Continuous functions defined on a sphere

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CONTINUOUS FUNCTIONS DEFINED ON A SPHERE

by

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D. G. M.
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CHAPTER I

INTRODUCTION

The problem of showing the existence of a circumscribing cube about any closed and bounded convex body in a 3-dimensional Euclidean space was solved by S. Kakutani\[1\] several years ago. In order to obtain this result he first had to prove the following result concerning continuous functions defined on the surface of a sphere.

THEOREM 1: Let $f(P)$ be a real valued continuous function defined on a 2-sphere $S^2$ in a 3-dimensional Euclidean space. Then there exists a triple of points $P_1$, $P_2$, $P_3 \in S^2$, perpendicular to one another, such that $f(P_1) = f(P_2) = f(P_3)$.

Here the "triple of points perpendicular to one another" means that the three vectors $OP_1$, $OP_2$, $OP_3$ from the center $O$ of the sphere $S^2$ to these three points $P_1$, $P_2$, $P_3$ are perpendicular in pairs.

In the proof of his theorem he used results from homotopy theory. As an application of this theorem, Kakutani then showed the existence of a circumscribing cube about any closed and bounded convex set in $\mathbb{R}^3$. Kakutani conjectured that both this theorem and its application could be extended to higher dimensions.

About a decade later Yamabe and Yujobo \[2\] generalized Kakutani's theorem to $n$-dimensions (THEOREM 2). By using the

\[1\] Numbers in square brackets refer to the numbers of the references in the list of references cited on page 22.
method of Kakutani, they showed the analogous result for circumscribing a closed and bounded convex set in $\mathbb{R}^n$.

Dyson [3] proved a result (THEOREM 3) about inscribing a square in the 2-sphere in a 3-dimensional Euclidean space so that for a continuous function defined on the sphere, the function would take on the same value at all the vertices of the square. However, Dyson was unable to extend his result to higher dimensions. The obvious generalization of his theorem to $(n+1)$-dimensions states that equal values of $f(P)$ occur at $2n$ points situated on $n$ perpendicular diameters of $S^n$. Dyson based his proof on the method of Yamabe and Yujobo, but technical difficulties arose for $n > 2$, which he was not able to overcome with this approach. The extension of Dyson's theorem to the $n$-sphere was accomplished by Yang [4] by means of a special homology theory. However, no attempt is made in this paper to present Yang's results.

Historically, a theorem of Borsuk [5] about continuous functions on spheres came before any of the theorems presented in this paper. It stated that for a continuous real valued function on an $n$-sphere there exists some pair of antipodal points where the function takes on the same value. This result also has an interesting application, in that it is used to provide a solution to the "Ham Sandwich Problem" [6,7].

The final chapter has an application of THEOREM 2 which is a stronger result than the application of Yamabe and Yujobo.
CHAPTER II
THEOREMS OF YAMABE-YUJOBO AND DYSON

The once unsolved problem of the possibility of generalizing Kakutani's theorem to higher dimensional cases was provided with an affirmative answer in the following theorem, known as Yamabe and Yujobo's theorem.

**THEOREM 2**: Let \( f(P) \) be a real valued continuous function on an \( n \)-sphere \( S^n \) with center \( 0 \), then there exist \( (n+1) \) points \( P_0, P_1, \ldots, P_n \) on \( S^n \) perpendicular to one another (which means that the vectors \( OP_0, OP_1, \ldots, OP_n \) are perpendicular in pairs) such that

\[
f(P_0) = f(P_1) = \ldots = f(P_n).
\]

**PROOF**: Let \( \mathbb{R}^m \) be an \((m+1)\)-dimensional Euclidean space with the origin 0 and with rectangular co-ordinate axes \( OE_1, OE_2, \ldots, OE_{m+1} \). The cartesian co-ordinates of a point \( P \) will be denoted by \((p_1, p_2, \ldots, p_{m+1})\), and the distance from 0 by \( \|P\| \). Let us denote by \( C \) the union of the concentric \( m \)-spheres whose common center coincides with 0 and whose radii are in the interval \([1,2]\).

Now we shall prove the following

**LEMMA 2A**: Let \( S_0 = \{ P \mid \|P\|=1 \} \) and \( S_1 = \{ P \mid \|P\|=2 \} \). If \( L \) is a closed set in \( C \) which intersects every continuous curve joining \( S_0 \) and \( S_1 \), then \( L \) contains \((m+1)\) points \( Q_0, Q_1, \ldots, Q_m \) such that

\[
\|Q_0\| = \|Q_1\| = \ldots = \|Q_m\|,
\]

and \( OQ_0, OQ_1, \ldots, OQ_m \) are perpendicular to one another.

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If \( m = 0 \), the lemma is obviously true. Let us assume that the lemma is true when the dimension of the space is \(< m + 1\).

