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Development of the theory of sets of points

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The University of Montana

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The Development

of

The Theory of Sets of Points

by

Raymond J. Garver

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Master of Arts.

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    A translation of IV above.
ordinal number which belongs to $F$. The ordinals of the second number class, $\omega, \omega + 1$, etc., are discussed, and the cardinal number of the second number class is defined to be $\aleph_1$.

The most interesting, if not particularly valuable, part of this article was the discussion of a class of numbers $\xi$ having the remarkable property that $\omega^\xi \geq \xi$. The smallest of these, $\xi$, is the limit $\omega^\omega$, where $\omega^0 = 1$, $\omega^1 = \omega$, $\omega^2 = \omega^\omega$, $\omega^3 = \omega^{\omega^\omega}$, and so on.

This completes the articles on the theory of point sets published before 1901 which were available, and in them most of the topics regarded today as a part of the subject have been discussed. A few points discussed in Hobson we have not met—but this has been either because the subject has been recently developed (as in the case of ordinary inner limiting sets, which date from 1903 *, systems of nets **, or the idea of the 'measure' of a set), or because articles discussing the subject were not available. ***

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* Due to W.H. Young.

** See note 3, page 29, of this paper.

*** For instance, while the Baire-Borel theorem was first proved in 1895, the proof appeared in the "Annales de l'Ecole Norm." Hobson, 107, contains a reference to an article by W.H. Young on the subject in "Proc. Lond. Math. Soc. (1) XXV, 367; which would be before 1901, but the article is actually found in Vol XXXV (1902). Sets of the first and second category were first examined by Baire, in the "Annali di Mat.", (3) III, 65, an article which probably appeared just before or after 1901, though I can find no date. Jordan's measure of a set of points dates at least from 1897—see Hobson, 183, and Journal de Mat. (4) Vol VIII.
We will see that $\alpha$ is not equal to $\omega$, and is always greater than $\omega$.

"By $\omega$ we understand the order-type of a well-ordered series $(e_1, e_2, \ldots, e_v)\ldots$ in which $e_v$ is less than $e_{v+1}$, and where $v = 0, 1, 2, \ldots \ldots$"

"If in an ordered set $M$ all order relations become inverted, then we obtain an ordered set which we will call the inverse of $K$, and will represent by $\overline{M}$. If $\overline{M} = \alpha$ then $\overline{M} = \overline{\alpha}$.

Cantor then considers the three special order types, $\omega + \omega$ (or $\pi$), $\eta$ (the order-type of all rational numbers greater than 0 and less than 1 in their natural order), and $\Theta$ (the order-type of the linear continuum $X$).

$\omega$ is characterised by several interesting relations: 1) $1 + \omega = \omega$, 2) $\omega + 1 = \omega$, 3) $\omega + \omega = \omega$, 4) $\omega \cdot \omega = \omega$. $\eta$ is characterised by 1) $\eta = \overline{\omega} = \omega$, 2) $\eta$ is everywhere dense, 3) $\eta$ has no highest or lowest element. Also

$\eta$ is a perfect type, and has the property that it contains a set $R$ of order-type $\eta$ as a part set, such that between any two optional elements of $X$, elements of $R$ lie.

Article XXXV, in Volume 49 of the Annalen, is not of any great importance. Cantor again defines well-ordered sets $\bar{\eta}$, and defines segments and remainders of well-ordered sets. The order-type of a well-ordered aggregate $F$ we call the

--- See Hobson, 204 and ???.

**-- See Hobson, 213.

***-- See Hobson, 211.

****-- See Hobson, 208.

*****-- See Hobson, 209.

--- See Hobson, 211, and page 18 of this paper.

### See Hobson, 213.
when "a certain order-arrangement prevails between its elements, such that of any two elements, m and n, one takes the higher, and one the lower, rank...... To each simply ordered set \( M \) belongs a fixed order type, which we will denote by \( \bar{M} \), and by which we will understand the concept obtained when we abstract from the set \( M \) the quality of the different elements, but retain the order of precedence among them."

Two sets \( M \) and \( N \) are similar, and have the same order type, if we can set up a one-to-one correspondence between them so that \( m_1 \) and \( m_2 \) correspond to \( n_1 \) and \( n_2 \), and the order relation of \( m_1 \) to \( m_2 \) in \( M \) is the same as that of \( n_1 \) to \( n_2 \) in \( N \). This is expressed by \( M \sim N \). (The corresponding expression for equivalence, we remember, was \( M \sim N \).

We obtain \( \bar{M} \) from \( M \) by a second abstraction.

"With finite sets the order-type offers no particular interest. All simply ordered sets with the same cardinal number are similar, and have the same order-type. Finite order types are subject to the same rules as finite cardinal numbers, and we can use for them the same symbols-- 1, 2, 3, ...... , even though they are conceptually different from the cardinal numbers.

However, it is very different with transfinite order-types, for to one transfinite cardinal number \( \alpha \) correspond infinitely many different types of simply ordered sets, which in the aggregate constitute a 'type-class'. A cardinal number \( \aleph \) is common to all types belonging to the class; therefore we call it the type class \( [\aleph] \). The simplest of these is the type class \( [\aleph_0] \).

We must differentiate from the cardinal number \( \aleph \), which fixes the type class \( [\aleph] \), that cardinal number \( \aleph \), which in turn is fixed through the type class \( [\aleph] \); it is the cardinal number which belongs to the type class \( [\aleph] \) inasmuch as it represents a well-defined set whose elements are the different types with the cardinal no. \( \aleph \).

--- See Hobson, 204 and 218.
For ten years after this last article was published (1885), no more papers on point sets appeared, with the exception of Article XXXIII, which has already been considered. * It was not until 1895-97 that Cantor published "Beiträge zur Begründung der transfiniten Mengenlehre", ** in Volumes 46 and 49 of the Annalen. These articles are largely concerned with a more advanced and abstract treatment of two topics already familiar to us—cardinal numbers, and ordinal numbers and order types. Chapter IV of Hobson is an adaptation, and in large part almost a word for word translation, of these two papers by Cantor. In the first part of Hobson's book, he at least re-arranges and changes to some slight extent Cantor's original material, but in Chapter IV, he copies it almost exactly as written.

The cardinal number of an aggregate M is defined in the first of these two articles as the concept obtained "when from the set M we abstract the quality of the different elements and the order in which they are given." *** This cardinal number we will denote by M. He gives a set of simple theorems on finite cardinal numbers which we will not take time to quote, and then defines N (Aleph-zero) as the cardinal number of the set of the positive integers. **** N has these properties:

1) It is greater than each finite number M.
2) It is the smallest transfinite cardinal number.
3) N + 1 = N.
4) N + N = N; N N = N; N N = N; N N = N; N N = N.

He then passes from cardinal to ordinal numbers. A set is simply ordered *****

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* - See page 30 above.
*** - See Hobson, 188. See also Hobson's criticism of this definition, page 239.
**** - See Hobson, 194.
***** - See Hobson, 203.
Theorem J - If \( P \) is a separarte set, then it is of the first cardinal number (if it is an infinite set), and there is a smallest number \( \alpha \) of the first or second number class such that \( P^\alpha = 0 \).

We have then in this case:

\[ P = \sum_{\alpha=0}^\infty P^\alpha \cdot a \ldots \leq \]

Theorem K - If \( P \) is enumerable, without being a separarte set, * then there is a smallest number \( \alpha \) of the first or second number class such that \( P^\alpha \) is a set dense in itself. Call this \( U \), and call \( \sum_{\alpha=0}^\infty P^\alpha \cdot a \), \( R \). Then

\[ P = U + R, \]

where \( R \) is either 0 or an enumerable separarte set.

