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A LONGITUDINAL STUDY OF STUDENT'S REPRESENTATIONS FOR DIVISION OF FRACTIONS

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Abstract: *The representations that students use as part of their mathematical problem solving can provide us with a window into their grasp of the concepts they are exploring and developing. In this paper, the author indicates how these representations can evolve over time and enrich the understanding of division of fractions, often thought to be the most difficult of elementary school mathematical topics. The results of this research suggest that when appropriate problems are provided for students, in a meaningful context, they can demonstrate understanding of division of fractions that is durable over time, and that they are able to flexibly move back and forth between and among representations, choosing what they deem to be appropriate forms for a particular situation.*

Keywords: elementary school; division of fractions; representations; problem solving; longitudinal research;

Introduction and Theoretical Framework

This research was designed to investigate two intertwined issues: the manner in which students build powerful ideas about fractions, division of fractions, in particular, and the importance of having students build, use, and connect different types of representations for these ideas. Specifically, this study investigates how the same group students developed and made sense of several different types of representations as part of their investigation into concepts involving division of fractions. Also under study was how these investigations helped them to avoid some of the common difficulties frequently experienced by others.

Difficulties with Fractions and Division of Fractions

The difficulties that many students have experienced while solving problems involving fractions have been well documented (for example: Davis, Alston, and Maher, 1991; Davis, Hunting, and, Pearn, 1993; Steffe, Cobb and von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards and Cobb, 1983; Tzur, 1999;). Therefore, it is of great importance to find ways to help students overcome these difficulties. Ma (1999) states that division is the most complex of the mathematical operations and that fractions are the most complicated numbers to deal with in arithmetic. Therefore, she considers division of fractions to be the most difficult topic in elementary mathematics, for both teachers and their students. As a case in point, she notes that only forty-three percent of the United States teachers in her study were able to perform the computation successfully and only one out of twenty-three teachers was able to give a correct

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representation for a problem involving division of fractions. Specifically, she found that it was common to confuse dividing by a unit fraction with dividing by the whole number in the denominator; that is, dividing by one-half was often confused with dividing by two. She also found it common to confuse division by a fraction with multiplication by a fraction, for example, dividing by one-half and multiplying by one-half. In some cases, there was also confusion about dividing by one-half, multiplying by one-half and dividing by two.

In order to explore the types of errors that students make in dividing fractions, we can refer to the work of Tirosh (2000), who states that these errors can be classified into three categories. She refers to the work of other researchers (Ashlock, 1990; Barash & Klein, 1996; Fischbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1989; Hart, 1981; Kouba,1989; Tirosh, Fischbein, Graeber, & Wilson, 1993) who have studied elements of each of these categories. These categories are: 1) algorithmically based mistakes, 2) intuitively based mistakes and 3) mistakes based on formal knowledge.

Algorithmically based mistakes are errors made in the computational process (Ashlock, 1990; Barash & Klein, 1996). Tirosh indicates that this type of error is often made when students are taught the algorithmic procedure and confuse a step in the procedure. An example of such an error might be taking the reciprocal of the dividend instead of the divisor.

Intuitively based mistakes are errors based upon misconceptions associated with the operation of division. Most children only understand the partitive model of division and therefore cannot understand how it would be possible for one to divide a dividend by a larger divisor. In the partitive model of division, one is asked to divide a quantity into equal groups and then told to find how many in each group. For example, if I have twelve apples and I want to divide them equally among four friends, how many would each friend get? Kouba (1989) suggests three intuitive models for division based upon partitive division of whole numbers in her study. These are: 1) sharing by dealing, 2) sharing by repeated taking away and 3) sharing by repeated building up. These intuitive models are not easily extended to fractions. Since partitive division is commonly used to introduce the operation, it becomes a strong model for the operation. When models are initially constructed to serve in a specific context, remnants of those early models remain in generalized and transferred problems (Lesh, Lester & Hjalmarson, 2003). Consistent with this notion, Tirosh continues her discussion of intuitively based mistakes by saying that children's experience with the partitive model limits their ability to extend their understanding to division to fractions (Fischbein, Deri, Nello, & Marino, 1985; Graeber, Tirosh, & Glover, 1989; Tirosh, Fischbein, Graeber, & Wilson, 1993). This is especially true of problems where the divisor is larger than the dividend. Ott, Snook & Gibson (1991) note that textbooks and classroom examples further limit the experiences of students and their ability to extend their knowledge of partitive division to division of fractions. In order for students to be successful with division of fractions, they must also be familiar with the quotative model of division. This model originates with repeated subtraction. In this model, one is asked to formulate groups of a certain amount and to find out how many groups there would be. For example, if I have 12 apples and I want to give groups of four apples to my friends, how many friends will receive a group of apples?

The third type of error may be categorized as mistakes based on formal knowledge. These errors are based on misconceptions about the nature of fractions and misconceptions about the nature of operations. For example, a student might think that division is commutative and argue that $1 \div \frac{1}{2} = \frac{1}{2} \div 1$, which is equal to $\frac{1}{2}$ (Hart, 1981).

In describing children's early encounters with fractions, Burns (2000) states, "From their experiences, a number of ideas about fractions take shape informally in children's minds. However, children's understanding of fractions typically is incomplete and confused." (p. 223) Some examples that Burns cites as adult use of the language of fractions are, "I'll be back in three-quarters of an hour", "I need two sheets of quarter-inch plywood", and "The dishwasher is less than half-full." Children use this language as well when they say, "You can have half of my cookie", "Here, use half of my blocks" and "It's a quarter past one." Young children are often overheard talking about "the bigger half". Thus children come to school with some familiarity with fractions, but their knowledge is often incomplete.

