Waist-to-waist beam transport with quadrupole singlet doublet and triplet thin lenses

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WAIST-TO-WAIST BEAM TRANSPORT WITH QUADRUPOLE
SINGLET, DOUBLET, AND TRIPLET THIN LENSES

By

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B.A., University of Montana, 1969

Presented in partial fulfillment of the requirements for the degree of
Master of Arts
UNIVERSITY OF MONTANA
1971

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Date June 7, 1971
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CHAPTER I

INTRODUCTION

In the field of geometrical optics, the problem of forming images of point source objects with thin lenses has been solved long ago. A point source is a particular case of a zero emittance beam. A zero emittance beam is defined to be a beam with zero area in the $x-x'$ space. $x$ is a space axis normal to the direction of propagation ($z$) of the beam, and $x' = \frac{dx}{dz}$ is the divergence of the beam in the $x-z$ plane. Emittance, $\epsilon$, is defined to be $\frac{A}{\pi}$ where $A$ is the area of the beam in the two dimensional $x-x'$ space. A beam from a point source is a zero emittance beam with zero width in the space. A parallel beam is another zero emittance beam since $x' = 0$. Both of these cases have been treated in geometrical optics.

The case of non-zero emittance beams has been neglected until recently. This problem arose in connection with charged particle beams and quadrupole magnets. Quadrupoles acting on a charged particle beam are analogous to lenses acting on a beam of light. A problem of interest is that of waist-to-waist transport, since a beam has its minimum extent at a waist. A waist is defined to be the point along $z$ where the product of the beam half-width, $x_m$, and the beam
divergence, $x_m^* = \left(\frac{dx}{dz}\right)_{\text{max}}$, is a minimum. See also Figure 1. In a lens where a diverging beam is being made to converge, $x_m^*$ may be small making $x_m x_m^*$ a minimum and the beam may have its maximum extent at a waist. Waist-to-waist transport consists of using a waist as an object and forming a new waist as an image. Banford, and later Silbar, have solved the problem of waist-to-waist transport with one thin lens in a plane formed by the optical axis $(z)$ and a space axis $(x)$ normal to $z$. Lee-Whiting and Bezić have treated waist-to-waist transport with a thick lens, using principal planes.

This paper treats the case of waist-to-waist transport with two thin magnetic quadrupole lenses (a doublet). This problem is important because at least two lenses are needed to form a waist in two planes, that is, in the $x$-$z$ and $y$-$z$ planes. Some questions that are answered are:

1. What focal lengths are needed to produce a waist at a desired location?

2. What is the ratio of the final beam half-width to the initial half-width (magnification)?

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3. How many different focal lengths will produce waists at a given location? Do they have the same magnification?

4. Are there limits on possible image distances, and if so, what are they?

This paper is organized into four main parts. The first deals with a phase space and its matrix representation, the waist-to-waist transformation matrix, quadrupole magnets, and also reviews the results for a single thin lens. The second part describes the doublet problem. The third section is devoted to the problem of the symmetric doublet and gives results of numerical calculations. A symmetric doublet is a doublet with the focal length of the second lens equal to the negative of the focal length of the first lens. Finally, in the last part of the paper, the symmetric triplet is considered. A symmetric triplet is a triplet with the focal length of the first and third lenses equal to the negative of twice the focal length of the second lens. Some numerical results are also given for the symmetric triplet.
CHAPTER II

BACKGROUND

In this section a phase space is defined and the phase space ellipse is discussed. It is then shown how this ellipse lends itself to a matrix formalism and the beam matrix is defined. Then the waist-to-waist transformation is derived. Finally quadrupole magnets and the waist-to-waist single lens problem are discussed. This section is a condensation of work by A. P. Banford, R. R. Silbar, K. C. Steffen, S. Penner, G. E. Lee-Whiting and N. Bezic, and K. L. Brown.

Phase Space Ellipse

A cartesian coordinate system is used and z is chosen as the optical axis. A plane containing the z axis, say the x-z plane, is utilized. Let \( x' = \frac{dx}{dz} \). The x-x' plane is called a phase space. This is not the customary definition. A phase space is usually defined as a coordinate and velocity space. But since \( v = \frac{dx}{dt} = \frac{dx}{dz} \frac{dz}{dt} = x' \frac{dz}{dt} \), the two definitions are compatible if \( \frac{dz}{dt} \) is a constant, that is, if particles are not accelerated along the z-axis. A point \((x_1, x'_1)\) in the x-x' plane represents the trajectory of a particle displaced a distance \( x_1 \) from the central trajectory and having a divergence \( x'_1 \) from it.
The beam is assumed to have some finite half-width, $x_m$, and to consist of particles having a divergence with absolute value from zero to some maximum value, $x^*_{m}$, for a fixed $z$. The beam then occupies an area in the $x$-$x'$ phase space bounded by $x_m$ and $x^*_m$. This area is approximated by an ellipse, the ellipse of minimum area circumscribing the occupied area. This is the basic assumption in beam transport theory. It is also assumed that the beam is symmetric in the $x$-$z$ plane so that the ellipse will be centered at the origin. An equation of an ellipse centered at the origin is