Let \( L_\varepsilon \) denote a closed connected \( m \)-dimensional manifold, the points of which are at distances \(< \varepsilon \) from \( L \), such that \( L_\varepsilon \) also intersects any curve joining \( S_2 \) and \( S_1 \). We want such a manifold because later we shall want to connect two points on it with a continuous curve. To show the existence of such an \( L_\varepsilon \), we use the compactness of \( L \), which is a closed and bounded set in \( \mathbb{R}^m \). If we take spherical neighborhoods of radii \(< \varepsilon \) of each point in \( L \), we will have an open covering of \( L \). We choose \( \varepsilon \) to be smaller than the minimum distance between \( L, S_2, \) and \( S_1 \). These sets are all closed and bounded, so they are a positive distance apart. By the compactness of \( L \), there exists a finite subcollection of this collection of spherical neighborhoods that also covers \( L \). If in the finite subcollection there are any spheres that are properly contained within another sphere of the finite collection, then eliminate these spheres from the collection. The union of this finite number of spheres has the same property as \( L \), since \( L \) is contained in it. If this union forms just one connected set, then let \( L_\varepsilon \) be the union of the surfaces of these spheres. Since any curve joining \( S_2 \) and \( S_1 \) must intersect \( L \) in at least one point and there is a sphere from our finite collection that contains that point, then this curve must intersect the surface of this sphere. Therefore, \( L_\varepsilon \) has the same property as \( L \). Next, we consider the case where the union of the
finite number of spheres forms more than just one connected set. Of course, there will be only a finite number of such sets. We can take the spheres to be closed spheres so that the components that are not connected to one another are then a positive distance apart. Thus, we can fit a curve between any two of the components. So, if we can join \( S_o \) and \( S_1 \) by a curve that does not intersect one or more of these components, then we shall eliminate these components from any further discussion. However, at least one of the components must intersect every curve from \( S_o \) to \( S_1 \), for otherwise, we could fit a curve between them joining \( S_o \) and \( S_1 \) that does not intersect \( L \). Choose one of these components that intersects every curve from \( S_o \) to \( S_1 \) and we then have the above case of one connected set with the same property as \( L \).

The set \( L_\varepsilon \) is closed so there are two points, \( P(1) \) and \( P(0) \), in \( L_\varepsilon \) such that

\[
\sup_{P \in L_\varepsilon} ||P|| = ||P(1)||,
\]

\[
\inf_{P \in L_\varepsilon} ||P|| = ||P(0)||.
\]

Now we join \( P(0) \) and \( P(1) \) by a continuous curve \( P(t) \), \( 0 \leq t \leq 1 \), on \( L_\varepsilon \). For every \( t \) we have a point \( P(t) \) and for each such point we can rotate the axes about the origin such that the point \( P(t) \) lies on the \( OE_{m+1} \) axis. So for \( t \) we shall determine a rotation \( \psi \) such that

\[
\psi(P(t)) \in OE_{m+1},
\]

\[
OE_{m+1}, \ UF(t), \text{ and } OV \text{ denote the half lines beginning at } 0.
\]

---

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and such that \( q_t \) is a continuous function of \( t \). One way that \( q_t \) may be defined so that it is a continuous function of \( t \) would be to let the rotation be about the line through the origin and perpendicular to the plane \( \mu \) determined by \( OE_{m+1} \) and \( OP(t) \). The direction of rotation will be determined by choosing the rotation through the smaller of the two angles in \( \mu \) determined by \( OE_{m+1} \) and \( OP(t) \). This then uniquely determines the rotation in all cases except when the two angles are equal, namely, of magnitude \( \pi \) radians. In the latter case it does not matter which one of the two is chosen, but in this case as in all cases the direction chosen will be considered in the positive direction. Thus for each \( t \) a direction is determined. With \( q_t \) defined in this manner it is clear that \( q_t \) is a continuous function of the point \( P(t) \), and, since \( P(t) \) is a continuous function of \( t \), we conclude that \( q_t \) is a continuous function of \( t \).