Children also meet difficulty with fractions because they are unable to see a fraction as something to be counted as well as something that is a quantity. Conjectures have been made that a similar trend occurs at some early stage in the development of ideas about whole numbers as young children learn that a nickel and two pennies, three coins, is called seven cents. Difficulties with whole numbers seem to be overcome with greater ease than those involving fractions. In their study, Alston, Davis, Maher & Martino (1994) encountered a similar situation when students used Cuisenaire Rods® to build representational models for fractions. When a five-centimeter long rod was given the number name one, children were able to call a one-centimeter long white rod, one-fifth, and to call the length of two white rods, each one centimeter in length, two-fifths. However, when a single two-centimeter rod, replaced two one-centimeter rods, some of the children who were videotaped did not call the single rod two-fifths, because they did not see two objects.

Lamon (2001) attributes some of the difficulties students have with fractions to their limited ability to extend the meaning of a fraction to various interpretations. She states that a fraction, such as $^{3}/_{4}$ can be interpreted as 1) a part/whole comparison 2) an operator 3) a ratio or rate 4) a quotient or 5) a measure. She suggests that students be involved in a variety of activities that will enable them to experience the meaning of fraction in a wide range of ways.

Fortunately, many researchers have also documented instances in which students have successfully been able to build ideas relating to fractions (Bulgar, Schorr & Maher, 2002; Cobb, Boufi, McClain & Whitenack, 1997; Kamii & Dominick, 1997; Ma, 1999; Reynolds, 2005; Steencken, 2001; Steencken & Maher, 2002). In particular, in my previous work and work done with others (2002; 2003a; 2003b;) the conceptual development of ideas relating to division of fractions amongst fourth grade students participating in a teaching experiment, was reported. Within this teaching experiment, children experienced problem solving involving division of fractions prior to formal algorithmic instruction. Further, I reported that when the task and methodology used in the teaching experiment were replicated as part of the regular teaching practice in another classroom (my own), similar outcomes were achieved. It is this latter group of students, those who studied division of fractions as part of their regular classroom experience, with the author as their regular mathematics teacher, who are the subjects of this research.

Both the students in the above-mentioned teaching experiment and the subsequent classroom replication had essentially used the same three main strategies to solve a particular series of problems (Bulgar, 2002; 2003a; 2003b; Bulgar, Schorr & Maher, 2002). There were no strategies other than these three observed in either the classroom-based study or the teaching experiment. All three of the strategies were based upon existing counting schemes. (For a full description of these strategies, see Bulgar, 2002; 2003a; 2003b.)

These strategies consisted of the following:

- Reasoning involving natural numbers

- Reasoning involving measurement

- Reasoning involving fraction knowledge

The predominant solution method observed in the fourth grade teaching experiment, (Bulgar, 2002, 2003a, 2003b) consisted of reasoning involving natural numbers. Essentially, these students built representations that converted the meters to centimeters, thereby substituting the fraction division with natural number division, a topic generally prominent in fourth grade mathematics curricula in New Jersey (See NJ Mathematics Coalition and NJ State Department of Education, 2002). However this solution method was seen in the work of only one fifth grader in the replicated classroom study, and even when it did appear, there was a claim by the student that it was developed after the problem was solved using reasoning involving fraction knowledge (Bulgar, 2003a, 2003b). All of those students in the fifth grade who drew representations, created linear models to represent the division of a piece of ribbon into various-sized bows.

In an effort to underscore the difficulty that young students have with fractions, DeTurck (2005) has suggested that because the conceptualizations necessary for truly understanding fractions are too difficult for elementary school children, the topic should be eliminated entirely and postponed until much later in the curriculum. This is contrary to the Standards set forth by the National Council of Teachers of Mathematics (NCTM 2000), which suggest that basic fraction concepts be introduced as early as kindergarten.

Representations

The representations that children use when solving mathematical problems provide us with a gateway to understanding their thinking (NCTM, 2000). The representations that were used by the subjects of this study included concrete materials, such as Cuisenaire® rods, string, ribbons; drawings; and language to explain their thinking in their own words. Therefore, it is essential to examine closely some of the existing literature surrounding the very important notion of representations.

Cramer and Henry (2002) state that in studies surrounding The Rational Number Project (Bezuk & Cramer, 1989; Post, Ipke, Lesh & Behr, 1985), the most important pedagogical belief evoked is that most children need to use concrete models to represent fractions in order to build cognitive representations of fractions. In addition, these representations must be used over time. Other major beliefs that grew from this project are that children need to be engaged in discourse to strengthen their ideas and that conceptual knowledge must precede the formal use of algorithms. This underscores the importance of closely looking at the representations that the students in this study built and using the existing opportunity (described under methodology) to examine how their representations evolved over time.

In an effort to better understand the cognitive role of representations, one can look at the work of Speiser and Walter (2000), who, in agreement with Davis (1984), base their assertion on the previous work of Minsky (1975) and others, when they claim that mathematical knowledge is cognitively represented symbolically, often in the form of representations that are referred to as frames. When students think about a mathematical situation, they must first build a representation, which is usually done in the form of a mental representation. The building of

these representations can be assisted by the use of pencil and paper or manipulative materials. The construction of these representations is often so rapid and so instinctive that students are not aware that they have come into existence. Then they do a memory search or construction of relevant knowledge in order to proceed. The mapping that they construct between the data representation that was the input and the knowledge representation that was there gets checked and revised and ultimately is used to solve a problem (Davis & Maher, 1990; Davis, Maher & Martino, 1992). The data representations are saved in frames (Davis, 1984; Minsky, 1975), which carry information and are arranged hierarchically according to their stability (Minsky, 1975). There are several subframes, which together form a counting construction frame. The representations for these counting frames can easily be interchanged between integers and fractions, making a counting frame for fractions a natural extension of the one for integers (Speiser & Walter, 2000).

Speiser and Walter (2000) further state that it is the binary partitions subframe which is used to assist in the development of representations involving iterated processes, including counting. The binary subframe is a representation for a set that is partitioned into two subsets. For example, when we add or subtract we are connecting to this type of representation. A special type of binary subframe is one where there is only one element in one of the subsets. It is this special type of binary subframe that helps create representations for iterated processes. Partitioning of a unit into fractions requires that a student be able to take a unit comprised of subunits and operate on it while simultaneously dividing into equal partitions (Tzur, 1999).