Theorem L - If \( P \) is of higher than the first cardinal number, then there is a smallest number \( \alpha \) of the first or second number class, such that \( P^\alpha \) is dense in itself; this latter consists of a part \( V \) which is dense in itself and comprises all points of \( P \) which are so situated that in each neighborhood of them points of \( P \) are contained which are of higher than the first cardinal number, ** and of a part \( U \), which if it is not 0 consists of the remaining points of \( P^\alpha \), and is an enumerable set dense in itself. If \( R \) has the same significance as above, it is a separarte set, and we have

\[ P = R + U + V. \]

If we define \( U + V \) as \( P^\infty \)or \( Pi \), the total coherence of \( P \), and \( R \) as \( Pr \), the rest or residuum of \( P \),

\[ P = Pr + P^\infty \quad \text{or} \quad P = Pr + Pi. \]

If \( P \) is a closed set, \( P^\prime = P^\prime \), \( P^{(\prime)} = P^{(\prime)} \), \( P^{\prime \prime} a = P^{(\prime)} - P^{(\prime \prime)} \), \( P^{\prime \prime} = P^\infty = P^{(\prime \prime)} \), and \( U = 0. \)

** - It will then contain a component dense in itself.

*** - See Hobson, 123, on points of degree \( a, c \), or \( x \) in the set \( P^\alpha \).

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He then proceeds to a more general analysis of sets. Any set $P$ consists of a set of isolated points, forming an isolated and therefore an enumerable set $P_a$, and of limit points forming a set $P_c$. That is,

$$P = P_a + P_c,$$

$P_c$ being $D(P, P')$, and $P_a$ being an isolated set or $O$, in which case $P$ is dense in itself.

In the same way, $P_c$ consists of isolated points $P_{ca}$, and of limit points $P_{c^2}$. Thus

$$P_c = P_{ca} + P_{c^2}, \quad \text{and} \quad P = P_a + P_{ca} + P_{c^2}.$$  

By a continuation of this process we obtain

$$P = P_a + P_{ca} + P_{c^2}a + \ldots + P_{c^{v-1}}a + P_{c^v},$$

where $P_a, P_{ca}, P_{c^2}a, \ldots, P_{c^{v-1}}a$ are termed the 1st, 2nd, 3rd, ..., $v$th adherence of $P$, and $P_{c^v}$ is the $v$th coherence of $P$.

If $P_{c^v}$ exists for all values of $v$ of the first or second number class, we define $P_{c'^v}$ as $D(P, P_{c^v}, P_{c^2}a, \ldots, P_{c^{v-1}}a)$. And, in general, if $Y$ is any number of the first or second number class not a limit number, then $P_{c^Y} = (P_{c^{Y-1}})^c$, while if $Y$ is a limit number of the second number class, $P_{c^Y} = D(\ldots, P_{c^v}, \ldots)$, where $Y$ takes on all values less than $Y$. Thus

$$P = \sum P_{c^Y} a + P_{c^Y}$$

where $Y' = 0, 1, \ldots < Y$.

$\sum P_{c^Y} a$ is evidently a "separirte" set, while each dense in itself part of $P$ is also a part of $P_{c^Y}$.

We then have four theorems, which can be taken to follow theorem H stated above:

Theorem I - When $P$ is a separirte set, then a number $a$ of the first or second number class exists such that

$$P_{c^a} = P_{c^{a-Y}} = 0,$$

but when $P$ is not a separirte set, then such a number $a$ exists that $P_{c^a}$, and $P_{c^{a+1}}$, is an everywhere dense set.
of the linear continuum, and that any closed set has either the first power or the power of the continuum. And again he remarks, "I shall show in a future communication that this division into two classes holds also for sets of points not closed." In conclusion, he devotes a few pages to the idea of content, "a notion ..... indispensible to later articles that I promised your journal in Acta, II, and that I will send later."

Of these later articles to appear in the Acta, only one was ever published, and it in the German, which was contrary to the original intention. This article is concerned largely with an analysis of sets, and is similar to the discussion of Hobson, 123 ff. Cantor begins by quoting theorems D, E, F, G of the collection given on pages 25-6 of this article, and adds a theorem H to the set, which is very similar to theorems C' and E' of page 27:

**Theorem H:** If \( P \) is any closed set, then it consists of two essentially separate parts, \( R \) and \( S \) (either of which may be \( 0 \)), so that:

\[
P = R + S,
\]

where \( R \) is a "separable" set (nowhere dense in itself), and at most of the first cardinal number, while \( S \), if it is not \( 0 \), is a perfect set and therefore of the cardinal number of the linear continuum. In case \( P \) is finite or of the first cardinal number, \( S \equiv 0 \), and there is a certain smallest \( \alpha \) of the first or second number class such that,

\[
p(\alpha) \equiv 0.
\]

But if \( P \) is of higher than the first cardinal number, then \( S \) is a perfect set, and there is a certain smallest \( \alpha \) of the first or second number class, such that

\[
p^{(\alpha)} \equiv p^{(\alpha+\alpha)} \equiv S \equiv p^{(\alpha)}
\]

The isolated set \( R \) has the quality that, from some fixed \( \alpha \) on,

\[
D(R, R^{(\alpha)}) = D(R, R^{(\alpha+\alpha)}) = 0.
\]

--- Article XXX.
Further theorems on content appear in Harnack's article in Annalen, XXV, * which seems to be his only article directly on point sets, though several of his previous papers contained certain references to the subject. ** A set of content 0, or a discrete set, is defined as one such that "all its points can be enclosed in a finite number of intervals whose sum can be made optionally small." An everywhere dense set in an interval 1 has, on the other hand, its content equal to 1.

The following theorems then hold:

1) If $P'$ is discrete, $P$ is also discrete.
2) A set of the first species and $n$-th order is discrete.
3) If $\mathfrak{F}^{(1)}$ is discrete, $P$ is discrete.
4) If $P^{(d)}$ is discrete, for some $d$ of the second number class, $P$ is discrete.
5) Every reducible set is discrete.
6) The content of $P$ is always equal to the content of $P^{(d)}$.

Pasch also has a few pages on "Inhalt einer Punktränge", in his paper, "Über einige Punkte der Funktionstheorie", *** but it contains little of interest. He refers to Cantor's previous discussion of content, **** and his discussion is very similar in form, but with an almost entirely different system of notation.

Article XXVII, "De la Puissance des Ensembles parfaits de Points", Cantor's next contribution to the Acta, contains nothing of importance which does not also appear in the long article ***** in the Annalen of the same year, which we have already considered. He again shows that any perfect set of points has the power

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* Article XXXII

*** - Article XXXIII, Annalen, XXX, 142-44.
**** - Annalen, XXIII, 473-78. See page 28 of this article.
***** - Article XXV.
ever-so-small sub-interval, and whose content is yet different from 0."

An apparently independent investigation by Stolz appears in this same volume of the Annalen, * and his definition of content, for which he uses a different term, however, is more easily comprehended than Cantor's, and agrees more closely with that given in Hobson. ** It applies to linear sets:

"In the finite interval (a, b), let any point set x be defined. Now consider a system of divisions of this interval of such sort that each results through division of the sub-intervals of the preceding, *** and indeed according to such a rule that from a fixed division T_m on, all the following divide the given interval into parts each of which is less than a given length . We consider a system of infinitely many such divisions or partitions, T_1, T_2, .... T_n, .... and add together for each partition those intervals which contain points of the given set x; we thus obtain a series of sums S_1, S_2, .... S_n, .... such that S_1 ≥ S_2 ≥ .... ≥ S_n ≥ .... ≥ 0. Then there exists a finite limit value \( \lim_{n \to \infty} S_n = L \), where \( S_n \geq L \), and \( L \leq 0 \).

"L is independent of the particular system of divisions considered and is called the 'Intervallgrenze' of the set.

"If L = 0, the sets agree with those which Harnack has called 'discret' sets". **** Sets everywhere dense in (a, b) always have \( L = b - a \). *****

* * Article XXVI.

--- Page 154.

*** The convenient notation, "system of nets", for a system of divisions of this kind is due to de la Vallée Poussin. (1916).

**** See Annalen XXV, 241 (1885). I do not find the term in any earlier article of Harnack's, though the reference here must be to some earlier article.

***** In this lies the chief difference between the content of a set and its measure.
He then proves the two very important theorems:

1) All linear perfect sets have the same cardinal number—the cardinal number of the linear continuum.

