Topological rings

Merle Eugene Manis

*The University of Montana*

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TOPOLOGICAL RINGS

by

MERLE EUGENE MANIS

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Chairman, Board of Examiners

Dean, Graduate School

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M. E. M.
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INTRODUCTION

Topological rings have applications in several branches of mathematics, but they are also interesting in their own right. The theory of topological rings has been developed quite extensively, but largely within the more general concept of a topological module. In this paper, the concept of a module is ignored, and some of the elementary properties of topological rings are exhibited.

The results obtained are standard, but some are relatively inaccessible without a broader mathematical background than is assumed. A few properties common to all topological rings are exhibited in the second section, but the paper is largely concerned with topological rings of a more particular type: rings whose topology is generated by a decreasing sequence of ideals. We call this topology the $M_i$-topology. Theorem 2.14 states that this topology is a pseudo-metric topology and Theorem 2.13 exhibits a pseudo-metric.

Section 3 deals with the completion of a ring with $M_i$-topology whose topology is metric. The method used is essentially that used in completing the real numbers.

Section 4 deals with some elementary properties of the completion of a ring with $M_i$-topology.

It is assumed that the reader is familiar with basic
concepts of algebra as found in Jacobson\(^1\)\(^2\). We assume that rings are commutative; otherwise, the definition of ring, ideal, quotient ring, isomorphism, homomorphism, etc., will be as in [2]. Less familiar concepts are defined below or when needed.

**Definition 1.1:** An ideal \(A\) of a ring \(R\) is said to be finitely generated in case there exist elements \(x_1, x_2, \ldots, x_s\) in \(A\) such that any element \(y\) in \(A\) is of the form \(\sum r_i x_i\), \(r_i\) in \(R\). The \(x_i\)'s are said to generate \(A\), and we write \(A = (x_1, x_2, \ldots, x_s)\).

**Definition 1.2:** The \(n\)th power of an ideal \(M\) of a ring \(R\), denoted by \(M^n\), is that subset of \(R\) consisting of all finite sums of elements of the form \(x_1 x_2 \cdots x_n\), where \(x_i\) is in \(M\).

It is easily verified that \(M^n\) is an ideal contained in \(M\).

We assume the basic definitions of topological concepts such as subspace, product space, metric space, neighborhood, limit point, closure, etc., as are found in Hall and Spencer [1]. Some important definitions and theorems are stated below. Proofs of the theorems may be found in [1].

**Definition 1.3:** Let \(S\) be a point set and \(T\) a collection of subsets of \(S\) called open sets. Then \(S\) is said to be a topological space with topology \(T\) in case the collection \(T\) satisfies the following conditions:

1) Every point of \(S\) is contained in at least one subset of \(T\).

1. Numbers in square brackets refer to the numbers of the references cited.
2) The union of any collection of elements of $T$ is an element of $T$.

3) The intersection of any finite number of elements of $T$ is an element of $T$.

Definition 1.4: Let $S$ be a point set, $B$ a collection of subsets of $S$, and $T$ the collection of all subsets of $S$ that may be obtained by unions of sets in $B$. Then $B$ is said to be a basis for, or to generate, $T$.

Theorem 1.5: Let $S$ be a point set and $T$ a collection of subsets of $S$ called open sets. Let $B$ be a basis for $T$. Then $S$ is a topological space with topology $T$ if and only if the following conditions hold:

1) Every point of $S$ is contained in at least one element of $B$.

2) Given $U, V$ in $B$, and any point $p$ in $U \cap V$, there is an element $W$ of $B$ such that $p \in W \subset U \cap V$.

Definition 1.6: A topological space $S$ is said to be: $T_1$ if points of $S$ are closed sets of $S$; $T_2$, or Hausdorff, if given any two points $p$ and $q$ of $S$, there are disjoint open sets $U$ and $V$ of $S$ which contain $p$ and $q$ respectively; $T_3$, or regular, if given any closed set $F$ of $S$ and any point $p$ not in $F$, there are disjoint open sets $U$ and $V$ of $S$ which contain $F$ and $p$ respectively.

Definition 1.7: A mapping $f$ of a topological space $S$ into a topological space $H$ is said to be continuous in case the inverse image of any open set of $H$ is open in $S$. 

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Theorem 1.8: A necessary and sufficient condition that a mapping \( f \) of \( S \) into \( H \) be continuous is that \( f^{-1}(U) \) be open in \( S \) for every basis element \( U \) of \( H \).

Definition 1.9: A mapping \( f \) of \( S \) into \( H \) is said to be open in case the image of every open set of \( S \) is open in \( H \).

Definition 1.10: Let \( A \) and \( B \) be subsets of a ring \( R \). Then \( A \pm B \), \( AB \), and \( A^{(2)} \) will denote, respectively, the subsets \( \{a+b \mid a \in A, b \in B\} \), \( \{ab \mid a \in A, b \in B\} \), and \( \{a \bar{a} \mid a, \bar{a} \in A\} \).
II

TOPOLOGICAL RINGS

Definition 2.1: Let $R$ be a ring and also a topological space. If the mappings $F(x,y) = x - y$ and $G(x,y) = xy$ of the product space $R \times R$ into $R$, are continuous, then $R$ is said to be a topological ring.

Clearly, every ring is a topological ring with the discrete topology. More interesting examples are the ring of real numbers in the "interval" topology and the ring of complex numbers in the topology of the plane. The purpose of this section is to derive some elementary properties of topological rings and to consider some specific types of topological rings.

Theorem 2.2: Let $R$ be a topological ring. Then the mappings $f, g, \text{ and } h$, where $f(x) = b + x$, $g(x) = b - x$, $h(x) = bx$, are continuous for any fixed $b$ in $R$. The mappings $f$ and $g$ are open.

Proof: We will prove the theorem only for $f$, the proofs for $g$ and $h$ being analogous. Let $U$ be any open set of $R$, $F(x,y) = x - y$. The set $F^{-1}(U)$ is open since $F$ is continuous, thus is of the form $\bigcup \limits_{\alpha \in K} U_\alpha \times V_\alpha$, $K$ some index set, where $U_\alpha$, $V_\alpha$ are open sets of $R$. Let $L = \{ \alpha \} - b$ is in $V_\alpha$, $\alpha$ in $K$. If $x$ is in $\bigcup \limits_{\alpha \in L} U_\alpha$, then $f(x)$ is in $U$. Conversely, if $f(x)$ is in $U$, then $(x, -b)$ is in $F^{-1}(U)$ and $x$ is in $\bigcup \limits_{\alpha \in L} U_\alpha$. Thus $f^{-1}(U) = \bigcup \limits_{\alpha \in L} U_\alpha$, and $f$ is continuous. Since $f(U) = f'^{-1}(U)$, where $f'(x) = x + (-b)$, the mapping $f$ is open.
As a direct consequence of the fact that the above mappings \( f \) and \( g \) are open and continuous, we have the following corollary:

**Corollary 2.3:** Let \( R \) be a topological ring, \( b \) any element of \( R \). Then \( b+U \) and \( b-U \) are open if \( U \) is open, closed if \( U \) is closed.

**Definition 2.4:** A collection \( B(a) \) of neighborhoods of a point \( a \) in a topological space \( S \) with topology \( T \), is said to be a basis at \( a \) if and only if, given any neighborhood \( U \) of \( a \) is \( S \), there exists a \( V \) in \( B(a) \) such that \( V \) is contained in \( U \).

Clearly, a collection \( B \) of subsets of \( S \) is a basis for the topology \( T \) of \( S \) if and only if \( B \) contains a basis at every point of \( S \).

**Theorem 2.5:** Let \( R \) be a ring with topology \( T \) such that \( U+a \) is open for any \( a \) in \( R \), \( U \) in \( T \). If \( B(0) \) is a basis at zero, then \( B(a)=\{U+a \mid U \text{ in } B(0)\} \) is a basis at \( a \) and \( B=\{U+a \mid U \text{ in } B(0), a \text{ in } R\} \) is a basis for \( T \).