Maher, Davis and Alston (1993) studied one student, Brian, over a period of time. They used videotapes of him doing problem-solving activities involving fractions and noted that as he worked with a partner, he was fluent in the use of different types of representations, using diagrams and concrete objects to help him solve problems. He represented his ideas in great detail. For example, when solving a problem involving the sharing of pizza, Brian used pattern blocks to build models and assigned specific students' names to the pattern block representations. After studying Brian in fifth and sixth grade, it was concluded that he insisted upon making sense of the models that he constructed rather than relying upon the ideas of others. The researchers also note the significance of an appropriate classroom environment, one that enhances students' opportunities to be engaged in thoughtful mathematics.

In their study, Watson, Campbell and Collins (1993) examined how four fraction problems were solved by children from kindergarten to grade ten to analyze the work of children's use of images, reality and experience. They found a developmental progression in the iconic reasoning, the ability to reason involving images and drawings, was developed in building ideas about fractions. They say that there is a connection between the development of iconic reasoning and of concrete symbolic reasoning. As a result of their findings, they urge schools to incorporate more problems that would give children an opportunity to develop their iconic reasoning so that the development of concrete symbolic reasoning can be supported. They also say that more study of this issue is needed in the form of teaching experiments.

In her examination of the various subconstructs of fractions described above, Lamon (2001) noted that they lend themselves to different types of representations and that significant understanding of these subconstructs is related to the use of continuous, discrete and unitary, solitary or composite representations. Area models or regions, sets of discrete objects, and the linear models such as number lines are the models most commonly used to represent fractions in

the elementary and junior high school (Behr & Post, 1992). Consistently, Van de Walle (2004) suggests that when studying fractions students should be provided with opportunities to be engaged in activities that employ a variety of representations.

The ability to use a variety of representations for the same concept, in this case, fractions, requires flexible thinking. Researchers such as Warner, Alcock, Coppolo and Davis (2003) and Warner and Schorr (in progress) emphasize that a critical aspect of mathematical flexibility is the ability of students to use multiple representations for the same idea and to link, extend and modify those representations to a broader range of situations, involving a more extensive array of models. Since the goal of instruction in this study was not merely to have students retrieve facts or procedures, or to display understanding only for very specific situations or for limited time periods, the notion of mathematical flexibility is of significant relevance. This type of mathematical flexibility is particularly important if students are to use knowledge across a wide spectrum of ideas. Fosnot and Dolk (2001) note, "The generalizing across problems, across models, and across operations is at the heart of models that are tools for thinking." (p.81). They report on a class in New York City wherein a third grade teacher provided students with three different contexts that lent themselves to different models while all three resulted in the same answer. In each case the children produced different representations that were closely linked to the context. Fosnot and Dolk go on to state that it is easy for students to notice that the answers are the same but that the important issue is for them to see the connections among and between the representations to develop a generalized framework for the operations.

Lamon (2001) also addresses the issue of students developing their own representations as opposed to adapting their teacher's representations, since the latter is not an indication of understanding. She goes on to say that:

> When a student truly understands something in the sense of connecting or reconciling it with other information and experiences, the student may very well represent the material in some unique way that shows his or her comfort with the concepts and processes. (P. 156)

From this, one can infer that being able to move flexibly among and between representations for division of fractions designates a deeper understanding of the concept.

It is important to address that representations can be both internal and external. Goldin and Shteingold (2001) note the importance of the distinction between internal and external representations as well as the significance of the connection between internal and external representations as a fundamental element of teaching and learning. They add that the internal models or representations that one builds are not observable, but can only be inferred from students' interaction with materials, discourse and /or the external representations they create. These researchers go on to say:

> Through interaction with structured external representations in the learning environment, students' internal representational systems develop. The students can then generate new external representations. Conceptual understanding consists in the power and flexibility of the internal representations, including the richness of the relationships among different kinds of representation. (P. 8)

Before continuing, it is important to distinguish between the conceptual models that are embodied in the representational media that students use, and the cognitive models that reside inside the minds of learners (Lesh & Doerr, 2003). In this work, both will be addressed, with an

emphasis on the nature of the cognitive models that are born out in the representational media that the students use, especially as evidenced in their mathematical flexibility. The nature of the models that the students have built in terms of their mathematical flexibility, not just during or shortly after the instruction took place, but rather over a more extended period of time is documented here. Also addressed is flexible thought in the context that follows, because it is relevant to this study.

Research Questions

The primary question under study in this research is, given appropriate conditions, how do students extend, modify, revise and refine their representations involving division of fractions over time? Additionally, how do they demonstrate flexibility so that they can fluidly move back and forth between and among representations, being selective about what models or representations will help them to create meaningful problem solutions without experiencing some of the many difficulties normally encountered by others?

A significant goal of this study is to better understand how a group of students extended, modified, revised, refined and ultimately generalized their ideas relating to division of fractions during the school year following their initial experience with problem solving activities related to this topic. This is done with a focus on mathematical flexibility, the nature of the representations that were used and the evolution of these representations during the following school year. In particular, the focal point is on how students initially used linear models, how these models evolved into discrete area models and how these students moved easily to continuous linear models when they found them to be more appropriate.

The research that frames this study includes investigations of fractions and more specifically division of fractions. This topic traditionally has involved much complexity and difficulty for students. Additionally, in order to trace the evolution of the representations of a particular population, it was necessary to examine studies regarding representations in general and specifically related to fractions.

Methods and Procedures

Background, Setting and Subjects

In the fall of 1993, a teaching experiment, including the study of fractions, was conducted in a small suburban New Jersey public school, under the direction of Carolyn Maher of Rutgers University, and other researchers from the University[1]. Fourth grade was selected for this teaching experiment because it is the year prior to the one in which students are traditionally taught fraction algorithms in New Jersey. The premise of this study was that the fraction knowledge that these fourth grade students built could therefore be attributed to the work done within the project rather than to classroom instruction. It was expected that careful monitoring of the children's development of ideas would give the researchers insight into how students built mathematical knowledge about fractions. The instructional design of this teaching experiment was the model for the regular classroom instruction in the classes that are the subject of the study reported upon here. (Some salient elements of the classroom instructional practices are described below, but for further descriptions of the instructional practices used in the teaching experiment see Bulgar 2002; Reynolds, 2005; Steencken 2001.)