$$\gamma x^2 - 2\alpha xx' + 2\beta x'^2 = \epsilon. \quad (1)$$

Now Liouville's theorem states that the particle density in phase space is constant if the particles move in an external magnetic field or in a general external field of conservative forces. This has the consequence that the area of the ellipse is constant, which, together with normalization of this area to $\pi \epsilon$, gives a constraint on the four ellipse parameters:

$$\beta \gamma - \alpha^2 = 1. \quad (2)$$

From the definition of emittance, $\epsilon$ is the emittance of the beam.

At a waist $x_m x'^*_{m}$ is a minimum. Now $4x_m x^*_{m}$ is the area of a rectangle surrounding the ellipse. From Figure 1 it can be seen that this area is $4\epsilon \sqrt{\pi}$. Thus for $x_m x^*_{m}$ to be a minimum, $\beta \gamma$ must be a minimum for constant emittance. But

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From Equation (2) \( \beta \sigma = 1 + \alpha^2 \), so \( \beta \sigma \) is a minimum when \( \alpha = 0 \). This implies that the beam ellipse is upright at a waist.

Now Equation (1) could be written

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}x_i x_j = 1 \quad \text{or} \quad a_{11}x_1^2 + a_{12}x_1 x_2 + a_{21}x_2 x_1 + a_{22}x_2^2 = 1. \quad (3)
\]

This agrees with Equation (1) if \( x_1 = x, \ x_2 = x', \ a_{11} = \frac{f}{\epsilon}, \ a_{12} = -\frac{\alpha}{\epsilon}, \ a_{21} = -\frac{\alpha}{\epsilon}, \ \text{and} \ a_{22} = \frac{\beta}{\epsilon}. \) This suggests a matrix formalism. Define

\[
\sigma^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{f}{\epsilon} & -\frac{\alpha}{\epsilon} \\ -\frac{\alpha}{\epsilon} & \frac{\beta}{\epsilon} \end{pmatrix}.
\]
\[ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \beta \epsilon & \alpha \epsilon \\ \alpha \epsilon & \gamma \epsilon \end{pmatrix}. \] (4)

\( \sigma \) is called the beam matrix, and is always symmetric.

Comparison of Equation (4) with Figure 1 shows that \( \sigma_{11} = x_m^2 \) and \( \sigma_{22} = x_m^*2 \). At a waist where the ellipse is upright \( \alpha = 0 \) so the matrix is diagonal and can be written

\[ \sigma = \begin{pmatrix} x_m^2 & 0 \\ 0 & x_m^*2 \end{pmatrix}. \] (5)

**Waist-to-Waist Transformation**

Now consider a transformation to take an initial beam matrix, \( \sigma_i \), to some final matrix \( \sigma_f \). Define the superscript \( T \) to denote a transpose matrix, defined by \((M^T)_{ij} = M_{ji}\).

Equation (3) can be written in matrix notation as

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1. \] (6)

If we define \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \), then

\[ x^T \sigma^{-1} x = 1. \] (7)

For an orthogonal, and therefore non-singular, matrix \( R \), \( RR^{-1} = 1 = R^T(R^T)^{-1} \), so \( x^T R^T(R^T)^{-1} \sigma^{-1} R^{-1} Rx = 1 \) and

\[ \begin{pmatrix} x^T R^T \end{pmatrix} (R \sigma R^T)^{-1} (Rx) = 1. \] (8)

Define \( R \) to be the transformation that changes phase space coordinates \((x_1, x^*_1)\) at \( z_i \) to coordinates \((x_f, x^*_f)\) at \( z_f \).

Thus if \( x_1 = \begin{pmatrix} x_1 \\ x_1^* \end{pmatrix} \) and \( x_2 = \begin{pmatrix} x_f \\ x_f^* \end{pmatrix} \), then

\[ Rx_1 = x_2. \] (9)

Let Equation (7) describe the ellipse at position \( z_i \). Then
Equation (9) can be written \( (x_1^T R^T) \left( R \sigma R^T \right)^{-1} (R x_1) = 1 \).