2) All closed linear sets (containing an infinite number of points) have either the first cardinal number, or the cardinal number of the linear continuum. *

"Later", he concludes, "this theorem will be extended to non-closed ** linear sets and to n-dimensional sets."

This article also contains Cantor's first remarks on the content of a set of points. *** He considers an n-dimensional set, and defines the content **** as \( \int_{\mathbb{R}^n} \), denoting it by \( I(P \text{ in } G_n) \). For linear sets, the definition has been formulated more conveniently by Stolz and Harnack, and we will later consider their articles in more detail. Cantor however proves several theorems which we should note:

1) \( I(P \text{ in } G_n) = I(P' \text{ in } G_n) = I(P'' \text{ in } G_n) \)

Theorem I) ***** When \( P \) is a reducible set, \( I(P) \) is always 0.

Theorem II) When \( P \) is not reducible, there is always a perfect set \( S \) which has the same content as \( P \).

"A perfect set \( P \) can have its content equal to 0, but only when it is nowhere everywhere dense... Also there are perfect sets which are everywhere dense in no

--- See Hobson, 118.

***- This theorem was extended by Hausdorff (1916) to cover a large class of non-closed sets, which he called Borel-sets, but has not been extended to cover all non-closed sets.

****- See page 14 of this paper--reference from Annalen XXI, 58.

*****- Inhalt.

*****- The notation for these three theorems is Cantor's.
Theorem C' - If $P$ is any closed set of the first cardinal number, then there is always a smallest number $\alpha$ of the first or second number class such that

$$p^{(\alpha)} < 0,$$

that is, a closed set of the first cardinal number is always reducible.

Theorem E' - If $P$ is any closed set of greater than the first cardinal number, then we can analyze $P$ into two sets $R$ and $S$, such that $S$ is a perfect set, and $R$ a set of the first cardinal number; also $(F')$ there exists a smallest number $\alpha$ of the first or second number class such that $p^{(\alpha)} \equiv S$.

"It is also important", he continues, "to notice the case where a set $P$ has such a quality that it is a divisor of its derivative $P'$, that is,$$
\text{D}(P, P') \equiv P.$$

In this case we shall call $P$ dense in itself."

"If $P$ is dense in itself, $P'$ is always a perfect set."

A set $P$ can be dense in itself without being everywhere dense ***, and a set can be everywhere dense in an interval $H$ without being dense in itself, though only in case it contains points outside the interval $H$.

"There is also the case where a set $P$ has such a quality that no part of it is dense in itself, in this we call the set $P$ a 'separirte' set."

"Isolated sets form a particular class of separirte sets. Further, all closed sets of the first cardinal number are separirte sets, as they would not otherwise be reducible, **** as are also the sets $R$ appearing in the theorems E, E'. To the separirte sets also belong sets of the type $P = D(P, P^n)$, for any set $P$."
Annalen. Here he first collects his own theorems A, B, C, and Bendixson's D, E, F, G, of Acta, II, writing them as six theorems. * Taken together, they form an important and compact set of theorems on enumerable and unenumerable sets. ** He adds another theorem:

**Theorem F**—If P is a point set such that its first derivative $P^\prime$ has a higher cardinal number than the first, then there is always a smallest number $\alpha$ of the first or second number class, such that

$$P^{(\alpha)} = P^{(\alpha+1)} = \ldots = P^{(n)}.$$

It is remarkable that up to this time Cantor has never found it necessary to introduce the terms "closed set" and "set dense in itself" (or some similar terms to express the same properties), but such is the case. He says here: ***

"To complete the theorems of the preceding paragraphs, as well as to conveniently lead to new investigations, I must now fix certain new definitions and new names. ........

"If a point set such that its derivative $P^\prime$ is contained in it, that is, if

$$D(P, P^\prime) = P^\prime,$$

we will call P a closed set. For any set P whatever, the set $M(P, P^\prime)$ is a closed set. Each set which is the derivative of another set is itself a closed set, and the inverse of this theorem also holds. If P is a closed set, then $P^\prime$ is a part of P, and $P \subseteq Q + P^\prime$, where Q is an isolated set, and therefore enumerable."

For closed sets, theorems C, E, and F above can be changed slightly:

* He combines theorems D and E, into one, which he calls D. Bendixson's theorem F he calls E.

** See pages 14, 19, 20, 21, 24, 25 for the development of this group of theorems.

*** Page 469.
Theorem D- If \( P' \) has a power greater than the first, there always exist points which belong to \( P' \) for every value \( \gamma \) of the first or second number class.

Theorem E- If \( P'^{(n)} \) is the set of all the points of theorem D, \( P'^{(n)} \) is a perfect set.

Theorem F- If \( P' - P'^{(n)} \) is denoted by \( R \), then \( R \) has the first power.

Theorem G- There exists a \( \gamma \) of the first or second number class such that \( D(R, R'^{(n)}) = 0 \).

Another interesting theorem, the proof of which is rather complicated, appears in Bendixson's article:

"If \( P \) is a perfect set, such that no part of it forms a continuous space, and situated in a continuous space of one dimension, the set \( P \) can always be expressed as the first derived set of an isolated set, situated also in a space of one dimension."

In an article appearing in Acta, V, "Beweis eines Satzes aus der Mannigfaltigkeitslehre", Edward Phragmén gives an alternate proof of Bendixson's theorem F:

"If \( P \) is an arbitrary set, and \( n \) the first number of the third class of Cantor, then \( P' - P'^{(n)} \) is of the first cardinal number." **

Cantor's next article in the Acta was preceded slightly by the sixth installment of his long paper, **** "Uber unendliche, lineare Punktmannigfaltigkeiten", in the

* These four theorems follow Cantor's theorems A, B, C. Cantor had already proved D, E, and F, but G was first proved by Bendixson. See pages 21 and 24 of this paper.

** See Article XXIX. As the theorem is stated here, \( P' \) might be enumerable, in which case \( P'^{(n)} \) would vanish.

*** Article XXVII (1884)

**** Article XXV (1884)
vious articles or is contained in the Annalen (Article XXV) of the same year. The first of them, Article XXIII, contains three theorems, at least two of which we have already met:

Theorem A—A set of points having the first power cannot be an enumerable set.

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vious articles, or is contained in the Annalen (Article XXV) of the same year. The first of them, Article XXIII, contains three theorems, two of which we have already met:

Theorem A- A set of points having the first power cannot be a perfect set.

Theorem B- If \( P \) is a set of points such that its derived set of order \( \alpha \) vanishes, \( \alpha \) belonging to the first or second number class, then the first derived set \( P' \), and the set \( P \) itself, are of the first power, unless one of them is finite. *

Theorem C- If \( P \) is a set of points such that its first derived set is of the first power, then there exists numbers \( \alpha \) of the first or second number class such that

\[ P(\alpha) \subseteq \emptyset \]

and of all these numbers, there is one which is the smallest. **

This theorem C is proved here for the first time, it being stated without proof in Article XIII. The proof depends on the relation-

\[ P' = \sum_{\alpha=1}^{\infty} (P(\alpha) - P(\alpha+1)) \cdot P^\alpha \]

where \( \alpha \) is any positive whole number of the first or second number class, and on theorem A above. If \( P' \) is enumerable, the perfect set \( P^\alpha \) must vanish, and the theorem follows at once.

Cantor also points out the error in his theorem of page 21 of this article, in which he divided an unenumerable derivative \( P' \) of a set \( P \) into two sets \( R \) and \( S \), and claimed for \( R \) the property of being a reducible set. That this last is not necessarily true was first shown by Bendixson, who has an article (Article XXIV) in this same volume of the Acta. Bendixson sums up the results of his investigations in four theorems:

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* * See page 19 of this article.

** See page 20-1 of this article.

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concerning his researches on the theory of sets. We believe we shall render our readers a service by first reproducing here in French the principal articles which he has already published on the subject; they are indispensable to the understanding of the new ones which are to follow, and which will be published in French. * The translation has been reviewed and corrected by-the author."