In the school years 2000 – 2001 and 2001 – 2002, a unique opportunity presented itself. While investigating the work of the students in the above-mentioned research study, the author had the opportunity to teach the same mathematical ideas to a second group of students who were in fifth grade, and later on, in sixth grade (also taught by the author). Although the content explored during these two years cover a wide range of topics, this paper addresses only ideas related to division of fractions. The particularities of this situation allowed the author to document the growth of these ideas over the course of two school years in the context of everyday teaching in the company of the students' regular mathematics teacher (as opposed to a project led by visiting University researchers). This will be more clearly described below.

The students being studied attended a small private parochial school in New Jersey that attracts children from several surrounding communities. This academically heterogeneous class consisted of 13 girls. These students had experienced a very traditional[2] classroom-learning environment prior to the fifth grade. They were used to being told whether or not their answers were correct and being shown procedures for doing mathematics. The predominant goal of mathematics had been to get the right numerical answer. In contrast, upon entering fifth grade and being taught mathematics by the author, the students were encouraged to take responsibility for convincing others that their solutions were correct and they were expected to write about their thinking on a regular basis. They began doing mathematics with block scheduling, meeting for one 40-minute period per week and two 80-minute periods per week. Discourse was of great importance. Responsive questioning took place to encourage mathematical thinking by attempting to elicit verbalization of mathematical thought (See Goldin & Shteingold, 2001 above). Predominantly, students worked in pairs or triads and collaboration was promoted. The classroom community was one in which students' ideas were always highly respected. Alternate strategies were encouraged, shared and discussed. The students were invited to discuss their thinking and to submit ideas in writing or via email. The goal was to achieve deep understanding of the mathematics embedded in the problems that the students experienced. Students were not taught algorithms. When they recognized patterns and could justify that these patterns were valid for the examples that they observed, they created generalizations, which they would apply to future problems. (See Bezuk & Cramer, 1989; Cramer and Henry,2002; Post, Ipke, Lesh & Behr, 1985, above.) A fundamental characteristic of the instructional environment was the facilitation of mathematical problem solving. This was based on the premise that students needed to be engaged in mathematical activities that promote understanding (Cobb, Wood, Yackel & McNeal, 1993; Davis & Maher, 1997; Klein & Tirosh, 2000; Maher, 1998; NCTM, 2000; Schorr, 2000; Schorr & Lesh, 2003). There was a strong effort by the teacher and assistants, when present, not to lower the cognitive demand of the problem solving activities so that the goal remained that students be "doing math" as described by Stein, Smith, Henningsen, and Silver, (2000). Therefore, conditions established during the fifth grade were set up to create a classroom community in which student inquiry and discovery were of paramount importance. Once this community was established, students actively participated and remained engaged in the work they did, often posing extensions and hypothetical situations to the problems they were assigned, indicating that their thinking and their personal goals went beyond just getting a numerical solution.

In the second year of this study, when the students were in the sixth grade, one of the original students had left the school, but another new student had joined, thus leaving the classroom population the same in number. The routines and classroom community that had been instituted the previous year remained in place, so that the semester began with the expectation that

mathematics class would involve inquiry and discovery. Students were enthusiastic and eager to continue with the type of work they had been engaged in during the previous year.

Data Collection

The primary data that were examined for this study consist of artifacts of actual student work, which were collected during two school years, when the subjects were in the fifth and sixth grades. These data pertained to the study of division of fractions. After the work was collected, written notes from the teacher were attached to some of the work, usually in the form of questions. When work was returned, students had the opportunity to answer these questions before the final submission of their papers. The written work was examined qualitatively for its relationship to representations used to solve a variety of problems involving division of fractions.

In addition, during the time that these students were introduced to division of fractions in the fifth grade, there were two mathematics education graduate students present, serving as assistants. They interacted with the students, questioning them about what they were doing and listening to their explanations. They were familiarized with the type of learning environment that was the norm of this class and therefore knew not to interfere with children's thinking. Their field notes are also included in the data collection. One of these graduate students was with the class all semester, as part of a university practicum experience. The other visited because of interest in the particular topic of division of fractions. Because of the author's association with the university, faculty and graduate student visitors had stopped by on various occasions. Also, other teachers from the school had come in to observe the teaching of mathematics. Therefore, these students were used to having other adults in the classroom as they worked. When the class was in the sixth grade, the classroom was located in a somewhat isolated supplemental trailer, so visitors from the school faculty were very rare. During the sixth grade activity reported here there was no one present other than the students and the author.

Tasks and Tools

During the fifth grade, students were assigned the task called Holiday Bows.[3] In this task they were provided with a meaningful context for understanding division of a natural number by a fraction. This topic is part of the fifth grade mathematics curriculum and appears in most fifth grade mathematics textbooks. The task involved finding out how many bows of several fractional lengths could be made from various sizes of ribbon. For example, one of the questions was how many bows, each one-third meter in length, could be made from a piece of ribbon that is six meters in length. Students had access to actual ribbons, pre-cut to the specified sizes, meter sticks, string and scissors to help them form concrete representations of their thinking. (See Fig. 1) This was the students' first classroom introduction to division of fractions.

HOLIDAY BOWS

    (1)   Red ribbon comes packaged in 6 meter lengths;
    (2)   Gold ribbon comes packaged in 3 meter lengths;
    (3)   Blue ribbon comes packaged in 2 meter lengths; and
    (4)   White ribbon comes packaged in 1 meter lengths.

Bows require pieces of ribbon that are different lengths. Your job is to find out how many bows of particular lengths can be made from the packaged lengths for each color ribbon.