Using Equation (9),
\[
x_2^T (R \sigma R^T)^{-1} x_2 = 1. \tag{10}
\]
The equation for the ellipse at \( z_f \) is \( x_2^T \sigma_f^{-1} x_2 = 1 \). This implies \( \sigma_f^{-1} = (R \sigma R^T)^{-1} \) or
\[
\sigma_f = R \sigma R^T. \tag{11}
\]

It is necessary to know what \( h \) is needed to transform a given matrix \( \sigma_i \) into a desired matrix \( \sigma_f \). The transformation is made in four steps. First \( \sigma_i \) is diagonalized and then converted to a unit matrix. Then it is transformed to another diagonal matrix and finally to the desired matrix. This corresponds to taking an initial ellipse, rotating it to an upright position, transforming it to a unit circle, transforming the circle to a new upright ellipse, and finally rotating the new ellipse to its desired orientation. However, when the ellipse is a unit circle, it can be rotated through an arbitrary angle without being changed. This arbitrary rotation also enters the matrix formalism.

A symmetric matrix can be diagonalized by an orthogonal \((\Theta_i^T \Theta_j = 1)\) transformation, \( \Theta_i \sigma_i \Theta_i^T = \Lambda_i \) and \( \Theta_f \sigma_f \Theta_f^T = \Lambda_f \), where \( \Lambda_i \) and \( \Lambda_f \) are diagonal. The diagonal matrices can be transformed into unit matrices: \( A_f \Lambda_f A_f^T = 1 \), \( A_i \Lambda_i A_i^T = 1 \), where \( A_{k'l'} = (A^T)_{k'l'} = \delta_{kl} \Lambda_{k'}^{-1} \). Now a degree of freedom enters when the matrix is a unit matrix since if \( Z Z^T = 1 \) then \( Z_1 Z_1^T = 1 \). Combining these transformations gives
\[ \omega_f = (\Theta_f^T A_f^{-1} Z A_i \Theta_i) \sigma_i (\Theta_f^T A_f^{-1} Z A_i \Theta_i)^{-1}. \]

Then from Equation (11), \( R = (\Theta_f^T A_f^{-1} Z A_i \Theta_i) \). All the needed matrices can be found except \( Z \), which can be written in its most general form as \( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \). For waist-to-waist transport \( \Theta_i \) and \( \Theta_f \) are both 1 since both initial and final ellipses are upright. Then \( R = A_f^{-1} Z A_i \).

Let \( \sigma_i = \begin{pmatrix} x_m^2 & 0 \\ 0 & x_m^2 \end{pmatrix} \) and \( \sigma_f = \begin{pmatrix} x_m^2 & 0 \\ 0 & x_m^2 \end{pmatrix} \). Then

\[ A_i = \begin{pmatrix} x_m^{-1} & 0 \\ 0 & x_m^{-1} \end{pmatrix} \) and \( A_f^{-1} = \begin{pmatrix} x_m & 0 \\ 0 & x_m \end{pmatrix} \), so

\[ R = \begin{pmatrix} M \cos \theta & M \sin \theta \\ -M \sin \theta & M \cos \theta \end{pmatrix}. \] (12)

where \( c = \frac{x_m}{x_m} \) = source collimator length and \( M = \frac{x_m}{x_m} \) = magnification. Equation (12) is the waist-to-waist transform.

**Quadrupole Magnets: Thin Lens Approximation**

Quadrupole magnets are considered next. Figure 2 shows the direction of the force on a charged particle moving through the magnet.

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given by the equation \( F = q(\nabla \times \mathbf{E}) \) where \( q \) is the charge, \( \nabla \) is velocity, and \( \mathbf{E} \) is the magnetic induction. A positively charged particle travelling in the \( +z \) direction in the \( y-z \) plane is deflected toward the \( z \)-axis, while a particle in the \( x-z \) plane is deflected away from the \( z \)-axis. Such a lens is converging in the \( y-z \) plane and diverging in the \( x-z \) plane. Thus a single thin lens cannot form coincident waists in the \( x-z \) and \( y-z \) planes. If the approximation that \( \frac{dx}{dz} \) and \( \frac{dv}{dz} \) are small compared to 1 is made, then the matrix transformations for charged particles in the lens are

\[
\begin{pmatrix}
\cosh kL & \frac{1}{k} \sinh kL \\
\frac{k}{\sinh kL} & \cosh kL
\end{pmatrix}
\]

for the diverging plane and

\[
\begin{pmatrix}
\cos kL & \frac{1}{k} \sin kL \\
\frac{k}{\sin kL} & \cos kL
\end{pmatrix}
\]

for the converging plane.

where \( k = \left[ \frac{q}{p} \left( \frac{\partial F_x}{\partial y} \right) \right]^{\frac{1}{2}} \). Here \( q \) is equal to the charge of the particle, \( p \) is equal to the momentum of the particle, and \( L \) is the effective length of the magnet, that is, the length of an ideal magnet which has no fringe field.\(^5\)

If a point source is placed at the primary focal point of a lens, the beam will emerge from the lens as a parallel beam. For a parallel beam incident on a lens, the beam will converge to a point at the secondary focal point.