Then follow, in the next hundred pages, Articles XIV to XXII, inclusive, which (with the exception of Article XVI, which merely leads up to XVII and is in itself not essential in the theory of sets) are translations of Articles IV, VI, II, VII, X, XI, XII, and XIII, originally published in German in Crelle's journal and in the Annalen. However, they cannot be dismissed as mere translations—Cantor has evidently revised them rather carefully before publishing them in French, and for this reason they are worthy of notice on their own account. We have already** noted two places where incorrect or ambiguous statements in the original articles were corrected for the French translation, and it is very possible that others exist. Also several of the articles were abridged somewhat, especially the translation of Article XIII, which we saw contained so much irrelevant matter. In the translation, Article XXII, this is omitted, and the article is thus shortened more than a third, without omitting anything of value. Thus it would seem that a reader who did not wish to read both the German and French articles, yet wished to do some research on the subject, might do better to read the French articles, remembering however when, and in what order they were first published.

The "new articles" on the subject which appear in the Acta are unfortunately not particularly important—they were probably written more as a favor to Mittag-Leffler than for any other reason. Most of the material in them is taken from pre-

--- This series of new articles consists only of Article XXIII (Acta II), Article XXVII (Acta IV), and Article XXX (Acta VII)—the third of which is in German.

**- See Page 8, note 1, and page 11, note 3.
themselves, though it is more convenient to define these two latter qualities first, and then define a perfect set as one being both closed and dense in itself. Perfect sets, as Cantor points out, are by no means always everywhere dense. * The similar question as to whether a set of the second species is necessarily everywhere dense has been raised and answered in the negative previously. ** The latter is the more general statement, since a perfect set is by definition a set of the second species, while a set of the second species is not in general a perfect set, though its first derivative, if it is not enumerable, will contain a perfect set as component. Cantor gives an example of a perfect non-dense set, defined analytically, in an appendix to the article under discussion. ***

Cantor's sixth article in the series, "Über unendliche, lineare Punktmannigfaltigkeiten", appeared in Annalen, XXIII, (1884), but before this, the second volume of the Acta Mathematica, Mittag-Leffler's journal, was published—a volume which contains a collection of a great part of the writings on point sets which had appeared up to 1883, together with two new papers on the subject. The series of articles is preceded by an editors note: ****

"M. George Cantor has been kind enough to promise us a series of new articles

* * * * - Page 590. The example is reproduced in Hobson, 118, with slight changes in symbols. H. J. S. Smith (Article V, page 147-8) gives a similar example, which he uses however in an entirely different connection. His article dates back to 1875, when the theory of point sets was hardly under way.
Cardinal number can also be made the basis of a certain analysis of sets—

"The sets of points P can now be divided according to the cardinal number of the first derivatives into two classes. If \( P^{(1)} \) has the cardinal number of the first number class, then, as has already been stated, there is a first whole number of the first or second number class for which \( P^{(k)} \) disappears. But if \( P^{(1)} \) has the cardinal number of the second number class, then \( P^{(1)} \) can always, and in only one way, be divided into two sets, so that

\[
P^{(1)} \cong R+S,
\]

where \( R \) and \( S \) are of very different quality.

"R is so constituted that it is capable, through a continuation of the derivative process, of a continual reduction, even to annihilation, so that there is always a whole number \( \gamma \) of the first or second number class such that

\[
R^{(\gamma)} \cong 0. **
\]

"Such a set \( R \) I will call a reducible set.

"S, on the other hand, is so constituted that the derivative process produces no change in it, that is, \( S = S^{(1)} = S^{(\gamma)} \). Such a set \( S \) I shall call a perfect set." This definition of a perfect set precedes anything on closed sets or sets dense in

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** Cantor of course is not justified in using this expression—he means, and should say, "the cardinal number of the continuum."

** In Acta II (Article XXIV) Bendixson gives an example to show that this is not always the case. \( R \) is always an enumerable set, and \( D(R, R^{(\gamma)}) = 0 \), for a certain \( \gamma \) of the first or second number class, but \( R \) is not necessarily a reducible set. In case \( R \) happens to be a closed set, which Cantor at first presumed to be the case always, (that is, he considered it to be expressible as the derivative of another set), then \( R^{(\gamma)} \cong 0 \), for some \( \gamma \).
"If \( P \) is a point set whose first derived set \( P^{(1)} \) has the cardinal number of the first number class, then there are whole numbers\( \alpha \) of the first or second number class for which \( P^{(\alpha)} \equiv 0 \), and of such numbers\( \alpha \), there is one smallest."

Cantor's discussion of the different forms of definition of a real number has already been mentioned, and need not concern us further here. Following this, he returns again to the matter of cardinal number, making the statement:

"The investigation and fixing of the cardinal number of a continuous set \( G_n \) in an \( n \)-dimensional domain reduces to the question of fixing the cardinal number of the linear continuum in the interval \((0, 1)\). I hope to soon be able to show, through a strict proof, that this cardinal number is no other than that of our second number class. From this it would follow that each infinite set of points has either the cardinal number of the first number class, or the cardinal number of the second number class."

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* Cantor says, "The proof of this theorem I will publish shortly, in consequence of a friendly challenge of my friend, Prof. Mittag-Leffler of Stockholm, in the first volume of the new mathematical journal edited by him." This journal is the Acta Mathematica, and the theorem actually appears in Vol. II, 409. See Article XXIII.

** Pg. 563-70.

*** Page 2 ff. of this paper.

**** On account of the theorem proved in Crelle, Vol. 84, and quoted on page 7 ff. of this article.

***** The theorem has never been proved—see Hobson, 224. \( c \), the cardinal number of the linear continuum may, so far as is known, be either equal to, or greater than, the cardinal number of the second number class. And even if it were known that \( c \) is equal to the cardinal number of the second number class, the next statement would not necessarily follow. Cantor seems to assume here that no cardinal number between \( a \) and \( c \) exists—see note on page 10 of this paper.
element of the set; also, after each individual element (if it is not the last), another fixed element immediately follows, and after each finite or infinite set of elements, another fixed element immediately follows. Well-ordered sets will be ascribed the same ordinal number when a simple reciprocal correspondence of the same can be set up such that, if E and F are any two elements of one set, and E' and F' the corresponding elements of the other, then the position of E relative to F in the first set is the same as the position of E' relative to F' in the second."

He continues,

"The essential difference between finite and infinite sets now appears—a finite set, no matter what order of succession we give its elements, has the same ordinal number, while on the other hand an infinite set will as a rule have different ordinal numbers according to the order of succession of the elements. The cardinal number of a set is, as we have seen, an attribute independent of the ordering; the ordinal number of a set depends on the mode of ordering, if we are dealing with infinite sets."

This discussion leads up to the following theorem:

"Each set of the cardinal number of the first class is enumerable through numbers of the second number class *, and only through such, and in fact the set can always be given such a succession of its elements that it becomes enumerable, for this particular succession, through an optional number of the second number class."

The use of transfinite ordinals also makes it possible to combine Theorems II, III, IV, and V of the preceding article—theorems having to do with enumerable sets—into one theorem:

"If P is a set whose derivative \( P^{(\alpha)} \) vanishes for some \( \alpha \) of the first or second number class, then is \( P^{(0)} \), and also P itself, an enumerable set."

The converse of this theorem also holds:

* That is, has its ordinal number or Anzahl equal to some particular number of the second number class.
Certain of the peculiar arithmetic properties of \( w \) are also commented upon. *

\[ 1+w \text{ is equal to } w, \text{ but } w+1 \text{ is not equal to } w. \]

\[ 2w \text{ is equal to } w, \text{ as is also } 1+2w, \text{ hence } w \text{ can be regarded as either an odd or an even number. } \]

\[ w^2, \text{ however, is different from } w. \]

A more detailed discussion of the addition and multiplication of transfinite ordinals, and in general a more advanced treatment of ordinals, appears in a later article. **

That an idea so novel and abstract as that of transfinite ordinals should be so completely and fully worked out by a single man in his first article on the subject is indeed impressive testimony to the genius of that man. Such testimony appears again and again in the works of Cantor, but never more convincingly than here.