I. White Ribbon	Ribbon Length of Bow	Number of Bows
1 meter	1/2 meter	
1 meter	1/3 meter	
1 meter	1/4 meter	
1 meter	1/5 meter	
II. Blue Ribbon	Ribbon Length of Bow	Number of Bows
2 meters	1/2 meter	
2 meters	1/3 meter	
2 meters	1/4 meter	
2 meters	1/5 meter	
2 meters	2/3 meter	
III. Gold Ribbon	Ribbon Length of Bow	Number of Bows
3 meters	1/2 meter	
3 meters	1/3 meter	
3 meters	1/4 meter	
3 meters	1/5 meter	
3 meters	2/3 meter	
3 meters	3/4 meter	
IV. Red Ribbon	Ribbon Length of Bow	Number of Bows
6 meters	1/2 meter	
6 meters	1/3 meter	
6 meters	1/4 meter	
6 meters	1/5 meter	
6 meters	2/3 meter	
6 meters	3/4 meter	

Figure 1. The Holiday Bows Task

During the beginning of the sixth grade, the students worked on the task called Tuna Sandwiches. This task was created by the author with the intention of being similar in structure to Holiday Bows, which had been experienced during the previous year. Students had access to various materials such as Unifix Cubes®, pattern blocks, Cuisenaire Rods®, paper, pencils, graph paper, meter sticks, string, scissors at all times during mathematics class and were encouraged to choose what they deemed appropriate and helpful as needed. The task follows directly below.

> Mr. Tastee's restaurant serves four different kinds of sandwiches. A junior sandwich contains 1/4 lb of tuna; a regular sandwich contains 1/3 lb of tuna; a large sandwich contains 1/2 lb of tuna and a hero sandwich contains 2/3 lb of tuna. Tuna comes in cans that are 1lb, 2 lb, 3lb and 5 lb. How many of each type of sandwich can you make from each size can? Find a clear way to record your information. You will need to write a letter to the restaurant owner, Mr. Tastee, and give him your findings.

One of the goals in creating the Tuna Sandwiches problem was for it to lend itself to be represented by an area model rather than a linear model, as was the case with Holiday Bows. That is, the intention was for the fractions to be based on a portion of a region, rather than a portion of a length as is the case in a linear model. Fosnot and Dolk (2001) state that just because we create a problem with certain models in mind, we cannot be assured that these models will be used by students. By creating a problem that was intended to be fundamentally similar in structure to the Holiday Bows yet embodied in a different type of representation, an area model, the notion of flexibility could be explored as well as an examination of the durability of the knowledge the students had demonstrated during the previous year.

In both the fifth and the sixth grades, these tasks and the whole class sharing and discussion associated with them were followed by problems that required students to find the values of expressions involving division of fractions using only symbolic notation. In the fifth grade, these problems consisted of only a natural number divided by a fraction. The problems were assigned upon completion and discussion of the Holiday Bows task. Students were asked to find the value of any two of the following three expressions:

- $5 \div \frac{1}{3}$
- $12 \div \frac{3}{4}$
- $7 \div \frac{2}{3}$

In the sixth grade, two problems involving only symbolic notation were assigned to the students approximately six weeks after they began working on the Tuna Sandwiches task. The problems, assigned one at a time, were to find the value of the expressions directly below.

- $2 \div \frac{3}{4}$
- $\frac{5}{8} \div 2\frac{1}{2}$

The second of these expressions was the students' first exposure to finding the quotient of a common fraction divided by a mixed numeral. As stated in the theoretical framework (See Tirosh, 2000 above), this type of division involving fractions is especially difficult for students to understand. There were three significant goals of the assignment of the problems involving only symbolic notation during the sixth grade. The first was to see whether or not the knowledge demonstrated in the past regarding division of fractions was durable. The second was to see if the

knowledge was flexible enough to be able to be extended and applied to a problem involving division of a common fraction by a mixed numeral and thirdly, to see if the students would make use of previously employed representations when the specific context was removed.

Results and Discussion

Prior to their work on division of fractions, the fifth grade students had done extensive problem solving with fractions using manipulative materials. They were encouraged to construct models to support their solutions, draw their models and to justify those solutions in terms of the representations. For example, on November 16, 2000, students were given the following problem.

> Which is greater, $^2/_3$ or $^3/_4$? By how much? Build a model to solve the problem and explain how your model can be used to find a solution.

At this early time in the semester, students were most familiar with Cuisenaire Rods® as a manipulative material for building models. All of the pairs were able to solve the problem using this linear model. In Figure 2, we see Samantha and Eve's solution, which articulates a clear understanding of the problem.[4]

Q: Which is greater $^2/_3$ or $^3/_4$?

A: $^3/_4$ is greater by $^1/_{12}$.

M [models]:

P		P		P	
Dg			Dg		
Lg		Lg	Lg		Lg
W	W	W	W	W	W	W	W	W	W	W	W

Lg		Lg		Lg	
P			P		W

E [Explanation]: 2 Dg = 1. 1 Lg = $^1/_4$. 1P = $^1/_3$. We took 3 Lg ($^3/_4$) and 2P ($^2/_3$) and put them one on top of the other and saw that 3 Lg ($^3/_4$) are bigger. 12W = 2Dg (1). If you put 1W ($^1/_{12}$) next to the 2P ($^2/_3$) then they are the same lenghth as the 3 Lg ($^3/_4$). So that is how we figured out our answer.

Fig. 2. Samantha and Eve's solution.

Holiday Bows and Applications – Fifth Grade

In late May of the same school year, the fifth graders worked on the Holiday Bows problem for two double (80 minute) periods on consecutive days. They had received no prior formal instruction regarding division of fractions and were not told that this problem was relevant to that topic. They worked collaboratively in groupings of their choice, but each submitted an individually completed chart (See fig. 1) and an individually written explanation of the solutions.