**Proof:** Let \( B(0) \) be a basis at zero and \( U \) an open set containing \( a \). Then \( U-a \) contains zero, hence a set \( V \) of \( B(0) \). But then \( V+a \) is a set of \( B(a) \) which is contained in \( U \); thus \( B(a) \) is a basis at \( a \).

Corollary 2.3 and Theorem 2.5 indicate that when studying the properties of a topological ring \( R \), it is often only necessary to consider basis sets containing zero. This is exemplified in the following theorem.
Theorem 2.6: Let $R$ be a ring and $B(o)$ a collection of subsets of $R$ each containing zero. Let $B$ be as defined above and $T$ the collection of subsets of $R$ generated by $B$. Then $R$ is a topological ring with topology $T$ and $B$ is a basis at zero if and only if the collection $B(o)$ satisfies the following conditions:

1) The intersection of any two sets of $B(o)$ contains a third set of $B(o)$.

2) If $U$ is any set of $B(o)$, then there is a set $W$ in $B(o)$ such that $W-W$ and $W^{(2)}$ are in $U$.

3) If $U$ is any set of $B(o)$, $a$ any element of $U$, $b$ any element of $R$, then there exists a set $W$ in $B(o)$ such that $W+a$ and $Wb$ are contained in $U$.

Proof: Suppose first that $R$ is a topological ring with topology $T$ and $B(o)$ a basis at zero. Trivially, $B(o)$ must satisfy condition 1). If $U$ is any set of $B(o)$, $a$ any element of $U$, the intersection $U \cap U-a$ of basis elements contains $0$ and therefore must contain a set $W'$ of $B(o)$. But then $W'+a$ is contained in $U$.

Let $f(x)=xb$, $U$ be in $B(o)$. Then $f^{-1}(U)$ is open since $f$ is continuous, it contains $0$; thus contains a set $W''$ of $B(o)$. Then $W''b$ is contained in $U$. The intersection $W'' \cap W'$ must contain a set $W$ of $B(o)$, and this $W$ satisfies condition 3).

Let $F(x,y)=x-y$, $G(x,y)=xy$. Since the mappings $F$ and $G$ are continuous, the sets $F^{-1}(U)$ and $G^{-1}(U)$ must be open.
in $\mathbb{R}^{RXR}$ for $U$ in $B(o)$, thus must be of the form $S=\bigcup_{\alpha}(U_{\alpha}XV_{\alpha})$, $K$ some index set, $U_{\alpha}$ and $V_{\alpha}$ sets of $B$. Since $(0, 0)$ is in $S$, $S$ must contain a set $U_{\alpha}XV_{\alpha}$, where $U_{\alpha}$ and $V_{\alpha}$ are sets of $B(o)$. A set $W$ of $B(o)$ contained in the intersection $U\cap U_{\alpha}\cap V_{\alpha}$ (for both $F^{-1}$ and $G^{-1}$) must satisfy condition 2).

Suppose that $B(o)$ satisfies 1), 2) and 3). We will show that $B$ generates a topology and $B(o)$ is a basis at zero. Clearly $B$ satisfies condition 1) of Theorem 1.5. Let $c$ be in $(U+a)\cap(V+b)$, $U, V$ in $B(o)$. Without loss of generality, we can assume $c=0$. Then $-a$ is in $U$, $-b$ is in $V$ so that by condition 3), there are sets $W_1, W_2$ in $B(o)$ such that $W_1 \subset U$, $W_1- aCU$ and $W_2 \subset CV$, $W_2- bCV$. By condition 1) there is a set $W$ of $B(o)$ contained in $W_1 \cap W_2 \subset W_1 \cap W_2=((W_1-a)+a)\cap ((W_2-b)+b) \subset (U+a)\cap(V+b)$. Thus $B$ generates a topology by Theorem 1.5.

Any neighborhood of zero must contain a basis set $U+a$, $U$ in $B(o)$, that contains zero. But then $-a$ is in $U$, and there is a set $V$ in $B(o)$ such that $V \subset U$ and $V- aCU$. Then $V=(V-a)+a \subset U+a$; thus $B(o)$ is a basis at zero.

Once it is shown that the mappings $F$ and $G$ above are continuous in this topology, the theorem will be proved.

Suppose $U$ is in $B(o)$ and that $x-y$ is in $U$. By hypothesis there exist sets $V_1$ and $V_2$ in $B(o)$ such that $V_1+x-y$ is in $U$ and $V_2-x$ is in $V_1$. Then $V_2-x=(V_2+x)-(V_2+y) \subset U$ and thus $(V_2+x)(V_2+y)$ is in $F^{-1}(U)$. Let $I=\{\alpha \mid x_{\alpha}-y_{\alpha}$ is in $U\}$, $V_{\alpha}$ a set derived as $V_2$ is above. Then $F^{-1}(U)=\bigcup_{\alpha \in I}(V_{\alpha}+x_{\alpha})(V_{\alpha}+y_{\alpha})$.
and $F^{-1}(U+a) = \bigcup_{\alpha \in \Lambda} (V_\alpha + x_\alpha + a)(V_\alpha + y_\alpha)$. Since the set is open, the mapping $F$ is continuous.

Suppose $U$ is a set of $B(o)$ and $xy$ is in $U+a$. Then $xy = u+a$, $u$ in $U$. By hypothesis, there exist sets $V_i$, $i=1,2,\ldots,6$, in $B(o)$ such that: $V_1 + u$ is in $U$; $V_2 + V_2$ is in $V_1$; $V_3 + V_3$ is in $V_2$ (in particular $V_3 + V_3 + V_3$ is in $V_1$); $V_4 x$ is in $V_3$; $V_5 y$ is in $V_3$; $V_6^{(2)}$ is in $V_3$.

Let $W$ be the intersection of $V_4$, $V_5$ and $V_6$. Then $W^{(2)}+Wx+Wy$ is in $V_1$, and $W^{(2)}+Wx+Wy+u+a = W^{(2)}+Wx+Wy+xy = (W+x)(W+y) \subseteq U+a$. Thus $(W+x)X(W+y)$ is in $G^{-1}(U+a)$. Let $L = \{ \alpha | x_\alpha y_\alpha \text{ is in } U+a \}$, then $G^{-1}(U+a) = \bigcup_{\alpha \in \Lambda} (W_\alpha + x_\alpha)X(W_\alpha + y_\alpha)$, where $W_\alpha$ corresponds to $W$ above. Since $G^{-1}(U+a)$ is open, $G$ is continuous. Since both $F$ and $G$ are continuous, $R$ is a topological ring.

Theorem 2.7: Let $R$ be a topological ring and $B(o)$ a basis at zero. Let $A = \bigcap_{U \in B(o)} U$. Then $A$ is an ideal of $R$.

Proof: Let $a$ and $b$ be two elements of $A$. For any $U$ in $B(o)$, there is a set $W$ in $B(o)$ such that $W-W$ is in $U$. But since $a$ and $b$ are in $W$, $a-b$ is in $U$. Thus $a-b$ is in $A$, and it follows that $A$ is an additive group. Let $x$ be any element of $R$. For any $U$ in $B(o)$, there is a $W$ in $B(o)$ with $Wx$ in $U$. Since $a$ is in $W$, $ax$ is in $U$, hence in $A$, and $A$ is an ideal.

The intersection in Theorem 2.7 is the closure of the point $0$; thus a topological ring is $T_1$ if and only if the intersection is the zero ideal. Since a topological ring
is regular (Theorem 2.8), and a regular $T_\bot$ space is Hausdorff, $R$ is Hausdorff if and only if the intersection is zero. Since a field has no proper ideals except $(0)$, a topological field must always be Hausdorff if it has at least one proper open set.

Theorem 2.8: A topological ring is regular.