Let \( \Sigma \) be the hyperplane \( \{ P \mid p_{m+1} = 0 \} \), and let
\[
H^{m-1}(P) = \{ Q \mid \|Q\| = \|P\|, \; OQ \perp OP \}.
\]
Then \( q_t(H^{m-1}(P(t))) \) is in \( \Sigma \) for all \( t, \; 0 \leq t \leq 1 \). Let \( C' \) be the intersection of \( \Sigma \) and \( C \), and let \( S'_0 \) and \( S'_1 \) be \( S_0 \cap \Sigma \) and \( S_1 \cap \Sigma \), respectively. To a point \( y \in C' \), we choose \( t = \|y\| - 1 \) and then consider the point \( x \) determined by
\[
x = q_t(y \cap q_t(H^{m-1}(P(t)))).
\]
So in this manner for each \( y \in C' \) there corresponds a point \( x \in C \), and we shall call this correspondence the function \( x = q(y) \).
With \( \varphi(y) \) defined in this manner, we now wish to verify that it is a continuous function of \( y \). To begin with, \( \varphi_t \) is a rotation equal in magnitude to \( \varphi \) and rotates about the same line as \( \varphi \) but in the opposite direction, so it is clear that \( \varphi_t \) is a continuous function of \( P(t) \) and, therefore, a continuous function of \( t \). Then it follows from the definition of a continuous function that for any \( x_0 \in C \) and any given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) so that when \( |t - t_0| < \delta \) then \( |x - x_0| < \varepsilon \). If \( |t - t_0| < \delta \) it follows that \( ||y|| - ||y_0|| < \delta \). Furthermore, since \( \varphi_t \) is continuous there also exists a \( \delta' \) such that when

\[
|\{(\varphi_t \cap \varphi_t^{-1}(P(t)))\} - \{(\varphi_t \cap \varphi_t^{-1}(P(t_0)))\}| < \delta'
\]

then \( |x - x_0| < \varepsilon \). This says that the two points \( \{(\varphi_t \cap \varphi_t^{-1}(P(t)))\} \) and \( \{(\varphi_t \cap \varphi_t^{-1}(P(t_0)))\} \) are close together and therefore \( \angle y_0y \) must be small; in fact, it must be an angle less than \( \theta \) where \( \theta = \delta' \) (\( \theta \) is in radians and \( \delta' \) is the minimum radius of a point in \( C \)). Thus \( |y - y_0| < \delta + \delta' \), hence \( \varphi(y) \) is a continuous function of \( y \).

Moreover, if \( ||y|| = ||z|| \), then \( ||\varphi(y)|| = ||\varphi(z)|| \); and if \( ||y|| = ||z|| \) and \( Oy \) is perpendicular to \( Oz \), then so is \( O\varphi(y) \) perpendicular to \( O\varphi(z) \). These results easily follow because of the way the value of \( t \) is determined, and the nature of \( \varphi(y) \).

The closed set \( \varphi'(L_\varepsilon) \) intersects any continuous curve \( \gamma \) joining \( S_0' \) and \( S_1' \) in \( C' \) because \( \varphi(S_0') \) and \( \varphi(S_1') \) are joined by \( \varphi(\gamma) \), the former of them being inside or on \( L_\varepsilon \), the latter being outside or on \( L_\varepsilon \). Therefore, by the induction assumption \( \varphi'(L_\varepsilon) \) contains \( m \) points \( R_1', R_2', \ldots, R_m' \) such that
\[ \|R_1\| = \|R_2\| = \cdots = \|R_m\| = t_0 + 1, \]
and \(OR_1, OR_2, \ldots, OR_m\) are perpendicular to one another. Hence

\[ \|P(t_0)\| = \|\varphi(R_1)\| = \|\varphi(R_2)\| = \cdots = \|\varphi(R_m)\|, \]

and \(OP(t_0), O\varphi(R_1), O\varphi(R_2), \ldots, O\varphi(R_m)\) are also perpendicular to one another.

Let us put \(P(t_0) = Q_0[\varepsilon], \varphi(R_1) = Q_1[\varepsilon], \ldots, \varphi(R_m) = Q_m[\varepsilon]\).

We now take a sequence of \(\varepsilon_n\) converging to zero. For each \(n\) we have an \((m+1)\)-tuple,

\[
\left\{ Q_0[\varepsilon_1], Q_1[\varepsilon_1], \ldots, Q_m[\varepsilon_1] \right\}
\]

\[
\left\{ Q_0[\varepsilon_2], Q_1[\varepsilon_2], \ldots, Q_m[\varepsilon_2] \right\}
\]

\[
\vdots
\]

\[
\left\{ Q_0[\varepsilon_n], Q_1[\varepsilon_n], \ldots, Q_m[\varepsilon_n] \right\}
\]

Since the sequence of points is in a closed and bounded set \(C\), we can take a subsequence which converges in the first column to say \(Q_0\). Then from this subsequence we can choose another subsequence that converges in the second column to say \(Q_1\), and continuing this for \(m+1\) subsequences, we obtain the limit points \(Q_0, Q_1, \ldots, Q_m\). Clearly every \(Q_i \in L\), since \(L\) is closed. Also

\[ \|Q_0\| = \|Q_1\| = \cdots = \|Q_m\|, \]

and the \(OQ_i\)'s are perpendicular to one another. Thus the lemma is proved.