Seven of the 13 students constructed representations, which they included with their solution explanations. All of the students who drew representations used linear models. The specific solutions of several students are worth mentioning. Some, such as Sarah, drew discrete linear models for each meter of ribbon. She used the solution method, *reasoning involving fractions* (See Bulgar 2002, 2003a, 2003b) to find out how many bows, each ½ meter in length could be made from six meters of ribbon (see above). In this solution method, one recognizes that ½ means one out of two equal parts that comprise the whole and therefore for each whole there will be two parts. The number of units (in this case meters) is therefore multiplied by two to find the number of halves in the entire amount. Using this solution method, Sarah states that in each of the six meters there would be two bows so in six meters there would be 6 x 2 bows or 12 (See Figure 3). Though the statement of the problem alludes to the ribbon being a continuous piece, six meters in length, her diagram indicates discrete representations for each of the six meters.
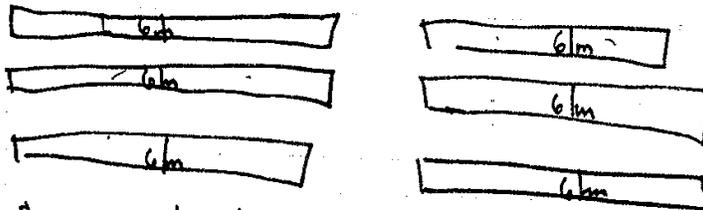
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(Continued)

& counted how many sections there where & there was 15 and that made 15 bows.

<u>Red</u>:

Q: How many bows can you make out of diffrent stzes to make 6 meters?

Ex: I took 6 meter!

& counted the sections & got 12 & each bow is sopossed to be ½ long so you can make 12 bows. You can also do 6 × 2 = 12.

Fig. 3. Sarah's explanation for how many bows, each ½ meter in length can be made from 6 meters of ribbon.

In contrast, Nicole drew a continuous linear representation for each of the ribbon lengths. She also used *reasoning involving fractions* indicating that in each meter there would be two bows, each ½ meter in length. However, unlike Sarah, who viewed the problem multiplicatively, Nicole imposed additive structures, adding two bows for each additional meter of ribbon. One might say that Nicole's solution makes use of *reasoning involving measurement* (See Bulgar, 2002; 2003a; 2003b) as well. In this solution method, students create a measurement tool, such as a piece of string, as long as the length of bow and then place it along the length of ribbon repeatedly, counting the number of times it fits on the ribbon. Though Nicole did not create such a tool, the additive structure of her solution implies that she is cognitively placing ½ meter pieces along each meter and counting them. She is thereby making use of an internal or mental model for the measurement tool, counting the number of times the tool could be placed along the given number of meters of ribbon. As stated by Goldin and Shteingold (2001), internal models are borne of and developed through the use of external models. These students previously had experience with the use of Cuisenaire® rods and apparently created an internal model based upon the external measurement models they had used in the past.

Pg. 1

x   11.

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Q: How many bars can you make out of a certain amount of _____?

E/a: What I did for the 1 meter ribbons for 1 meter are: If you have 1 meter + the length of a bar is ½ a meter then obviously it's 2... did the same thing for 1/3 = 3, 1/4 = 4, 1/6 = 5.

: m.: | 1 meter |
for example: | ½ meter | ½ m |

E/a: What I did for 2 meters is: Since for 2 is ½ of 2 I add the #'s for 1. For example: for the ½ meter an 1 meter I got 2. So I added 2 + 2 = 4. So that was my answer.

m.: | 2 m. |
| ½ m | ½ m | ½ M | ½ M |

E/a: What I did for 3 m. is: I added what I got for 3 m. 1 m + 2 m.
For example: ½ of 1 meter was 2
½ of 2 meters was 4 so I added the together 2 + 4 = 6.

m.: | ½ M | ½ M | ½ M | 3 M | M | ½ M |

Figure 4. Nicole's solutions.

Both Gabriella and Stephanie did not draw representations, but each used *reasoning involving fractions* and multiplication. They both wrote lengthy and detailed explanations for their solutions. Stephanie wrote the following to explain her solution for how many bows, each $^2/_3$ meter in length could be made from two meters of ribbon.

> …with 2 m. each meter had 3 thirds and 2m. would be 3 x 2 = 6 [thirds], so 6 would be the answer [for the number of bows $^1/_3$ meter in length that could be made from 2m of ribbon]. But 6 is to much even though 2m. x $^2/_3$ [she circled the 3 in the denominator] = 6. I thought about that leftover $^2/_3$ [she circled the 2 in the numerator]. Then I realized that if you can divide 2 ÷6 = 3! That would be the answer because if you minus ½ from the 6 it equals 3. If you have 6 and you – ½ from every $^1/_3$ of 6, it = 3.

Stephanie has confused the language for subtraction, which she refers to as "minus", and division. She has also reversed the dividend and the divisor when she states that 2÷6=3, which is a common misconception observed in students of this age. However, she has evidently understood the inverse relationship created by enlarging the size of the bow and getting fewer bows. Understanding that as the divisor is doubled, the quotient divided in two is a very complex notion. Yet, it appears to be very clear to Stephanie from her explanation. Additionally, though she has confused the dividend and the divisor in her symbolic notation, her explanation indicates that she understands the role of each of the numbers conceptually.

Samantha clearly and concisely explains how she used *reasoning involving measurement* to solve the problems (See Figure 5). However, she does not make any mention of how she created the measurement tool. Most students did this by cutting a one meter piece of string and then folding it equally into the number of parts needed for the unit fractions. That is, to create a measurement tool that was $^1/_3$ meter in length, they would fold the meter length of string into three equal parts and cut it on the folds. When creating measurement tools for non-unit fractions, several began with the unit fraction tool. For example, to create a measurement tool that is $^2/_3$ meter in length, one would cut two $^1/_3$ pieces, place them end to end and then cut another piece of string the length of the combined pieces.

S

Q: How many bows of diffrent sizes can you make of diffrent size ribbon?

A: I figured all the answeres out out by putting the string next to the ruler and finding the "Ribbon ≠ Length of Bow" and seeing how many strings I could get to fit to that length.

