Numerical solution of the general surface wave problem

Jimmie Duane Dent

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NUMERICAL SOLUTION OF THE GENERAL SURFACE WAVE PROBLEM

by

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B.A., University of Montana, 1972

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A numerical technique was developed to solve the hydrodynamic equations associated with the free surface motion of a fluid. The technique was based on a unique procedure using integral equations to solve Laplace's equation. Laplace's equation became the governing equation of the fluid in the limit of incompressibility, irrotationality, and inviscid behavior. The dynamics of the system were governed by a non-linear differential equation that became the boundary condition for Laplace's equation on the surface.

The advantages of this method were that arbitrary shapes of both the surface and the reservoir container present only minimal problems, therefore general motions over general bottom profiles were easily modeled. Additional advantages were that computations were required only on the boundary of the fluid and a Courant condition limiting the maximum time step interval did not exist.

The method was applied to the run-up of a solitary wave on a vertical wall with good results when compared to other studies. However, application to the problem of a wave shoaling on a plane beach yielded somewhat unsatisfactory results due to certain inaccuracies in the current procedure.
TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>ii</td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. THE HYDRODYNAMICS OF WAVE MOTION</td>
<td>3</td>
</tr>
<tr>
<td>Kinematics</td>
<td>3</td>
</tr>
<tr>
<td>Velocity Potential</td>
<td>5</td>
</tr>
<tr>
<td>Equation of Motion</td>
<td>6</td>
</tr>
<tr>
<td>Conversion of Mass</td>
<td>8</td>
</tr>
<tr>
<td>Boundary Conditions</td>
<td>9</td>
</tr>
<tr>
<td>The Equation Set</td>
<td>11</td>
</tr>
<tr>
<td>III. SOLVING THE EQUATIONS OF WAVE MOTION</td>
<td>12</td>
</tr>
<tr>
<td>The Procedure</td>
<td>12</td>
</tr>
<tr>
<td>Initial Conditions</td>
<td>13</td>
</tr>
<tr>
<td>Summary</td>
<td>14</td>
</tr>
<tr>
<td>IV. THE SOLUTION OF LAPLACE'S EQUATION</td>
<td>16</td>
</tr>
<tr>
<td>Preliminaries</td>
<td>16</td>
</tr>
<tr>
<td>Integral Equation Formulation</td>
<td>17</td>
</tr>
<tr>
<td>Numerical Solution of the Integral Equation</td>
<td>21</td>
</tr>
<tr>
<td>Treatment of the Singularities</td>
<td>23</td>
</tr>
<tr>
<td>Analytic Integration of $\int \frac{3}{\alpha n} \log(\rho) , ds$</td>
<td>25</td>
</tr>
<tr>
<td>Result</td>
<td>27</td>
</tr>
<tr>
<td>V. NUMERICAL APPROXIMATION OF THE FREE SURFACE</td>
<td>28</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cubic Splines</td>
<td>28</td>
</tr>
<tr>
<td>Parametric Representation</td>
<td>29</td>
</tr>
<tr>
<td>Spline Derivation</td>
<td>30</td>
</tr>
<tr>
<td>Tri-diagonal Equation Solution</td>
<td>32</td>
</tr>
<tr>
<td>Geometrical Properties of the Free Surface</td>
<td>33</td>
</tr>
<tr>
<td>VI. RESULTS</td>
<td>35</td>
</tr>
<tr>
<td>FORTRAN Code</td>
<td>35</td>
</tr>
<tr>
<td>Solitary Wave</td>
<td>36</td>
</tr>
<tr>
<td>Wave Run-up on a Vertical Wall</td>
<td>38</td>
</tr>
<tr>
<td>Shoaling of a Solitary Wave on a Plane Beach</td>
<td>40</td>
</tr>
<tr>
<td>VII. SUMMARY</td>
<td>41</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>43</td>
</tr>
<tr>
<td>APPENDIXES</td>
<td>52</td>
</tr>
<tr>
<td>A. General surface wave problem FORTRAN IV code</td>
<td>52</td>
</tr>
<tr>
<td>B. Illustration of the output of the surface wave code</td>
<td>64</td>
</tr>
</tbody>
</table>

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## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. File ICDAT, initial conditions for a typical solitary wave</td>
<td>45</td>
</tr>
<tr>
<td>2. Results of solitary wave run-up study</td>
<td>45</td>
</tr>
</tbody>
</table>

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## LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Solitary wave</td>
<td>46</td>
</tr>
<tr>
<td>2.</td>
<td>Solitary wave profiles after reflection from vertical boundaries</td>
<td>46</td>
</tr>
<tr>
<td>3.</td>
<td>Solitary wave oscillating between vertical walls</td>
<td>47</td>
</tr>
<tr>
<td>4.</td>
<td>Solitary wave run-up and reflection from a vertical wall</td>
<td>48</td>
</tr>
<tr>
<td>5.</td>
<td>Wave run-up on a vertical wall (R/h)</td>
<td>49</td>
</tr>
<tr>
<td>6.</td>
<td>Wave run-up on a vertical wall (R/d)</td>
<td>49</td>
</tr>
<tr>
<td>7.</td>
<td>Solitary wave shoaling on a plane beach</td>
<td>50</td>
</tr>
<tr>
<td>8.</td>
<td>Solitary wave shoaling on a plane beach</td>
<td>50</td>
</tr>
<tr>
<td>9.</td>
<td>Solitary wave shoaling on a plane beach showing the numerical instability</td>
<td>51</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

Waves on the surface of a liquid is a phenomenon that is familiar to everyone. In fact the analogy to water waves is quite often invoked in the description of wave motion encountered elsewhere in physics. However, as in most problems relating to the real world, the equations describing fluid motion are quite complex. These equations are further complicated by the unknown position of the free surface and the relatively complex boundary conditions to be applied there. As a result, numerical techniques are mandatory to find solutions to problems that would otherwise prove to be unmanageable.

Detailed in this paper is a computational procedure for solving the general problem of waves propagating on the surface of a liquid over an arbitrary bottom profile. Although some approximations to the physical properties of the fluid will be imposed, there will be practically no restrictions on the scale or relative size of the resulting motion. What will be considered is the motion of a so-called perfect fluid. That is, the fluid is assumed to be incompressible and inviscid, and its motion is assumed to be irrotational. These assumptions, although somewhat restrictive, are quite reasonable for most water wave motions. However, breaking or curling of
waves is precluded because of the discontinuities that the resulting turbulence produces. What results is a procedure of quite general applicability, enabling a very large range of important wave phenomena, from tidal motions to small scale wind-generated waves, to be examined over a large domain of differing bottom topographies.
CHAPTER II

THE HYDRODYNAMICS OF WAVE MOTION

The derivations following in this chapter are standard and generally can be found in any hydrodynamics text, the first 15 pages of Stoker's book Water Waves is a good example.

Kinematics

The basic hydrodynamics system which will be solved is that of a reservoir, denoted $C_1$, filled with a liquid such as water which has an unconstrained free surface, denoted $C_2$ and described by a function $\eta(x,z,t)$.

It is first necessary to distinguish between the two common methods of describing fluid motion. The Eulerian system is

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referenced to a particular coordinate system fixed outside the moving fluid. The parameter denoting which part of the fluid that is being described is then the outside grid point \((x,y,z)\). Time derivatives, denoted \(\frac{\partial}{\partial t}\), are thus taken while maintaining the fixed position \((x,y,z)\) as the fluid flows by.

In the Lagrangian system each parcel of fluid is given its own arbitrary identity as is described by a function \(\xi\), which it keeps as the fluid flows through time. A property of the fluid is thus described as a function of the particular particle of interest as it moves with the fluid. The particle path, for example, is given by \(x(\xi)\). Therefore, the Lagrangian time derivative is just the total time derivative, \(\frac{d}{dt}\), taken for fixed \(\xi\), while following the fluid motion.

The relation between the two systems can be seen by first noting that

\[
\frac{d}{dt} x = v \quad \text{and} \quad \frac{d}{dt} t = 1.
\]

Hence, for an arbitrary function \(F(x,t)\),

\[
\frac{d}{dt} F(x,t) = \frac{dx}{dt} \cdot \nabla F + \frac{\partial}{\partial t} F \frac{dt}{dt}
\]

\[
= v \cdot \nabla F + \frac{\partial F}{\partial t}.
\]

This is the resulting general relation between the Eulerian and the Lagrangian time derivatives.

Applying this relation, for example, to the function \(\eta(x,z,t)\) that describes the position of the free surface yields:
\[
\frac{d}{dt} n(x,z,t) = \mathbf{v} \cdot \nabla \eta + \frac{\partial \eta}{\partial t}
\]

or

\[
v_y = v_x \frac{\partial \eta}{\partial x} + v_z \frac{\partial \eta}{\partial z} + \frac{\partial \eta}{\partial t} \quad (\text{II-2})
\]

where \(v_x, v_y, \) and \(v_z\) are respectively the \(x, y, \) and \(z\) components of the velocity of a particle on the surface, and \(\frac{\partial \eta}{\partial x}\) and \(\frac{\partial \eta}{\partial z}\) are the perpendicular slopes of the surface. This equation is thus the Eulerian equivalent of the general kinematic relation expressed in Lagrangian coordinates as

\[
\frac{dx(\xi)}{dt} = v_x \quad \text{and} \quad \frac{dy(\xi)}{dt} = v_y \quad . \quad (\text{II-3})
\]

It is either of these, equations (II-2) or (II-3), that may be integrated to determine the motion of the free surface from a knowledge of the time dependence of the velocity components. The use of which equation depends upon the convenience that one of the coordinate systems might have for a particular application. More will be said later about this decision as it relates to the water wave problem.

