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Abstract: A set with a binary operation is a fundamental concept in algebra and one of the most fundamental properties of a binary operation is associativity. In this paper the authors discuss binary operations on a three-element set and show, by an inclusion-exclusion argument, that exactly 113 operations out of the 19,683 existing operations on the set are associative. Moreover these 113 associative operations are accounted for by means of their operation tables.

Keywords: binary operations; associativity; inclusion-exclusion principle; recreational mathematics

1. Introduction

It may catch someone by surprise that the number of distinct binary operations on a set of only three elements is as large as 19,683. To prove this is an easy calculation; there are 3 different answers for each of the 9 seats of a 3×3 operation table so the number of distinct operations is 3^9. The number of commutative operations on the set can be calculated in a similar way, resulting in $3^6 = 729$ operations. The number of operations for which there exists an identity in the set is $3\cdot 3^4 = 3^5 = 243$ and for the same number of operations there exists a zero. This calculation is straightforward, but no easy calculation and no educated guess, seems to give the answer to the following question:

"How many binary operations on a three element set are associative?"

The objective of this paper is to answer this question. In other words to decide how many of the 19,683 different binary operations on a three-element set are associative. The authors have deliberately chosen to use algebraic concepts and tools to arrive at their conclusion rather than programming a computer to do the task. Overall the argument is inclusive-exclusive and operations are grouped together according to the algebraic properties they share. Initially the 243 operations for which there exists an identity in the set are studied, followed by the 243 operations where there exists a zero and finally the remaining 19,215 operations for which there neither

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exists an identity nor a zero (18 contain both) are studied. For the last mentioned operations further classification is needed.

We point out that the number of operations, the number of commutative operations and the number of operations containing an identity/zero is no harder to calculate for an *n*-element set than for a set of three elements. These numbers are $n^{(n^2)}$, $n^{\frac{n(n+1)}{2}}$ and $n^{(n-1)^2+1}$ respectively. To the interested reader we recommend the following references for further reading: [**1**], [**2**].

2. General Concepts

A *binary operation* (hereafter referred to only as an *operation*) on a set *S* is a rule that assigns to each ordered pair (*a*, *b*), where *a* and *b* are elements of *S*, exactly one element, denoted by *ab*, in *S*. A set *S* with an operation is said to be *closed* under the operation and in general if *H* is a subset of *S* then *H* is *closed* if *ab* is in *H* for all *a* and *b* in *H*. Quite often an operation on a set is referred to as multiplication. Throughout this section each of the sets *S* and *S′* will be closed under operation.

An operation on *S* is *commutative* if $xy = yx$ for every *x* and *y* in *S*. An operation on *S* is *associative* if $x(yz) = (xy)z$ for every *x*, *y* and *z* in *S*. A *semigroup* is a set *S* with an associative operation. If *a* is an element of a semigroup and *n* is a natural number then a^n is defined to be the product $aaa...a$, of *n* factors. An element *x* of *S* is said to be an *idempotent* if $xx = x$, an element *e* of *S* is said to be an *identity* if $ex = x$ and $xe = x$ for all *x* in *S*, and an element *z* of *S* is said to be a *zero* if $zx = z$ and $xz = z$ for all *x* in *S*.

An *isomorphism* between *S* and *S′* is a one-to-one function φ mapping *S* onto *S′* such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all *x* and *y* in *S*. If there exists an isomorphism between *S* and *S′*, then *S* and *S′* are said to be *isomorphic*, denoted $S \approx S'$.

An *anti-isomorphism* between *S* and *S′* is a one-to-one function φ mapping *S* onto *S′* such that $\varphi(xy) = \varphi(y)\varphi(x)$ for all *x* and *y* in *S*. If there exists an anti-isomorphism between *S* and *S′*, then *S* and *S′* are said to be *anti-isomorphic*, denoted $S \approx a\ S'$.

3. Useful Theorems

The following theorems about sets, *S* and *S′*, closed under operations are well known and easy to prove:

Theorem 1 If an identity exists in *S* then it is unique.

Theorem 2 If a zero exists in *S* then it is unique.