Proof: Let $R$ be a topological ring, $F$ a closed set and $p$ a point not in $F$. Let $B(o)$ be a basis at zero. Since $F(x) = x - p$ is open, continuous and one-to-one, we can assume without loss of generality that $p = 0$; then there is a set $U$ in $B(o)$ disjoint from $F$, a set $W$ in $B(o)$ with $W - W$ in $U$. Suppose $a$ is in $W/W + b$. Then $a = w + b$, both $a$ and $w$ in $W$. Then $b = a - w$, and $b$ is in $W$; thus $W/W + b = \emptyset$ if $b$ is not in $U$. The sets $W$ and $\bigcup_{x \in F} W + x$ are two disjoint open sets containing 0 and $F$ respectively, thus $R$ is regular.

Let $R$ be a topological ring with basis $B(o)$ at zero and $A$ any ideal of $R$. Let $f$ be the natural homomorphism of $R$ onto $R/A$. Consider the collection $B(\hat{o}) = \{f(U + A) | U \in B(o)\}$, of subsets of $R/A$. Let $f(U + A)$, $U$ in $B(o)$, be a set of $B(\hat{o})$, $W$ a set of $B(o)$ such that $W - W$ and $W(2)$ are contained in $U$. Then $f(W + A) - f(W + A) = f(W - W + A) \subseteq f(U + A)$, and $f(W + A) - f(W(2) + (W + A) (W + A)) \subseteq f(W(2) + A) \subseteq f(U + A)$. Thus $B(\hat{o})$ satisfies condition 2) of Theorem 2.6.

Proofs that $B(\hat{o})$ satisfies condition 1) and 3) of Theorem 2.6 are analogous to the proof above. Thus $R/A$ is a topological ring with the collection of sets $B(\hat{o})$. 

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as a basis at zero. Note that if $A$ is the ideal of Theorem 2.7, $R/A$ is a Hausdorff space.

Theorem 2.9: Let $R$ be a topological ring, $A$ any ideal of $R$. Then the natural homomorphism $f$ of $R$ onto the topological ring $R/A$ (defined above) is open and continuous.

Proof: Let $B(o)$ be a basis at zero, $U+b$ an arbitrary set of $B$. Since $f$ is a homomorphism, $f(U+b)=f(U)+f(b)$, and since $f(U)$ is open by definition, $f(U+b)$ is open. Thus the mapping $f$ is open.

Basis sets of $R/A$ are of the form $f(U)+f(b)$, where $U$ is in $B(o)$; we have $f^{-1}(f(U)+f(b)) = \bigcup_{x \in U} x+A+b+A = \bigcup_{x \in U} x+A+b = U+A+b = \bigcup_{y \in U} U+y$, so that $f$ is continuous, since this set is open.

Corollary 2.10: Let $R$ be a topological ring, $A$ the ideal of Theorem 2.7. Then the natural homomorphism $f$ of $R$ onto $R/A$ is an open, continuous mapping of $R$ onto a Hausdorff space.

It is customary in the study of topological rings to consider only those rings which are Hausdorff spaces, and to replace a ring $R$ with the ring $R/A$ of Corollary 2.10 if it is not Hausdorff.

Theorem 2.11: Let $R$ be a topological ring with basis $B(o)$ at zero. Then the closure $\overline{A}$ of a subset $A$ of $R$ is $\bigcap_{U \in B(o)} (U+A)$.

Proof: Suppose that $x$ is in $U+A$ for every $U$ in $B(o)$. For each fixed $U$ in $B(o)$ there is a set $W$ in $B(o)$ such that $W-W$ is in $U$. Then $x$ is in $W+a$ for some $a$ in $A$, $x-a$ is
in $W$, and $a-x$ is in $-W$. But, since $-W$ is in $U$, $a$ is in $U+x$. Since the sets $U+x$ constitute a basis at $x$, every neighborhood of $x$ contains a point of $A$; thus $x$ is in $\overline{A}$. Conversely, if $x$ is in $\overline{A}$, then every neighborhood $U+x$, $U$ in $B(o)$, contains a point of $A$. An argument similar to that above shows for every $U$ in $B(o)$, $x$ is in $U+a$ for some $a$ in $A$, thus that $x$ is in $\bigcap_{a \in \omega} (U+A)$.

Let $R$ be a ring, and let $S = \{M_\alpha | \alpha \in L\}$ be a linearly ordered collection of ideals of $R$ with $M_\alpha \subseteq M_\beta$ if $\alpha \geq \beta$. We have: (1) the intersection of two ideals of $S$ is an ideal of $S$; (2) $M_\alpha - M_\beta = M_\alpha$, $M_\alpha (2) \subseteq M_\alpha$; (3) $M_\alpha + a = M_\alpha$ for any $a$ in $M_\alpha$; $M_\alpha b \subseteq M_\alpha$ for any $b$ in $R$. Thus, by Theorem 2.6, the cosets $M_\alpha + b$, $\alpha$ in $L$, $b$ in $R$, generate a topology in $R$, and $R$ is a topological ring with this topology.

Theorem 2.12: The sets of $S$ (thus all basis sets) of the above topology are both open and closed.

Proof: The sets are open by definition. But, by Theorem 2.11, $M_\alpha = \bigcap_{\alpha \in L} (M_\alpha + M_\alpha) = \bigcap_{\alpha \in L} M_\alpha = M_\alpha$; thus the sets are also closed.

In the remainder of this paper, we will always assume that the collection $S$ of ideals is countable, that $L$ is a subset of non-negative integers and that $M_0 = R$. We shall consider the collection $S$ as given and designate the resulting topology the "$M_1$-topology" of $R$. An interesting special case occurs if $M_1 = M^1$, for some given ideal $M$ in $R$ (here $M^1$ is the $1$th power of the ideal $M$; see Definition 1.2).
Suppose $R$ is a topological ring with a $M_1$-topology. Let $S(x,y) = \{ n \mid x-y \text{ is in } M^n \}$. For some fixed $q$, $0 < q < 1$, we define a real valued function $d$ on the set $R \times R$ by the formula $d(x,y) = \inf \{ q^n \mid n \in S(x,y) \}$.

Theorem 2.13: The function $d$ has the following properties:

1) $d(x,y) \geq 0$,
2) $d(x,y) = 0$ if and only if $x-y$ is in $\bigcap_{n=0}^{\infty} M_n$,
3) $d(x,y) = d(y,x) = d(x-y,0)$,
4) $d(x,y) \leq \max \{ d(x,z), d(y,z) \}$ $\leq d(x,z) + d(y,z)$,
5) $d(ax,ay) \leq \min \{ q^n d(a,0), d(x,y) \}$.

Proof: 1), 2) and 3) are obvious from the definition of $d$. If $d(x,y) = 0$, 4) is trivial. Otherwise, for some $n$, $x, y$ is in $M_n$ but not in $M_{n+1}$. If $x-z$ and $y-z$ are in $M_s$, then $x-y$ is in $M_s$, $n \geq s$, and 4) follows. If either $d(a,0)$ or $d(x,y)$ is zero, 5) is trivial, so assume that $a$ is in $M_t$ but not in $M_{t+1}$, and that $x-y$ is in $M_n$ but not in $M_{n+1}$. Let $t = \max(r,n)$. Then $a(x-y)$ is in $M_t$, and 5) follows.

From properties 1), 3) and 4), it follows that $d$ is a pseudo-metric on $R$, and $\bigcap_{n=0}^{\infty} M_1 = (0)$ if and only if $d$ is a metric. For any $\epsilon > 0$, let $N_\epsilon(a) = \{ x \mid d(a,x) < \epsilon \}$. Then $N_\epsilon(a) = M_n + a$, where $n$ is an integer such that $q^n < \epsilon \leq q^{n-1}$. Since, in $R$ the basis sets of the $M_1$-topology and of the pseudo-metric topology generated by $d$ are identical, we have:

Theorem 2.14: $R$ is a pseudo-metric space in its
If \( \bigcap_{n=0}^{\infty} M_1 = (0) \), the \( M_1 \)-topology is a metric topology.