Now let us reflect back to the statement of Lemma 2A for a moment. We can consider a point \(P\) in the set \(C\) as a
point in the topological product space $J \times S^m$, where $J=[1,2]$ and $S^m$ is the $m$-dimensional unit sphere. Then $P$ can be represented as an ordered pair $(j,s)$ for $j \in J$, $s \in S^m$, and with $j=j(P)$ and $s=s(P)$ both being continuous functions of the point $P$ in $J \times S^m$. For example, take $s(P)$ to be the same for all points on a given ray, and $j(P)$ to be the same for all points on an $m$-sphere of a given radius in the interval $[1,2]$ and center $0$. In these terms Lemma 2A says that there exist points $P_0, P_1, \ldots, P_m$ such that

$$j(P_0)=j(P_1)=\ldots=j(P_m),$$

and the $s(P_i)$'s are perpendicular.

This space $J \times S^m$ is topologically equivalent to the space $I \times S^m$, where $I=[0,1]$ and $S^m$ is again the $m$-dimensional unit sphere. A point $P$ in $I \times S^m$ is represented then by $(t,s)$, $t \in I$, $s \in S^m$, where $t=t(P)$ and $s=s(P)$ are continuous functions of $P \in I \times S^m$. Thus, making use of this equivalence, the next lemma is a restatement of Lemma 2A.

**Lemma 2B**: Let $S^m_0$ and $S^m_1$ be the sets $\{ P \mid P \in I \times S^m, \ t(P)=0 \}$ and $\{ P \mid P \in I \times S^m, \ t(P)=1 \}$, respectively. If $L$ is a closed set in $I \times S^m$ which intersects every continuous curve that joins $S^m_0$ and $S^m_1$, then $L$ contains $(m+1)$ points $Q_0, Q_1, \ldots, Q_m$ such that

$$t(Q_0)=t(Q_1)=\ldots=t(Q_m),$$

and such that the points $s(Q_i)$'s are perpendicular to one another.

We will make use of this lemma in the proof of the main
theorem that now follows. For a real valued function $f(x)$ on $S^n$, there exist two points, $x(1)$, $x(0)$, with

$$\sup_{x \in S^n} f(x) = f(x(1)),$$
$$\inf_{x \in S^n} f(x) = f(x(0)).$$

These points exist because $f(x)$ is a continuous function defined on a closed and bounded set, and therefore attains its maximum and minimum on the set. We join $x(0)$ and $x(1)$ by a continuous curve $x(t)$ on $S^n$, $0 \leq t \leq 1$. We may consider $S^n$ as the unit sphere in an $(n+1)$-dimensional space $\mathbb{R}^{n+1}$ with origin $O$. For a point $P$ in $\mathbb{R}^{n+1}$ the co-ordinates of $P$ are $(p_1, p_2, \ldots, p_{n+1})$ and $||P||$ is the distance of $P$ from the origin. Again let $\mathcal{H}$ denote the hyperplane $\{P \mid p_{n+1} = 0\}$ and $E_i (0 \leq i \leq m)$ denote the point whose $i$-th co-ordinate is equal to 1 and the other co-ordinates are zero. As in the proof of Lemma 2A, let $H^{n-1}(P) = \{ Q \mid ||Q|| = ||P||, OQ \perp OP \}$.

We take again a rotation of the axis $\mathcal{R}_i$ such that

$$\mathcal{R}_i(x(t)) = E_{n+1},$$
\(\mathcal{R}_i\) being continuous. Then $\mathcal{R}_i(H^{n-1}(x(t))) \in \mathcal{H}$ for every $t$. Let $S^{n-1}$ be $S^n \cap \mathcal{H}$. Let us consider the topological product $I \times S^n$ of $I = [0, 1]$ and $S^n$. A point $P \in I \times S^n$ is represented by $(t, u)$, $t \in I$, $u \in S^n$ and where $t = t(P)$, $u = u(P)$ are both continuous.

We define $S^{n-1}_0$ and $S^{n-1}_i$ to be the sets $\{ P \mid P \in I \times S^{n-1}; t(P) = 0 \}$ and $\{ P \mid P \in I \times S^{n-1}; t(P) = 1 \}$, respectively.

