Y^{-1}(s)| ds < K \quad \text{for} \quad t > t_0. \]
Suppose that $f$ satisfies the inequality
\[ |f(t,x)| \leq \alpha |x| \quad \text{for} \quad t > t_0 \]
where $\alpha < 1/K$. Then the zero solution of (1) is asymptotically stable.

**Proof:** By the preceding lemma, $Y(t) \to 0$ as $t \to \infty$ and in particular there is a positive number $M$ such that $|Y(t)| \leq M$ for all $t \geq t_0$. Let $x(t)$ be a solution of (1) such that $|x(t_0)| \leq \frac{c(l-\alpha K)}{2M}$. By the variation of constants formula
\[ x(t) = Y(t)x(t_0) + \int_{t_0}^{t} Y(t)Y^{-1}(s)f[s,x(s)]ds \]
Let $v(t) = \sup_{t_0 \leq s \leq t} |x(s)|$. Then
\[ |x(t)| \leq |Y(t)||x(t_0)| + \int_{t_0}^{t} |Y(t)Y^{-1}(s)||f[s,x(s)]||ds \leq M|x(t_0)| + \int_{t_0}^{t} |Y(t)Y^{-1}(s)|\alpha |x(s)| ds \leq M|x(t_0)| + \alpha K v(t). \]
Since $v(t)$ is increasing, $|x(s)| \leq M|x(t_0)| + \alpha K v(s)$
\[ \leq M|x(t_0)| + \alpha K v(t) \quad \text{if} \quad t_0 \leq s \leq t. \]
Thus $v(t) \leq M|x(t_0)| + \alpha K v(t)$ and since $1-\alpha K > 0$, $v(t) \leq M|x(t_0)|/(1-\alpha K)$. In particular
\[ |x(t)| \leq M|x(t_0)|/(1-\alpha K) \leq c/2. \]
By the same argument given...
in (5.1) this shows that \( x(t) \) is defined for all \( t > t_0 \).

It also shows that the zero solution of (1) is stable since

\[ |x(t_0)| < \min\{e(1-\alpha K)/M, c(1-\alpha K)/2M\} \]

implies that \( x(t) \) is defined for \( t > t_0 \) and \( |x(t)| \leq M|x(t_0)|/(1-\alpha K) < \epsilon \) for \( t > t_0 \).

It remains to show that \( |x(t)| \to 0 \) as \( t \to \infty \). Let

\[ u = \limsup_{t \to \infty} |x(t)| \].

Suppose that \( u > 0 \). Choose a number \( \alpha \) so that \( \alpha K < \alpha < 1 \). There is a value \( t_1 > t_0 \) such that for all \( t > t_1 \), \( |x(t)| \leq u/\alpha \). Thus \( |f[t, x(t)]| \leq \alpha |x(t)| \leq au/a \) for \( t > t_1 \). Hence for \( t > t_1 \)

\[ x(t) = Y(t)x(t_0) + Y(t) \int_{t_0}^{t} Y^{-1}(s) f[s, x(s)] ds + \int_{t_1}^{t} Y(t) Y^{-1}(s) f[s, x(s)] ds \]

so that

\[ |x(t)| \leq |Y(t)||x(t_0)| + |Y(t)|| \int_{t_0}^{t_1} Y^{-1}(s) f[s, x(s)] ds| + K\alpha u/a \]

Since \( Y(t) \to 0 \) as \( t \to \infty \), we have

\[ u = \limsup_{t \to \infty} |x(t)| \leq K\alpha u/a < u , \]

a contradiction. Thus \( u = 0 \) and \( x(t) \to 0 \) as \( t \to \infty \). Thus the zero solution of (1) is asymptotically stable and the proof is complete.