**Velocity Potential**

The irrotationality condition, which physically means that there are no eddies or vortices, is expressed by the property that the curl of the velocity everywhere vanishes, i.e.

\[
\mathbf{\nabla} \times \mathbf{v} = 0
\]

This in turn implies that the circulation of the fluid is also zero. Also, for an inviscid fluid, as long as the free surface remains
simply connected, the circulation can be shown to be a constant of the motion (Stoker, p. 7). Hence, it need only be shown that the curl of the velocity vanishes everywhere at some time to ensure that it will vanish everywhere for all time. This may easily be done, for example, by assuming that the wave progresses onto a surface that was initially at rest.

Thus given the vanishing of the curl of the velocity, a necessary and sufficient condition for a scalar velocity potential to exist, will be satisfied. Irrotationality, therefore, allows the velocity to be expressed as the gradient of a scalar function.

\[ \mathbf{v} = -\nabla \phi \]  

(II-4)

**Equation of Motion**

By now applying Newton's second law to an elemental mass of fluid, the equation of motion governing the fluid flow can be obtained.

\[ \rho \frac{d\mathbf{v}}{dt} = F_p + F_g + F_d \]

Where \( \rho \) is the mass density of the fluid, \( F_p \), \( F_g \), and \( F_d \) are respectively the forces per unit volume due to pressure gradients, gravity, and any external driving forces. Assuming that the driving force is derivable from a potential function, as is the force of gravity, substitution into the force equation gives:

\[ \rho \frac{d\mathbf{v}}{dt} = -\nabla p - \rho \nabla \Omega - \rho \nabla \Psi \]

where \( p \) is the pressure, \( \Omega \) is the gravity potential, and \( \Psi \) is the
driving force potential. After expanding the Lagrangian time derivative and substituting the velocity potential, the equation becomes:

\[ \rho \frac{\partial}{\partial t} (\nabla \phi) + \rho \nabla \cdot \nabla \psi = -\nabla p - \rho \nabla \Omega - \rho \nabla \psi \]

which can further be simplified by noting that since \( \nabla \times \nabla = 0 \)

\[ \nabla \cdot \nabla \psi = \nabla \psi \cdot \nabla = \nabla (\frac{1}{2} \psi^2) \]

Using this substitution, the assumption of incompressibility making \( \rho \) a constant, and the knowledge that the two operators \( \nabla \) and \( \frac{\partial}{\partial t} \) commute, the equation of motion finally becomes:

\[ \nabla \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + \frac{p}{\rho} + \Omega + \psi \right) = 0 \]

or, after integrating

\[ \frac{\partial \phi}{\partial t} = - \frac{1}{2} v^2 - \frac{p}{\rho} - \Omega - \psi + F(t) \]

where \( F(t) \), the constant of integration, will be absorbed into the velocity potential term. This is possible, without loss of generality, because the velocity depends only upon the spatial gradient of the potential (Stoker, p. 10).

Substituting \( g y \) for the gravity potential and dropping the external force term, the equation can be written in the form generally referred to as Bernoulli's equation:

\[ \frac{\partial \phi}{\partial t} = - \frac{1}{2} v^2 - \frac{p}{\rho} - g y \quad (\text{II-5}) \]

This is the dynamical equation in Eulerian coordinates that the fluid flow must satisfy. A Lagrangian equivalent to this equation can also be written by using the relation given in Equation (II-1)
to substitute for $\frac{\partial \phi}{\partial t}$ to yield

$$\frac{d\phi}{dt} - \mathbf{v} \cdot \nabla \phi = -\frac{1}{2} \mathbf{v}^2 - \frac{p}{\rho} - g y$$

Then, after expanding the $\nabla \phi$ term to $v^i_x \hat{i} + v^j_y \hat{j} + v^k_z \hat{k}$, the equation becomes

$$\frac{d\phi}{dt} - (v^2_x + v^2_y + v^2_z) = -\frac{1}{2} \mathbf{v}^2 - \frac{p}{\rho} - gy$$

which then gives for the Lagrangian form of Bernoulli's equation

$$\frac{d\phi(\xi)}{dt} = \frac{1}{2} v(\xi)^2 - \frac{p}{\rho} - g y(\xi) \quad (II-6)$$

As will be noted by examination, Bernoulli's equation is a non-linear partial differential equation in two independent variables, $\phi$ and $p$. Hence even if the equation were linear, it does not contain enough information to uniquely determine the fluid motion. Another constraint on the fluid is necessary. This is usually provided by the equation of continuity.

Conservation of Mass

The equation of continuity is a statement of the conservation of mass. That is, the change in mass in some enclosed volume $V = \int d^3 x$ is equal to the amount of mass flowing through the boundary $d\sigma$ of $V$ at each instant in time, i.e.

$$\frac{\partial}{\partial t} \int \rho \, d^3 x = \oint \rho \, \mathbf{v} \cdot d\sigma$$

Interchanging integration and differentiation on the left and applying the divergence theorem on the right gives

$$\int \frac{\partial \rho}{\partial t} \, d^3 x = \int \mathbf{v} \cdot (\rho \mathbf{v}) \, d^3 x$$
And since the volume element is arbitrary, the integrands can be set equal to each other, resulting in the equation of continuity in Eulerian form.

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \mathbf{v})$$

For an incompressible fluid, since $\rho$ is a constant, the continuity equation further simplifies to

$$\nabla \cdot \mathbf{v} = 0$$

Using the definition of velocity potential, it is thus seen that the constraint on the fluid motion takes the form of Laplace's equation in $\phi$.

$$\nabla \cdot \mathbf{v} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \quad (II-7)$$

**Boundary Conditions**

The use of Laplace's equation makes possible a nice procedure for determining the fluid motion. For through Laplace's equation, $\phi$ can be determined everywhere within the fluid by knowing only its values or its derivatives at the boundary of the fluid at each instant in time (Courant and Hilbert, p. 223). The spatial derivatives then give the velocity components and hence through Equation (II-2) or (II-3) the motion. It remains then to apply the previously derived equation of motion to the boundaries to determine the boundary conditions on $\phi$.

It is first observed that the pressure term in Bernoulli's equation is a constant over the free surface. It may then be absorbed into the velocity potential term as was the constant of
integration previously, so that on the surface the Eulerian form of
Bernoulli's equation becomes
\[ \frac{\partial \phi}{\partial t} = -\frac{1}{2} v^2 - g n \] (II-8)

Similarly for the Lagrangian form
\[ \frac{d\phi(\xi)}{dt} = +\frac{1}{2} v^2(\xi,t) - g y(\xi,t) \] (II-9)

Where \( y(\xi,t) \) now refers to a point on the fluid surface.

Bernoulli's equation now has the form of a differential
condition in the velocity potential. Hence knowing the initial value
of \( \phi \) and the time history of \( v \) and \( n \) the value of \( \phi \) at time \( t=t'=t+\Delta t \) may be obtained by a straightforward integration. Given an
initial \( n \), subsequent time values of this function may be found by
integrating Equation (II-2) or (II-3), with the function \( v \) coming
from the value of \( \phi \) just prior to \( t=t' \) through Equation (II-4).
This results in a real time dependent condition on \( \phi \) at the surface
that can be used as a boundary condition in the solution of Laplace's
equation. The initial values of \( \phi \) and the surface profile will be
the parameters that initiate the integration process and thus will
be the parameters that determine the resulting motion.

At the fluid-reservoir interface the boundary condition is
simply that there is no fluid transported through the interface,
that is, that the normal velocity of the fluid must vanish.
\[ 0 = v_n = -\frac{\partial \phi}{\partial n} = -\nabla \phi \cdot n \] (II-10)

Where \( n \) is the unit normal to the reservoir pointing outward from
the fluid.
The Equation Set

The hydrodynamical fluid equations thus governing the motion of a liquid with a free surface are:

The Laplacian of the velocity potential

\[ \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \]  

(II-11)

with Bernoulli's equation providing the dynamic boundary condition at the surface, the integrated form of either

\[ \frac{\partial \phi}{\partial t} = -\frac{1}{2} v^2 - g \eta \]  

(II-12a)

or

\[ \frac{d\phi(x,t)}{dt} = +\frac{1}{2} v(x,t)^2 - g y(x,t) \]  

(II-12b)

and the no flow through condition at the reservoir boundary.

\[ \frac{\partial \phi}{\partial n} = \nabla \phi \cdot n = 0 \]  

(II-13)

As initial conditions to Bernoulli's equation, the surface profile, \( y(x,t) \) or \( \eta \), and the values of \( \phi \) on the surface must be given at time zero. Finally the surface must obey the kinematic equation.

\[ \frac{\partial \eta}{\partial t} = v_y - v_x \frac{\partial \eta}{\partial x} - v_z \frac{\partial \eta}{\partial z} \]  

(II-14a)

or

\[ \frac{dx(\xi)}{dt} = v_x, \quad \frac{dy(\xi)}{dt} = v_y \]  

(II-14b)
CHAPTER III

SOLVING THE EQUATIONS OF WAVE MOTION

The Procedure

From the results of the previous section it is seen that in order to determine the motion of the free surface at any instant in time, that is, to integrate either of Equations (II-14), a knowledge of the three components of the surface velocity at this time is required. This knowledge, obtained from the spatial derivatives of the velocity potential, can be secured by solving Laplace's equation, subject to the aforementioned boundary conditions appropriate during this time interval. It is this equation, with its attending dynamic boundary condition, that makes analytic methods impossible and hence numerical methods mandatory if any type of generality in the motion is desired.