If the incoming and outgoing rays are extended into the lens as shown in Figure 3, the intersections of these rays form a plane, called in the first case the entrance or 1st principal plane and in the second case the exit or 2nd principal plane. The focal length ($f$) is the distance from the primary focal point to the entrance principal plane and $f'$ is the distance from the secondary focal point to the exit principal plane. If the media on both sides of the lens are the same then $f' = f$. The entrance principal plane of the lens is located a distance $t_i$ after the magnet entrance and the exit principal plane is located a distance $t_e$ before the magnet exit. The focal length is denoted by $f$. Fenner has shown that in the converging plane

$$\frac{1}{f_c} = k \sin kL$$

(13)
\[ t_i = t_e = \frac{(1 - \cos kL)}{k \sin kL}, \quad (14) \]

and in the diverging plane

\[ \frac{1}{f_D} = -k \sinh kL \]

and

\[ t_i = t_e = \frac{(\cosh kL) - 1}{k \sinh kL}. \quad (16) \]

A quadrupole lens is thin if the two principal planes coincide. This requires both planes to coincide at the center of the lens. Then \( t_i = t_e = \frac{L}{2} \). Equations (14) and (15) can be written as power series \( t_i = t_e = \frac{L}{2}(1 - \frac{1}{12}k^2L^2 + \ldots) \) for the converging plane and \( t_i = t_e = \frac{L}{2}(1 + \frac{1}{12}k^2L^2 + \ldots) \) for the diverging plane. Then the thin lens approximation is valid if \( k^2L^2 \ll 1 \). Expanding Equations (13) and (15) in power series gives \( \frac{1}{f_c} = k^2L(1 - \frac{1}{6}k^2L^2 + \ldots) \) for the converging plane and \( \frac{1}{f_D} = -k^2L(1 + \frac{1}{6}k^2L^2 + \ldots) \) for the diverging plane. Then for a thin lens \( f_c = -f_D = \frac{1}{k^2L} \). So with thin lenses all the bending can be considered to take place at the center of the lens and the lens can be represented by a matrix that just changes the trajectory of the particle. \( \begin{pmatrix} x' \\ x' \\ \end{pmatrix} = B \begin{pmatrix} x \\ x_0 \\ \end{pmatrix} \), where \( B \) equals \( \begin{pmatrix} 1 & 0 \\ \frac{1}{L} & 1 \end{pmatrix} \). The drift space (no lens) transform can be shown to be \( \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \) where \( z \) is the length of the drift. A thin lens system can be represented by a drift-focus-drift. So the transform

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\(^7\) Banford, op. cit., p. 39.

\(^8\) Penner, op. cit.
from a point a distance \( l \) from a lens of focal length \( f \) to a point a distance \( L \) beyond the lens is

\[
R = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}.
\tag{17}
\]

Equation (17) can be written

\[
R = \begin{pmatrix} 1 - \frac{L}{f} & f + 1 - \frac{fL}{f} \\ \frac{1}{f} & 1 - \frac{f}{f} \end{pmatrix}.
\tag{18}
\]

waist-to-Waist Transform for Singlet

In order to accomplish waist-to-waist transport with one thin lens, the thin lens transform Equation (17) must equal the waist-to-waist transform Equation (12). Equating the elements of these matrices gives

\[
\cos \theta = \frac{1}{f} (1 - \frac{L}{f}), \tag{19a}
\]

and

\[
\cos \theta = \mathcal{W}(1 - \frac{f}{f}). \tag{19b}
\]

\[
\sin \theta = \frac{1}{\mathcal{W}} (l + L - \frac{fL}{f}), \tag{20a}
\]

and

\[
\sin \theta = \frac{\mathcal{W}}{f}. \tag{20b}
\]

Equations (19a) and (19b) can be solved for \( L \) in terms of \( f \), \( l \), and \( \mathcal{W} \) to give
\[ L = f + M^2(l - f). \]  \hspace{1cm} (21)

Equations (19b) and (20b) can be solved for \( M^2 \) to give
\[ M^2 = f^2 \left[ (f - l)^2 + c^2 \right]^{-1}. \]  \hspace{1cm} (22)

Substitution of Equation (22) into Equation (21) yields
\[ L = \frac{f(l^2 - fl + c^2)}{(l - f)^2 + c^2}. \]  \hspace{1cm} (23)