Explanation: ~~I figured out all the numbers~~ I think my method works because when you measure the string to the right length and see how many strings you can measure it to, you can get an answer.
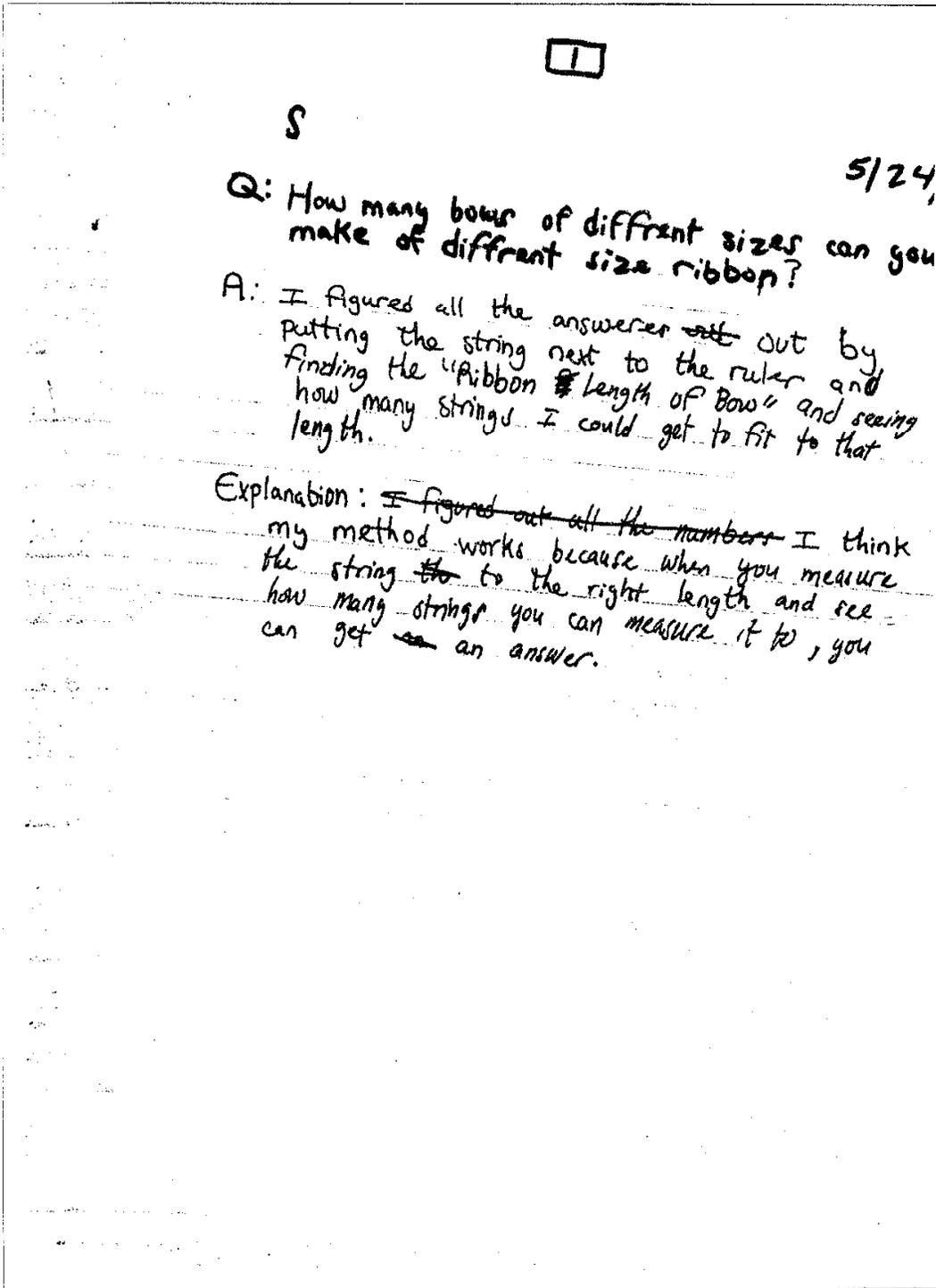
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Figure 5. Samantha's solution

Several of the students indicated that they found the problems involving a non-unit divisor to be more difficult. In her description of how many bows, each $^2/_3$ meter in length could be made from two meters of ribbon, Brooke appears to have an internal representation of each meter discretely as well as each $^1/_3$ of a meter in each bow. She does not draw a representation, but she writes the following.

> you take the ribbin which is 2m. long & cut it into half each is a m. and each half cut into 3 pieces and each 2 piece is a bow so you get 3 [bows].

We get a window into the internal representation that Brooke has created by examining her external representation (See Goldin & Shteingold, 2001 above) and the language she uses in her explanation. After she deduces that there are six 1/3-meter bows in the two meters of ribbon, she cognitively connects two such pieces to form a bow that is made of 2/3 meter of ribbon.

In this population, Olivia was the only one to refer to *reasoning involving natural numbers* (See Bulgar 2002, 2003a, 2003b) and she did not come up with this type of solution initially. At the conclusion of her written work she says the following.

> I figured out a shorter way to explain this & it makes more sense. It works as follows: 1 meter = 100 centimeters. You could change the amount of meters you have into centimeters. Thus, let's say you have to make bows each 1/2 of a meter. Figure out how many centimeters = 1/2 of a m. 50 centimeters = 1/2 of a m because half of 100 is 50. Then see how many times 50 goes into 100. However many times 50 goes into 100 is how many bows you can make with each bow 1/2 of a m. & with 1 m. You can also do this with 1/3 of a m. or 1/4… as long as you change 1/3 or 1/4 of a meter into a # amount of centimeters. You can also do this with 2 or 3m… of string as long as you change 2 or 3m… into centimeters. I think this works because you have to figure out how many 1/3rds or 1/4th s of a m. go into 1 m. That is saying the same thing as a certain # of centimeters go into 100 or 200 or 300. Or you could do 1 ÷ 1/4 and you would get 4. That is the same thing as 100 ÷ 25 = 4. They both = the same thing which proves they both work.