Theorem (5.2) has several important corollaries and applications. First, if the linear equation (2) is uniformly asymptotically stable, by theorem (4.7) there exist positive constants \( M, \alpha \) such that \( |Y(t)Y^{-1}(s)| \leq Me^{-\alpha(t-s)} \) for \( t_0 < s < t \). It follows that \( Y(t) \) satisfies the condition of (5.2) with \( K = M/\alpha \). Thus the condition on \( Y(t) \) in (5.2) is stronger than asymptotic stability but weaker than uniform asymptotic stability.
Lyapunov's First Theorem: Let $A$ be a constant matrix and suppose that the system $x' = Ax$ is asymptotically stable. Then there is a positive constant $b$ such that if $B(t)$ is a continuous matrix and satisfies $\|B(t)\| < b$ for $t > t_0$, then the system $x' = Ax + B(t)x$ is asymptotically stable.

Proof: Since $x' = Ax$ is a.s., by theorem (4.5) it is also u.a.s., and therefore $Y(t) = e^{tA}$ satisfies the conditions of (5.2). The result now follows since for $0 < b < 1/K$, and $\|B(t)\| < b$, $f(t,x) = B(t)x$ satisfies the conditions of (5.2).

As an application, we consider the differential equation

$$y'' + 2uy' + k^2y = 0 \quad (u>0, k\not=0).$$

Such an equation arises for physical systems which describe simple harmonic motion but are placed in a medium which offers resistance to the motion. Recall from chapter 3, page 18, that the above equation is equivalent to the system $x' = Ax$ where $A = \begin{bmatrix} 0 & 1 \\ -k^2 & -2u \end{bmatrix}$. We shall assume that $k^2 > u^2$ and let $w = \sqrt{k^2 - u^2}$. Then from chapter 3, page 19, the fundamental matrix $Y(t) = e^{tA}$ is given by

$$e^{tA} = \frac{e^{-ut}}{w} \begin{bmatrix} w\cos(wt) + u\sin(wt) & \sin(wt) \\ (-w^2-u^2)\sin(wt) & w\cos(wt) - u\sin(wt) \end{bmatrix}$$

Now $Y(t)Y^{-1}(s) = e^{tA}e^{-sA} = e^{(t-s)A} = Y(t-s)$. 

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It now follows from (1.8) that \( |Y(t)Y^{-1}(s)| = |Y(t-s)| \leq M e^{-u(t-s)} \)
where \( M \) is the maximum of the two numbers \( (w+u+1)/w \) and \( (w^2+u^2+w+u)/w \). Hence (5.2) is applicable with \( K = M/u \).

We say that \( f(t,x) = o(|x|) \) uniformly in \( t \) as \( |x| \to 0 \) if for each \( \varepsilon > 0 \) there is a corresponding \( d > 0 \) such that \( \frac{|f(t,x)|}{|x|} < \varepsilon \) for \( t > t_0 \) and \( |x| < d \).

Corollary 1: Let \( Y(t) \) satisfy the conditions of (5.2) and suppose that \( f(t,x) = o(|x|) \) uniformly in \( t \) as \( |x| \to 0 \). Then the zero solution of (1) is asymptotically stable.

Proof: There is a constant \( d > 0 \) such that \( \frac{|f(t,x)|}{|x|} < a < 1/K \) for \( |x| < d \). Thus \( |f(t,x)| < a |x| \) for \( |x| < d \). The proof of (5.2) may now be applied, the only difference being that the constant \( c \) is replaced by \( d \).

As an example, consider the Bernoulli equation
\[
x' = a(t)x + b(t)x^n
\]
where \( a(t) \) and \( b(t) \) are continuous for \( t > t_0 = 0 \) and \( n > 2 \).

The fundamental matrix for the linear equation \( x' = a(t)x \) is the scalar \( Y(t) = \exp \left[ \int_0^t a(s)ds \right] \). Thus the conditions on \( Y(t) \) are satisfied if \( \int_0^t |Y(t)Y^{-1}(s)| \, ds = \int_0^t \exp \left[ \int_0^s a(u)du \right] \) is bounded. If in addition \( b(t) \) is bounded, \( f(t,x) = b(t)x^n \) satisfies the condition of the corollary and the zero solution is asymptotically stable.