Thus given a beam with source collimator length \( (c) \), an
object distance \( (l) \), a lens with focal length \( (f) \), the
image distance \( (L) \) can be found. The equation can be solved
for \( f \) to give
\[ f = \frac{2Ll + l^2 + c^2 \pm \sqrt{(l^2 + c^2)^2 - 4L^2c^2}}{2(L + l)}. \]

For \( f \) to be real, \((l^2 + c^2)^2 - 4L^2c^2 > 0\). Thus the maximum
image distance is \( L_{\text{max}} = \frac{l^2 + c^2}{2c} \). So if \( L = \frac{l^2 + c^2}{2c} \),
there is one focal length that will produce a waist at \( L \).
If \(-\frac{l^2 + c^2}{2c} < L < \frac{l^2 + c^2}{2c}\), there will be two focal lengths
that will produce the waist, but the waists will be of
different magnification. A negative \( L \) means that the image
is on the same side of the lens as the object and is called
a virtual waist. A waist at a positive \( L \) is called a real
waist.
CHAPTER III

DOUBLET

In this section the methods previously developed are applied to the problem of waist-to-waist beam transport with two thin lenses. The matrix transform is written and from that an equation for the image distance in terms of the object length, focal lengths, lens separation, and source collimator length is derived. The resulting equation is solved for the focal length of one lens in terms of the other and some conditions on image distances and focal lengths are obtained.

Matrix Transformation

The two-lens system is set up as shown in Figure 5.

![Figure 5 Doublet]

The transform from A to B is given by

\[
R = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/t_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}
\]

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which reduces to
\[
R = \left( 1 - \frac{t}{f_1} - \frac{L}{F} + \frac{tL}{F_1} + \frac{tL}{F_2} \right) \left( 1 - \frac{t}{f_2} - \frac{tL}{F_2} \right)
\]

Penner has shown that two thin lenses can be represented as a thick lens with focal length defined by
\[
\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{t}{f_1 f_2}.
\]

Substitution of this relation into Equation (24) leads to
\[
R = \left( 1 - \frac{t}{f_1} - \frac{L}{F} - \frac{tL}{F_1} - \frac{tL}{F_2} \right) \left( 1 - \frac{t}{f_2} - \frac{L}{F} \right)
\]

Equation for Focal Length

If Equation (26) is to represent waist-to-waist transport it must be equal to Equation (12). This leads to the following relations:

\[
\cos \Theta = \frac{1}{F} \left( 1 - \frac{t}{f_1} - \frac{L}{F} \right)
\]

\[
\cos \Theta = \frac{1}{F} \left( 1 - \frac{t}{f_2} - \frac{L}{F} \right)
\]

\[
\sin \Theta = \frac{1}{c_m} \left( l + L + t - \frac{tL}{F_1} - \frac{tL}{F_2} - \frac{L}{F} \right)
\]

\[
\sin \Theta = \frac{c_m}{F}.
\]

Equations (27) and (28) give
\[
L = F(1 - \frac{t}{f_1}) + M^2(l - F + \frac{tF}{F_2}).
\]

Equations (28) and (30) and \(\sin^2 \Theta + \cos^2 \Theta = 1\) yield
\[
\eta^2 = \frac{F^2}{c_m^2} \left[ c^2 + (F - l - \frac{tF}{F_2})^2 \right]^{-1}.
\]

Substituting Equation (32) into Equation (31),
\[
L = F \left[ (l^2 + c^2)(1 - \frac{t}{f_1}) - tF(1 - \frac{t}{f_2}) - F + 2tl \right] \left[ c^2 + (F - l - \frac{tF}{F_2})^2 \right]^{-1}.
\]
This equation tells where a given doublet forms a waist from an initial waist position and source collimator length. However, often one has a system where \( l, L, t, \) and \( c \) are fixed and wants to know what focal lengths give a waist at a certain location. To solve Equation (33) for \( f_1 \), Equation (33) is rearranged and Equation (25) is used to get a quadratic equation for \( f_1^*: \\
\begin{align*}
(c^2L-c^2f_2-l^2f_2+tf_2^2-t^2f_2+lf_2^2-2ltf_2+Ltf_2^2+L^2t^2L-2Lt^2f_2 \\
-2ltf_2^2+2lt^2f_2+2L^2f_2^2-2Ll^2t-2Lt^2f_2+4Lt^2f_2-2Lt^2)^f_1+e^2L^2f_2 \\
+c^2L^2-2c^2Ltf_2+c^2tf_2^2-c^2t^2f_2^2+l^2tf_2^2-l^2t^2f_2^2+L^2f_2^2+L^2t^2 \\
-2L^2tf_2 = 0. 
\end{align*}
\end{equation}

This equation can be solved by the quadratic formula. Therefore, for an arbitrary \( f_2 \) there are zero, one, or two real values of \( f_1^* \) depending on the value of the discriminant (D).