Put

$$Q_u(P) = \Psi(P), \text{ and }$$
$$F(P) = f(x(t)) - f(\Psi(P)).$$
For these functions the value of $t$ is determined by the point $P$ in $I \times S^{n-1}$, since $t=t(P)$. For $P_1$ and $P_2$ such that $t(P_1)=t(P_2)$, if $u(P_1)$ is perpendicular to $u(P_2)$, then $\Psi(P_1)$ is perpendicular to $\Psi(P_2)$. Although $u(P)$ is a point on $S^{n-1}$, $\Psi(P)$ may not be, but it will be on $S^n$ since it is just a rotation about the origin. We have defined $x(t)$ as being on $S^n$ and $f(x)$ is a continuous function on $S^n$ by hypothesis, so the difference
\[ f(x(t(P))) - f(\Psi(P)) = F(P) \]
is a continuous real valued function. Let the set of zero points of $F(P)$ be $K$. We chose the point $x(1)$ to be a point where $f(x)$ takes on its largest value on $S^n$, therefore $F(P) \geq 0$ for $P \in S_{i}^{n-1}$. Likewise, $F(P) \leq 0$ for $P \in S_{o}^{n-1}$. Thus any curve which is drawn from $S_{o}^{n-1}$ to $S_{i}^{n-1}$ intersects $K$. And with this we have shown that the conditions of Lemma 2B are satisfied, so we can conclude that $K$ contains $n$ points $P_1, P_2, \ldots, P_n$ such that
\[ t(P_1)=t(P_2)=\ldots=t(P_n)=t^* \]
and such that the $u(P_i)$'s are perpendicular to one another. Since each of the $P_i$'s is in $K$, $F(P_i)=0=f(x(t^*)) - f(\Psi(P_i))$. Thus
\[ f(x(t^*))=f(\Psi(P_1))=f(\Psi(P_2))=\ldots=f(\Psi(P_n)) \]
and $\Psi(P_1), \Psi(P_2), \ldots, \Psi(P_n)$ are perpendicular since the $u(P_i)$'s are perpendicular and the $t(P_i)$'s are all equal. Also, $Q(x(t_*))$ is perpendicular to the hyperplane containing the set of $u(P_i)$'s, and therefore $x(t_*)$ is perpendicular to the set of $\Psi(P_i)$'s. Thus we have completed the proof of
the Yamabe-Yujobo Theorem.

For the particular case of \( n=2 \), we have THEOREM 1 as a corollary.

The next theorem is similar to Kakutani's theorem in that it pertains to a 3-dimensional case.

**THEOREM 3 (Dyson):** Let \( S^2 \) be the surface of a sphere, center \( Z \), in Euclidean 3-space \( \mathbb{R}^3 \), and let \( f(x) \) be a continuous real valued function defined on \( S^2 \). Then there exist four points \( x_1, x_2, x_3, x_4 \) on \( S^2 \) forming the vertices of a square with center \( Z \), such that

\[
f(x_1) = f(x_2) = f(x_3) = f(x_4) .
\]

**PROOF:** In order to complete the proof, we first consider two lemmas. The proof of Lemma 3B contains the essential part of the whole argument, namely, the application of the Yamabe-Yujobo method.

**LEMMA 3A:** Let \( \Lambda \) be a closed and bounded set of points in a Euclidean plane \( \mathbb{R}^1 \) such that there is a point \( Z \) not belonging to \( \Lambda \) and such that every continuous curve joining \( Z \) to infinity intersects \( \Lambda \). Let \( \Lambda' \) be the reflection of \( \Lambda \) in \( Z \). Then \( \Lambda \) and \( \Lambda' \) have at least one point in common.

Given any positive \( \varepsilon \), we can find a simple closed curve \( \Lambda_\varepsilon \), which satisfies the same conditions as \( \Lambda \), such that every point of \( \Lambda_\varepsilon \) is of distance less than \( \varepsilon \) from \( \Lambda \). To show the existence of \( \Lambda_\varepsilon \), we can use a technique similar to the determination of \( L_\varepsilon \) in Lemma 2A. Then if the lemma is proved for \( \Lambda_\varepsilon \), we can do the same for each \( \varepsilon \) in a decreasing sequence.
in which \( \epsilon_n \) goes to zero as \( n \) goes to infinity. For each \( n \) we would have a point \( x_n \) such that it is a common point of \( A_{\epsilon_n} \) and \( A'_{\epsilon_n} \). This gives a sequence of points \( \{x_n\} \) as an infinite subset of a compact set \( A^* \), where \( A^* \) is the closed and bounded set all of whose points are of distance less than or equal to \( 2\epsilon_1 \) from \( A \) (\( \epsilon_1 \) is the largest \( \epsilon \) in the decreasing sequence \( \{\epsilon_n\} \)). Therefore \( \{x_n\} \) must have an accumulation point, which will be in \( A \), since \( A \) is closed. Thus the lemma is true also for \( A \). So we need only prove it for a \( A \) which is a simple closed curve. Let \( P_1 \) and \( P_2 \) be points of \( A \) at the least and greatest distance from \( Z \). If \( P_1 \) or \( P_2 \) belongs to \( A' \) the lemma is true. If not, \( P_1 \) must lie inside \( A' \) and \( P_2 \) outside. But \( P_1 \) and \( P_2 \) are joined by a continuous arc of \( A \). This arc must cut \( A' \) in at least one point. In order to verify this statement, let us assume that this arc did not cut \( A' \). Now consider the reflection of this arc in \( Z \), this reflected arc then will not cut \( A \). The point \( P'_1 \) (the reflection of \( P_1 \) in \( Z \)) will lie inside of \( A \) and \( P'_2 \) outside of \( A \).