A classical theorem of stability theory is the following which is easily proven using the preceding results.
Theorem: Let $A$ be a constant matrix and suppose that the system $x' = Ax$ is asymptotically stable. Let $g(x)$ be continuous for $|x| \leq c$ and suppose $g(x) = o(|x|)$ as $|x| \to 0$. Then the zero solution of $x' = Ax + g(x)$ is asymptotically stable.

This follows immediately from corollary 1 since $x' = Ax$ is u.a.s. The motion of a simple pendulum is governed by an equation of the form $y'' + \sin y = 0$. This equation does not take into account the fact that there is friction and air resistance which slow the pendulum down. If we assume that these forces are proportional to the angular velocity $y'$, the equation becomes $y'' + ay' + \sin y = 0$, where $a > 0$ is the constant of proportionality. The theorem above may be used to show that the zero solution of this equation is a.s.

Making the transformation $x_1 = y$, $x_2 = y'$, we obtain the system

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_2 - \sin x_1
\end{align*}
$$

This may be written as $x' = Ax + g(x)$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ x_1 - \sin x_1 \end{bmatrix}$. The equation $x' = Ax$ is a.s. (see remarks following theorem (4.9) page 29).

As $x_1 \to 0$, then \[ \frac{|g(x)|}{|x|} = \frac{|x_1 - \sin x_1|}{|x|} \leq \frac{|x_1 - \sin x_1|}{|x_1|} \to 0 \] as $|x| \to 0$. Thus by the above theorem, the zero solution is a.s.
As a second application, consider the system

\[
x'_1 = ax_1 + bx_2 + P(x_1) + Q(x_2) \\
x'_2 = cx_1 + dx_2 + R(x_1) + S(x_2)
\]

where \(P, Q, R,\) and \(S\) are polynomials in their indicated arguments. Writing \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \) \(g = \begin{bmatrix} P+Q \\ R+S \end{bmatrix},\) the above system becomes \(x' = Ax + g(x).\) Suppose that \(x' = Ax\) is a.s.

If the lowest order terms of the polynomials are of degree at least two, then \(g(x) = o(|x|)\) as \(|x| \to 0\) and hence the zero solution is a.s.

The following is an example which shows that the condition on \(Y(t)\) in (5.2) cannot in general be reduced. Even if \(f(t,x) = o(|x|)\) uniformly in \(t\) as \(|x| \to 0\), the a.s. of (2) need not imply the a.s. of the zero solution of (1). To see this, consider the Bernoulli equation \(x' = a(t)x + x^2\) where \(a(t) = -1/(t+1).\) Let \(t_0 = 0.\) The linear equation \(x' = a(t)x\) has the solutions \(x(t) = x_0/(t+1).\) Here, \(Y(t) = 1/(t+1) \to 0\) as \(t \to \infty\) and the linear equation is a.s. It is also uniformly stable.

On the other hand, the nonlinear equation \(x' = a(t)x + x^2\) has the solutions \(x(t) = x_0 \frac{x_0}{(t+1)[1-x_0 \log(t+1)]}.\) Now for any \(x_0 > 0,\) \(x(t)\) is undefined at the time \(t = e^{1/x_0} - 1.\) Hence the zero solution of the nonlinear equation is not even stable.

The conditions on \(Y(t)\) in (5.2) are satisfied if (2) is uniformly asymptotically stable. But in this case, the conditions on \(f\) may be reduced. Consider the condition
i) There is a positive constant $r$ such that $|f(t, x)| \leq r|x|$ for $t \geq t_0$ and $|x| \leq c$. Also for each $\varepsilon > 0$, there is a corresponding pair of positive numbers $\delta = \delta(\varepsilon)$, $T = T(\varepsilon)$ such that $|f(t, x)| \leq \varepsilon |x|$ for $t \geq T$ and $|x| \leq \delta$.