Theorem 3 In a set *S* of more than one element, an identity and a zero have to be distinct.

Theorem 4 If there exists an isomorphism between *S* and *S′* and the operation on *S* is associative then the operation on *S′* is also associative.

Theorem 5 If there exists an anti-isomorphism between *S* and *S′* and the operation on *S* is associative then the operation on *S′* is also associative.

Theorem 6 If S is a finite set, the operation on S is defined by an operation table A and A^{T} (the transposition of A) defines an operation on the set $S' = S$, then there exists an anti-isomorphism between S and S'.

4. Example

Consider a set $S = \{a, b\}$ with two elements. The number of different binary operations on this set is 16. The corresponding operation tables are listed below:

1	*a*	*b*
a	*a*	*a*
b	*a*	*a*

2	*a*	*b*
a	*a*	*a*
b	*a*	*b*

3	*a*	*b*
a	*a*	*a*
b	*b*	*a*

4	*a*	*b*
a	*a*	*a*
b	*b*	*b*

5	*a*	*b*
a	*a*	*b*
b	*a*	*a*

6	*a*	*b*
a	*a*	*b*
b	*a*	*b*

7	*a*	*b*
a	*a*	*b*
b	*b*	*a*

8	*a*	*b*
a	*a*	*b*
b	*b*	*b*

9	*a*	*b*
a	*b*	*a*
b	*a*	*a*

10	*a*	*b*
a	*b*	*a*
b	*a*	*b*

11	*a*	*b*
a	*b*	*a*
b	*b*	*a*

12	*a*	*b*
a	*b*	*a*
b	*b*	*b*

13	*a*	*b*
a	*b*	*b*
b	*a*	*a*

14	*a*	*b*
a	*b*	*b*
b	*a*	*b*

15	*a*	*b*
a	*b*	*b*
b	*b*	*a*

16	*a*	*b*
a	*b*	*b*
b	*b*	*b*

By studying the operation tables we come to the following conclusions:
Eight operations are commutative. See tables 1, 2, 7, 8, 9, 10, 15 and 16.
Eight operations are associative. See tables 1, 2, 4, 6, 7, 8, 10 and 16.
Six operations are both commutative and associative. See tables 1, 2, 7, 8, 10 and 16.
For four operations there exists an identity in S. See tables 2, 7, 8 and 10.
For four operations there exists a zero in S. See tables 1, 2, 8 and 16.
For two operations there exist both an identity and a zero in S. See tables 2 and 8.

For the purpose of finding the number of associative operations it is not necessary to study all sixteen tables, we can group together isomorphic and anti-isomorphic tables (see Section 3). For this purpose we use the one-to-one function from S onto S' that interchanges a and b. The result is the following: $1 \approx 16$, $2 \approx 8$, $3 \approx a\,5$, $3 \approx 12$, $3 \approx a\,14$, $4 \approx a\,6$, $5 \approx a\,12$, $5 \approx 14$, $7 \approx 10$, $9 \approx 15$, $11 \approx a\,13$, $12 \approx a\,14$ which cuts the number of tables to be studied down to seven. See tables 1, 2, 3, 4. 7, 9 and 11.

5. Operations on a Three-Element Set

As mentioned in the introduction, the number of possible binary operations on a set of three elements is 19,683. Of these 729 are commutative and for 243 different operations there exists an identity, and likewise for 243 operations there exists a zero. We now proceed to answer the question: How many associative operations exist on a set of three elements?

For a three-element set S proving associativity for a given operation amounts to verifying 27 different equations:

$(xy)z = x(yz)$, where x, y and z are elements of S.

A single counterexample suffices to show that a given operation is not associative. Clearly counterexamples need not to be unique. When referring to a particular equation involving the elements x, y and z of S, we shall simply refer to it as xyz. It can easily be seen that if one of the elements x, y and z is an identity or a zero then the equation $(xy)z = x(yz)$ holds.

The analysis of the associative operations on a three-element set $S = \{a, b, c\}$ will now be divided into two steps. First we assume the existence of an identity or a zero, since this considerably increases the likelihood of an associative operation. Subsequently operations for which there neither exists an identity nor a zero will be discussed.