Then if we connect \( P'_1 \) to \( Z \) with a straight line and connect \( P'_2 \) to infinity with a straight line in the direction away from \( Z \), we will have connected \( Z \) to infinity with a continuous curve that does not intersect \( A \). And this is a contradiction of the conditions of the lemma. The lemma is proved.

The word "continuum", whenever it appears in Lemma 3B and thereafter, means a closed arcwise connected region. We now proceed to the next lemma.
LEMMA 3B. Let $L$ be a closed and bounded set of points in a Euclidean 3-space $R^3$, such that there is a point $Z$ not belonging to $L$, such that every continuous curve joining $Z$ to infinity intersects $L$. Let $L'$ be the reflection of $L$ in $Z$ and $D$ the intersection of $L$ and $L'$. Suppose that $D$ is a continuum. Then there exist two points $x_1$ and $x_2$ in $D$ equidistant from $Z$ and subtending a right angle at $Z$.

Let $P_1$ and $P_2$ be the points in $D$ at the least and the greatest distances from $Z$, respectively. Since $D$ is a continuum we can join $P_1$ and $P_2$ with a continuous arc $¥$ in $D$. We can extend $¥$ to a continuous arc

$$\Delta = \lambda_1 + ¥ + \lambda_2,$$

where $\lambda_1$ is the line segment $ZP_1$ and $\lambda_2$ is the half-line joining $P_2$ to infinity in the direction away from $Z$. The arc $\Delta$ can be parameterized by denoting its points by $P(t)$, $t$ being a real variable going from 0 to $\infty$ and $P(t)$ being a continuous function of $t$, $P(0)=Z$ and $P(t) \to \infty$ as $t \to \infty$.

For each value of $t$ let $H(t)$ be a circle with center $Z$ in a plane perpendicular to $ZP(t)$ and with radius $|ZP(t)|$.

Suppose $\Pi$ is a fixed plane through $Z$ and $K(t)$ is the circle in $\Pi$ with center $Z$ and radius $t$. Next we can define a map that maps $K(t)$ onto $H(t)$ by a rotation about $Z$ combined with a change of scale in the ratio $|ZP(t)|:t$. One way in which such a mapping could be accomplished would be to first rotate the plane of the circle $K(t)$ into the plane of the circle $H(t)$ using the line of intersection of the two planes as an axis. Then make the necessary change of scale radially.
As \( t \) goes from 0 to \( \infty \) the whole plane \( \mathcal{H} \) is mapped into the set of points \( \mathcal{H} \), the union of all the circles \( H(t) \). This set \( \mathcal{H} \) will be a 2-dimensional infinite surface with possibly some "wrinkles" centered about \( Z \). Let \( \Lambda \) be the set of points in \( \mathcal{H} \) which are mapped into points of \( L \). Thus \( \Lambda \) is the inverse image of this continuous mapping of the closed and bounded set \( L \cap \mathcal{H} \), and therefore \( \Lambda \) is closed and bounded. Furthermore, \( \Lambda \) satisfies all the conditions of Lemma 3A. For if there existed a continuous arc in \( \mathcal{H} \) joining \( Z \) to infinity without cutting \( \Lambda \), then the map of this arc would be a continuous arc in \( \mathcal{H} \) joining \( Z \) to infinity without cutting \( L \), which is a contradiction.

Applying Lemma 3A we conclude that there exists a point \( y \) in common to both \( \Lambda \) and \( \Lambda' \). So the point \( y \) and the reflection \( y' \) of \( y \) in \( Z \) both belong to \( \Lambda \). Since \( y \) and \( y' \) are equidistant from \( Z \) and diametrically opposite then there is a \( K \)-circle through \( y \) and \( y' \). Let \( K(t_0) \) be this circle. Our mapping we have described maps \( K(t_0) \) onto \( H(t_0) \) and thus maps \( y \) and \( y' \) onto two points \( x_1 \) and \( x'_1 \) on \( H(t_0) \). These points \( x_1 \) and \( x'_1 \) are reflections of each other in \( Z \) and both belong to \( L \), since they are images of points in \( \Lambda \). Therefore, they both belong to \( L \) and to \( L' \) and since \( D \) was the intersection of \( L \) and \( L' \), they both belong to \( D \). Now let \( x_2 \) be the point \( P(t_0) \). Since \( x_2 \) is located on \( H(t_0) \) and because of the way in which the circle \( H(t_0) \) was defined, the points \( x_1 \) and \( x_2 \) are equidistant from \( Z \) and subtend a right angle at \( Z \). Now all we have to show is that \( x_2 \) is in \( D \). Since \( x_1 \) is in \( D \) the distance
\[|Zx_4| \text{ is not less than } |ZP_1| \text{ nor greater than } |ZP_2|. \] Therefore \(x_2\) belongs to the part \(\gamma\) of the curve \(\Delta\) and not to \(\lambda_1\) or \(\lambda_2\). Thus \(x_2\) lies in \(D\), and the lemma is proved.