Theorem (5.3) Suppose there exist positive constants $K$ and $a$ such that $|Y(t)Y^{-1}(s)| \leq Ke^{-a(t-s)}$ for $t \geq s \geq t_0$. Suppose also that $f(t, x)$ satisfies the condition (i) above. Then the zero solution of (1) is asymptotically stable.

Proof: We may assume that $K > 1$ and $r > a/K$. Let $0 < \varepsilon < a/K$, $\delta = \delta(\varepsilon) < c$, $T = T(\varepsilon)$ as in condition (i). Let $x(t)$ be a non-continuable solution of (1) such that

(a) $|x(t_0)| < (\delta/2K^2)e^{(a-Kr)(T-t_0)}$

Let $t^* = \sup \{t \mid |x(s)| < \delta/2, t_0 \leq s < t\}$. By the variation of constants formula, $x(t) = Y(t)Y^{-1}(t_0)x(t_0) + \int_{t_0}^{t} Y(t)Y^{-1}(s)f[s, x(s)]ds$.

Thus for $t_0 \leq t < t^*$, $|x(t)| \leq Ke^{-a(t-t_0)}|x(t_0)| + \int_{t_0}^{t} Ke^{-a(t-s)}r|x(s)|ds$.

Multiplying by $e^{a(t-t_0)}$, $e^{a(t-t_0)}|x(t)| \leq K|x(t_0)| + \int_{t_0}^{t} Kr|x(s)|ds$. By Gronwall's inequality, $e^{a(t-t_0)}|x(t)| \leq K|x(t_0)| \exp \left(\int_{t_0}^{t} Krds\right) = K|x(t_0)|e^{Kr(t-t_0)}$.

Therefore,

(b) $|x(t)| \leq K|x(t_0)|e^{(Kr-a)(t-t_0)}$ for $t_0 \leq t < t^*$.

Suppose that $t^* \geq T$. Using (a) and (b) and the fact that $Kr-a > 0$, $|x(t)| < \delta/2K$ for $t_0 \leq t < t^*$. By (2.4), $x(t)$ is defined at least on the interval $[t_0, t^*]$ and $|x(t^*)| < \delta/2K < c$. By (2.3), $x(t)$ is defined in a neighborhood to the right of $t^*$. But $|x(t^*)| < \delta/2K$. 

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and the continuity of $|x(t)|$ imply that $|x(t)| < \delta/2K$ in a neighborhood to the right of $t^*$. This contradicts the definition of $t^*$. Therefore, $t^* > T$. For $T < t < t^*$,

$$x(t) = Y(t)Y^{-1}(T)x(T) + \int_T^t Y(t)Y^{-1}(s)f[s,x(s)]ds$$

and since $|f[t,x(t)]| \leq \varepsilon|x(t)|$ for $T < t < t^*$,

$$|x(t)| \leq Ke^{-\alpha(t-T)}|x(T)| + \int_T^t Ke^{-\alpha(t-s)}\varepsilon|x(s)|ds$$

Thus $e^{\alpha(t-T)}|x(t)| \leq K|x(T)| + \int_T^t e^{\varepsilon Ke^{\alpha(s-T)}}|x(s)|ds$ and by

Gronwall's inequality $e^{\alpha(t-T)}|x(t)| \leq K|x(T)|e^{\varepsilon K(t-T)}$ or

$$|x(t)| \leq K|x(T)|e^{(\varepsilon K-\alpha)(t-T)}.$$ Since $|x(T)| < \delta/2K$,

(c) $|x(t)| < (\delta/2)e^{(\varepsilon K-\alpha)(t-T)}$

But $\varepsilon K-\alpha < 0$ so $t^* = +\infty$. Hence $|x(t)| < \delta/2$ for all $t \geq t_0$ and since $\delta$ can be taken arbitrarily small, the zero solution of (1) is stable. Finally from (c), $|x(t)| \rightarrow 0$ as $t \rightarrow +\infty$ since $\varepsilon K-\alpha < 0$. Thus the zero solution of (1) is asymptotically stable and the proof is complete.