We have already remarked (Theorem 2.12) that the basis sets of the \( M_1 \)-topology are both open and closed. If \( M_n = (0) \) for some \( n \), the resulting topology is the discrete topology in which all sets are both open and closed. If \( M_n \) is not zero for any \( n \), but \( \bigcap_{n=0}^{\infty} M_1 = (0) \), then points are closed sets which are not open, and their complements are open sets which are not closed.

By varying the choice of the sequence of \( M_1 \)'s in the same ring, it is possible to have radically different topologies. For example, consider the ring \( \mathbb{I} \oplus \mathbb{I}/(4) \), where \( \mathbb{I} \) is the ring of integers. Denote the elements of the ring by \([x,y] \), with \( x \) in \( \mathbb{I} \) and \( y \) in \( \mathbb{I}/(4) \). We will consider the topologies generated by three sequences of ideals, each sequence consisting of the powers of one of the principal ideals \( M_1 = ([2_0,0]) \), \( M_2 = ([0,2]) \), and \( M_3 = ([2,1]) \).

We have \( \bigcap_{n=0}^{\infty} M_1^n = ([0,0]) \), but \( M_1^n \neq ([0,0]) \) for any \( n \); thus the \( M_1 \)-topology of the ring is metric and not discrete. \( M_2^n = ([0,0]) \), and the \( M_2 \)-topology of the ring is discrete. \( \bigcap_{n=0}^{\infty} M_3^n = ([0,1]) \), and the \( M_3 \)-topology is not \( T_1 \).

1. Those familiar with uniform spaces (J. Kelley [3]) will recognize from Theorem 2.6 that a topological ring always has a uniform topology. Since a uniform space is pseudo-metrizable when its uniformity has a countable basis [3], Theorem 2.14 could be strengthened.
III
CAUCHY SEQUENCES AND COMPLETIONS

In this section it will be assumed that the ring $R$ is a metric space, with the metric $d$ and $M_1$-topology. Sequences will be denoted by symbols $\hat{x}, \hat{y}, \hat{a}$, etc., and their indexed elements by $x_n, y_n, a_n$, etc.

Definition 3.1: A sequence $\hat{x}$ in a metric space $R$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there is an integer $N$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n > N$.

Definition 3.2: A sequence $\hat{x}$ in a metric space $R$ is said to converge with limit $a$, if for every $\epsilon > 0$, the neighborhood $N_\epsilon(a)$ contains all but a finite number of the indexed elements of $\hat{x}$. Then we write $\lim \hat{x} = a$.

It is easily verified that a sequence $\hat{x}$ in a metric space converges only if it is a Cauchy sequence.

Definition 3.3: A metric space $R$ is said to be complete if every Cauchy sequence of $R$ has a limit in $R$.

Definition 3.4: Two Cauchy sequences $\hat{x}, \hat{y}$, are said to be equivalent if $\lim d(x_n, y_n) = 0$. We write $\hat{x} = \hat{y}$ in case $\hat{x}$ is equivalent to $\hat{y}$.

The following standard results for Cauchy sequences will be assumed:

1) If $\hat{x} = \hat{y}$, $\hat{y} = \hat{z}$, then $\hat{x} = \hat{z}$.

2) If $\hat{y}$ is a subsequence of $\hat{x}$, then $\hat{x} = \hat{y}$.

3) If $\hat{x}$ is convergent and $\lim \hat{x} = a$, $\lim \hat{x} = b$, then
4) If \( \hat{x} = \hat{y} \) and \( \hat{x} \) converges, then so does \( \hat{y} \), and 
\[
\lim \hat{x} = \lim \hat{y}.
\]

Cauchy sequences in the ring \( R \) have several additional properties due to the structure of \( R \). If \( \hat{x} \) and \( \hat{y} \) are Cauchy sequences of \( R \), we define the sequences \( \hat{x}+\hat{y} \) and \( \hat{x}\hat{y} \) as the sequences with indexed elements \( x_n+y_n \) and \( x_ny_n \), respectively.

Theorem 3.5: If \( \hat{x} \) and \( \hat{y} \) are Cauchy sequences of \( R \), then the sequences \( \hat{x}+\hat{y} \) and \( \hat{x}\hat{y} \) defined above are also Cauchy sequences.

Proof: Given \( \epsilon > 0 \), there is an integer \( N \) such that 
\[
d(x_n,x_m) < \frac{\epsilon}{2}, \quad d(y_n,y_m) < \frac{\epsilon}{2}, \quad \text{whenever } n,m > N.
\]
But then, 
\[
d(x_n+y_n,x_m+y_m) \leq d(x_n-x_m,y_n-y_m) \leq d(x_n-x_m,0) + d(y_n-y_m,0) = d(x_n,x_m) + d(y_n,y_m) < \epsilon,
\]
by Theorem 2.13. Also by Theorem 2.13, 
\[
\min \left\{ d(x_n,0), d(y_n,0) \right\} \leq \min \left\{ d(x_n,x_m), d(y_n,y_m) \right\} < \epsilon.
\]
Thus \( \hat{x}+\hat{y} \) and \( \hat{x}\hat{y} \) are Cauchy sequences.

Theorem 3.6: If \( \hat{x}, \hat{y} \) and \( \hat{z} \) are Cauchy sequences of \( R \) and \( \hat{x} = \hat{z} \), then \( \hat{x}+\hat{y} = \hat{z}+\hat{y} \) and \( \hat{x}\hat{y} = \hat{z}\hat{y} \).

Proof: We have 
\[
d(x_n+y_n,z_n+y_n) = d(x_n-z_n,0) = d(x_n,z_n)
\]
and 
\[
d(x_ny_n,z_ny_n) \leq \min \left\{ d(y_n,0), d(x_n,z_n) \right\} \leq d(x_n,z_n)
\]
by Theorem 2.13. Thus, since \( \lim_{n} d(x_n,z_n) = 0 \), \( \lim_{n} d(x_n+y_n,z_n+y_n) = 0 \), and \( \lim_{n} d(x_ny_n,z_ny_n) = 0 \), and the desired result follows.

The strong triangular inequality, 
\[
d(x,y) \leq \max \left\{ d(x,z), d(z,y) \right\},
\]
gives rise to a correspondingly strong Cauchy condition.
**Theorem 3.7:** A sequence \( \hat{x} \) in \( \mathbb{R} \) is a Cauchy sequence if and only if \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \).

**Proof:** If \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \), then for any \( \varepsilon > 0 \) there is an integer \( N \) such that \( d(x_n, x_{n+1}) < \varepsilon \) whenever \( n > N \). But we have \( d(x_n, x_m) \leq \max \{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \ldots, d(x_{m-1}, x_m) \} \), thus \( d(x_n, x_m) < \varepsilon \) if \( m, n > N \), and \( \hat{x} \) is a Cauchy sequence. Conversely, if \( \hat{x} \) is a Cauchy sequence, as a special case of the Cauchy condition we must have \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \).

We define the symbol \( \sum_{n=1}^{\infty} a_n \) as \( \lim_{n \to \infty} \sum_{k=1}^{n} a_k \), whenever the latter limit exists. If the limit exists, the series is said to converge. The following theorem is an immediate consequence of Theorem 3.7 and the definitions.

**Theorem 3.8:** A necessary condition that \( \sum_{n=1}^{\infty} a_n \) converge in \( \mathbb{R} \) is \( \lim_{n \to \infty} a_n = 0 \). If \( \mathbb{R} \) is complete, the condition is also sufficient.

**Definition 3.9:** Let \( \mathbb{R} \) be a metric space. A metric space \( \hat{\mathbb{R}} \) is said to be a completion of \( \mathbb{R} \) if and only if the following conditions are satisfied:

1) \( \hat{\mathbb{R}} \) is complete,
2) \( \hat{\mathbb{R}} \) contains \( \mathbb{R} \) as a subspace,
3) \( \mathbb{R} \) is dense in \( \hat{\mathbb{R}} \), that is, the closure of \( \mathbb{R} \) in \( \hat{\mathbb{R}} \) is \( \hat{\mathbb{R}} \),
4) the metric for \( \hat{\mathbb{R}} \), restricted to \( \mathbb{R} \), is the metric for \( \mathbb{R} \).