The numerical technique employed here for solving Laplace's equation is particularly well suited for the surface wave problem. It makes use of the property that the boundary conditions uniquely determine the potential and its spatial derivatives at all points within the fluid, and hence on the boundary itself (Courant and Hilbert, p. 223). Consequently the problem is formulated so that variables on the boundary only are calculated. This saves having to compute the interior variables which, for conventional finite difference techniques, are normally needed but are not pertinent.
to wave motion. In this case the variable that must be determined is the normal velocity of the surface. The tangential velocities are already known from the differentiation of the boundary condition on \( \phi \) with respect to perpendicular arc lengths along the surface. These three perpendicular velocity components may then be transformed into the three components of the velocity in the \( x, y, \) and \( z \) directions by a suitable rotation of the coordinate axes.

Being able to solve Laplace's equation for the components of the surface velocity at each instant in time allows the position of the surface to be integrated forward in time. Also then the new boundary condition at the surface, through Bernoulli's equation, can be updated. Hence, the conditions are again appropriate for the new velocities to be calculated through Laplace's equation. This results in a numerical procedure for marching the position of the free surface of the fluid through time. It should be noted here that, as will be explained later, the mathematical and computational machinery is available to get inside the fluid to calculate those parameters, such as velocity and pressure, that may be desired.

**Initial Conditions**

The final information that is required to make the problem work is the initial condition to be applied to Bernoulli's equation. In other words, the initial values of \( \phi \) and the surface profile. The profile must be given, but the function \( \phi \) is most readily calculated from a given knowledge of the initial velocity of the surface.
Given an initial value of the normal velocity, \( \frac{\partial \phi}{\partial n} \) along the surface, Laplace's equation can be solved using this as the surface boundary condition. The solution of the resulting Neumann problem will then determine a value of \( \phi \) along the surface. This value can then be used as an initial value to be inserted into Equation (11-12) along with the given initial conditions on the velocity and position, to determine the initial boundary value of \( \phi \). Thus a value of the velocity potential to be inserted as the first boundary condition on \( \phi \) at the surface can be determined.

**Summary**

An outline of the computational procedure used to determine the surface wave motion is now given below.

I. Assume given at time \( t \) an initial value of the normal velocity, \( v_n = \frac{\partial \phi}{\partial n} \). Also assume given, the initial position of the surface.

II. Solve Equation (II-11) \((\nabla^2 \phi = 0)\) for the velocity potential, on the surface, subject to Equation (II-13) \((\frac{\partial \phi}{\partial n} = 0)\) at the reservoir boundary, and \( \frac{\partial \phi}{\partial n} = v_n \) on the surface.

III. Differentiate \( \phi \) along the surface to find the tangential velocity components. Then rotate the coordinate system to find \( v_x \), \( v_y \), and \( v_z \).

IV. Integrate the kinematic Equation (II-14) over the interval \( dt \), using values of the old surface profile and the new found velocity components to find the new position of the surface at time \( t + dt \).
CHAPTER IV

THE SOLUTION OF LAPLACE'S EQUATION

Preliminaries

The real effort in the solution of the surface wave problem is derived from having to solve Laplace's equation at each time interval. The rest of the procedure is rather straightforward finite differencing and numerical integration. It is thus the solution of the mixed boundary value problem of Laplace that makes up the bulk of the numerical code.

The numerical method used to solve Laplace's equation is of a unique design. Rather than finite differencing, the method is to approach the solution through an integral equation. It is well known that integration is a better defined and more accurate numerical process than is differentiation, which often has inherent stability problems. The real value of the integral equation procedure however is evidenced when complicated contours or boundary regions are being modeled. Finite differencing necessitates a numerical interpolation technique to treat those parts of the boundary that may not fall on one of the differencing grid mesh points. This interpolation procedure is even further complicated in this case by the presence of the moving free surface boundary (see, for example, Chan and Street). As will be seen, the integral equation procedure suffers none of these difficulties. Finally, as was pointed out

16
earlier, finite differencing requires that the velocity potential be computed at each of the internal grid points, whereas the integral equation formulation only requires computations to be performed at the fluid boundary. Therefore, for the particular problem of surface wave motion, the integral equation approach enjoys distinct numerical advantages over traditional finite differencing methods.

**Integral Equation Formulation**

While generally applicable to the full three dimensional problem, the numerical integral equation formulation for the solution of Laplace's equation will, because of compactness and time limitations, be discussed not only in two dimensions. All equations and results derived in this two dimensional coordinate system have direct analogies in three dimensions (see, for example, Chapter IV, Courant and Hilbert); hence the conversion to a full 3-D coordinate system could be carried out in a quite straightforward manner.

The integral equation technique for the solution of the two dimensional boundary value problem starts with a given contour \( C \), enclosing a two dimensional region \( \sigma \). It is necessary to find a function \( \phi \) which satisfies \( \nabla^2 \phi = 0 \) within \( \sigma \) and also satisfies a set of given boundary conditions on \( C \).

Using Green's secondary identity in two dimensions (Courant and Hilbert, p. 256).

\[
\iint_{\sigma} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d\sigma = \oint_{C} (\phi \nabla \cdot \nabla \psi - \psi \nabla \cdot \nabla \phi) \cdot n \, ds \quad (IV-1)
\]
where \( \mathbf{n} \) is again the outward normal to \( C \) and \( ds \) is an element of arc length along \( C \). First let \( \phi(x,y) \) be the desired solution to Laplace's equation, i.e.

\[
\nabla^2 \phi = 0
\]

Then let \( \psi = \log(\rho) \) where \( \rho \) is now the magnitude of the vector from a point \((\xi,\eta)\) on the contour \( C \) to a point \((x,y)\) in \( \sigma \).

\[
\rho = (x - \xi) \hat{i} + (y - \eta) \hat{j}
\]

so that

\[
\rho \equiv |\rho| = [(x - \xi)^2 + (y - \eta)^2]^{1/2}
\]

\( \psi \) is therefore also a solution of Laplace's equation except possibly in the immediate neighborhood of \((x,y)\) where \( \log(\rho) \) becomes unbounded.

The left side of Equation (IV-1) thus reduces to zero at all points within \( \sigma \) except around \((x,y)\). Let this region around \((x,y)\) become an infinitesimally small circle of radius \( \varepsilon \), and then let the integration be performed.

\[
\iint \phi \nabla^2 \psi \, d\sigma = \iint_{\varepsilon} [\phi \nabla^2 \psi] d\sigma
\]

\[
= \phi(x,y) \iiint_{\varepsilon} [\nabla \cdot \nabla \psi] d\sigma
\]

\[
= \phi(x,y) \oint_{\varepsilon} [\mathbf{n} \cdot \nabla \psi] ds
\]

\[
= \phi(x,y) \oint_{\varepsilon} [- \frac{\partial \psi}{\partial \rho} (- \rho \, d\omega)]
\]

\[
= \phi(x,y) \oint_{\varepsilon} d\omega
\]

Use was made of the two dimensional divergence theorem, the fact that \( \rho = \varepsilon \), and that \( \mathbf{n} \) is in the direction of \(-\rho\).
$\omega$ is then defined to be the angle formed by the vector $-\rho$ and the $x$ axis as the integration is carried out around $ds$. That means

$$\iint \phi \sqrt{\psi} \, d\sigma = 2\pi \phi(x,y) \quad (x,y) \text{ in } \sigma \quad \text{(IV-4)}$$

that since the total angle subtended by $-\rho$ is $2\pi$ as the integration around $ds$ is taken. If $(x,y)$ happens to be on $C$, then the integration is performed over the domain of a semicircle of radius $\varepsilon$, and the result being a total angle of $\pi$ being subtended by $-\rho$. Thus:

$$\iint \phi \sqrt{\psi} \, d\sigma = \pi \phi(x,y) \quad (x,y) \text{ on } C \quad \text{(IV-5)}$$

Finally if the point $(x,y)$ coincides with a corner that makes an angle $\alpha$, integration gives

$$\iint \phi \sqrt{\psi} \, d\sigma = \alpha \phi(x,y) \quad (x,y) \text{ on } C \text{ corner} \quad \text{(IV-6)}$$

It is therefore seen that for a point $(x,y)$ in the interior of $\sigma$ Equation (IV-1) becomes
\[
\phi(x,y) = \frac{1}{2\pi} \int_C (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} \, ds
= -\frac{1}{2\pi} \int_C \left[ \log(\rho) \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \log(\rho) \right] ds
\]  

(IV-7)

This is an equation that can be numerically integrated for any point \((x,y)\) and will yield the solution to Laplace's equation provided that all along the boundary \(C\) the two functions \(\phi\) and \(\frac{\partial \phi}{\partial n}\) are known. As part of the formulation of the problem, that is, given the boundary conditions, the value of one but not both of these functions will be known at each point of \(C\). Thus a method of determining the other boundary function must be found.

Consider Equation (IV-1) with the point \((x,y)\) now on the boundary \(C\). Using the result of Equation (IV-5), the integral becomes

\[
\phi = -\frac{1}{\pi} \int_C \left[ \log(\rho) \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \log(\rho) \right] ds \quad .
\]  

(IV-8)

The value of the left side of the equation is now just a particular element of the boundary function \(\phi\). What results is a Fredholm integral equation that can be solved to determine the boundary function \(\phi\) or \(\frac{\partial \phi}{\partial n}\) not given by the original boundary condition (Kantorovich and Krylov, p. 97). Once this unknown boundary function is found, it, along with the given condition, may be used in Equation (IV-7) to find the value of the potential function anywhere within \(\sigma\). Also to be noted is that Equation (IV-7) may be spatially differentiated, the differentiation only operating on the \(\log(\rho)\) terms under the integral, thus producing an equation for the determination of the directional derivatives of the potential.
Getting back to the water wave problem where it is necessary to determine $\frac{\partial \phi}{\partial n}$ on the free surface, integral Equation (IV-8) can be seen to be the only equation that must be solved.