**Conditions on Focal Lengths of Lenses**

After some simplification,
\begin{align*}
D = (-4c^2f_2^4-24c^2t^2f_2^2+16c^2tf_2^3+16c^2t^3f_2-4c^2t^4)L^2 \\
+ (-8c^2tf_2^2+24c^2t^2f_2^3-24c^2t^3f_2^2+8c^2t^4f_2)L + (-4c^2t^2f_2^4 \\
+8c^2t^3f_2^3-4c^2t^4f_2^2) + f_2^4(c^2t^2)^2). 
\end{align*}

For real solutions for \( f_1^* \) to exist, \( D \geq 0 \). This reduces to
\begin{align*}
-4c^2(f_2-t)^4L^2-8c^2tf_2(f_2-t)^3L-4c^2t^2f_2^2(f_2-t)^2+f_2^4(c^2f^2)^2 \\
\geq 0. 
\end{align*}

Now if \( f_2 = t \), then \( f_2^4 \geq 0 \), so \( f_2 = t \) will always give an \( f_1^* \) to get a waist.
If \( f_2 \neq t \), the solution for Equation (36) is

\[
\frac{2ct f_2(f_2-t)+f_2^2(c^2+f^2)}{-2c(f_2-t)^2} \leq I \leq \frac{2ct f_2(f_2-t)-f_2^2(c^2+f^2)}{-2c(f_2-t)^2}. \quad (37)
\]

From the first half of Equation (37):

\[
(2ct+2cL+c^2+f^2)f_2^2 + (-2ct^2-4cLt)f_2 + 2cLt^2 \geq 0.
\]

Solving for \( f_2 \) gives:

1. If \( L > \frac{2c^2}{c^2+f^2} \), there are no restrictions on \( f_2 \).

2. If \( \frac{-c^2-f^2-2ct}{2c} < L < \frac{c^2+f^2}{2(c^2+f^2)} \), then

\[
f_2 \geq \frac{ct(t+2L)}{2c(t+L)} + \frac{\sqrt{c^2t^2-2cL(c^2+f^2)}}{2c(t+L) + (c^2+f^2)}
\]

or

\[
f_2 \leq \frac{ct(t+2L)}{2c(t+L)} - \frac{\sqrt{c^2t^2-2cL(c^2+f^2)}}{2c(t+L) + (c^2+f^2)}.
\]

3. If \( L = \frac{-c^2-f^2-2ct}{2c} \), then \( f_2 \geq \frac{t(c^2+f^2+2ct)}{2c(2ct-c^2-f^2)} \).

4. If \( L < \frac{-c^2-f^2-2ct}{2c} \), then \( ct(t+2L)+\sqrt{c^2t^2-2cL(c^2+f^2)} \)

\[
\leq f_2 \leq \frac{ct(t+2L)-\sqrt{c^2t^2-2cL(c^2+f^2)}}{2c(t+L) + (c^2+f^2)}.
\]

From the second half of Equation (37):

\[
(2ct + 2cL - c^2 - f^2)f_2^2 + (-2ct^2 - 4cLt)f_2 + 2cLt^2 \leq 0.
\]

The equation can be solved for \( f_2 \):

A. If \( L < \frac{-c^2}{2(c^2+f^2)} \), there are no acceptable values of \( f_2 \).

B. If \( L > \frac{c^2+f^2-2ct}{2ct} \) and \( L > \frac{-c^2}{2(c^2+f^2)} \), then

\[
\frac{ct(t+2L)-\sqrt{c^2t^2+2cL(c^2+f^2)}}{2c(t+L)-(c^2+f^2)} \leq f_2 \leq \frac{ct(t+2L)+\sqrt{c^2t^2+2cL(c^2+f^2)}}{2c(t+L)-(c^2+f^2)}.
\]

C. If \( L = \frac{c^2+f^2-2ct}{2ct} \), then \((ct-c^2-f^2)f_2 \leq \frac{t}{2}(2ct-c^2-f^2)\).

D. If \( \frac{-c^2}{2(c^2+f^2)} \leq L \leq \frac{c^2+f^2-2ct}{2ct} \), then
\[ f_2 \geq \frac{ct(t + 2L) - \frac{t\sqrt{c^2t^2 + 2ct(c^2 + l^2)}}{2c(t + L)} - (c^2 + l^2)}{2c(t + L)} \quad \text{or} \quad \]
\[ f_2 \leq \frac{ct(t + 2L) + \frac{t\sqrt{c^2t^2 + 2ct(c^2 + l^2)}}{2c(t + L)} - (c^2 + l^2)}{2c(t + L)} . \]