5.1. Operations for which there exists an identity or a zero

5.1.1. Operations for which there exists an identity

Let us assume that a is an identity. This determines 5 of the 9 places in a 3×3 operation table and leaves 81 different possibilities of filling in the remaining 2×2 subtable. For 16 of those subtables the subset $\{b, c\}$ is closed with respect to the operation. According to previous discussion of associative operations in a two element set (Section 4) one can find a counterexample to associativity for 8 of those 16 operations and such an example will also give a counterexample to the associativity of the 3×3 operation table. The 8 remaining 2×2 operation tables show an associative operation and a being an identity assures associativity of the 3×3 table. These 8 associative operations are shown below:

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*b*	*b*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*b*	*c*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*c*
c	*c*	*b*	*c*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*c*
c	*c*	*c*	*b*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*c*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*c*	*b*
c	*c*	*b*	*c*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*c*	*c*
c	*c*	*c*	*c*

Now let us consider the operations where the 2×2 subtable is not closed, in other words the subset $\{b, c\}$ is not closed. There are 4 possibilities: a can appear exactly once, exactly two times, exactly three times or four times in the subtable. We investigate each case separately.

i) a appearing exactly once

The number of corresponding 3×3 tables is 32 and only two of them turn out to be associative:

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*a*	*c*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*b*	*a*

In both tables *a* is on the diagonal which must be the case since otherwise *cbc* or *bcb* give a counterexample to associativity. In the table on the left, *c* is a zero, and in the other one, *b* is a zero. Again this must be the case since otherwise counterexamples to associativity can be found among: *bcc*, *ccb*, *cbb*, *bbc*.

ii) *a* appearing exactly two times
The corresponding 3×3 tables are 24 and only one of them turns out to be associative:

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*c*	*a*
c	*c*	*a*	*b*

In all other cases a counterexample may be found among: *bbc*, *ccb*, *cbb*, *bcc*.

iii) *a* appearing exactly three times
The corresponding 3×3 tables are 8 and none of them associative. Counterexamples may be found among: *bbc*, *ccb*.

iv) *a* appearing four times
The corresponding 3×3 table is unique and not associative. Counterexample *bbc*.

It has now been shown that exactly 11 operations on *S* for which *a* is an identity, are associative and the respective operation tables have been listed. The corresponding operation tables where *b* is an identity are likewise 11, and additional 11 tables have *c* as an identity. One can conclude (see Section 3) that the number of associative operations on *S* for which there exists an identity (*a, b* or *c*) amounts to 33.

5.1.2. Operations for which there exists a zero

Let us assume that *a* is a zero. We first consider operations where the subset $\{b, c\}$ is closed and as before, when *a* was assumed to be an identity, this gives exactly 8 associative operations:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*b*
c	*a*	*b*	*b*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*b*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*b*
c	*a*	*c*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*c*	*b*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*c*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*b*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*c*
c	*a*	*c*	*c*

Next let us consider the operations where the 2×2 subtable is not closed with respect to $\{b, c\}$. As before there are 4 possibilities: *a* can appear exactly once, exactly two times, exactly three times or four times in the subtable. We again investigate each case separately.

i) *a* appearing exactly once
The corresponding 3×3 tables are 32 and only two of them turn out to be associative. These are the tables:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*c*	*a*

In all other cases counterexamples to associativity can be found among: *bbb*, *bbc*, *bcb*, *bcc*, *cbb*, *cbc*, *ccb*, *ccc*.

ii) *a* appearing exactly two times
The corresponding 3×3 tables are 24 and 5 of them turn out to be associative:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*a*	*a*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*a*	*a*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*a*	*a*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*a*	*c*	*a*

In other cases counterexamples can be found among the same conditions as given in i).

iii) *a* appearing exactly three times
The corresponding 3×3 tables are 8 and 4 turn out to be associative:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*a*	*a*	*a*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*a*
c	*a*	*a*	*a*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*a*	*b*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*a*	*c*

Counterexamples for the other 4 tables may be found among: *bbc*, *bcc*, *cbb*, *ccb*.

iv) *a* appearing four times
The corresponding 3×3 table is unique and clearly associative:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*a*	*a*

It has been shown that exactly 20 operations on *S*, where *a* is a zero, are associative. The respective operation tables have been listed. The corresponding operation tables where *b* is a zero are likewise 20 and additional 20 tables have *c* as a zero. One can conclude (see Section 3) that the number of associative operations on *S* for which there exists a zero (*a, b* or *c*) amounts to 60.