**Numerical Solution of the Integral Equation**

To solve Equation (IV-8) let the boundary $C$ be broken into two parts. Let $C_1$ denote the part of the boundary that is in contact with the reservoir and hence has the boundary condition $\frac{\partial \phi}{\partial n} = 0$. Let $C_2$ denote that part of the boundary corresponding to the free surface with the boundary condition given by Equation (II-12). Now subdivide the entire boundary into discrete segments, having $N_1$ subintervals on $C_1$, and $N_2$ subintervals on $C_2$. Denote these individual segments by $S_i$, where $i = 1$ to $N$ and $N = N_1 + N_2$. Equation (IV-8) can then be written:

$$\phi \simeq -\frac{1}{\pi} \sum_{i=1}^{N} \int_{S_i} [\log(\rho) \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \log(\rho)] \, ds$$  \hspace{1cm} (IV-9)

If the subintervals are made small enough, the functions beneath the integral will be nearly constant and hence may approximately be removed from the integral.

$$\phi \simeq -\frac{1}{\pi} \sum_{i=1}^{N} \left\{ [\log(\rho)]_i \left( \frac{\partial \phi}{\partial n} \right)_i - \phi_i \left( \frac{\partial}{\partial n} \log(\rho) \right)_i \right\} \, ds_i \ .$$  \hspace{1cm} (IV-10)

The subscripted variables refer to some average value, say the value at the center of the subinterval, and $ds_i$ is the arc length over that subinterval. This procedure will be recognized as merely being an application of the open two point Newton-Coates quadrature rule.
to contour integration (Pennington, p. 143). Of course higher order quadrature rules do exist, but for simplicity the two point rule will be used here. It should be noted though that for the case where the contour does not have any sharp corners, so that the unit normal is a continuous function, it is known that the two-point quadrature for solving contour integrals has the same accuracy as any of the higher order quadrature formulas (Isaacson and Keller, p. 340).

Equation (IV-10) thus becomes an equation in N unknowns, those values of \( \phi_i \) on \( C_1 \) (\( i = 1, N_1 \)), and the values of \( \left( \frac{\partial \phi}{\partial n} \right)_i \) on \( C_2 \) (\( i = N_1 + 1, N \)). Known are the N values of \( \left[ \log(\rho) \right]_i \) and \( \left[ \frac{3}{\partial n} \log(\rho) \right]_i \), the values of \( \phi_i \) on \( C_2 \) and the values of \( \left( \frac{\partial \phi}{\partial n} \right)_i \) on \( C_1 \). If Equation (IV-10) is now consecutively applied to each of the N subintervals, that is, if \((x, y)\) is allowed to assume in turn the position of the center of each subinterval, a system of N equations will result.

The value of \( \phi \) on the left side of Equation (II-10) will coincide with either a sought or a known boundary value of \( \phi_i \). Letting \( j \) denote which of the N positions that \((x, y)\) has taken on the contour, the N equations in N unknowns take the form:

\[
\phi_j = - \frac{1}{\pi} \sum_{i=1}^{N} \left[ \log(\rho) \right]_{ij} \left( \frac{\partial \phi}{\partial n} \right)_{i} - \phi_i \left[ \frac{3}{\partial n} \log(\rho) \right]_{ij} ds_i \quad (IV-11)
\]

An independent system of linear equations of this kind, for which there are an equal number of equations and unknowns, may usually be solved. Thus the \( N_2 \) unknown values of \( \frac{\partial \phi}{\partial n} \) on the surface may be determined along with the \( N_1 \) values of \( \phi \) at the reservoir boundary by inverting the linear system of equations.
A careful look at Equation (IV-11) however, reveals that there are apparent singularities in the \( \log(p) \) and the \( \frac{\partial}{\partial \eta} \log(p) \) terms when the indices coincide. A way must be found to treat these two terms to make the linear system complete.

**Treatment of the Singularities**

First look at the \( \log(p) \) term. Going back to Equation (IV-9) it is seen that \( \log(p) \) is part of an approximation for \( \int_{S_j} \log(p) \, ds \) when \((x, y)\) is at the center of the \( S_j \) interval. Taking this integral and assuming that \( S_j \) is small so that \( ds \) can be approximated by \( dp \), the integral becomes:

\[
\int_{S_j} \log(p) \, dp = \int_{-ds/2}^{ds/2} \log(p) \, dp
\]

\[
= p \left[ \log|p| - 1 \right]_{-ds/2}^{ds/2}
\]

\[
= ds_j \left[ \log(ds_j/2) - 1 \right] \quad (IV-12)
\]

The first apparent singularity is thus resolved.

For the second singularity it is necessary to look at the mathematical limit of \( \frac{\partial}{\partial \eta} \log(p) \) as \( p \) goes to zero. First define a local coordinate system in the neighborhood of the apparent singularity so that \( x \) is tangent to, and \( y \) is perpendicular to, \( C \) at that point, as shown below. Then expanding the equation of the contour, represented in this coordinate system as \( y(x) \), in a Taylor series about \( y=0 \) and noting that \( y(0)=0 \) and \( \frac{dy}{dx}(0)=0 \), \( y(x) \) becomes...
\[ y(x) = y(0) + x \frac{dy}{dx}(0) + x^2/2 \frac{d^2y}{dx^2} + \ldots \]

\[ = \frac{x^2 d^2y}{dx^2} + \ldots \]

Now rewriting \( \frac{\partial}{\partial n} \log(\rho) \) in this coordinate system making use of the definition of \( \rho \), Equation (IV-3), and the fact that the normal is just in the opposite direction from \( y \), yields the following result,

\[ \frac{\partial}{\partial n} \log(\rho) = \frac{\partial}{\partial y} \log[(x^2 + y^2)^{1/2}] \]

\[ = -\frac{1}{2} (x^2 + y^2)^{-1} \frac{\partial}{\partial y}(x^2 + y^2) \]

\[ = -\frac{y}{x^2 + y^2} \]

\[ = -\frac{x^2 d^2y(0)}{2 dx^2} \{x^2[1 + \frac{x^2}{4} (\frac{d^2y(0)}{dx^2})^2 + \ldots]\} + \text{higher order terms} \]

which in the limit becomes

\[ \lim_{\rho \to 0} \frac{\partial}{\partial n} \log(\rho) = \lim_{x \to 0} -\frac{1}{2} \frac{d^2y}{dx^2}(0) / [1 + \frac{x^2}{4} (\frac{d^2y(0)}{dx^2}) + \ldots] + \ldots \]
\[ = -\frac{1}{2} \frac{d^2 y}{dx^2}(0) = \frac{1}{2} k \]  

(IV-13)

where \( k \) is the curvature of \( C \) at the point \( j \), where \( \rho = 0 \).

Having now taken care of the apparent singularities, Equation (IV-11) becomes:

\[ (-\pi + \frac{1}{2} k \int ds_j) \phi_j + \sum_{i=1}^{N} \phi_i \left[ \frac{\partial}{\partial n} \log(\rho) \right]_{ij} ds_i = \]

\[ \sum_{i \neq j}^{N} \phi_i \left[ \log(ds_j/2) - 1 \right] + \sum_{i=1}^{N} \left( \frac{\partial}{\partial n} \right)_i \log(\rho) \right]_{ij} ds_i. \]  

(IV-14)

\( j=1,N \)

This has, as mentioned before, the form of a linear system of independent equations, the inversion of which can be carried out now by any suitable linear equation solving routine, such as Gaussian elimination (Pennington, p. 341).

**Analytic Integration of** \( \int \frac{\partial}{\partial n} \log(\rho) \) \( ds \)**

It will be shown that the expression that results in the term

\[ \phi_i \left[ \frac{\partial}{\partial n} \log(\rho) \right]_{ij} ds_i \]

in Equation (IV-14) can be better approximated by a direct analytic integration rather than the approximation given above. Assuming that \( \phi \) is nearly constant, as before, so that it may again be pulled out of the integral, the source of the above term as given by Equation (IV-9) may be written...
\[
\int_{s_1} \phi \frac{\partial}{\partial n} \log(\rho) \, ds \geq \phi \int_{s_1} \frac{\partial}{\partial n} \log(\rho) \, ds.
\]

It is this last expression that can be integrated analytically, by using the relation

\[
\nabla \rho \cdot n = \left\{ \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} \right\} [(x-\xi)^2 + (y-\eta)^2]^{1/2} \cdot n
\]

\[
= \frac{1}{\rho} [(x-\xi) \hat{i} + (y-\eta) \hat{j}] \cdot n
\]

\[
= \frac{1}{\rho} \rho \cdot n
\]

so that

\[
\phi \int_{s_1} \frac{\partial}{\partial n} \log(\rho) \, ds = \phi \int_{s_1} \frac{1}{\rho} \nabla \rho \cdot n \, ds
\]

\[
= \phi \int_{s_1} \frac{1}{\rho^2} \rho \cdot n \, ds
\]

\[
= \phi \int_{s_1} \cos \theta \rho \, ds
\]

where \( \theta \) is the angle formed by \(-\rho\) and \(n\). This result in turn can be rewritten using the relation

\[
\frac{\cos \theta}{\rho} \, ds = d\omega
\]

which is derived by noting that as ds becomes infinitesimally small in the following diagram \( \rho \, d\omega = ds \cos \theta \).
Then

\[ \phi_i \int_{s_i} \frac{\partial}{\partial n} \log(\rho) \, ds = \phi_i \int_{s_i} \, d\omega \]

\[ = \phi_i \, d\omega_i \]  \hspace{1cm} (IV-15)

**Result**

The result is that the integral term from Equation (IV-9) can be approximated more accurately by \( \phi_i \, d\omega_i \) where \( d\omega_i \) is the angle subtended at \((x_j, y_j)\) by the arc of \( ds_i \). Therefore, Equation (IV-14) can be better written

\[ (-\pi + \frac{1}{2} k_j \, ds_j) \phi_j + \sum_{i=1, i \neq j}^{N} \phi_i \, d\omega_i = (\frac{\partial \phi}{\partial n})_j [\log(ds_j/2)-1] \]

\[ + \sum_{i=1, i \neq j}^{N} (\frac{\partial \phi}{\partial n})_i [\log(\rho)]_{ij} \, ds_i \]  \hspace{1cm} (IV-16)

It is this equation that is then inverted for the direct determination of the \( N_2 \) values of \( \frac{\partial \phi}{\partial n} \) on the surface.
CHAPTER V

NUMERICAL APPROXIMATION OF THE FREE SURFACE

Cubic Splines

Having found a numerical method for the solution of Laplace's equation, it is now necessary to suitably approximate the profile of the free surface. This is necessary because certain geometrical properties of the free surface, namely the arc length, the curvature, and the normal direction, must be known to the routine solving Laplace's equation. Also the midpoints of the curve between adjacent points must be known for the analytic integration that was discussed in the last section. These factors must be known for the reservoir boundary as well, but because this boundary remains stationary, it should be an easy matter to describe it by a set of known piecewise smooth functions. Therefore it is necessary to accurately describe the free surface profile given only the position of the surface at a discrete number of points.