So then \( f_2 \) must satisfy both one of the relations 1, 2, 3, 4 and one of the relations A, B, C, D. Notice that Condition A limits the range of values of \( L \) for waist-to-waist transport. However, it does not limit real waists since \( \frac{-ct^2}{2(c^2 + l^2)} \leq 0 \). It is possible to find a doublet that will form a waist a distance \( L \) from the second lens by choosing \( f_2 \) to satisfy the appropriate conditions for this \( L \) and then solving Equation (34) for \( f_1 \). The magnification is given by Equation (32).
CHAPTER IV

SYMMETRIC DOUBLET

The formulas in the previous section simplify for the case of the symmetric doublet, that is, two thin lenses for which \( f_2 = -f_1 \).

Substitution of \( f_2 = -f_1 \) into Equation (34) leads to

\[
(t+L)f_1^4 + (t^2+2Lt)f_1^3 + (t^2L-2Lt-c^2t-l^2t-2t^2)f_1^2 \\
+ (-2Lt^2+c^2t^2+l^2t^2)f_1 + c^2L^2 + Lt^2t^2 = 0. \quad (38)
\]

Descartes' rule of signs says that there are, at most, two positive roots.

Equation (38) can be solved by computer methods. One method is to use trial values of \( f_1 \) and look for the function to change sign. A sign change means that there is a root between the two trial values of \( f_1 \). The region in which to look for solutions is determined by the conditions on \( f_2 \) given at the end of Chapter 3 and the values of \( f_1 \) for which the first term dominates the rest of the equation.

For numerical values let \( l \) and \( L \) range from one to three meters, \( t \) range from 0.5 to 2 meters, and \( c \) range from 0.1 to 1.0 m/rad.

Figure 6 shows results for \( c = 1.0 \) as \( l, L, \) and \( t \) vary, with \( l = L \). No positive roots were found. Formation of a waist distant \( L \) from the second lens in both the \( x-z \) and...
y-z planes implies that both positive and negative solutions to Equation (38) must exist, since $f_c = -f_D$ for a thin lens. So with the range of values in Figure 6 it is impossible to form a waist at the same location in both the x-z and y-z planes.

Figure 7 shows solutions for $f_1$ for $c = 0.1$ as $l$, $L$, and $t$ vary, with $l = L$. This time positive and negative roots were found. This figure also indicates that there is a minimum value of $l$ and $L$ for which positive solutions exist and that this value increases with increasing $t$.

Figure 8 shows positive values of $f_1$, for $l = L = 3.0$, $t = 1.0$ as $c$ varies from 0.1 to 0.4. As $c$ increases, the values of $f_1$ converge and vanish.
Figure 6
Focal Lengths for Symmetric Doublet
Figure 7: Focal Lengths for Symmetric Doublet
Figure 3
Positive Focal Lengths for Symmetric Doublet

\[ f = L = 3.0 \]
\[ t = 1.0 \]
CHAPTER V

SYMMETRIC TRIPLET

This section treats the case of the symmetric triplet. A symmetric triplet is three thin lenses in which the focal length of the first (f) is the same as the focal length of the third (f). The focal length of the second is \(-\frac{1}{2}f\), and the separation (t) of the first and second is the same as the separation of the second and third lenses. The matrix transformation is written down and an equation for the focal lengths is derived. Finally some numerical results are given. Figure 9 shows a symmetric triplet.

![Symmetric Triplet Diagram](image)

**Figure 9**
Symmetric Triplet

The matrix transformation from A to B is

\[
R = \begin{pmatrix}
1 & L \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{t} & 1
\end{pmatrix}
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\frac{2}{t} & 1
\end{pmatrix}
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\frac{1}{t} & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

which can be simplified to

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Focal Length Equation

Waist-to-waist transport yields the transformation

\[
R = \begin{pmatrix}
\frac{M}{M} \cos \theta & cM \sin \theta \\
-\frac{1}{cM} \sin \theta & \frac{1}{M} \cos \theta
\end{pmatrix}.
\]  

(12)

Equating the elements of the two matrices gives four equations. Solving the equations from the upper left and lower right elements for \( \cos \theta \) gives

\[
\cos \theta = \frac{1}{M} \left(1 - \frac{2t^2}{f^2} - \frac{2tL}{f^2} + \frac{2t^2}{f^2} \right) = M \left(1 - \frac{2tL}{f^2} - \frac{2t^2}{f^2} + \frac{2t^2}{f^2} \right). \quad (39)
\]