As an application of (5.3), if (2) is uniformly asymptotically stable, if $B(t)$ is a matrix which is continuous for $t \geq t_0$, and if $B(t) \rightarrow 0$ as $t \rightarrow +\infty$, then the linear system

$$x' = A(t)x + B(t)x$$

is asymptotically stable.
Now consider the equation

(1) \[ x' = f(t,x) = [f_1(t,x), \ldots, f_n(t,x)] \]

where \( f \) satisfies the following conditions:

i) \( f \) is continuous for \( t > t_0, \ |x| < c \)

ii) \( f(t,0) = 0 \) for all \( t > t_0 \)

iii) for each point \((t_1,x_1)\) with \( t_1 > t_0 \) and \( |x_1| < c \),
the equation (1) has at most one solution \( x(t) \)

which satisfies \( x(t_1) = x_1 \).

The following is a method to determine the stability behavior of the zero solution of (1) called the second method of Lyapunov. For brevity, let \( S \) denote the set of points \((t,x)\) such that \( t > t_0, \ |x| < c \). Consider a real valued function \( V = V(t,x) = V(t,x_1, \ldots, x_n) \) with the following properties:

a) \( V \) is defined and continuous on \( S \).

b) \( V(t,0) = 0 \) for all \( t > t_0 \).

c) \( V \) has all it's first order partial derivatives \( \partial V/\partial x_1, \ldots, \partial V/\partial x_n, \partial V/\partial t \)
defined and continuous on \( S \).

Such a function is called positive semidefinite if \( V \geq 0 \) on \( S \).

A function \( W = W(x_1,x_2,\ldots,x_n) \) with the properties listed above but independent of \( t \) is said to be positive definite if \( W \geq 0 \) and \( W = 0 \) if and only if \( x = 0 \). The function \( V \) is said to be positive (negative) definite if there is a positive definite
function $W$ such that $V \geq W$ on $S$. The function $V$ is said to have an infinitesimal upper bound if for each $\varepsilon > 0$ there is an $h > 0$ such that $|V| < \varepsilon$ for all $t \geq t_0$, $|x| < h$. For example $V(t,x) = (x_1 + \ldots + x_n) \sin t$ has an infinitesimal upper bound. Through out the remainder of this section, $V$ is assumed to have the properties $a, b,$ and $c$. Define a function $V'$ on $S$ as follows:

$$V' = V'(t,x) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} f_j(t,x) + \frac{\partial V}{\partial t}$$

If $x(t)$ is any solution of (1) on an interval, we may consider the function $v(t) = V(t,x(t))$. By the chain rule

$$v'(t) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} x_j(t) + \frac{\partial V}{\partial t} = V'[t,x(t)].$$

Thus $V'$ is the derivative of $V$ along the solution $x(t)$.

**Theorem (5.4)** Suppose that $V$ is positive definite and that $V \leq 0$ on $S$. Then the zero solution of (1) is stable.

**Proof:** Since $V$ is positive definite, there is a positive definite function $W(x) = W(x_1, \ldots, x_n)$ such that $V \geq W > 0$ for $t \geq t_0$, $|x| < c$, $x \neq 0$. Since $V \geq 0$ on $S$, $V' \leq 0$ on $S$. Let $0 < \varepsilon < c$ and consider the set $I = \{ x \mid \varepsilon \leq |x| \leq c \}$. $I$ is compact and since $W$ is continuous on $I$, it attains its minimum value $u$ on $I$. Since $W > 0$ on $I$, $u > 0$. Since $V$ is continuous and $V(t,0) = 0$, there exists a number $\delta$, $0 < \delta < \varepsilon$, such that $V(t_0, x) < u$ if $|x| < \delta$. Let $x(t)$ be a solution of (1) such that
It will be shown that \( x(t) \) is defined for all \( t \geq t_0 \) and satisfies \( |x(t)| < \epsilon \) for \( t \geq t_0 \). It may be assumed that \( x(t) \neq 0 \) for any \( t \) where it is defined because the assumption (iii) would then imply that \( x(t) = 0 \) for all \( t \). Let \( t_1 \) be the right end point of the maximal interval of definition of \( x(t) \).