A method of making this approximation is to use a construction called a cubic spline, which is, to fit an individual cubic polynomial between each pair of given adjacent points, then to match the first and second derivatives at each point where two of these cubics meet. Thus each piecewise cubic polynomial has four conditions to satisfy which can be used to evaluate its four constants. A continuous function results that has both first and second derivatives.
being continuous also. Therefore this function, which again is just a piecewise polynomial, may be differentiated analytically, first to determine the slope, and then again to find the curvature at any point on the surface (Alberg, Nilson and Walsh, Chapter 1).

**Parametric Representation**

To retain the full generality of the motion, a parametric representation of the surface must be chosen. That is, both the values of $x$ and $y$ on that surface must be given as a function of another variable, for example arc length $S$ along the surface. This will be recognized as being equivalent to the Lagrangian method of representing a fluid, where the identifying parameter in this case would be $S$.

The reason for choosing this representation is to allow the surface profile to become vertical or even to start to curl over, as it does just before a wave breaks. For these cases the function $y(x)$ becomes multivalued and hence is not uniquely defined. The Eulerian representation will therefore be undefined at these points. Thus the second of Equations (11-12) and (11-14) will be the proper equations to use with this parametric representation.

For simplicity the parametric variable $t$ is chosen to be the index that numbers the individual subscripted surface points $x_i$ and $y_i$. This is done rather than using the arc length, because the data set is then uniformly spaced in one unit intervals. The surface will therefore be described by the functions $x(t)$ and $y(t)$ where $t$ has
the range 1 to n and when \( t = t_i = i \), \( x(t_i) \) and \( y(t_i) \) are just the data points \( x_i \) and \( y_i \).

**Spline Derivation**

The spline fit of equally spaced data is particularly simple to construct. It will be done now for the variable \( y(t) \), with the equations for \( x(t) \) being completely analogous. The procedure is actually carried out in a reverse order. That is, let \( Z_{y1}, Z_{y2}, ..., Z_{yn} \) be the values of the second derivative at the given points \( t_i \) (\( i = 1, n \)). Making the second derivative linear, and hence continuous, yields for the interval \([t_i, t_{i+1}]\)

\[
y'' = \frac{d^2y}{dt^2} = Z_{yi} \left( \frac{t_{i+1} - t}{h_i} \right) + Z_{y(i+1)} \left( \frac{t - t_i}{h_i} \right) \quad (IV-17)
\]

where \( h_i = t_{i+1} - t_i = 1 \).

Integrating this expression twice then gives

\[
y' = \frac{dy}{dt} = -Z_{yi} \left( \frac{(t_{i+1} - t)^2}{2} \right) + Z_{y(i+1)} \left( \frac{(t - t_i)^2}{2} \right) + c_1 \quad (IV-18)
\]

and

\[
y = Z_{yi} \left( \frac{(t_{i+1} - t)^3}{6} \right) + Z_{y(i+1)} \left( \frac{(t - t_i)^3}{6} \right) + c_1 t + c_2 \quad , \quad (IV-19)
\]

where \( c_1 \) and \( c_2 \) are constants of integration that can be evaluated by noting that the curve passes through \( y_i \) when \( t = t_i \) and \( y_{i+1} \) when \( t = t_{i+1} \). Evaluating these constants and then substituting into (IV-18) and (IV-19) yields, for the equation of the curve,
and for the first derivatives

\[
y'(t_i) = -\frac{Z_{yi}}{2} + \frac{Z_{yi+1}}{2} + (y_{i+1} - y_i) - \frac{Z_{yi+1} - Z_{yi}}{6} \quad \text{(IV-21)}
\]

It remains to determine the values of \( Z_{yi} \) and \( Z_{yi+1} \). This is done by matching the slope at \( t_i \) given by Equation (IV-21), with the slope at \( t_i \) given by the equation over the preceding interval \( (t_{i-1}, t_i) \). The two equations for \( y' \) are then

\[
y'(t_i+) = -\frac{Z_{yi}}{3} - \frac{Z_{yi+1}}{6} + y_{i+1} - y_i
\]

and

\[
y'(t_i-) = \frac{Z_{yi-1}}{6} + \frac{Z_{yi}}{3} + y_i - y_{i-1}
\]

Equating these two expressions yields the equation

\[
Z_{yi-1} + 4Z_{yi} + Z_{yi+1} = 6(y_{i+1} - 2y_i + y_{i-1}) \quad \text{(IV-22)}
\]

There are \( n-2 \) of these equations, one for each \( i = 2, 3, \ldots, n-1 \), for determining the \( n \) quantities \( Z_i \), \( i = i, n \). Thus 2 more conditions must be specified. These two conditions are usually conditions specified on \( Z_{yi} \) and \( Z_{yn} \) or on the values of the slope at each end point. Since for the surface wave problem neither the curvature nor the slope is known at the endpoints, the values of \( Z_{yi} \) and \( Z_{yn} \) are arbitrarily set to zero.
Thus a system of n-2 equations results, from which the n-2 values of $Z_{yi}$ can be found. This system of equations has the useful property that each equation has only three unknowns in it, the diagonal element and its adjacent elements to each side. A very efficient algorithm exists for inverting tridiagonal systems of this kind.

**Tri-Diagonal Equation Solution**

Given the tri-diagonal system, Equation (IV-22), let a recursion relation with two constants E and F be defined by

$$Z_{yi} = E_i Z_{y(i+1)} + F_i$$  \hspace{1cm} (IV-23)

or

$$Z_{y(i-1)} = E_{i-1} Z_{yi} + F_{i-1}$$  \hspace{1cm} (IV-24)

Substituting this last expression into Equation (IV-22) to eliminate the $Z_{yi-1}$ term yields after rewriting in the form of Equation (IV-23)

$$Z_{yi} = -\frac{1}{E_{i-1} + 4} Z_{y(i+1)} + \frac{1}{E_{i-1} + 4} [6(y_{i+1} - 2y_i + y_{i-1}) - F_{i-1}] .$$

From this a recursion relation for the constants E and F can be identified.

$$E_i = -\frac{1}{E_{i-1} + 4}$$  \hspace{1cm} (IV-25)

$$F_i = E_i [F_{i-1} - 6(y_{i+1} - 2y_i + y_{i-1})]$$  \hspace{1cm} (IV-26)

Thus by first determining the constants $E_i$ and $F_i$ for $i = 2, n-1$ using Equations (IV-25) and (IV-26) the $Z_{yi}$ for $i = 2, n-1$ can be found using Equation (IV-23).
Geometrical Properties of the Free Surface

With the spline fit constants $Z_{y1}$ and $Z_{x1}$ the parametric equations of the surface are given by (IV-20) for $y(t)$ and $x(t)$. The geometric properties of the surface that are needed can then be found. First to be noted is that for $y = y_i$ and $x = x_i$ Equation (IV-21) for the first derivatives become

$$\begin{align*}
y'(t_i) &= y_{i+1} - y_i - (2Z_{y1} + Z_{y(i+1)}) \\
x'(t_i) &= x_{i+1} - x_i - (2Z_{x1} + Z_{x(i+1)}) \quad \text{(IV-27)}
\end{align*}$$

The unit normal components $n_x$ and $n_y$ at $t_i$ can then be written

$$\begin{align*}
n_{xi} &= -\frac{y'(t_i)}{(y'(t_i)^2 + x'(t_i)^2)^{1/2}} \quad \text{(IV-28)} \\
n_{yi} &= \frac{x'(t_i)}{(y'(t_i)^2 + x'(t_i)^2)^{1/2}} \quad \text{(IV-29)}
\end{align*}$$

The curvature at $t_i$ can be found by making use of the 2nd derivatives $Z_{x1}$ and $Z_{y1}$:

$$k(t_i) = \frac{y'(t_i) Z_{x1}(t_i) - x'(t_i) Z_{y1}(t_i)}{[x'(t_i)^2 + y'(t_i)^2]^{3/2}} \quad \text{(IV-30)}$$

Also an approximation for $ds$ at $t_i$ can be made

$$ds_i = [x'(t_i)^2 + y'(t_i)^2]^{3/2} \quad \text{(IV-31)}$$

Finally the position of the surface midway between two data points can be found by substituting $t = t_{i+1/2}$ into Equation (IV-20) for
both the $x$ and $y$ functions: both $(t_{i+1} - t)$ and $(t - t_i)$ then become equal to $1/2$ and the equation that results is

$$y(t_{i+1/2}) = -\frac{1}{16}(Z_{yi} + Z_{y(i+1/2)}) + \frac{1}{2}(y_{i+1} + y_i)$$  \hspace{1cm} \text{(IV-32)}$$

with a similar equation for $x(t_{i+1/2})$.