**Theorem 3.10:** Every metric space has a completion.
Proof: Let $R$ be a metric space with metric $d$. Consider the space $\hat{R}$ of equivalence classes of all Cauchy sequences of elements in $R$. We will use the symbols $\hat{x}$, $\hat{y}$, $\hat{a}$, etc., to denote both a particular Cauchy sequence and the equivalence class in $\hat{R}$ which it determines. The formula $\hat{x}=\hat{y}$ then will indicate that $\hat{x}$ is equivalent to $\hat{y}$ and that the equivalence classes in $\hat{R}$ determined by $\hat{x}$ and $\hat{y}$ are identical. We will show, with suitable definitions and conventions, that $\hat{R}$ is a completion of $R$.

The function $d(\hat{x},\hat{y})=\lim_{n\to\infty} d(x_n, y_n)$ is well defined (since $\{d(x_n, y_n)\}$ is a Cauchy sequence of real numbers if $\hat{x}$ and $\hat{y}$ are Cauchy sequences) and is easily seen to be a metric on $R$ (Hall and Spencer [1]). We use this metric to generate a topology on $\hat{R}$.

Every element $a$ in $R$ determines a Cauchy sequence $\hat{a}$ in $\hat{R}$ with the property $a_n=a$ for every $n$. $\hat{a}$ in turn determines an equivalence class of $R$. Note that $\hat{a}=\hat{b}$ if and only if $a=b$. We embed $R$ in $\hat{R}$ by indentifying each $a$ in $R$ with the equivalence class determined by $\hat{a}$. As defined, $R$ is seen to satisfy condition 4) and 2) of Definition 3.9.

Let $\hat{x}$ be an arbitrary Cauchy sequence from an equivalence class of $\hat{R}$. We will show that $\hat{x}$ is the limit of a Cauchy sequence (of sequences) $\hat{y}$ of $\hat{R}$ with elements of the embedded $R$, thus that $R$ is dense in $\hat{R}$.

Let $K$ be the $n$th element of the sequence $\hat{x}$. Define
the sequence \( \hat{r}_n \) as a sequence in which every indexed element is the element \( K \), and \( \hat{r} \) as a sequence with elements \( \hat{r}_n \). Then \( \hat{r} \) is a Cauchy sequence of \( \hat{R} \). Now \( \lim_{n \to \infty} d(\hat{x}, \hat{r}_n) = \lim_{n \to \infty} (\lim_{r \to \infty} d(x_r, x_n)) = 0 \); thus \( \hat{r} \) converges to \( \hat{x} \). Thus \( R \) satisfies condition 3) of Definition 3.9.

The proof will be complete once it is shown that \( \hat{R} \) is a complete metric space. Let \( \hat{x} \) be a Cauchy sequence in \( \hat{R} \) with indexed elements \( \hat{x}_n \). From each equivalence class \( \hat{x}_n \), we choose a representative Cauchy sequence with indexed elements \( x_n, i \). We have:

\[
\hat{x}_1: x_{1,1}, x_{1,2}, x_{1,3}, \ldots \\
\hat{x}_2: x_{2,1}, x_{2,2}, x_{2,3}, \ldots \\
\vdots
\]

Consider the sequence \( x_n \). For each \( n \), there exists an integer \( N(n) \) such that \( d(x_{n,N(n)}, x_n, i) < 1/n \) for every \( i > N(n) \), since \( \hat{x}_n \) is a Cauchy sequence of \( R \). For each \( n \), let \( a_n = x_{n,N(n)} \), and define the constant sequence \( \hat{a}_n \). This gives a new sequence \( \hat{a} \) of elements of \( \hat{R} \).

By construction, \( d(\hat{a}_n, \hat{x}_n) < 1/n \). Thus \( \hat{a} \) and \( \hat{x} \) are equivalent Cauchy sequences, and, if either converges, both converge, and the limits are the same.

Since \( \hat{a} \) is a Cauchy sequence, for every \( \epsilon > 0 \) there exists an integer \( N \) such that \( d(a_n, a_m) < \epsilon \), whenever \( n, m > N \). Consider the sequence \( \hat{y} \) with indexed elements \( y_n = x_{n,N(n)} \); \( \hat{y} \) is a Cauchy sequence of \( R \). Since
We will call the space \( \hat{R} \), as constructed above, the metric completion of \( R \).

**Theorem 3.11:** If \( \hat{R} \) is the metric completion of \( R \) with the \( M_1 \)-topology, then the metric \( \hat{d} \) has the following properties:

1) \( \hat{d}(\hat{x},\hat{y}) \leq \max \{ \hat{d}(\hat{x},\hat{z}),\hat{d}(\hat{y},\hat{z}) \} \),

2) \( \hat{d}(\hat{x},\hat{y}) = \hat{d}(\hat{x}-\hat{y},0) \),

3) \( \hat{d}(\hat{x}\hat{z},\hat{x}\hat{y}) \leq \min \{ \hat{d}(\hat{x},\hat{o}),\hat{d}(\hat{y},\hat{x}) \} \).

**Proof:** Given Cauchy sequences \( \hat{x}, \hat{y}, \hat{z} \) of the ring \( R \), we have:

\[
\lim_{n} d(\hat{x},\hat{y}) = \lim_{n} d(x,y) = \lim_{n} d(x_n,y_n) \leq \lim_{n} \max \{ d(x_n,z_n), d(y_n,z_n) \} = \max \{ \lim_{n} d(x_n,z_n), \lim_{n} d(y_n,z_n) \} = \max \{ \hat{d}(\hat{x},\hat{z}), \hat{d}(\hat{y},\hat{z}) \} ;
\]

\[
\hat{d}(\hat{x},\hat{y}) = \lim_{n} d(x_n,y_n) = \lim_{n} d(x_n-y_n,0) = \hat{d}(\hat{x}-\hat{y},0) ;
\]

\[
\hat{d}(\hat{x}\hat{z},\hat{x}\hat{y}) = \lim_{n} d(x_ny_n,x_nz_n) \leq \lim_{n} \min \{ d(x_n,0) , d(y_n,z_n) \} = \min \{ \hat{d}(\hat{x},\hat{o}), \hat{d}(\hat{y},\hat{z}) \} .
\]

**Theorem 3.12:** If \( \hat{R} \) is the metric completion of \( R \) with the \( M_1 \)-topology, then \( \hat{R} \) is a ring, with the operations of addition and multiplication as defined in Theorem 3.5.

**Proof:** Theorem 3.5 shows that \( \hat{R} \) is closed under the operations, and Theorem 3.6 gives uniqueness of the operations. The other properties of a ring follow directly from the fact that \( R \) is a ring.
Given a set $S$ in $\hat{R}$, the symbol $\overline{S}$ will denote the closure of $S$ in $\hat{R}$. Note that it may be the case that $\mathbb{R} = \hat{\mathbb{R}}$.

**Theorem 3.13:** If $A$ is an ideal (subring) of $\mathbb{R}$, then $\overline{A}$ is an ideal (subring) of $\hat{\mathbb{R}}$. If $B$ is an ideal (subring) of $\hat{\mathbb{R}}$, then $\overline{B}$ is an ideal (subring) of $\hat{\mathbb{R}}$.

**Proof:** Let $A$ be a subring of $\mathbb{R}$. Then $\hat{x}$, $\hat{y}$ are in $\overline{A}$ if and only if $\hat{x}$ and $\hat{y}$ are (equivalent to) Cauchy sequences of elements of $A$. But then $\hat{x}\hat{y}$ and $\hat{x}-\hat{y}$ are also Cauchy sequences of elements from $A$, thus $\overline{A}$ is a subring. If $A$ is an ideal in $\mathbb{R}$, $\hat{z}$ any element of $\hat{\mathbb{R}}$, then the product $\hat{z}\hat{z}$, of Cauchy sequences is a Cauchy sequence of elements from $A$, thus $\overline{A}$ is an ideal.