The other matter of importance in this article is a discussion of cardinal number and ordinal number, leading up to several theorems.

***To each well-defined set belongs a fixed cardinal number, and two sets will be ascribed the same cardinal number if they can be put in one-to-one correspondence.

"With finite sets the cardinal number coincides with the ordinal number of the elements. With infinite sets, a fixed cardinal number, independent of their ordering, will be ascribed them.

"The smallest cardinal number of infinite sets, as is easily proved, must be ascribed to those sets which can be placed in one-to-one correspondence with the first number class.'"

He turns aside here for a moment to define a well-ordered set- ****

****"By a well-ordered set is understood a well-defined set in which the elements are joined through a certain given succession, according to which there is a first
We then postulate a second principle of generation (Any infinite sequence of numbers such that there is no greatest number shall be followed by a new number, which is ordinally greater than all the numbers of the sequence, but which has no number immediately preceding it), with whose aid we define a new number $w$, which follows all the positive integers. Now using the first generation principle, we obtain the numbers $w + 1$, $w + 2$, $\ldots$, $w + v$; and, with the aid of the second principle, the number $w + 2$. Similarly we obtain $w \cdot 3$, $w \cdot 3 + v$, $\ldots$, $w \cdot M$, $\ldots$, $w^2$, $w^3$, $w^4$, and, in general, numbers of the form $v \cdot w^M + v \cdot w^{M-1} + \ldots + v \cdot w + v \cdot k$.

where $M$, $v$, $v$, $\ldots$, $v$ can have any positive whole number values.

The second principle will give us a number $w^w$, following all such numbers, and similarly we get $w^w$, $\ldots$, and so on in never ending succession. Numbers which can be thus formed by these two principles of generation are defined as numbers of the second class (II). Cantor then postulates the existence of a new number $\Omega$, following all such numbers, and forming the first number of an even more extensive third number class. Here he also proves two theorems:

1) The set of numbers which precede any particular number of the second number class is enumerable. **

2) The complete set of numbers of the second number class is not enumerable, that is, has a cardinal number different from that of the first class. ***

**- Here Cantor uses $2w$ in this connection, but in Article XXXIV, 503, he uses $w^2$. See also Hobson, page 89. $2w$ is equal to $w$, by definition of multiplication of transfinite numbers, and hence should not be used here.

***- Hobson, 91.

****- Hobson, 92.
in the nature of a justification of, almost as he himself says, an apology for, certain innovations (such as the introduction of transfinite ordinals) which he wished to make, in the face of much adverse criticism from other mathematicians. However, the non-philosophical part of the article contains material of the highest importance—and perhaps the most important is that on transfinite ordinal numbers. Cantor says, *

"The continuation of these investigations depends on an extension of the idea of real, whole numbers—and indeed an extension which, I believe, has never been even suggested before this.

"The dependence in which I see myself placed on this extension of the number idea is so great that, without it, it would scarcely be possible for me to take the smallest step forward in the theory of sets; there might in this circumstance be found a justification of, or if necessary, an apology for, my introduction of apparently strange ideas into my discussion. For the extension in question is a matter of the continuation of the real whole number series into the realm of the infinite; perilous as this may appear, I express not only the hope but the firm conviction that this extension will, with time, come to be regarded as quite simple, suitable, and natural. But I by no means conceal that I am placing myself, with this undertaking, in a certain opposition to widespread views on mathematical infinity and to frequently represented views on the nature of the number magnitude."

He then defines the two principles of generation "with the help of which the new positive transfinite numbers will be defined".

"The formation of the finite whole numbers ** (numbers of the first number class) depends on the principal of adding unity to a number already formed. The number of the numbers of the 1st class formed in this manner is infinite, and among them is no

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* - Page 545.

** - Page 577.
here that this is perhaps the one important point in which Cantor's ideas or methods have been improved upon to any extent. The idea of the "content" of a set of points, as formulated by Cantor, Harnack, Pasch and Stolz * in particular, has been almost entirely replaced by the more useful notion of "measure", which was developed after 1901. The difference of course is merely a matter of whether or not we allow an infinite number of enclosing intervals. However, to return to Cantor's reference,

"In investigations which *messrs. Du Bois Reymond ** and Harnack *** have made in generalizing certain theorems of integral calculus, linear point sets have been used which have the quality that they can be enclosed in a finite number of intervals, so that the sum of all the intervals is optionally small.

"In order that a linear set may possess this quality it is evidently necessary that it be everywhere dense in no ever-so-small interval, yet this last condition is not sufficient to confer upon a set the mentioned property. A necessary condition is given in

Theorem VI- If a linear point set P contained in an interval (a, b) is such that its derivative P' is enumerable, then it is always possible to enclose the points of P in a finite number of intervals with a sum optionally small."

Cantor's next article (Article XIII) is a long, rather disconnected discussion, which is as much philosophical in nature as it is mathematical. A good deal of the philosophical matter would appear to be almost unintelligible to the reader with only an average knowledge of German, and can indeed be entirely omitted here--it is largely

--- The development of this idea is contained in Articles XXV, XXVI, XXXII, XXXIII, which will be discussed later.

--- This is probably in Article VIII or IX, though I was unable to find the exact place.

--- Annalen XIX, 235 to 279, especially 238 ff.
as he gives it is too long to reproduce profitably; it rests essentially on the
fact that in any ever-so-small interval real numbers appear which are not in a given
everywhere dense set. Cantor seems more interested in the philosophical side of the
theorem than in the mathematical, emphasizing particularly the possibility of con-
tinuous motion in a non-continuous space.

In Article XII appears for the first time an important set of theorems concerning
the enumerability of sets under certain conditions. Introductory to the theorems,
Cantor defines an isolated set $Q$ as one for which $D(Q,Q') = 0$. It is then not a
difficult matter to prove

Theorem I - Each isolated set is enumerable. *

Now if from any set $P$ we remove the set $D(P,P')$, the remainder $Q$ is by definition
an isolated set, and hence enumerable. That is,

$$P = Q + D(P,P').$$

Now if $P'$ is enumerable, $D(P,P')$, which is a part of $P'$, is at most enumerable, hence
$Q + D(P,P')$, and hence $P'$ is enumerable. We have then

Theorem II - If the derivative $P'$ of a set $P$ is enumerable, then $P$ is also
enumerable.

By an extension of the above method of proof we easily arrive at

Theorem III - Each point set of the first species and n-th order is enumerable.

Theorem IV - Each point set $P$ of the second species for which $P^{(n)}$ is enumerable is
itself enumerable.

Theorem V - Each point set $P$ of the second species for which $P^{(n)}$ is enumerable is
itself enumerable, $n$ being a number of the form $\omega + \alpha$

We also find here Cantor's first reference to any metrical property of a point
set, which contains the germ of the idea of the "content" of a set. We might mention

* - See Hobson, 99.
to others, merely pointing out that the ideas of limit point, derived set, everywhere-denseness, and of cardinal number, are applicable immediately to sets in domains of $n$ dimensions.

There next appears a review and continuation of the discussion of cardinal number, which leads up to the following theorem on sets of intervals, a theorem "which has many beautiful applications in the Theory of Numbers and Theory of Functions."

"In an $n$-dimensional, everywhere infinitely extended, continuous space $A$, let there be defined an infinite number of $n$-dimensional continuous sub-spaces, separate from each other, and at most adjoining on their boundaries; the set of all these sub-spaces is always enumerable."

The particular case of this theorem for which $n$ is equal to 1 is most important in the study of linear sets of points. The proof for this case is very simple, * and yet it can be made the basis for the proof of the general theorem. ** Cantor proves the general theorem first, and in a manner a little different from, and more difficult than, that used by Hobson, though the latter seems to be only an adaptation of the former.