These relations, Equations (IV-28) to (IV-32), supply the values that are needed to carry out the numerical solution of the mixed boundary value problem that results from the mathematical description of a perfect fluid.
FORTRAN Code

A FORTRAN IV code built around the procedure detailed in Chapter IV for solving Laplace's equation was put together according to the outline presented in the Summary section of Chapter III. This code in its entirety is reproduced in Appendix 1. Some of the features of this code not already covered will now be described.

The code is broken into a number of subroutines, with the main routine being that part of the code that takes care of all the bookkeeping and does the hydrodynamics. It is this section that integrates Bernoulli's equation and the kinematic equations for the points on the surface. This data along with other pertinent data is then written into several different disc files which when the code exits can be accessed to determine the parameters that have been calculated for a particular run.

It is into this main portion of the code that the variables for the particular run are entered. Needed at the time of execution are the time increment, number of time increments, and the number of spatial intervals on the various parts of the boundary. The initial conditions that are also needed are assumed stored in a disc file under the name ICDAT and consist of a number n giving the number of points on the surface, the initial time and interval number, and a
code number to tell whether the initial condition is on the potential or on the normal velocity. Then finally the initial data in the form of the values of x, y, PHI or PHIEN for each of the n points on the surface. Upon exiting, the code writes the current values of these same parameters in exactly the same format into another data file with the name FCDAT, so that if necessary the code can be continued from where it left off by transferring the contents of the file FCDAT into ICDAT.

The various subroutines that are needed by the main program consist of: 1) a routine SURFIT, to calculate the various geometrical properties of the free surface as was detailed in the last chapter, 2) a routine RSVBD, to calculate the coordinates and geometrical configuration of the reservoir boundary, 3) a routine LAPLCE, to solve Laplace's equation using the integral equation procedure that was explained in Chapter IV, 4) a routine GELG which is a subroutine taken from the IBM subroutine library to solve the resulting set of simultaneous linear equations by Gaussian elimination, and finally 5) a routine SPLCON to determine the spline constants $Z_{x_1}$ and $Z_{y_1}$ used in SURFIT, and $Z_{\phi_1}$ used in finding $\frac{\partial \phi}{\partial s}$. Thus a code was put together in this manner to model surface wave behavior. To test the numerical methods a solitary wave was selected to model.

Solitary Wave

A solitary wave is in general a non-linear single symmetric disturbance of height h above an undisturbed surface of depth d.
The wave, over a smooth flat bottom, translates without change of shape or velocity. Waves of this form have been observed in nature and have been found to have a time independent form approximated by the classical formula of Boussinesq (1872):

\[ n = d + h \text{sech}^2 \left[ \left( \frac{x - x_0}{2d} \right) \left( \frac{3h}{d} \right)^{1/2} \right] . \]  

(VI-1)

for \( h/d \leq 0.40 \) and where \( x_0 \) is the position of the center of the wave. Latitone (1960) then showed by a second order perturbation analysis that the speed of such a wave could be given by

\[ c = (gd)^{1/2} \left[ 1 + \frac{1}{2} \left( \frac{h}{d} \right) - \frac{3}{20} \left( \frac{h}{d} \right)^2 + \ldots \right] . \]  

(VI-2)

Where \( g \) is the acceleration of gravity.

Using this information, the initial conditions for the position and the normal velocity of a solitary wave can be calculated. This is easily accomplished since only conditions on the surface itself must be found. Since the wave is traveling with uniform velocity \( c \) parallel to the undisturbed surface, the normal velocity of the wave profile must simply be \( v_{\text{normal}} = c \cdot n = cn_x \). Thus both the initial shape and normal velocity are easily computed. A simple code was written following this prescription to generate the initial condition file ICDAT for the surface wave problem. A typical example of the results is given in Table 1. A solitary wave was thus generated. The undisturbed liquid depth was chosen to be \( d \) in units of \( g \), and the horizontal distance was then scaled to this depth again in units of \( g \).
As a first test of the surface wave code, a solitary wave was generated and allowed to traverse a section of undisturbed liquid over a uniform bottom as illustrated in Figure 1. The wave was then allowed to reflect off the vertical boundary at the end of the section and retrace its path back to the starting point. In Figure 2 the wave profile after reflection is compared with the profile as it was initially generated. As can be seen the comparison is qualitatively quite good. Twenty points were used to model the surface, with a total of 26 points being used over the reservoir. A time increment of \( t = .1 \) seconds was used. Also tried were \( t = .2 \) seconds. There was little difference in comparing the results using the first two values, however the run with the time increment of \( t = .2 \) seconds produced a wave of smaller amplitude upon reflection. For this reason most succeeding applications of the code were run with a time increment of \( t = .1 \) seconds.

As a further test the wave was allowed to continue on in the reverse direction to be reflected from the other boundary and again returned to the starting point. The resulting profile is also shown in Figure 2. A summary of the wave motion is shown in Figure 3, and Appendix 2 contains a summary of the output parameters for this run.

**Wave Run Up On A Vertical Wall**

As a quantitative study of the results produced by the surface wave code, the run up of a solitary wave on a vertical wall as illustrated in Figure 4 was studied. The run up \( R \), of a wave is
defined as the height above the undisturbed liquid level that a wave achieves when it intercepts an obstacle, in this case a vertical wall. Several experimental and computational studies of this problem have been made and are available for comparison. Street and Camfield (1966) ran a series of experiments in a two-dimensional wave channel to study solitary wave deformation. Some of the results of that study pertain to wave run up on a vertical wall and Van Dorn (1966), making use of this data suggested the equation

\[
\frac{R}{h} = 2.0 + \frac{h}{d}
\]  

(VI-3)

to describe the run-up. This equation with the data that led to it are plotted along with the results of this paper's and another's numerical study in Figure 5. Here the plot is \(R/h\) as a function of \(h/d\). The non-linearity of the problem is emphasized by the amount that \(R/h\) is greater than 2, which would result from the linear superposition of two waves traveling in opposite directions. A more sensitive plot of wave run-up is shown in Figure 6 where \(R/d\) is plotted as a function of \(h/d\). In both cases the results of the runs made by the integral equation method closely parallel the results of the other numerical study. Table 2 contains the data plotted in Figures 5 and 6. The other numerical study is the SUMMAC code which was developed at Stanford University by Chan et al. (1967). It is a modified marker and cell method of solving the Navier-Stokes equation and relies heavily on finite differencing techniques. The surface wave code that has been described in this paper is much simpler and a much faster code than is the SUMMAC code, and is hence
better suited for many applications. Also there is no stability
criterion for the integral equation method, which for the SUMMAC
code limits the maximum time step increment by a Courant condition
related to the spatial interval size. Here the time step increment
is only constrained by the velocity and relative accuracy that is
desired. However as the code is now written, another accuracy
limitation restricts the run-up problem to cases for which \( h/d < .5 \)
and even further for accurate reflections to \( h/d < .33 \). The SUMMAC
code however, accurately treats cases up to at least \( h/d \leq .7 \). Within
these restrictions though the integral equation method proves to be
quite workable for the run-up problem.

**Shoaling of a Solitary Wave on a Plane Beach**

A preliminary study was done for a wave shoaling on a plane
beach inclined at a slope of .05. Results of several runs are
illustrated in Figures 7 and 8. Qualitatively the motion appears
approximately correct, as the wave approaches the beach the leading
dge of the wave steepens and finally breaking results. However
the amplitude growth of the wave is too slow. The wave does not
rise to the correct level and consequently the onset of breaking
is delayed. This result is attributed to the inaccuracy of the
solution of Laplace's equation, since an increase in the number of
intervals increases the rate of amplitude growth. A numerical insta-
bility however limits the total number of points that can be usefully
applied to the surface, and hence the total accuracy suffers. Growth
of this numerical instability can readily be observed in the figures.
CHAPTER VII

SUMMARY

This paper has outlined a numerical technique to solve the hydrodynamic equations associated with free surface motion. Results of the wave run up on a verticle wall have proved the feasibility of the procedure. Although results of the shoaling of a wave on a plane beach have shown that refinements can still be made. Of first necessity is the need to increase the accuracy of the solution of Laplace's equation. This is most readily done by using a higher order integration formula. Thus fewer points on the surface will be needed, so that the growth of numerical instabilities will be minimized. In this vein a better spline procedure needs to be derived since this is most likely the source of the instability. Finally the treatment of the non-linear Bernoulli's equation should be improved. An iteration procedure would probably be needed to get an accurate representation. Again, the feasibility of the procedure outlined in this paper has been proved. With the improvements outlined above a useful tool for studying wave motion should result.