5.1.3. Operations for which there exists an identity and a zero

The number of operations on S having both an identity and a zero is readily seen to be 18; there are 3 possibilities for choosing the identity (a, b or c) for each one of these, 2 possibilities for choosing the zero (see Section 3) and finally 3 possibilities (a, b or c) to fill in the remaining seat of the operation table. To show that each one of these 18 operations is associative we assume, without loss of generality, that a is the identity and b the zero. Then the only condition for associativity that needs to be checked is ccc which holds in all three cases; $cc = a$, $cc = b$, $cc = c$:

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*b*	*a*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*b*	*b*

	a	*b*	*c*
a	*a*	*b*	*c*
b	*b*	*b*	*b*
c	*c*	*b*	*c*

Summary of Section 5.1.

Section 5.1.1 is devoted to operations where there exists an identity in S. The total number of these is 243; 81 operation for each of a, b or c being the identity. The number of associative operations having a for an identity is shown to be 11 and accordingly the total number of associative operations with an identity (a, b or c) is 33. Section 5.1.2 is devoted to the case where there exists a zero in S. As for the identity the total number of operations with a zero is 243; 81 for each of a, b or c being the zero. The number of associative operations having a for a zero is shown to be 20 and hence the total number of associative operations on S with a zero is 60. In Section 5.1.3 the number of associative operations with both an identity and a zero is counted as 18 and an argument given that each one of these operations is associative. We summarize our results of Section 5.1: The number of associative operations on S with an identity, a zero or both an identity and a zero in S is 75:

$$33 + 60 - 18 = 75.$$

5.2. Operations for which there neither exists an identity nor a zero

Having considered all possible operations for which there exists an identity or a zero we will from now on, assume that there neither exists an identity nor a zero in S. The following theorem is of great help.

Theorem If an operation on $S = \{a, b, c\}$ is associative then S contains an idempotent.

Proof: Consider the elements a, a^2, a^3 and a^4 of S. These, of course, are at most three different elements of S so two of them are equal and one of the following six cases must hold true:

i) $a^2 = a$ and a is an idempotent.

ii) $a^3 = a$ and a^2 is an idempotent as $(a^2)^2 = a^4 = a^3a = aa = a^2$.

iii) $a^4 = a$ and a^3 is an idempotent as $(a^3)^2 = a^6 = a^4a^2 = aa^2 = a^3$.

iv) $a^3 = a^2$ and a^2 is an idempotent as $(a^2)^2 = a^4 = a^3a = a^2a = a^3 = a^2$.

v) $a^4 = a^2$ and a^2 is an idempotent

vi) $a^4 = a^3$ and a^3 is an idempotent as $(a^3)^2 = a^6 = a^4a^2 = a^3a^2 = a^4a = a^3a = a^4 = a^3$.

This theorem on the existence of an idempotent in a three-element semigroup is a special case of a theorem proven by Frobenius in an article published 1895 [**3**]. In the article Frobenius shows that if S is a semigroup, a an element of S and the subsemigroup $\{a, a^2, a^3, a^4 \ldots\}$ is finite then this subsemigroup will contain exactly one idempotent. In 1902 E. H. Moore showed that some power of each element in a finite semigroup is an idempotent. See [**4**].

According to the theorem above one can, when looking for associative operations in a three-element set $S = \{a, b, c\}$, assume the existence of an idempotent in S and we begin by assuming that a is an idempotent.

Given the condition $aa = a$ there are 9 possibilities for the first line of an operation table: *a a a*, *a a b*, *a a c*, *a b a*, *a b b*, *a b c*, *a c a*, *a c b*, *a c c*. Three of these, *a a b*, *a c b*, *a c a*, can be immediately eliminated by counterexamples to associativity found among *aac*, *aab* so we are left with 6 possibilities: *a a a*, *a a c*, *a b a*, *a b b*, *a b c*, *a c c* and we will refer to those by numbers 1 to 6.