The lower left and upper right elements give

\[
\sin \theta = \frac{1}{cM} \left(\frac{2t}{f^2} - \frac{2t^2}{f^2} \right) = \frac{1}{cM} \left(\frac{1}{f^2} + \frac{2t^2}{f^2} - \frac{2t^2}{f^2} \right). \quad (40)
\]

Solving Equation (39) for \( \omega^2 \) and substituting into Equation (40) yields, after some simplification:

\[
(l + 2t + L)f^6 + 2t^2f^5 + \left(-8t^2L - 4t^2L - 4tL - 2t^2L + 4t^2 - 2t^2 - 2c^2t\right)f^4
\]

\[
+ \left(4t^4L + 2t^2L + 4t^4L + 2t^2L + 8t^4L + 4t^4L + 4c^2t^2L\right)f^3
\]

\[
+ \left(-8t^6L - 8t^6L - 4t^4L - 8t^6L - 8t^6L - 8t^6L - 8t^6L - 8t^6L - 8t^6L - 8c^2t^2L\right)f
\]

\[
= 0. \quad (41)
\]

Numerical Results

Figures 10, 11, 12 show plots of some numerical solutions of Equation (41). The plots were drawn from the data points shown. Figure 10 shows values of \( f \) for \( \lambda = L \), \( \lambda \) and \( L \) range from 1.0 to 3.0, \( \lambda = 0.5 \), and \( c = 1.0 \).
Four solutions were found except at \( J = 1.0 \) where one more solution was found. Figures 11 and 12 show values of \( f \) for \( J = L \). \( J \) and \( L \) range from 1.0 to 3.0 and \( t = 1.0 \). In Figure 11 \( c = 1.0 \) and in Figure 12 \( c = 0.1 \). The plots are very similar except that two more roots were found for \( J = 2.0 \) in Figure 12. In fact, the upper and lower lines are identical in the two graphs. From this, one can conclude that two of the solutions of Equation (41) do not depend on \( c \).

By examining the two upper lines in all three plots, it may be seen that where \( L = J = 2t \), \( f = t \) is a solution. By substitution of \( f = t \) into Equation (41) it is seen that \( f = t \) is a solution if \( J + L = 4t \); \( J \) need not equal \( L \). The magnification for the waist formed by \( f = t \) is \( \pm 1 \), so by choosing \( J + L = 4t \) and \( f = t \) the transformation is a unit transformation and the initial waist is reproduced. Also at \( L = J = 2t \), \( f = 2t \) is also a solution but the transformation is not a unit transformation. Substitution of \( f = 2t \) into Equation (41) shows that \( f = 2t \) is a solution for \( L = J = 2t \).
Figure 10

Focal Lengths for Symmetric Triplet
for \( l = L, t = 0.5, c = 1.0 \)
Figure 11
Focal Lengths for Symmetric Triplet
for \( f=1 \), \( t=1.0 \), \( c=1.0 \)
Figure 12

Local Lengths for Symmetric Triplet

for $f=L$, $t=1.0$, $c=0.1$
CHAPTER VI

SUMMARY

When this study was begun, the hope was that simple useful algebraic expressions for waist-to-waist transport would be derived. Unfortunately, the algebra became cumbersome and this hope was not realized.

The methods developed for the singlet thin lens were extended to the doublet and triplet. For the doublet equations were derived that enable one to choose lenses that will produce a waist at a desired location. It was found that \( f_2 \), the focal length of the second lens, must obey certain inequalities for values of \( f_1 \) to exist. A lower limit on image distances was found, as was an expression for the magnification. It was also found that for a given \( f_2 \) there are, at most, two values of \( f_1 \) that will produce a waist. The work was done in only one plane, \( x-z \), but the equation in the other plane, \( y-z \), is identical in form. Since \( f_x = -f_y \) for each quadrupole lens, equations for \( f_{1x} \) and \( f_{1y} \) can be set equal to each other with the appropriate minus sign, and the solutions for \( f_{2x} \) and \( f_{2y} \) for waists in two planes can be obtained.

For the symmetric doublet it was found that there are at most four values of \( f_1 \) that give waists at a desired location. Also for the object distance set equal to the
image distance, the source collimator length and image
distance determine whether or not positive solutions exist.

For the symmetric triplet the equation for the focal
lengths giving waist-to-waist transport is of the sixth
degree; and all six roots may be real. It was found that if
the sum of the object distance and the image distance is
four times the lens separation, the focal length set equal
to the lens separation gives a waist with magnification \( \frac{1}{2} \). Also, if the above conditions are satisfied and the object
distance is equal to the image distance, then another solution
is one where the focal length equals twice the lens sepa-
ration.

The author gratefully acknowledges the assistance of
Dr. Mark J. Jakobson in the preparation of this paper and
the research that preceded it.
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