In this article Cantor also proves another remarkable and interesting theorem which does not appear in Hobson or in any other article which I read. Let us consider a point set $(M)$, which is everywhere dense in an $n$-dimensional, continuous, connected domain $A$, and which is also an enumerable set. "Let us conceive the set $M$ removed from the domain $A$, and the then remaining domain denoted by $U$, then the remarkable theorem exists that, for $n \geq 2$, the domain $U$ does not cease to be continuously connected, that is, two points, $N$ and $N'$ of the domain $U$ can always be joined by a continuous line which belongs, with all its points, to $U$, so that it contains no points of $M$." The proof

* See Hobson, 97.

** See Hobson, 98.
derivative of \( P \) of order \( \omega \). If this set has a derivative, we denote it by \( P^{(\omega)} \), and by using these two methods of forming derivatives, we obtain derived sets of the general form:

\[
P^{(\omega^r + m, s^{-1} + \ldots)}
\]

where all the \( n \)'s are finite, and \( \omega \) may be finite or infinite. *

For sets of the first species, \( P^{(\omega)} = 0 \), and also, if \( P^{(\omega)} = 0 \), the set is of the first species. Hence the relation \( P^{(\omega)} = 0 \) completely characterizes a set of the first species.

Cantor also indicates a simple method of forming a set of the second species, for which \( P^{(\omega)} \) is equal to a single point. This particular interval is also in no interval everywhere dense, thus answering in the negative a question already raised in Article VII as to whether a set of the second species is necessarily everywhere dense. **

In Article XI, Cantor first discusses the possibility of extending the ideas and theorems already defined for linear sets to sets in 2, 3, or \( n \) dimensions. "It can be assumed beforehand", he says, "that most of the qualities and relations appearing in connection with linear sets can be proved, with obvious modifications, for sets in 2, 3, or \( n \)-dimensional domains." And he is satisfied to leave most of this extension **

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* This discussion precedes anything on transfinite ordinals. The logical order (and that followed by Hobson) would seem to be one in which transfinite ordinals were introduced first.

** Several examples of sets of the second species are given in Article XXVIII, (Mittag-Leffler), pg. 56ff, and most of these are quoted in Hobson, 96. We have here a set for which \( P^{(\omega)} \) consists of the single point zero, one for which \( P^{(\omega^\omega)} \) consists of a finite number of points, and one for which \( P^{(\omega)} \) exists. This last should be written \( P^{(\omega^2)} \), as \( 2w \) (by definition of the rule for multiplying with transfinite ordinals) \( \geq w \), while \( w \cdot 2 \neq w \). Hobson writes this correctly when he quotes the example.
The cardinal number of a set is also discussed here, in a manner similar to that of the previous discussion of Article VI. He considers again enumerable sets, giving as examples the aggregates of all rational and of all algebraic numbers, and sets of the power of the continuum, and concludes, "Whether these two classes are the only ones in which point sets fall, shall not yet be examined. On the other hand, we shall prove that these two are in reality different classes." Then follows again the proof of Article IV, page 260, and of Hobson, page 81.

Article X, a continuation of VIII, was published in 1880. It is concerned largely with the introduction of certain symbols and definitions, whose purpose is to facilitate the statement of later theorems, etc. Thus we find introduced the expressions \( P \leq Q \), \( P \equiv \{P_1, P_2, P_3, \ldots \} \), \( M \{P_1, P_2, \ldots \} \), \( D \{P_1, P_2, \ldots \} \), whose meaning we need not stop to discuss. Also he adopts 0 as a symbol to express the absence of points, without adopting one to use in case a set consists of the single point 0 or zero. The equivalence of two sets is expressed by \( P \sim Q \).

This article also contains the introduction of the idea of transfinite derivatives. "If \( P \) belongs to the second species, then will \( P' \) be composed of two essentially different sets, \( Q \) and \( R \), so that \( P' \equiv \{Q, R\} \), of which \( Q \) consists of those points of \( P \) which are lost by proceeding far enough in the series \( P', P'', P''', \ldots \); the other, \( R \), comprises those points which remain contained in all members of the series \( P', P'', P''', \ldots \)." R then is defined through the relation:

\[
R \equiv D(P', P'', P''', \ldots)
\]

This set \( R \) will now be expressed through the symbol \( P^{(\omega)} \) and will be called the

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He later replaces this by \( P \equiv P_1 + P_2 + P_3 + \ldots \). See Article XII, 51.

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He uses script letters for the \( M \) and \( D \), which seems unnecessary.

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This symbol is later replaced by \( P^{(\omega)} \). Even in the translation of this Article which appears in Acta II (1883) (Article XIX), he uses \( P^{(\omega)} \), though in Annalen XXI (1881) (Article XII), he still uses \( P^{(\omega)} \).

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later to the exact solution of this question." *

With Article VII, we come to the first discussion devoted solely to point sets. "punktmannigfaltigkeiten", Cantor called them first, a term hardly designed to attract readers. He later adopted the more convenient term "punktmenge". The material of this article is largely a collection of what had already appeared in II, IV, and VI. The derivative idea is again explained, and is extended to cover the case where "the series of derived sets of \( P \), that is, the series \( P', P'' \), ..... continues without end." In this case we call \( P \) a set of the second species. He also points out "that all points of \( P'' \), \( P''' \), ..... are also points of \( P' \), while a point belonging to \( P' \) is not necessarily a point of \( P'' \), though he does not as yet use this quality to define a closed set. A set everywhere dense in an interval \((a, b)\) is here defined for the first time, and the fact that such a set \( P \) has for its derivative (or at least for a part of its derivative) all the points of \((a, b)\) is mentioned. The conclusion is then drawn that a set of the first species cannot possibly be everywhere dense as an everywhere dense set is clearly of the second species. The question as to whether, conversely, a set of the second species must be everywhere dense is, for the time being, left open. **

* - A statement similar to this appears perhaps half a dozen times in Cantor's writings, sometimes with the note that he had not yet proved it, and sometimes as though it had been proved. The statement has been proved to be true for a very large class of linear point sets (see note on page 28 below), but has not been proved for linear sets in general.

** - Cantor's term for "everywhere dense" is "überalldicht". Du Bois Raymond uses "pantaschische" or "apantaschische" in a similar sense. (See Article VIII, 287-8). "i call a 'pantaschische' distribution of a point set in an interval such a distribution that in each smallest extent of the interval, points which belong to the set are met with." (See also Article IX, 127-8).
We have then to prove the theorem—

II A variable number $e$ which can take on all irrational values in the interval $(0, 1)$ can be put in one-to-one correspondence with a variable number $x$, which can take on all real values in the interval $(0, 1)$, so that to each value of $e$, one and only one value of $x$ corresponds, and conversely.

From this our original theorem follows, a theorem which simplifies point set theory immensely; in that in many cases we can replace a p-dimensional set by a linear set without loss of generality.

In this article also appears the proof of the theorem that the aggregate of all rational numbers (in particular those between 0 and 1, including 0 and 1) is enumerable. This of course follows from the theorem, already proved, that all real algebraic numbers form an enumerable aggregate, but the proof here is different, and the theorem deserves a place of its own. In Hobson, it precedes the other, more general theorem, which is probably the better order.

As has already been mentioned, this article (VI) is concerned with Sets of Real Numbers, and not with Sets of Points. However, two references appear which indicate that Cantor was at least beginning to formulate his theory of point sets. On page 248 he defines a "linear" aggregate of real numbers as one "consisting of real, different numbers, so that one and the same number appears in each linear aggregate not oftener than once as an element." And at the close of the article he states,

"We now consider the question "into what classes linear sets are divided, and what is the number of these classes, if we group in different classes sets of different cardinal number, and in the same class sets of the same cardinal number." The number of such classes, he concludes, is "equal to two". "There are then, with linear sets, only two different cardinal numbers, corresponding to these two classes; we return

*— See Hobson, 80.
To prove this theorem, we start with the known theorem that each irrational number greater than 0 and less than 1 can be represented by an infinite continued fraction,

\[ e = \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \cdots}}} = \left(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\right) \]

where \(\alpha_1, \alpha_2, \ldots, \alpha_n\) are positive, whole numbers.