Applications of such a code are obvious. The aforementioned shoaling of a wave on a beach or another obstacle is the first application that comes to mind. The deformation of a wave as it travels over obstacles below the liquid surface however is the area
in which this code excels, since the procedure for solving Laplace's equation works just as well for bottoms of arbitrary geometry as it does for smooth flat bottoms. Thus the motion of a wave as it traverses the continental shelf and impinges on a breakwater or submerged reef could be modeled. Many other applications can also be imagined.
REFERENCES


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TABLE 1. File ICDAT, initial conditions for a typical solitary wave. (h/d = .1)

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<td>.210</td>
</tr>
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TABLE 2. Results of solitary wave run-up study

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
FIGURE 1 Solitary Wave (h/d = .3)

FIGURE 2 Solitary Wave profiles after reflection from vertical boundaries (h/d = .1)
FIGURE 3 Solitary Wave oscillating between vertical walls (h/d = .1)
FIGURE 4 Solitary Wave run-up and reflection from a vertical wall (h/d = .1)
**FIG. 5** WAVE RUN-UP ON A VERTICAL WALL ($R/h$)

**FIG. 6** WAVE RUN-UP ON A VERTICAL WALL ($R/d$)

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FIG. 7 Solitary Wave shoaling on plane beach (h/d = 0.1)

FIG. 8 Solitary Wave shoaling on plane beach (h/d = 0.2)
FIGURE 9 Solitary Wave shoaling on plane beach, showing the numerical instability. ($h/d = .4$)
APPENDIX A

General surface wave problem FORTRAN IV code
*/
1 5250
2 CONTROLLER: D +SUPSOL
3 UNITS FOR
4 WORKARM
5 PROGRAM SUP(supdat,tape5=supdat,pldat,tape6=pldat,icdat,
6 tape7=icdat,icdat,tape8=fcdata)
7 C PROGRAM FOR THE SOLUTION OF THE GENERAL WATER WAVE PROBLEM
8 C
9 C PARAMETERS
10 C N1,N2,NB-HO. OF POINTS ON SURFACE
11 C N1,N2,NB-HO. OF POINTS ON RESERVOIR SIDES AND BOTTOM
12 C NT-TOTAL NO. OF POINTS
13 C KDS-CURVATURE AND ARC LENGTH
14 C C1,C2,C3,C4-COMponents of the normal direction
15 C XHFP-COORDINATE OF MIDPOINT BETWEEN X(I) AND X(I+1)
16 C YHFP-COORDINATE OF MIDPOINT BETWEEN Y(I) AND Y(I+1)
17 C PHI,PHIEN-POTENTIAL AND NORMAL DERIVATIVE OF THE POTENTIAL
18 C X1,X2,Y1,Y2-HEIGHT OF SURFACE AT X-U AND X=1
19 C TAPC=TOTAL NO. OF POINTS ON A UNIT SPACING IN THE X DIRECTION
20 C INITIAL WAVE PROFILE AND WATER DEPTH READ IN FROM DATA FILE
21 C ICRT ALONG WITH THE INITIAL VALUES OF PHI OR PHIEN (4E16.8)
22 C IN HO. IF PHI GIVEN IND=1 IF PHIEN GIVEN
23 C FINAL VALUES OF THE SAME DATA ARE WRITTEN INTO FILE FCDOC
24 C WHEN PROGRAM EXITS
25 C COORDINATES OF SURFACE POINTS AND MIDDPOINTS ARE WRITTEN INTO
26 C FILE PLDAT AT EACH TIME STEP TO BE USED LATER IN A PLOTTING
27 C FOMF, SUPNAT FILE CONTAINS INFORMATION ABOUT THE
28 C SURFACE AT EACH TIME INTERVAL
29 C
30 C XH=F(X(C200),Y(C200),K(200),DS(200),NXC(100),NXC(100),Y0,Y1,N)
31 C YH=F(X(C200),Y(C200),K(200),DS(200),NXC(100),NXC(100),Y0,Y1,N)
32 C
33 C WAVE PROBLEM PARAMETERS
34 READ(7,002) H,IT,IND
35 002 FORMAT(2X,3I3)
36 801 FORMAT(2X,3I3)
37 802 FORMAT(2X,3I3)
38 803 FORMAT(1X,28HENTER TIME INCREMENT)
39 804 FORMAT(1X,28HENTER NO. OF TIME INCREMENTS)
40 805 FORMAT(1X,28HENTER TIME INCREMENT)
41 C CREATION OF DATA FILES
42 CALL DEVICE(CHCREATE,CHSUPDAN)
43 CALL DEVICE(6HCREATE,6HPDAT)
44 CALL DEVICE(6HCREATE,6HFCDAT)
45 C CALL FOR PROBLEM PARAMETERS
46 READ(7,002) H,IT,IND
47 801 FORMAT (2X,3I3)
48 802 FORMAT (2X,3I3)
49 803 FORMAT (1X,28HENTER TIME INCREMENT)
50 804 FORMAT (F10.0)
51 805 FORMAT (1X,28HENTER NO. OF TIME INCREMENTS)
52 C INITIAL WAVE SHAPE AND POTENTIAL INSERTED HERE
53 DO 801 I=1,N
54 READ(7,787)X(I),Y(I),PHI(I),PHIEN(I)
55 787 FORMAT (2X,8E16.8)
56 C0=SFD+Y(I)
57 801 CONTINUE
58 WRITE (59,809)
59 809 FORMAT (2X,3I3)
60 C SFD IS THE APPX. STILL WATER SURFACE HEIGHT
61 /
BEGINNING OF LOOP FOR EACH ADDITIONAL TIME INTERVAL

CALL SURFIT TO GET SURFACE PROFILE DATA
CALL SURFIT TO SOLVE LAPLACE EQUATIONS FOR PHIEN ON SURFACE
CALL SPLCON FOR SPLINE CONSTANTS USED TO FIND PHIES
CALL SPLCON PHIES
CALL SPLCON PHIES
CALL SPLCON PHIES
121 \( \phi(i) = \phi(i) - (\phi(i) + (2 \cdot \phi(i) + (\phi(i) + \phi(i)))/6)/\phi(i) \)

122 \( \phi(i) = \phi(i) + \phi(i) + \phi(i) \)

123 \( \phi(i) = \phi(i) + (\phi(i) + \phi(i)) \)

124 \( \phi(i) = \phi(i) + \phi(i) + \phi(i) \)

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151 \( \phi(i) = \phi(i) + \phi(i) + \phi(i) \)

152 \( \phi(i) = \phi(i) + \phi(i) + \phi(i) \)

153 \( \phi(i) = \phi(i) + \phi(i) + \phi(i) \)

154 \( \phi(i) = \phi(i) + \phi(i) + \phi(i) \)
SUBROUTINE SURFIT
C ....C1(200),Y(200),K(200),DS(200),XNC(100),YNC(100),Y0,Y1,N
C ....C1(200),YHFP(200),NT
C ....C2(200),ZY(100)
C.
C..Call SPLCOM(X,Y,Z)
C..Call SPLCOM(Y,H,ZY)
C
DO 610 1-2,16
9 YT=-(Y(I)+X(I-1)-(2.*Z(I)+Z(I-1))/6.
10 YT=-(Y(I)+Y(I-1)-(2.*Z(I)+Z(I-1))/6.
11 X(I)=SRT(YT*YT+YT)
12 X(I)=Y(I)+DS(I)
13 N(I)=XT/DS(I)
14 NP(I)=NTH(Z(I)-XT*ZY(I))/DS(I)**3
15 NP(I)=NTH(Y(I-1)-Y(I)+Z(I))/2.-Z(I+1)/16.
16 NP(I)=NTH(Y(I-1)+Y(I)+Z(I))/2.-Z(I-1)/16.
17 XT=-(X(2)+X(I)+2.*Z(I)+Z(2))/6.
18 YY=-(Y(2)+Y(I)+2.*Z(I)+Z(2))/6.
19 DS(I)=SRT(XT*YT+YT)
20 XNC(I)=XT/DS(I)
21 YNC(I)=XT/DS(I)
22 K(I)=YTH(Z(I)-XT*ZY(I))/DS(I)**3
23 Y(I)=Y(I)+X(I)-1.*XNC(I)/YNC(I)
24 Y(I)=Y(I)+X(I)-1.*XNC(I)/YNC(I)
25 NT=0.
26 NT=NT+1.
27 NT=NT+.999999999
28 NT=NT+Y0
29 RETURN
30 END
SUBROUTINE RVED
C SUBROUTINE TO DETERMINE THE RESERVOIR POINTS AND CONFIGURATION
C UNIT WIDTH RESERVOIR WITH VERTICAL SIDES OF HEIGHT Y0, Y1
C REQUIRED X.HI,HI,HN,NI,Y0,Y1
C CALCULATES X,Y.D.S.K.XHFP,YHFP FOR THE RESERVOIR
C CONTAINS/C1,X(200),Y(200),K(200),DS(200),XNC(100),YNC(100),Y0,Y1,N
C CALL/C2/XHFP(200),YHFP(200),NT
C CALL/C3/PH1,H(200),PH1(200)
C CALL/C4/H1,H2,NB

REAL K

VB=N1
VB=NB
NEW=NI+NB2-NB
VB=0.
X(1)=20.*Y(1)+.6+26.*X(1)*YNC(1)/YNC(1)/(1.+20.*YNC(1)/YNC(1))
Y(1)+.6+26.*X(1)*YNC(1)/YNC(1)
Y=PH1
Y=XHFP(1)=0.
20 CALCULATION OF COORDINATES OF POINTS ON CONTAINER BOTTOM
DO 503 I=H+H1+1,H+H1+NB
503 Vi=I-H1-N1
X(I)=(1./2.*VN1)+(VI-1.)/VN1+.6
Y(I)=0.
X(I)=Y1*(1.-VI-1.)/VN1-1./2.*VN1)
504 Vi=I-H1-N1
DO 904 I=H+H1+1,NT
504 Vi=I-H1-N1
Y(I)=YH(I-1)+20.+.6
504 Vi=0.
DO 904 Vi=I-VH(I-1)*YN(I-1)./2.
504 XHFP(I)=Y(I)
504 XHFP(I)=Y(I)/2.
CONTINUE
DO 247 I=H+1,HT-1
504 XHFP(I)=X(I)+X(I+1))/2.
504 YHFP(I)=Y(I)+Y(I+1))/2.
CONTINUE
43 CONTAINER BETWEEN POINTS
DO 247 I=H+1,HT-1
504 XHFP(I)=X I
504 YHFP(I)=Y I
504 XHFP(I)=H+1
504 YHFP(I)=0.
504 XHFP(I)=H+1
504 YHFP(I)=0.
504 XHFP(I)=H+H1+NB2-NB
504 YHFP(I)=0.
504 XHFP(I)=H+1
504 YHFP(I)=Y0
504 RETURN

END
SUBROUTINE LAPLCE(X,Y,K,DS,PHI,PHIEN,XHFP,YHFP,N,NT)

SUBROUTINE TO SOLVE LAPLACE EQUATION

IMPLICIT X,Y,K,DS,PHI OR PHIEN,XHFP,YHFP,N,NT

N IS THE NO. OF POINTS THAT HAVE PHI AS A
BOUNDARY CONDITION (MUST BE FIRST N) AND IT IS
THE TOTAL NUMBER OF POINTS NOT-N POINTS THEN HAVE THE
BOUNDARY CONDITION ON PHIEN.