We will show (Theorem 3.15) that $\hat{\mathbb{R}}$ is a complete topological ring. Then, using the above reasoning on $\hat{\mathbb{R}}$, the remainder of the theorem is immediate.

In particular, $M_n$ is an ideal of $\hat{\mathbb{R}}$ for each $n$.

**Theorem 3.14:** The collection of cosets $M_n + \hat{a}$, $a$ in $\mathbb{R}$, $n = 1, 2, \ldots$, is a basis for the topology of $\hat{\mathbb{R}}$.

**Proof:** For any $x, y$ in $\mathbb{R}$, the value of $d(x, y)$ is either $0$ or $q^n$, for some integer $n$. Since $\hat{d}(\hat{x}, \hat{y}) = \lim_n d(x_n, y_n)$, it follows that $d(x, y)$ is either $0$ or $q^n$, for some integer $n$. It is clear, then, that given $\epsilon > 0$, $q^n < \epsilon \leq q^{n-1}$, for any $\hat{x}$ in $\hat{\mathbb{R}}$ we have $N_\epsilon(\hat{x}) = \{ \hat{y} | \hat{d}(\hat{x}, \hat{y}) \leq q^n \}$. Thus $N_\epsilon(\hat{x})$ is closed.

Since $\mathbb{R}$ is dense in $\hat{\mathbb{R}}$, there is some $a$ in $\mathbb{R}$ such that
â is in \( N_\epsilon(\hat{x}) \). If \( \hat{y} \) is in \( N_\epsilon(\hat{x}) \), then \( \hat{d}(\hat{a},\hat{y}) \leq \max \{ \hat{d}(\hat{x},\hat{y}), \hat{d}(\hat{a},\hat{x}) \} < \epsilon \), and thus \( N_\epsilon(\hat{x}) = N_\epsilon(\hat{a}) \).

Clearly \( R \cap N_\epsilon(\hat{a}) = M_n + a \), and since \( R \) is dense in \( \hat{R} \), \( M_n + a \) is dense in \( N_\epsilon(\hat{a}) \) and \( \overline{M_n + a} = N_\epsilon(\hat{a}) \). Since any sequence of \( M_n + a \) is of the form \( \hat{x} + \hat{a} \), where \( \hat{x} \) is a sequence from \( M_n \), \( \overline{M_n + a} = \overline{M_n} + \hat{a} \). Thus the collection of cosets is the same as the collection of \( \epsilon \) neighborhoods and thus is a basis for the topology of \( \hat{R} \).

Theorems 2.6 and 3.14 prove the following theorem.

Theorem 3.15: The ring \( \hat{R} \), the metric completion of the ring \( R \) in its \( M_1 \)-topology, is a topological ring. The topology of \( \hat{R} \) is the \( \overline{M_1} \)-topology.

Definition 3.16: Let \( R \) be a topological ring with a metric topology. The ring \( \hat{R} \) is said to be the completion of \( R \) if and only if the following conditions hold:

1) \( \hat{R} \) is a metric space and is the completion of the metric space \( R \),

2) \( \hat{R} \) is a topological ring in its metric topology,

3) \( \hat{R} \) contains \( R \) as a subring.

Theorem 3.17: If the ring \( R \) is a metric space in its \( M_1 \)-topology, then \( R \) has a completion.

Proof: The ring \( \hat{R} \) of Theorem 3.12 is a completion of \( R \).
IV
SOME PROPERTIES OF COMPLETIONS

In this section, the symbol \( R \) will always denote a topological ring that is a metric space with some given \( M_1 \)-topology, \( \hat{R} \) the metric completion of \( R \). If the \( M_1 \)-topology of \( R \) is generated by the powers of some ideal \( M \), we will denote the resultant topology as the \( M \)-topology of \( R \), otherwise, no mention of the topology will be made. As in Section III, \( \overline{S} \) will denote the closure in \( \hat{R} \) of a subset \( S \) of \( R \).

Definition 4.1: A Noetherian ring is a commutative ring with identity in which every ideal is finitely generated (see Definition 1.1).

Definition 4.2: A local ring is a Noetherian ring \( R \) which has a unique proper maximal ideal \( M \).

It can be shown ([4]) that a necessary and sufficient condition that a Noetherian ring \( R \) be a local ring is that the non-units of \( R \) form an ideal \( M \). It can also be shown ([4]) that \( \bigcap_{n=0}^{\infty} M^n = (0) \). As an example of a local ring, consider, for a fixed prime \( p \), the ring \( I_p \) of rational numbers of the form \( x/y \), \( x, y \) integers and \( y \) not divisible by \( p \). The non-units of \( I_p \) are exactly those elements of the form \( px/y \), and these elements form an ideal \( M \) of \( I_p \). Any subset of \( I_p \) not contained in \( M \) must contain a unit, thus cannot be a proper ideal.
thus M is a unique maximal ideal of \( I_p \), and \( I_p \) is a local ring.

**Lemma 4.3:** Let \( R \) be a ring with identity, \( A \) an ideal of \( R \). If \( A \) is finitely generated, then so is \( A^n \) for every \( n \).

**Proof:** Let \( A = (x_1, x_2, \ldots, x_r) \). Clearly every element of \( A^n \) is of the form \( \sum b_j x_1^{a_j} x_2^{b_j} \cdots x_r^{c_j} \), with \( b_j \) in \( R \) and the exponents on each term summing to \( n \). Also, every element of this form must be in \( A^n \); thus \( A^n \) is finitely generated by the set \( \{ x_1^{a_j} x_2^{b_j} \cdots x_r^{c_j} | a + b + \cdots + c = n \} \).

Suppose the ring \( R \) is a topological ring with \( M \)-topology. Along with the natural topology of \( R \) generated by the ideals \( \overline{M^n} \) (Theorem 3.14), the \( \overline{M} \)-topology of \( \hat{R} \) arises naturally. The two topologies will be the same if the ideals \( \overline{M^n} \) and \( \overline{M}^n \) are the same.

An element of \( \overline{M}^n \) is a finite sum of elements of the form \( \hat{x}_1 \hat{x}_2 \cdots \hat{x}_n \), where \( \hat{x}_i \) is a Cauchy sequence from \( M \). The products, and thus the finite sums, are Cauchy sequences from \( M^n \), and we have \( \overline{M}^n \subset \overline{M}^n \). Since \( \overline{M} \supset M^n \), \( \overline{M} \supset M^n \) and thus \( \overline{M} = \overline{M^n} \).

Consider an arbitrary element \( \hat{y} \) of \( \overline{M}^n \). The indexed elements of \( \hat{y} \) are finite sums of elements of the form \( x_1 x_2 \cdots x_n \), with \( x_i \) in \( M \); however, the number of terms in the sum need not be constant. For some \( r \), an element \( \hat{z} \) of \( \overline{M}^n \) is the sum of \( r \) terms of the form \( \hat{x}_1 \hat{x}_2 \cdots \hat{x}_n \), and any indexed element of \( \hat{z} \) is the sum of exactly \( r \) terms of
the form \(x_1x_2\ldots x_n\). Thus it is conceivable that the two ideals are not always equal.

If the two topologies are not the same, the \(\overline{M}\)-topology is the stronger in the sense that every set that is open in the natural topology of \(\hat{R}\) is also open in the \(\overline{M}\)-topology. This follows from the observation that \(\overline{M} = \bigcup M^n + a\). The following theorem shows that the topologies are the same in one important case.

**Theorem 4.4:** If \(R\) has an identity and \(M\) is finitely generated, then the natural topology of \(\hat{R}\), the completion of \(R\) with \(M\)-topology, is the \(\overline{M}\)-topology.