To each positive irrational number \(e\) greater than 0 and less than 1 corresponds a fixed infinite series of positive whole numbers \(\alpha_1, \alpha_2, \ldots, \alpha_n\); and conversely, each such series fixes a certain irrational number \(e\) between 0 and 1.

Now let \(e_1, e_2, \ldots, e_m\) be \(n\) independent variable magnitudes of which each can take on all irrational values in the interval \((0, 1)\), and each of these only once. * We put

\[ e_1 = \left(\alpha_{11}, \alpha_{12}, \alpha_{13}, \ldots, \alpha_{1n}, \ldots\right) \]

\[ e_m = \left(\alpha_{m1}, \alpha_{m2}, \alpha_{m3}, \ldots, \alpha_{mn}, \ldots\right) \]

These \(n\) irrational numbers determine an \(n+1\)st irrational number \(d\) between 0 and 1,

\[ d = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n, \ldots) \]

where we assume the following relations between the numbers \(\alpha\) and \(\beta\) -

\[ (1) \quad \beta_{(n+1)m+n} = \alpha_{nm} \left(\frac{\alpha_{n+1}, \ldots, \alpha_{m}}{\beta_{n+1}, \ldots, \beta_{m}}\right) \]

But also inversely if we start with an irrational number \(d\), we fix the series \(\theta_1\), and consequently fix the series of irrationals \(e_1, e_2, \ldots, e_m\), again using relation (1). From this theorem I results. ***

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*** This restriction is certainly incorrect. In Acta II-316 the theorem is correctly stated, omitting the "and each of these only once".

***-d is therefore defined by \((\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{m-1})\).

**** This theorem appears in Hobson, 84.
then the same cardinal number belongs to two sets only when the number of their
elements is equal. And if one set is part of another, its cardinal number is less
than the cardinal number of the other. But when we come to infinite sets, if "an
infinite series M is a part of another N, it can by no means be concluded that its
cardinal number is less than that of N. This conclusion is only justified when we
know that the cardinal number of M is not equal to that of N." He then redefines
enumerable sets, and states the two following theorems:

"If M is a set of the power of the positive, whole numbers (an enumerable set),
then each infinite part of M has the same power.

"If M', M'', M''', .... is a finite or simply infinite series of sets, each of
which is enumerable, then the set M, which results from joining M', M'', M''', ....
is also enumerable."

The principal part of this article, however, is devoted to the proof of the
theorem that the p-dimensional continuum has the same cardinal number as the one-
dimensional continuum. He states the theorem as follows:

"If x_1, x_2, .... x_n are n independent, variable real numbers, each of which
can assume all values in the interval (0, 1) *, and if t is another variable with the
same scope, then it is possible to make correspond the one magnitude t to the system
of n magnitudes x_1, x_2, .... x_n, so that to each fixed value of t corresponds a
fixed value system x_1, x_2, .... x_n, and conversely."

This theorem depends on two others, the first of which is:

I If e_1, e_2, .... e_n are n independent variables of which each can take on all
irrational values in the interval (0, 1), and if d is another variable with the same
scope, then it is possible to put in one-to-one correspondence the one magnitude d
and the system of n magnitudes, e_1, e_2, .... e_n.

** The more general theorem follows at once from this. See Hobson, 83, for a
transformation that will effect the generalization.
number with \( N > 1 \), let follow after it the two real algebraic numbers with \( N = 2 \), denoting these with \( w_1 \) and \( w_2 \), etc." 

This theorem can of course be immediately extended to cover the case of all algebraic numbers, real or complex. **

The second theorem in this article is that the aggregate of all real numbers in a given interval is not enumerable. The proof consists in assuming an ordering of the real numbers of the given interval in the form \( w_1, w_2, \ldots \), and then proving that other real numbers must exist in the interval (these being limit points of sets of \( w \)-points) which cannot appear in our original series. ***

Article VI is practically a continuation of IV. Here we are first introduced to the idea of cardinal number. "When two well-defined sets can be placed in one-to-one correspondence, simply and completely, we shall say that these sets have equal cardinal number, or that they are equivalent." **** When the sets under consideration are finite, the idea of cardinal number corresponds to that of number; *****

** The first few terms of the series are 0, 1, -1, 2, -2, \( \frac{1}{2} \), \(-\frac{1}{2} \), 3, -3, 1/3, -1/3, \( \frac{1}{3} \sqrt{2} \), \(-\frac{1}{3} \sqrt{2} \), \( \sqrt{2} \), \( 1 + \sqrt{3} \), \( 1 - \sqrt{3} \), \(-1 + \sqrt{3} \), \(-1 - \sqrt{3} \), etc. When he says \( \Phi (2) = 2 \), Cantor means that there are two numbers whose rank is 2 which we do not already have in our series; similarly \( \Phi (3) \). There are actually 7 real numbers which are roots of equations of rank 3, but 3 of them already appear in our series.

** This proof is reproduced in Hobson, pg. 80, with the exception that \( N \) is there put equal to \( n + |a_1| + |a_2| + \ldots + |a_N| \) and we are told to let \( N \) equal 3, 4, 5, \ldots successively. Hobson appears to desire to exclude 0 from the list, since if \( N \) is 2, we obtain the equation \( x^2 = 0 \), with root 0.

*** This proof also appears in Article VII, 5, and in Hobson, 81.

**** This idea is discussed further in many later articles, as we shall see.

***** Anzahl.
"By a limit point of a set P, I understand a point so placed that in each neighborhood of the same are an infinite number of points of P."*

Again,

"Accordingly it is easy to prove that a set consisting of an infinite number of points has at least one limit point." Here he omits the necessary restriction that the set be bounded.**

Otherwise the definitions of limit point and of derived sets of the 1st, 2nd, ...... n-th, order, are just as we have them today. The development of transfinite derivatives comes in a later article.

In Volume 77 of Crelle (Article IV) Cantor proves two theorems in connection with real numbers which are of fundamental importance in the theory of point sets. First, the aggregate of real algebraic numbers is shown to be enumerable, that is, they can be placed in one-to-one correspondence with the positive integers. To prove this, we consider an equation-

\[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0, \]

where all the a's are positive or negative integers. Then put

\[ N = n + |a_n| + |a_{n-1}| + \cdots + |a_1|, \]

where we may define N as the rank *** of the equation. Cantor continues,

"To each positive whole number value which we may assign N, there are only a finite number of real algebraic numbers with the rank N; the number of these may be denoted by \( \Phi(N) \)-for example, \( \Phi(1) = 1 \), \( \Phi(2) = 2 \), \( \Phi(3) = 4 \). Then the numbers of the aggregate can be ordered as follows; take as the first number \( w \), the one

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**- It is of course unnecessary to postulate an infinite number of points. In Article XI, pg. 114, he omits "an infinite number of" from his definition.

***- In Article XI, pg. 114, he adds this restriction.

***- Höhe.
and is equal to b."

In this same article*, Cantor also explains the other two definitions of a real number, those of Weierstrass and Dedekind. These should be read if only to compare them with Cantor's. Also in this connection, Articles I and especially III of the bibliography might well be read.

To return once more to the first article, Cantor continues,

"We now differentiate the points of a straight line by specifying their distances from a fixed point 0, with the + or - sign, according as the point in question lies in the positive or negative part of the line.

"If this distance has a rational ratio to unity, then it will be expressed through a number magnitude of the aggregate A; otherwise it is always possible to specify a series \( a_1, a_2, \ldots, a_n, \ldots \) which has the quality already expressed (that is, it is a Fundamentalreihe), and which has such a relation to the distance in question that the points of the line to which the distances \( a_1, a_2, \ldots, a_n, \ldots \) belong press infinitely near to the fixed point as \( n \) increases.

"This we express as follows—the distance from 0 to the fixed point is \( b \), where \( b \) is the real number corresponding to the series \( a_1, a_2, \ldots, a_n, \ldots \).

"To make complete the connection between the aggregate of real numbers and the points of the straight line, we must add an axiom—that also inversely to each number magnitude, a fixed point of the line belongs whose coordinate equals that number magnitude."