Suppose first that \( t_1 < +\infty \). By theorem (2.4) \( x(t) \) must be defined on \([t_0, t_1]\) with \( |x(t)| < \epsilon \) for \( t_0 < t < t_1 \). Hence,

\[
V[t, x(t)] - V[t_0, x(t_0)] = v(t) - v(t_0) = \int_{t_0}^{t} v'(s) \, ds = \int_{t_0}^{t} v'[s, x(s)] \, ds \\
\]

Therefore since \( x(t) \neq 0 \), \( 0 < V[t, x(t)] \leq V[t_0, x(t_0)] \).

Since \( |x(t_0)| < \epsilon \), there exist \( t_2, t_0 < t_2 < t_1 \) such that \( |x(t)| < \epsilon \) for \( t_0 < t < t_2 \). Suppose there is a value \( t^* \), \( t_2 < t^* < t_1 \) such that \( |x(t^*)| = \epsilon \). Then since \( V[t_0, x(t_0)] < u \), \( u > V[t_0, x(t_0)] > V[t^*, x(t^*)] > W[x(t^*)] > u \) a contradiction.

Therefore \( |x(t)| < \epsilon \) for \( t_0 < t < t_1 \). In particular \( |x(t_1)| < \epsilon < c \).

By (2.3) we may continue the solution past \( t_1 \), another contradiction. Therefore \( t_1 = +\infty \). It remains to show that \( |x(t)| < \epsilon \) for \( t > t_0 \). Since \( |x(t_0)| < \epsilon \), there exists \( t_2 > t_0 \) such that \( |x(t)| < \epsilon \) for \( t_0 < t < t_2 \). If for any \( t > t_0 \), \( |x(t)| > \epsilon \), then there is a smallest value \( t^* > t_2 \) such that \( |x(t^*)| = \epsilon \). Then \( |x(t)| < \epsilon < c \) for \( t_0 < t < t^* \). As before, \( 0 < V[t, x(t)] \leq V[t_0, x(t_0)] \).

Therefore since \( V[t_0, x(t_0)] < u \), we obtain the contradiction \( u > u \) as above. This completes the proof.
The differential system

\[ y''_i = -\frac{\partial W}{\partial y_i} \quad (i = 1, 2, 3) \]

may be interpreted as describing the motion of a unit mass in a potential field \( W = W(y_1, y_2, y_3) \).

Letting \( x_i = y_i \quad i = 1, 2, 3 \) and \( x_{3+i} = y'_i \quad i = 1, 2, 3 \)
the above system becomes

\[
\begin{align*}
    x'_1 &= x_4 \\
    x'_2 &= x_5 \\
    x'_3 &= x_6 \\
    x'_4 &= -\frac{\partial W}{\partial x_1} \\
    x'_5 &= -\frac{\partial W}{\partial x_2} \\
    x'_6 &= -\frac{\partial W}{\partial x_3}
\end{align*}
\]

Let \( V = W(x_1, x_2, x_3) + \frac{1}{2}[x_4^2 + x_5^2 + x_6^2] \). The last term is the kinetic energy and \( V \) is the total energy of the system. Then \( V' = 0 \) for all \( x \). Now \( V \) is positive definite if and only if \( W \) is positive definite. Thus if \( W \) is positive definite and \( f \) satisfies the conditions i, ii, and iii, (5.4) applies and the zero solution is stable.