Similarly, the corresponding 6 columns are the only possibilities for the first column of an operation table, and we will also refer to those by numbers 1 to 6. We have thus at most $6 \cdot 6 = 36$ possibilities for the first line and first column of an operation table, if the operation is to be associative and we assume $aa = a$. We refer to these partial tables by the numbers given to their line and column, for example partial table 2-4, which is shown below, consists of line no. 2 and column no. 4:

2-4	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*b*		

For simplification, rather than study these possibilities one by one, the possibilities that would lead to the same number of associative operation tables are grouped together and a representative chosen for each group.

Let f denote the isomorphism from S to S, that interchanges b and c. This isomorphism relates a table starting with line no. 2 (*a a c*) with a table starting with line no. 3 (*a b a*) and therefore each operation table starting with line no. 2 corresponds to a table that starts with line no. 3 under this isomorphism. A similar relation holds true for tables that start with lines no. 4 and no. 6, *a b b* and *a c c*. On the other hand lines no. 1 and no. 5, *a a a* and *a b c*, will be mapped onto themselves by this isomorphism. The same holds true for columns, so for the six lines (columns) and this isomorphism f one can list the relationship between lines (columns) in the following way referring to the lines (columns) by their numbers: $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 6$, $f(5) = 5$, *f*(6) = 4. In general, the isomorphism *f* relates the partial table denoted *x*-*y* to the partial table *f*(*x*)-*f*(*y*).

In this way *f* gives several examples of different partial tables that give rise to the same number of associative operations, for example 1-2 and 1-3, 1-4 and 1-6, 2-1 and 3-1, 2-2 and 3-3. Furthermore partial tables *x*-*y* and *y*-*x* will always result in the same number of associative operations since the partial table *y*-*x* is the transposition of the partial table *x*-*y* (see Section 3). According to these observations we can now classify the previously mentioned 36 possibilities into the following 13 classes which we label C1-C13:

C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13
1-1	1-2	1-4	1-5	2-2	2-3	2-4	2-5	2-6	4-4	4-5	4-6	5-5
	1-3	1-6	5-1	3-3	3-2	3-6	3-5	3-4	6-6	5-4	6-4	
	2-1	4-1				4-2	5-2	4-3		5-6		
	3-1	6-1				6-3	5-3	6-2		6-5		

Each class could be characterized by its algebraic properties, and partial tables of the same class will result in the same number of associative operation tables. For further inspection it is therefore sufficient to choose one representative of each class, and we choose the first partial table listed in each class. Indeed, since operations for which there exists an identity or a zero have been accounted for, one can skip class C1 (where *a* is a zero) and class C13 (where *a* is an idempotent).

Now consider classes C7, C9 and C12.The partial tables of their representatives are:

2-4	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*b*		

2-6	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*c*		
c	*c*		

4-6	*a*	*b*	*c*
a	*a*	*b*	*b*
b	*c*		
c	*c*		

It is easy to see that *aca* or *aba* gives a counterexample for each one of those partial tables so there are no associative operations to be found in these classes.

There are 8 classes left, each containing two or four partial tables:

	C2	C3	C4	C5	C6		C8		C10	C11		
	1-2	1-4	1-5	2-2	2-3		2-5		4-4	4-5		
	1-3	1-6	5-1	3-3	3-2		3-5		6-6	5-4		
	2-1	4-1					5-2			5-6		
	3-1	6-1					5-3			6-5		

As mentioned before, the following partial tables have been chosen to represent these classes:

1-2	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*		
c	*c*		

1-4	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*b*		
c	*b*		

1-5	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*b*		
c	*c*		

2-2	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*a*		
c	*c*		

2-3	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*a*		

2-5	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*c*		

4-4	*a*	*b*	*c*
a	*a*	*b*	*b*
b	*b*		
c	*b*		

4-5	*a*	*b*	*c*
a	*a*	*b*	*b*
b	*b*		
c	*c*		

Let us consider representative 1-2

For the partial table 1-2 one can choose *bb* and *bc* in the following *9* different ways which we number *1-9*:

1	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*c*		

2	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*c*		

3	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*c*
c	*c*		

4	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*c*		

5	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*b*
c	*c*		

6	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*c*		

7	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*a*
c	*c*		

8	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*b*
c	*c*		

9	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*c*
c	*c*		

Counterexamples to associativity can be found among *bba*, *bbc*, *bca* for all tables other than *1*, *4*, *6*. For these tables we point out that if the sought operation is to be associative one can use *cab* and *cac* to calculate *cb* = *c* and *cc* = *c*. This makes *b* an identity for table *6* and we refer to Section 5.1. Tables *1* and *4* turn out to give associative operations with neither an identity nor a zero:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*c*	*c*	*c*

Next consider representative 1-4

We use *bab* and *bac* to calculate *bb* = *b* and *bc* = *b* which gives the table:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*b*		

This partial table can be finished in two ways to give an associative operation:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*b*	*b*	*b*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*b*	*b*	*c*

There are no other possibilities to fill in partial table 1-4 without getting a counterexample to associativity among *cba*, *cca*, *cba*.

The remaining six representatives, 1-5, 2-2, 2-3, 2-5, 4-4, 4-5, can all be dealt with in a similar manner and we state the results:

Each of the following representatives lead to one associative operation: 1-5, 2-2, 4-4, 4-5. The corresponding operation tables are shown below:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*a*	*c*
b	*a*	*a*	*c*
c	*c*	*c*	*a*

	a	*b*	*c*
a	*a*	*b*	*b*
b	*b*	*a*	*a*
c	*b*	*a*	*a*

	a	*b*	*c*
a	*a*	*b*	*b*
b	*b*	*b*	*b*
c	*c*	*c*	*c*

Representative 2-3 does not lead to an associative operation and representative 2-5 does not lead to an associative operation without an identity or a zero.

Summary of Section 5.2.

Section 5.2 is devoted to finding the associative operations on *S* where there neither exists an identity nor a zero. First a theorem is proved which states the existence of an idempotent in *S* if the operation is associative, and thereafter *a* is assumed to be idempotent. Possible operation tables are grouped into 13 classes according to the first line and first column of each table and one representative chosen for each class. Two of the classes have been accounted for in Section 5.1 (existence of an identity/zero), and three classes are immediately shown to give no associative operations and thereby excluded. The remaining eight classes are studied, case by case, by considering elements of the four remaining seats of each operation table and the tables either included as associative or excluded. It turns out that associative operations are found in six of the eight classes and the result is shown below:

Representative of class	1-2	1-4	1-5	2-2	4-4	4-5
Associative operations for the representative	2	2	1	1	1	1
Number of partial tables in class	4	4	2	2	2	4
Total of associative operations	8	8	2	2	2	4

There are 26 associative operations altogether. Let us recall that *a* was assumed to be an idempotent ($aa = a$) and that neither an identity nor a zero exist. Obviously we get the same number of associative operations assuming that *b* is an idempotent and likewise by assuming that *c* is an idempotent. By writing down the 26 tables where *a* is an idempotent we can count that among them there are 18 tables where $bb = b$ and of these 18 there are 14 with $cc = c$. We summarize our results of Section 5.2: The number of associative operations on a three-element set for which there neither exist an identity nor a zero is 38:

$$26 + 26 + 26 - (18 + 18 + 18) + 14 = 38.$$

6. Conclusion

The conclusion of this article is that among the 19,683 different operations on a three-element set, $S = \{a, b, c\}$, there are exactly 113 operations which are associative, in other words there exist exactly 113 three-element semigroups. The article also reveals that 75 of these semigroups have an identity or a zero, but 38 have neither an identity nor a zero. Out of the 75 that have an identity or a zero, 18 have both, 15 have an identity but not a zero and 42 have a zero but not an identity.

The article lists 42 of the 113 operation tables for associative operations in a three-element set and points out how each of the remaining associative operations corresponds to one of the 42 operations given.

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