The particular idea concerning point sets which Cantor develops here is that of limit points and derived sets. As this is the first presentation of the matter, it is not surprising that a few of his statements here are modified slightly. For instance, he says,
smaller than a negative rational number.

"To express the existence of these three relations, we place respectively
\[ b = a, \quad b > a, \quad b < a. \]

"From these definitions it results as a consequence that if \( b \) is the limit of
the series \((l)\), then \( b - a_n \) becomes infinitely small as \( n \) increases, so that the
expression 'limit of the series \((l)\)' for \( b \) finds a certain justification.

"The totality of magnitudes \( b \) may be denoted by \( B \)."

In this manner Cantor first defines irrational numbers, and arrives at the
aggregate \( B \) of real numbers. Some such definition is essential before starting
the study of sets of points, since as Cantor says, when we speak of points, we
really have real numbers in mind.

A clearer and revised presentation of Cantor's definition of real numbers ap­
pers in Article XIII, pg. 567-8. Each set of rational numbers with the property
assumed above he calls a "Fundamentalreihe", and he "joins to it a number \( b \) defined
by it". He then shows that \( b \) can be \( =0, >0, <0 \), and after defining \( b \pm b', b-b' \),
through the proper combination of two fundamental series, finally states the con­
ditions for one real number being equal to, greater than, or less than, a second
real number. He then concludes,

"After all these preparations it can be proved that, when \( b \) is the number fixed
through a fundamental series \((a_\nu)\), then \( b - a_\nu \) (with increasing \( \nu \)), becomes smaller in
absolute value than each conceivable rational number, that is,
\[ \lim_{\nu \to \infty} a_\nu = b \]

"We must notice this in particular—the number \( b \) is not defined as the limit of
the terms \( a_\nu \) of a fundamental series, for this would involve the logical defect of
assuming the existence of the limit \( \lim_{\nu \to \infty} a_\nu \); rather we go at the matter inversely,
so that we start with our preceding definition of \( b \), with certain qualities and re­
lations to the rational numbers, and from this draw the inference that \( \lim_{\nu \to \infty} a_\nu \) exists

\( \nu \to \infty \)
have the quality that the difference $a_{n+1} - a_n$ as $n$ increases, becomes infinitely small, whatever the positive whole number $m$ may be; or, in other words, that for any arbitrary pre-assigned $\varepsilon$ (positive and rational), there exists a whole number $n_1$, such that $(a_{n+1} - a_n)^2 < \varepsilon$, when $n \geq n_1$, and when $m$ is an arbitrary positive whole number.

"This quality of such a series I express in the words, 'The series $a_1, a_2, \ldots$ has a fixed limit $b$.'"

"These words have no other meaning than the expression of that particular quality, and just as we unite with the series (1) a particular symbol $b$, it follows that by other such series, other symbols $b'$, $b''$, etc. are built.

"If a second series, $a'_1, a'_2, \ldots a'_n$, is given, which has a fixed limit $b'$, then we find that the two series (1) and (1') always have one of the three following relations, which are mutually exclusive. Either (1) $a_{n+1} - a'_n$ becomes infinitely small as $n$ increases, or (2) $a_{n+1} - a'_n$ remains, from some fixed $n$ on, greater than a positive rational number $\varepsilon$, or (3) $a_{n+1} - a'_n$ remains, from some fixed $n$ on, less than a negative rational number $\varepsilon$.

"When the first relation holds, I place $b = b'$, for the second, $b > b'$, for the third, $b < b'$.

"Also we find that a series (1) which has a limit $b$ has, to a rational number $a$, only one of the three following relations. Either 1) $a_{n+1} - a$ becomes infinitely small as $n$ increases, or 2) $a_{n+1} - a$ remains, from some fixed $n$ on, always greater than a positive rational number $\varepsilon$, or 3) $a_{n+1} - a$ remains, from some fixed $n$ on, always

**- The author certainly means $|a_{n+1} - a_n|<\varepsilon$.

**- In his later article XIII, pg. 568, Cantor takes a different point of view.

b is not defined as the limit, in fact the theory of limits is represented as being deduced from the theory of irrational numbers.
HISTORICAL DEVELOPMENT OF THE THEORY OF SETS OF POINTS

The history of the development of the Theory of Sets of Points until 1901 is essentially a review of the writings on the subject of one man, Georg Cantor. We find an occasional article on point sets by Bendixson, Phragmén, Stolz or Harnack, and a reference now and then to Du Bois Reymond, Mittag-Leffler, Pasch, and a few others, but practically everything was first due to Cantor.

And Cantor's first article did not discuss point sets for their own interest or importance, but only as a means toward proving a certain trigonometrical theorem. This article appeared in 1871. (See Article II) And a little later (1873-78) he published in Crelle's journal several articles on the Theory of Sets or Aggregates, with however practically no reference to point sets. We may surmise that it was not long before he began to realize the possibility of applying the method of his first article to this different subject matter, and to translate his Theory of Sets of Real Numbers into the clearer and more powerful Theory of Point Sets. In fact as soon as 1879 he published the first of the set of six articles, "Über unendliche, lineare Punktmannigfaltigkeiten", which contain practically all the fundamentals of the subject.

Let us go back for a moment to Cantor's first article on the subject. (II). He writes as follows—*

"The rational numbers form the foundation of wider ideas of a number magnitude; I will call them the aggregate A, zero being excluded.

"In this discussion of number-magnitudes in the wider sense, we shall be concerned with infinite series of rational numbers of the form a1, a2, ........, which

*— Page 123 ff.
M. Pasch- "Über einige Punkte der Functiontheorie".
Annalen-30- 132 to 154 (1887).
Pages 142-44 only of this article; "Inhalt einer Punktmenge".

G. Cantor- "Beiträge zur Begründung der transfiniten Mengenlehre", - I
Annalen-46-481 to 512 (1895).

G. Cantor- Same title as XXXIV- II.
Annalen-49-207 to 246 (1897).
G. Cantor - "Une Contribution a la Theorie des Ensembles".
Acta-2-311 to 328 (1883).
A translation of VI above.

G. Cantor - "Sur les series trigonometriques".
Acta-2-329 to 335 (1883).
A translation of Annalen-4-139 ff. which is not essential in itself, but merely leads up to Article XVII below.

G. Cantor - "Extension d’une theoreme de la Théorie des series trigonometriques".
Acta-2-336 to 348 (1883).
A translation of II above.

G. Cantor - "Sur les Ensembles infinis et linears de points"-I,II,III
Acta-2-345 to 356, 356 to 360, 361 to 371, 371 to 380 (1883).
Translations of VII, X, XI, XII above.

G. Cantor - "Fondements d’une Theorie generale des Ensembles".
Acta-2-381 to 408 (1883).
A translation of XIII above.

G. Cantor - "Sur divers theorems de la Théorie des Ensembles de Points".
Acta-2-409 to 414 (1883).

I. Bendixson - "Quelques theorems de la Théorie des Ensembles de Points".
Acta-2-415 to 429 (1883).

G. Cantor - Same title as VII-VI.
Annalen-23-453 to 458 (1884).

Stolz - "Uber einen zu einer unendliche Punktmengen gehörigen Grenzwerth".
Annalen-23-152 to 156 (1884).

G. Cantor - "De la Puissance des Ensembles parfaits de Points".
Acta-4-381 to 392 (1884).

G. Mittag-Leffler - "Sur la representation analytique des fonctions monogènes uniforme d'une variable indépendante".
Acta-4-1 to 79 (1884), especially page 56 ff.

E. Phragmen - "Beweis eines Satzes aus der Mannigfaltigkeitslehre".
Acta-5-47 to 48 (1884).

G. Cantor - "Über verschiedene Theoreme aus der Theorie der Punctmengen".
Acta-7-105 to 124 (1885).

E. Phragmen - "Über die Begrenzungen von Continua".
Acta-7-43 to 48 (1885).

A. Harnack - "Über den Inhalt von Punktmengen".
Annalen-25- 241 to 250 (1885).