Theorem (5.5) Suppose that \( V \) is positive definite and \( V' \) is negative definite. Suppose also that \( V \) has an infinitesimal upper bound. Then the zero solution of (1) is asymptotically stable.
Proof: There exist positive definite functions $W, W^*$ such that $V \geq W$ and $-V' \geq W^*$ on $S$. By (5.4), the zero solution of (1) is stable. Thus there exists $\delta > 0$, $0 < \delta < c$, such that if $x(t)$ is any solution of (1) with $|x(t_o)| < \delta$, then $x(t)$ is defined and satisfies $|x(t)| < c$ for all $t > t_o$. Since $V' < 0$ on $S$, $v(t) = V[t,x(t)]$ is a nonincreasing function. Suppose $\lim v(t) \neq 0$. Then since $v$ is nonincreasing, there is an $\varepsilon > 0$, $0 < \varepsilon < c$ such that $v(t) \geq \varepsilon$ for all $t > t_o$. Since $V$ has an infinitesimal upper bound, there is an $h > 0$, $0 < h < c$ such that $|x| < h$ implies $V(t,x) < (\varepsilon/c) |x|$ for all $t > t_o$. Thus we cannot have $|x(t)| < h$ for any $t > t_o$ since if this were the case, $v(t) = V[t,x(t)] < (\varepsilon/c) |x(t)| < \varepsilon h/c < \varepsilon$ contradicting the fact that $v(t) \geq \varepsilon$. Hence $|x(t)| \geq h$ for all $t > t_o$. Let $u$ be the minimum value of $W^*$ on the compact set $I = \{x| h \leq |x| \leq c\}$. Then $u > 0$ since $W^*$ is positive definite. Therefore since $h \leq |x(t)| < c$ for $t > t_o$, $W^*[x(t)] \geq u$ for $t > t_o$. But $-v'(t) = -V'[t,x(t)] \geq W^*[x(t)] \geq u$. Integrating from $t_o$ to $t$, $-v(t) + v(t_o) \geq \int_{t_o}^{t} uds = u(t-t_o)$. This implies that $v(t)$ is negative for all $t > t_o + v(t_o)/u$, a contradiction. Thus $\lim v(t) = 0$. Since $v(t) = V[t,x(t)] \geq W[x(t)] \geq 0$ $t \to \infty$ $\lim W[x(t)] = 0$. Now $W(x) = 0$ if and only if $x = 0$ and it follows that $x(t) \to 0$ as $t \to \infty$. This completes the proof.

There are many other types of stability. Those considered so far are related to the initial point $(t_o, x_o)$. This amounts to stability relative to errors in initial data.
Suppose a physical system is governed by the equation

\begin{equation}
 x' = f(t,x)
\end{equation}

In actuality, the system may encounter perturbations which
which are not accounted for in the equation (1). Hence
not only is the initial data in error, but the equation
itself is only an approximation. This leads to the concept
of "Total stability" which was introduced by J. G. Malkin.

Definition: A solution $x(t)$ of (1) is said to be **totally stable** if for each $\epsilon > 0$, there is a corresponding $\delta > 0$ such that if $g(t,x)$ satisfies the inequality $|g(t,x) - f(t,x)| < \delta$, then any solution $y(t)$ of the equation $y' = g(t,y)$ with $|y(t_0) - x(t_0)| < \delta$ is defined and satisfies $|y(t) - x(t)| < \epsilon$ for all $t > t_0$. [ $g(t,x)$ is assumed to be continuous and satisfy the same conditions as $f(t,x)$ with regard to existence and uniqueness of solutions]

Let $b(t)$ be a vector function which is continuous for $t > t_0$. As before, let $Y(t)$ be the fundamental matrix for the homogeneous system (2) such that $Y(t_0) = I$. The next theorem gives sufficient conditions for the total stability of the inhomogeneous equation $x' = A(t)x + b(t)$.