**Proof:** By Lemma 4.3, \(M^r\) must be finitely generated for every \(r\), so let \(M^r = (x_1^r, x_2^r, \ldots, x_n^r)\) in \(R\). We will show that \(M^r = (x_1^r, x_2^r, \ldots, x_n^r)\) in \(\hat{R}\), and then, since \(M^r\) contains these generators, \(\overline{M^r} = M^r\).

Let \(\hat{y}\) be an arbitrary Cauchy sequence from \(M^r\). Then, by definition of the metric and Theorem 3.7, we have \(y_{n+1} - y_n\) in \(M^{s(n)}\), where \(s(n)\) tends to infinity as \(n\) tends to infinity. Let \(t = s(n) - r\). Then, since \(M^{s(n)} = M^r M^t\), we can write \(y_{n+1} - y_n = \sum a_{n,j} x_j\), with \(a_{n,j}\) in \(M^t\). Since each \(y_n\) is in \(M^r\), we let \(y_1 = \sum b_{1,j} x_j\), and define inductively \(b_{n+1,j}\) as \(b_{n,j} + a_{n,j}\). Then by induction, \(y_n = \sum b_{n,j} x_j\).

Since \(b_{n+1,j} = b_{n,j} + a_{n,j}\), \(\lim d(b_{n+1,j}, b_{n,j}) = d(a_{n,j}, 0) = 0\); and the sequences \(b_{n,j}\) (indexed elements \(b_{n,j}\) are Cauchy sequences. It follows that
The right hand side of the equality goes to zero as $n$ tends to infinity; thus \( \hat{y} = \sum \hat{b}_j x_j \). Since \( \hat{y} \) was an arbitrary element of \( \overline{M} \), we have \( \overline{M} = \langle x_1, x_2, \ldots, x_n \rangle \).

Theorem 4.5: Let \( A \) be any ideal of \( R \) and \( f \) the (open, continuous) natural homomorphism of \( R \) onto the topological ring \( R/A \) (as defined in Theorem 2.9). If \( \lim_{n} a_n = a \) in \( R \), then \( \lim_{n} f(a_n) = f(a) \) in \( R/A \). Also, if \( R \) is complete, then \( R/A \) is complete.

Proof: The first statement follows readily from the continuity of \( f \). Suppose \( R \) is complete and let \( \hat{a}' \) be a Cauchy sequence in \( R/A \). Put \( b'_n = a'_n - a'_{n-1} \) for \( n \geq 2 \), and \( b'_1 = a'_1 \). Then by Theorem 3.7, \( \lim_{n} b'_n = 0 \), that is, \( b'_n \) is in \( M_{s(n)} + A/A \), where \( s(n) \) goes to infinity as \( n \) goes to infinity. Thus we can find, for each \( n \), \( b_n \) in \( M_{s(n)} \) such that \( f(b_n) = b'_n \). By construction, \( \lim_{n} b_n = 0 \), and since \( R \) is complete \( \sum_{n=1}^{\infty} b_n \) converges by Theorem 3.8. Suppose it converges to \( a \). Then \( \lim_{n} f(\sum_{n=1}^{\infty} b'_n) = \lim_{n} \sum_{n=1}^{\infty} b'_1 = \lim_{n} a'_n = f(a) \).

Thus \( R/A \) is complete.

Theorem 4.6: Let \( A \) be any ideal of \( \hat{R} \) and \( B = A \cap R \).

Let \( \hat{R}/A \) and \( R/B \) be the topological rings defined in Theorem 2.9. Then \( \hat{R}/A \) is (isomorphic to) a completion of \( R/B \).

Proof: The mapping \( B + a \rightarrow A + a \) of \( R/B \) into \( \hat{R}/A \) is one-to-one since distinct cosets \( B + a \) of \( R/B \) are contained...
in distinct cosets \( A+a \) in \( \hat{R} \). This mapping embeds the ring \( R/B \) in the ring \( \hat{R}/A \).

The topological rings \( R/B \) and \( \hat{R}/A \) both have a metric topology since their topologies are the \( M_1+B/B \) and \( M_1+A/A \)-topologies respectively. Since \( x-y \) in \( M_1+B \) implies that \( x-y \) is in \( M_1+A \), the metric on \( \hat{R}/A \) restricted to \( R/B \) (as an embedded ring) is the metric for \( R/B \).

Thus the metric space \( \hat{R}/A \) contains the metric space \( R/B \) as a subspace. \( \hat{R}/A \) is complete by Theorem 4.5, thus we need only show that \( R/B \) is dense in \( \hat{R}/A \). Let \( A+a \) be an element of \( \hat{R}/A \). Then there is a Cauchy sequence \( \hat{a} \) of elements of \( R \) that converges to \( a \). Then the sequence \( \{A+a_n\} \) of elements of \( R/B \) converges to \( A+a \).

If the ideal \( B \) is open in \( R \), the point zero is open in \( R/B \) and the topology of \( R/B \) is discrete. Thus \( R/B \) is complete, and \( R/B \) must then be isomorphic to \( \hat{R}/A \). This observation proves Corollary 4.7.

Corollary 4.7: The quotient rings \( R/M_1 \) and \( \hat{R}/M_1 \) are isomorphic.

Corollary 4.8: If \( A \) is an ideal of \( \hat{R} \) and \( A \cap R=0 \), then \( A=(0) \).

Proof: \( \hat{R} \) and \( \hat{R}/A \) must both be the completion of \( R \) by Theorem 4.6. Thus if \( a \in A \) and \( a \neq 0 \), there is a sequence \( \hat{x} \) of \( R \) that converges to \( a \) in the completion \( R \). But then \( \hat{x} \) converges to zero in the completion \( \hat{R}/A \), a contradiction to \( a \neq 0 \).
A topological space is said to be compact if every collection of open sets which cover the space contains a finite subcollection which also covers the space. In a metric space, this is equivalent to the statement that every infinite subset of the space has a limit point in the space ([1]). For this reason, a compact metric space is necessarily complete.

Theorem 4.9: Let \( \hat{R} \) be the completion of the ring \( R \) with \( M_1 \)-topology. A necessary and sufficient condition that \( \hat{R} \) be compact is that the quotient ring \( R/M_1 \) be finite for every \( i \).

Proof: Since \( R/M_1 \) is isomorphic to \( \hat{R}/\hat{M}_1 \) (Theorem 4.1) for each \( i \), an equivalent condition is that the latter quotient ring be finite for every \( i \). To simplify notation we will assume that \( R \) is complete, that is, \( R = \hat{R} \), and \( M_1 = \overline{M}_1 \).

Suppose that \( R \) is compact. For each \( i \), the collection of sets \( M_1 + a \), \( a \) in \( R \), is an open cover of \( R \), thus must contain a finite subcollection which covers \( R \). It follows that \( R/M_1 \) is finite.

Suppose that \( R/M_1 \) is finite for every \( i \). Since \( R \) is complete, it will be sufficient to show that every infinite sequence indexed by the positive integers has a Cauchy subsequence.

Let \( \{a_i\} \) be an infinite sequence. Since \( R/M_1 \) is finite, only a finite number of the cosets \( M_1 + a_i \), \( i = 1, 2, \ldots \),
can be distinct, that is, there is a subsequence \( \{a_{1,i}\} \) such that all the cosets \( M_1 + a_{1,i} \) are identical.

We make the inductive assumption that there is a subsequence \( \{a_{n,i}\} \) such that all the cosets \( M_1 + a_{n,i} \) are identical. Since \( R/M_{n+1} \) is finite, only a finite number of the cosets \( M_{n+1} + a_{n,i}, i = 1, 2, \ldots \), can be distinct. Thus we can choose a subsequence \( \{a_{n+1,i}\} \) of \( \{a_{n,i}\} \) such that all the cosets \( M_{n+1} + a_{n+1,i}, i = 1, 2, \ldots \), are identical.