CONTINUES COMPLEMENTARY BOUNDARY CONDITION PHI OR PHIEN

IF (PHI(1),Y(1),K(1),DS(1),XHFP(1),YHFP(1),PHI(1),PHIEN(1)
THEN DS(1)=0(200),0(200),TT(300000)

INPUT PARAMETERS:

INPUT OF CALCULATION OF MATRIX COEFFICIENTS

I IS THE INDEX OF THE FIELD POINT AND I IS THE INDEX
OF THE INTEGRATION POINT

DO 229 J=1,NT

IF (I.EQ.1) GO TO 48

DIFX=X(J)-X(I)

DIFY=Y(J)-Y(I)

PHOSO=DIFX*DIFY

RHO-SORT(PHOSQ)

LOGRH0=ALOG(RHO)

XHFP(I)=X(I)

YHFP(I)=Y(I)

XHFP(I+1)=X(I)+DIFX

YHFP(I+1)=Y(I)+DIFY

Rностей=(XHFP(I)-XHFP(J))*(XHFP(I)-XHFP(J))

Sностей=(YHFP(I)-YHFP(J))*(YHFP(I)-YHFP(J))

IF (COST.GT.1.) COST=1.

R(I)=-ACOS(COST)

IF (SINT.LT.0.) R(I)=-R(I)

D(I)=DS(I)/LOGRHO

GO TO 419

40 D(I)=DS(I)*(ALOG(DS(I)/2.)-1.)

P(I)=-P(I)+DS(I)/2.

41 C MULTIPY THE SUITABLE COEFFICIENTS BY PHI FOR THE SURFACE

AND BY PHIEN FOR THE CONTAINER AND THEN PUT INTO THE
RIGHT HAND SIDE SINGLE COLUMN MATRIX

DO 920 I=1,NT

920 Q(I)=P(I)*PHI(I)

921 NT=Q(I)*PHI(I)

DO 923 I=1,NT

923 Q(I)=Q(I)+Q(I)*PHI(I)

924 NT=NT

925 CONTINUE

CALL SIMULTANEOUS EQUATION SOLVER, UNKNOWN PHIEN ON

SURFACE WILL BE IN THE FIRST N ELEMENTS OF B

CALL GELGCB,NT,NT,1.00001,IER)

IF (IER.NE.0) WOT 59.811,IER

FORMAT (18X,10H 1ER = ,12)

50 C READ IN MATRIX OF UNKNOWN COEFFICIENTS FROM ROW

51 C STOP AL TERMS TO COLUMN STORAGE

DO 823 J=1,NT

823 J=J+(J-1)*NT

824 jjp=Q(I)

825 CONTINUE

CALL SIMULTANEOUS EQUATION SOLVER, UNKNOWN PHIEN ON

SURFACE WILL BE IN THE FIRST N ELEMENTS OF B

CALL GELGCB,NT,NT,1.00001,IER)

IF (IER.NE.0) WOT 59.811,IER

FORMAT (18X,10H 1ER = ,12)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
DO 251 I=1,N
         V(I,EB)=E(I)
DO 352 I=N+1,NF
RETURN
END
SUBROUTINE SPLCON(X,N,Z)
DIMENSION X(1),Z(1),E(100),F(100)
E(1) = 0.
5 DO 60 I=2,N+1
60  E(I) = 1./((Z(I-1)+4.)
7  D(I) = E(I)*((X(I+1)-2.*X(I)+X(I-1))*6.-F(I-1))
8  Z(I) = 0.
9  D0 610 J=1,N-2
10  J = J + 1
11  D610 = E(I)*Z(I+1)+F(I)
12  Z(I) = 0.
13  RETURN
14  END
15 C
16 C

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SUBROUTINE GELG(R, A, M, N, EPS, IER)

SOLVES A GENERAL SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS BY MEANS
OF GAUSSIAN ELIMINATION WITH COMPLETE PIVOTING.

DESCRIPTION OF PARAMETERS

R - THE M BY N MATRIX OF RIGHT HAND SIDES

ON RETURN R CONTAINS THE SOLUTION

A - THE M BY N COEFFICIENT MATRIX (DESTROYED)

N - THE NUMBER OF EQUATIONS IN THE SYSTEM

M - THE NUMBER OF RIGHT HAND SIDE VECTORS

EPS - AN INPUT CONSTANT WHICH IS USED AS RELATIVE

TOLERANCE FOR TEST ON LOSS OF SIGNIFICANCE

IER - RESULTING ERROR PARAMETER CODED AS FOLLOWS

IER = 0 - NO ERROR

IER = -1 - NO RESULT BECAUSE OF M LESS THAN 1 OR

Pivot Element at any elimination step = 0

IER = K - WARNING DUE TO POSSIBLE LOSS OF SIGNIFICANCE

INDICATED AT ELIMINATION STEP K+1 WHERE PIVOT

ELEMENT WAS LESS THAN OR EQUAL TO THE INTERNAL

TOLERANCE EPS TIMES ABSOLUTELY GREATEST ELEMENT

OF MATRIX A

INPUT MATRICES R AND A ARE ASSUMED TO BE STORED COLUMN-WISE

IN THE ARRAYS R AND A SUCCESSIVE PIVOT LOCATIONS, ON RETURN

SOLUTION MATRIX R IS STORED COLUMN-WISE TOO.

THE SUBROUTINE GIVES RESULTS IF THE NUMBER OF EQUATIONS M IS

GREATER THAN N AND PIVOT ELEMENTS AT ALL ELIMINATION STEPS

ARE DIFFERENT FROM 0. HOWEVER WARNING IER = K (IF GIVEN)

INDICATES POSSIBLE LOSS OF SIGNIFICANCE, IN CASE OF A WELL

CONDIT MATRIX A AND APPROPRIATE TOLERANCE EPS. IER = K MAY BE

INTERPRETED THAT MATRIX A HAS THE RANK K. NO WARNING IS

GIVEN IN CASE M = 1

DIMENSION A(1), R(1)

I = 1:23:23:1

SEARCH FOR GREATEST ELEMENT IN MATRIX A

1ER = 0

PIV = 0

I = 1:23:23:1

K = I - 1

IF (A(K, K) = 0) THEN

RETURN

DO 23

1ER = 1

RETURN

23 CONTINUE

I = 1

DO 43

L = I + 1

IF (I .GT. M) THEN

RETURN

22 CONTINUE

L = 1

DO 43

J = L + 1

IF (J .LE. M) THEN

23 CONTINUE

J = J + 1

43 CONTINUE

START ELIMINATION LOOP

LST = 1

DO 51 K = 1, M

I = L

51 CONTINUE

1ER = 0

RETURN

END
11 C AREA IS ROW INDEX, J+K COLUMN INDEX OF PIVOT ELEMENT
12 C PIVOT ROW REDUCTION AND ROW INTERCHANGE IN R.H.S. R
13 A= LST+M
15 M = PIDJ
16 R= RIVJ(N(L))
18 C IS ELIMINATION TERMINATED
20 IF (I-I0) 9, 15, 18
21 C COLUMN INTERCHANGE IN MATRIX A
23 L= LST+M
25 L= 12, 12, 10
27 J= J+1
29 N= N-1
31 N= N-1
33 R= RIVJ(N(L))
35 M = PIDJ
37 C Pivot interchange and pivot row reduction in matrix A
39 L= LST+M
41 L= 12, 12, 10
43 R= RIVJ(N(L))
45 M = PIDJ
47 C Save column interchange information
49 (LST) = J
51 C ELEMENT REDUCTION AND NEXT PIVOT SEARCH
53 PIV= 0
55 LST= LST + 1
57 J= 0
59 J= J + 1
61 M = I-J, LST, LEND
63 PIVI= A(I-J)
65 JST= 1+M
67 J= J + 1
69 J= J + 1
71 L= I-J, LST, LEND
73 L= LST + 1
75 L= I-1
77 R= RIVJ(N(L)) + PIVI(N(L))
79 T= ABS(R(L))
81 IF (T-PIV) 15, 15, 14
83 PIV= T
85 J= J + 1
87 J= J + 1
89 J= J + 1
91 J= J + 1
93 J= J + 1
95 J= J + 1
97 LST= LST + 1
99 C END OF ELIMINATION LOOP
101 C BACK SUBSTITUTION AND BACK INTERCHANGE
103 IF (M-I) 23, 22, 19
105 JST= NM-M
107 LST= R
109 J= J + 1
111 IF (J-1) 21, 2, M
113 JST= 1+I
115 JST= 1+J
117 L= LST-LST
119 T= X(J)
121 LL= J

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APPENDIX B

Illustration of the output of the surface wave code. (for figures 2-4)
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