**Theorem (5.6)** Suppose there is a constant $K$ such that
\[
\int_{t_0}^{t} |Y(s)Y^{-1}(s)| ds \leq K \text{ for } t > t_0.
\]
Then every solution of the inhomogeneous equation $x' = A(t)x + b(t)$ is totally stable.
Proof: By the lemma preceding (5.2), \( Y(t) \to 0 \) as \( t \to \infty \), and there is a constant \( M \) such that \( |Y(t)| \leq M \) for \( t > t_0 \).

If \( \epsilon > 0 \), let \( \delta = \epsilon / (M+K) \). Suppose that \( g \) satisfies the inequality \( |g(t,x) - A(t)x - b(t)| < \delta \). Let \( x(t) \) be any solution of the inhomogeneous equation, say

\[
x(t) = Y(t)x(t_0) + \int_{t_0}^{t} Y(t)Y^{-1}(s)b(s)ds.
\]

Let \( y(t) \) be a solution of \( y' = g(t,y) \) such that \( |y(t_0) - x(t_0)| < \delta \). \( y(t) \) is also a solution of the equation

\[
y'(t) = A(t)y(t) + [g(t,y(t)) - A(t)y(t)].
\]

Thus by the variation of constants formula

\[
y(t) = Y(t)y(t_0) + \int_{t_0}^{t} Y(t)Y^{-1}(s)[g(s,y(s)) - A(s)y(s) - b(s)]ds.
\]

Hence

\[
y(t) - x(t) = Y(t)[y(t_0) - x(t_0)] + \int_{t_0}^{t} Y(t)Y^{-1}(s)[g(s,y(s)) - A(s)y(s) - b(s)]ds.
\]

Therefore,

\[
|y(t) - x(t)| \leq |Y(t)||y(t_0) - x(t_0)| + \delta \int_{t_0}^{t} |Y(t)Y^{-1}(s)|ds
\]

\[
< M\delta + \delta K = \epsilon.
\]

As a final example, consider the scalar equations

i) \( x' = f(x) = -ax \quad (a > 0) \)

ii) \( y' = g(y) = -ay + \sum_{k=0}^{\infty} \left[ a_k \cos(ky/c) + b_k \sin(ky/c) \right] \)

First, the solutions of (ii) are uniquely determined by initial values and are defined for all \( t > 0 \). In fact, by the mean value theorem, for any two numbers \( y_1 \) and \( y_2 \), there is a number \( y \) such that \( g(y_1) - g(y_2) = g'(y)(y_1 - y_2) \). Since \( g' \) is bounded, \( g \) satisfies a Lipschitz condition and by theorem (2.2) the solutions of (ii) are uniquely determined by initial values.
and defined for all \( t \). Let \( x(t) = x_o e^{-at} \) be any solution of (i) and let \( y(t) \) be the solution of (ii) which satisfies \( y(0) = y_o \). Using theorem (5.6), the difference \( |y(t) - x(t)| \) may be estimated. The fundamental matrix for (i) is the scalar \( Y(t) = e^{-at} \) and the conditions of the theorem are satisfied with \( K = 1/a \). Also, we may take \( M = 1 \). Note that in the proof of the theorem, \( \delta \) is expressed as an explicit function of \( \epsilon \), namely \( \delta = \epsilon / (M + K) \). In particular let

\[
\delta = \max \left( |y_o - x_o|, \sum_{k=0}^{m} [a_k | + |b_k|] \right)
\]

Then \( |y_o - x_o| \leq \delta \) and also \( |g(x) - f(x)| \leq \delta \). Therefore, \( |y(t) - x(t)| \leq \epsilon = \delta (M + K) \). That is, \( |y(t) - x_o e^{-at}| \leq \delta (1 + 1/a) \).
BIBLIOGRAPHY