Thus for each positive integer \( n \), we can find a subsequence \( \{a_{n,i}\} \), each subsequence contained in the preceding subsequence, with the property that all the cosets \( M_1 + a_{n,i} \) are identical. Let \( b_n = a_{n,n} \). By construction, \( b_n - b_{n+1} \) is in \( M_n \) for each \( n \). Thus \( \lim_{n \to \infty} d(b_n, b_{n+1}) = 0 \), and \( \{b_n\} \) is a Cauchy subsequence by Theorem 3.7.

If \( R \) is a ring with identity, \( M \) a finitely generated ideal of \( R \), it is possible to give an algebraic completion (see [4]) of \( R \) with its \( M \)-topology. We will not give an algebraic completion here, but we will show that the metric completion \( \hat{R} \) is a homomorphic image of a certain derived ring.

Let \( x_1, i = 1, 2, \ldots, r \) be indeterminates over a ring \( R \) and let \( X^s = \{ \sum \alpha t_v x_1^a x_2^b \cdots x_r^c \mid t_v \text{ in } R, a + b + \cdots + c = s \} \). Let \( Q_s \) be an arbitrary element of \( X^s, s = 1, 2, \ldots \). Then the set \( P \) consisting of all elements of the form \( \sum_{s \geq 0} Q_s \) is a ring under the operations defined by \( \sum_{s \geq 0} Q_s + \sum_{s \geq 0} Q'_s = \sum_{s \geq 0} (Q_s + Q'_s) \).
and \( (\sum_{s=0}^{\infty} q_s)(\sum_{s=0}^{\infty} q'_s) = \sum_{s=0}^{\infty} (\sum_{u+v=s} q_u q'_v) \). The ring \( P \) is called a formal power series ring over \( R \).

Suppose \( M \) is a finitely generated ideal of \( R \) with generators \( a_1, i=1,2,\ldots,r \), and that \( \bigcap_{m=0}^{\infty} M^m = (0) \). Let \( q_s \) be the elements of \( M^S \) obtained by substituting \( a_i \) for \( x_i, i=1,2,\ldots,r \), in \( Q_S \). Clearly, all elements of \( M^S \) may be obtained this way. For any particular sequence of \( q_s \)'s, \( q_s \) goes to zero as \( s \) goes to infinity, thus the sequence \( \left\{ \sum_{s=0}^{m} q_s \right\} \) is a Cauchy sequence (Theorem 3.7). This gives a mapping of \( P \) into \( R \).

Lemma 4.10: The mapping \( f(\sum_{s=0}^{\infty} q_s) = \sum_{s=0}^{\infty} q_s \) of \( P \) into \( \hat{R} \) is onto.

Proof: We must show that every Cauchy sequence of \( R \) is equivalent to a Cauchy sequence of the form \( \left\{ \sum_{s=0}^{m} q_s \right\} \). Let \( \hat{x} \) be a Cauchy sequence of \( R \). Then for every integer \( s \), there exists an integer \( N(s) \) such that \( x_m - x_n \) is in \( M^S \) for all \( m,n > N(s) \). Choose \( N(1) > N(j) \) if \( i > j \), and consider the subsequence \( \{ x_{N(i)} \} \).

We have \( x_{N(s)} - x_{N(s+1)} \) in \( M^S \) for each integer \( s \). Set \( q_0 = x_{N(1)}, q_s = x_{N(s+1)} - x_{N(s)} \). The \( q_s \)'s are images under \( f \) of \( q_s \)'s in \( X^S \), and \( x_{N(n)} = \sum_{s=0}^{n} q_s \). That is, a subsequence of \( \hat{x} \) is of the desired form.

Theorem 4.11: If \( M \) is a finitely generated ideal of \( R \) and \( \bigcap_{m=0}^{\infty} M^m = (0) \), then \( \hat{R} \), the metric completion of \( R \) with \( M \)-topology, is a homomorphic image of a formal power series ring.
Proof: We will show that the map $f$ of Lemma 4.1 is a homomorphism. 

$$f(\sum_{i=0}^{\infty} Q_s + \sum_{i=0}^{\infty} Q'_s) = f(\sum_{i=0}^{\infty} (Q_s + Q'_s)) = \left\{ \sum_{i=0}^{\infty} (q_s + q'_s) \right\} = \left\{ \sum_{i=0}^{\infty} q_s \right\} + \left\{ \sum_{i=0}^{\infty} q'_s \right\} = f(\sum_{i=0}^{\infty} Q_s) + f(\sum_{i=0}^{\infty} Q'_s); 

f((\sum_{i=0}^{\infty} Q_s)(\sum_{i=0}^{\infty} Q'_s)) = \left\{ \sum_{i=0}^{\infty} (\sum_{j=0}^{\infty} q_s q'_j) \right\} = \left\{ \sum_{i=0}^{\infty} q_s \right\} \left\{ \sum_{i=0}^{\infty} q'_s \right\} = f(\sum_{i=0}^{\infty} Q_s)f(\sum_{i=0}^{\infty} Q'_s).$$

All but the second to the last equivalence follow directly from the definition. Let:

$$S_n = \sum_{i=0}^{\infty} \left( \sum_{u \geq v} q_u q'_v \right) = a_0 q_0' + q_1 a_0' + q_2 a_0' + \cdots + a_n q_0'$$

$$+ a_0 q_1' + a_1 q_1' + \cdots$$

$$+ a_0 q_2' + \cdots$$

$$\vdots$$

$$+ a_0 q_n'$$

$$S'_n = (\sum_{i=0}^{\infty} q_s)(\sum_{i=0}^{\infty} p_s) = a_0 q_0' + q_1 a_0' + q_2 a_0' + \cdots + a_n q_0'$$

$$+ a_0 q_1' + a_1 q_1' + \cdots + a_n q_1'$$

$$+ a_0 q_2' + \cdots$$

$$\vdots$$

$$+ a_0 q_n' + q_1 a_n' + \cdots + a_n q_n'.$$

Thus the sequences are not identical. However they are equivalent since $S'_n - S_n = x_n = \sum_{i=0}^{n+1} q_i q'_i$ is in $M^{n+1}$ for each $n$; hence $\lim_n d(S'_n, S_n) = \lim_n d(x_n, 0) = 0$.

This verifies that $f$ is a homomorphism. Note that if $M$ has $r$ generators, then $P$ may have as few as $r$ indeterminates over $R$.

It can be shown ([4]) that any power series ring over $
a Noetherian ring is also a Noetherian ring. Since any homomorphic image of a Noetherian ring is a Noetherian ring ([4]), we have:

**Theorem 4.11:** The completion of a Noetherian ring with $M$-topology is Noetherian.

**Lemma 4.13:** If $\tilde{R}$ is the completion of $R$ with $M$-topology, $u$ any unit of $R$, then every element of $\mathbb{M}+u$ is a unit.

**Proof:** We need only show that $m+1$, $m$ in $\mathbb{M}$, has an inverse in $R$, for then $u+m = (1+mu^{-1})u$ will also have an inverse. $1-\sum_{n=0}^{\infty} m^n$ is an inverse for $m+1$ in $R$.

**Theorem 4.14:** The completion of a local ring $R$ in its $M$-topology ($M$ maximal) is a local ring.

**Proof:** Let $a$ be any element of $\hat{R}$ that is not in $\mathbb{M}$. Since $R$ is dense in $\hat{R}$, there is some element $b$ of $R$ such that $\mathbb{M}+b = \mathbb{M}+a$ (see proof of Theorem 3.14). But $b$ is not in $M$; thus $b$ is a unit of $R$, $a$ is a unit of $\hat{R}$, and $\hat{R}$ is a local ring.

The result in Lemma 4.13 indicates that in general it will not be the case that $\bar{A} \cap R = A$ for arbitrary ideals $A$ of $R$. For example, consider the completion of the ring of integers with the $(2)$-topology. Lemma 4.13 shows that the closure of any ideal of the form $(2n+1)$ must be the whole space.
REFERENCES