Now we are ready to consider the proof of the theorem itself. We suppose now that the conditions of the theorem are satisfied, namely, \(f(x)\) is a real valued continuous function defined on the 2-sphere \(S^2\).

We first show that the theorem will be true for a continuous function \(f_0(x)\), if it is true for each function \(f_n(x)\) of a sequence \(\{f_n(x)\}\) of continuous functions tending uniformly to \(f_0(x)\) on \(S^2\). Let us then suppose that the theorem is true for \(f_n(x)\) for each \(n\). Thus for each \(n\) we have four points, \(x_{1n}, x_{2n}, x_{3n}, x_{4n}\), that have the properties states in the theorem, namely, that the value of the function is the same at all four points, and the points form the vertices of a square with center \(Z\). We shall consider all of these points as an infinite sequence of quadruples,

\[
\begin{align*}
( & x_{11}, x_{21}, x_{31}, x_{41} ) \\
( & x_{12}, x_{22}, x_{32}, x_{42} ) \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \\
( & x_{1n}, x_{2n}, x_{3n}, x_{4n} ) \\
 & \vdots \quad \vdots \quad \vdots \quad \vdots \\
\end{align*}
\]

Since the infinite set of points is in a closed and bounded set \(S^2\), we can choose a subsequence that converges in the first column to say \(x_{10}\). From this subsequence we can choose another subsequence that converges in the second column to
say \( x_{20} \), and we continue taking subsequences of subsequences obtain the points \( x_{30} \) and \( x_{40} \). This limit set of four points, \( x_{10}, x_{20}, x_{30}, x_{40} \), have the properties stated in the theorem for the limit function \( f_*(x) \).

We have left to show that we can find such a sequence of functions, so that for each function in the sequence the theorem is true.

First, let us define three sets. For a continuous function \( f(x) \) on \( S^2 \), we let \( A \) be the open set of points on \( S^2 \) for which \( f(x) > f(x') \), where \( x' \) is the reflection of \( x \) in \( Z \). Then \( A' \) is the open set of points on \( S^2 \) for which \( f(x) < f(x') \). Let \( O \) be the closed set of points for which \( f(x) = f(x') \). Thus \( S^2 = A + A' + O \).

In order to find the required sequence, we may choose a sequence of functions such that for each function of the sequence \( O \) consists of only a finite number of continua. For example, each function in the sequence may be finite sums of spherical harmonics approximating to the given function. Therefore, we need to show that the theorem can be proved for a function for which \( O \) consists of only a finite number of continua, since our sequence will consist of functions of this type.

Let \( f(x) \) be such a function, that is, \( f(x) \) is a real valued continuous function on \( S^2 \) for which \( O \) consists of \( m \) continua. We shall use an induction on \( m \) to prove that the theorem is true for each \( m \).
The case \( m=0 \) is trivial, since there is no function that satisfies the conditions of the theorem with \( m=0 \).

When \( m=1 \), \( C \) is a continuum. In this case, we may suppose without loss of generality that \( f(x)>0 \). Then we map \( S^2 \) onto a closed and bounded set of points \( L \) in the following way. Each point \( x \) is mapped into the point lying in the line \( Zx \) at a distance \( f(x) \) from \( Z \). The set \( C \) is mapped onto \( D \), the intersection of \( L \) with \( L' \). Since \( C \) is a continuum and the mapping is continuous, \( D \) is also a continuum. Therefore, all the conditions of Lemma 3B are satisfied, and so there exist points \( y_1 \) and \( y_2 \) in \( D \) equidistant from \( Z \) and subtending a right angle at \( Z \). Let \( x_1 \) and \( x_2 \) be the points on \( S^2 \) which are mapped onto \( y_1 \) and \( y_2 \). Then \( x_1 \) and \( x_2 \) are in \( C \), so \( f(x_1)=f(x_2) \) and \( x_1Zx_2 \) is a right angle. Therefore, if \( x_3 \) and \( x_4 \) are the reflections of \( x_1 \) and \( x_2 \) in \( Z \), the four points \( x_1, x_2, x_3, x_4 \) have the properties stated in the theorem.

Suppose now that \( m>1 \), and assume that the theorem is true for all functions for which \( C \) consists of \( k \) continua, where \( k<m \). Then \( C \) is separated into two parts by some connected region \( R \) belonging to \( A \) or \( A' \). The reflection \( R' \) of \( R \) in \( Z \) is a region of \( A' \) or \( A \) which also separates \( C \) into two parts. Hence without loss of generality we suppose that \( R \) belongs to \( A \). Of course, \( A \) will not necessarily be a connected set, so we want to take \( R \) to be one of the connected components of \( A \), that separates \( C \) into two parts. \( R \) is not simply connected, since it encloses a part of \( C \). The complement of \( R \) on \( S^2 \) is a closed set \( Q \) consisting of components.
Q₁, Q₂, ..., Q_j that are continua, pair-wise separated from one another, with 2 ≤ j ≤ m. Each Q_i intersects the boundary of R, which is included in G; hence, each contains a component of G.

One of the Q_i, say Q_1, contains R'. If now x is any point in Q_2, x is separated from R' by R and x' is separated from R by R', and therefore x' belongs to Q_1. Thus the Q_i of Q_2 in Z is included in Q_1 and separated from Q_j by R.

We now define a new function f°(x) as follows. Let T be the set of points in S^e which are at a distance less than ε from Q_z. For a suitable ε > 0, T is an open set including Q_2 and separated by a positive distance from Q_1, Q_2, ..., Q_j. Let

\[ f°(x) = \begin{cases} f(x) & \text{for } x \notin T, \\ g(x) & \text{for } x \in T, \end{cases} \]

where g(x) is a continuous function satisfying:

\[ g(x) > \text{Max} \{ f(x), f(x') \}, \]

and such that f°(x) is continuous; this is possible since the boundary of T lies in A. Since Q_2 is contained in Q_1, f°(x') = f(x') for x in T. The set of points C° for which f°(x) = f°(x') is identical with C minus the parts of C belonging to Q_1 and Q_2. Thus the theorem is true for f(x) if it is true for f°(x). But for f°(x) the number of continua in C is m-2 at most. Therefore, employing our induction hypothesis, the theorem is true for all m.

Thus we have shown that we can find a sequence in which each function satisfies the theorem, and with this we have completed the proof of the theorem.
CHAPTER III
AN APPLICATION

Kakutani showed that for any bounded and closed convex set in $\mathbb{R}^3$ there exists a circumscribing cube about it. A cube in $n$-dimensional Euclidean space is a set consisting of all those points $x=(x_1, x_2, \ldots, x_n)$ for which $a_i \leq x_i \leq b_i$ for each $i$, where the numbers $\{a_i\}$ and $\{b_i\}$ are such that $b_i - a_i$ has the same value $e$ for each $i$, or a rotation of a set of this form. The number $e$ is the length of an edge of the cube. A face of the cube is a subset consisting of all those points $x=(x_1, x_2, \ldots, c_i, \ldots, x_n)$, where $c_i$ is equal to either $a_i$ or $b_i$ for some $i$. A cube is said to circumscribe a set $K$ if the cube contains $K$ and each face of the cube contains at least one point of $K$. Yamabe and Yujobo generalized Kakutani's result to $\mathbb{R}^n$.

Now we should like to generalize further by removing the restriction of convexity, which also removes the property of connectedness of the set. However, we shall use Kakutani's original method and the Yamabe-Yujobo theorem.

THEOREM 4: Let $K$ be any closed and bounded set in $\mathbb{R}^{n+1}$. Then there exists a circumscribing cube around $K$.

PROOF: Let $S^n$ be an $n$-sphere in $\mathbb{R}^{n+1}$ with origin 0 of $\mathbb{R}^{n+1}$ as center. For any point $P \in S^n$, consider the two tangent planes to $K$ (parallel to each other) which are perpendicular to the vector $OP$. These planes may coincide if $K$ is a flat set. Let $f(P)$ be the vertical distance of these two planes. Now
if we can show that $f(P)$ is a continuous function on $S^2$, then by THEOREM 2 there exist $n+1$ points $P_0, P_1, \ldots, P_n$ of $S^n$ perpendicular to one another such that

$$f(P_0) = f(P_1) = \ldots = f(P_n)$$

It is clear that the corresponding tangent planes form a cube which circumscribes the set $K$.

For convenience, in the proof that $f(P)$ is a continuous function, we can consider the set as rotating about some point $Z$ in the convex hull of the set. The convex hull of a set is defined as the intersection of all the convex sets containing the set. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two parallel planes, whose normals always have a fixed direction, tangent to $K$. The set is closed and bounded so we may let $A$ be the greatest distance from $Z$ of any point in the set. For a rotation $\theta$ of the set about $Z$, the displacement of plane $\mathcal{C}_1$ cannot be more than $A|\theta|$; similarly for $\mathcal{C}_2$. So, given any $\epsilon > 0$, we can choose $\delta = |\theta| = \epsilon/(2A)$ and thus make

$$|f(P) - f(P_0)| < A|\theta| + A|\theta| = (2A)(\epsilon/(2A)) = \epsilon.$$

Therefore, $f(P)$ is a real valued continuous function on $S^n$. 

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REFERENCES