As some of the students moved from the unit fractions to the non-unit fractions, they had to adjust their strategies. Linda solved the problems that required division by a unit fraction using *reasoning involving fractions*, multiplying the number of meters by the number of bows of each size in a one-meter piece of ribbon. She assumed this method was no longer valid when faced with a non-unit fraction divisor and therefore changed her mental representation and employed the strategy of *reasoning involving measurement*. The following appeared in the field notes of one researcher who was present.

> When she got to the question of 6m ribbon and $^2/_3$m bow, she started measuring. I asked her why she didn't just use her multiplying method, she replied, "cause there's a 2 there not a 1 [in the numerator], so you can't do it, you can only do it when there's a 1, so I have to measure it if there's another number there." It's ironic how she understands that the 2 in the numerator makes her method invalid, but she doesn't understand why. *(C. Hayworth, unpublished notes, May 24, 2001)*

After the students had all completed the work on Holiday Bows and submitted their individual papers, the problems were discussed in a whole group setting with various students sharing their solutions. A week later, they were asked to find the value of any two of the following expressions, which involve only symbolic notation.

(1) $5 \div {}^{1}/_{3}$

(2) $12 \div {}^{3}/_{4}$

(3) $7 \div {}^{2}/_{3}$

Students were asked to solve the problems and explain the solutions. Most of them chose the unit fraction divisor for one of the problems. Every submitted solution was correct, though there were a variety of explanations. Some students used approaches that focused on procedural explanations with some variety among them. For example, Samantha offers the following for her incorrectly copied example $5 \div {}^{3}/_{4}$.

> you can get the answer by timzing (x) the number that you're dividing by. Like you would timez the <u>5</u> by the <u>4</u> and you would get <u>20.</u> And the you'de divide <u>20</u> by <u>3</u> which is $6\,{}^{2}/_{3}$.

Brooke writes the following.

> before you do $12 \div {}^{3}/_{4}$ you have to find out how many [fractional] parts there are so you do 12 x 4 which equals 48 then you do $48 \div 3$ which equals 16.

While it appears that these students are relying upon algorithmic solutions, they seem to be using "Procedures with Connections" (See Stein, Smith, Henningsen & Silver, 2000) in that they refer to dividing and breaking into fractional parts. That is, they initially find the number of unit fractions in the length of ribbon and then "join" these discrete unit fractional parts to create the desired non-unit fractional piece.

Sarah was the only one in the class who drew area representations for her solutions. The students had worked with pattern blocks for other fraction activities in the past, but they were not available when the students worked on Holiday Bows. Her explanations are noteworthy. She states the following for $5 \div {}^{1}/_{3}$.

> You are doing 5 x 3 = 15. You do 1 devided by 15 because it is only one third not three thirds.

Even though Sarah has confused the dividend with the divisor, which is not uncommon when children of this age use language to express their mathematical experiences, she indicates an understanding of the role of the number of thirds in the divisor. Therefore, it is expected that she would easily transition to dividing by a non-unit divisor. Regarding $12 \div {}^{3}/_{4}$, she writes the following.

> You do how many times can four go into every one of the 12 (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) which is doing 12 x 4 = 48. Now you have to do $48 \div 3$ & get 16.

Her discrete area representation consists of twelve circles divided into fourths with a sequential multiple of four written above each one.

A few of the students made some reference to ribbon or bows during their explanations. For example, Gabriella drew a continuous linear model and stated the following when explaining her solution for $7 \div {}^{2}/_{3}$.

> 2 goes into 3 3 time so I do 3 x 7 because I want to see how much times 3 can go into 7 meters. 3 x 7 = 21. Now I want to devide 21 by 2 because if you have 7m. ribbon you devide it into 3 parts and I can make 2. (Her representation appears here in her explanation with the notation "2 parts out of 3 parts". She then draws a standard division problem that

indicates that 21 ÷ 2 = 10 ½.) A half is because 1 part out of 2 is ½ and 2 ÷ 21 is 10 ½ .

Gabriella has also reversed the language of the dividend and the divisor, but she unmistakably understands the very difficult inverse relationship in division that when you divide by a larger dividend, you get a smaller quotient.

Stephanie, Olivia and Eve also make reference to understanding this inverse relationship. Eve states the following, which additionally indicates that she is thinking multiplicatively about the problem.

> Now I will explain 12 ÷ ¾ = 16. There are 48 one thirds in 12. ¾ is 3 times as much as $^1/_3$. The answer will be three times less than 48 which is (48 ÷ 3) 16.

Stephanie indicates her understanding of the inverse relationship when she states the following.

> The second one was 12 ÷ ¾ so like the other problem you do 12 x 4 because you ask how many times does ¾ go into 12. It is 48 so now you divide by 48 because you aren't asking how many times does ¼ go into 12 but ¾ . So ¾ is more ribbon. Now you divide 48 ÷ 3 which is 16.

Stephanie has mentioned ribbon, which seems to indicate that although she did not draw any representation, she is referring back to an internal model that she has created, one that is a continuous linear model. Here the external representation in the form of language is leading us to believe that she has constructed an internal model of the ribbon to help her solve the problem.

Olivia also explained her solution in terms of the inverse relationship. She states the following in her explanation of 7 ÷ $^2/_3$ = 10 $^1/_2$.

> First you must find out how many thirds are in one. You must find this out because you can't find out how many $^2/_3$ are in 7 if you can't find out how many $^1/_3$ go into 7. $^1/_3$ goes into one, three times because $^3/_3$ are 3 pieces put together to equal one. $^1/_3$ is one of these 3 pieces. Once you know that you have to multiply 7 x 3. You multiply because $^1/_3$ goes into one three times, & you need to find out how many of these threes go into 7. 7 x 3 = 21. 21 would be how many $^1/_3$ go into 7. You must find out how many $^2/_3$ go into 7. $^2/_3$ is more than $^1/_3$, so therefore you would get less ribbons… you divide 21 in half or by 2. 21 divided by 2 is 10 ½ & that is your answer..

Olivia has gone into great detail to justify her solution and has also referred to the ribbons, alluding to her use of a cognitive model based on her work with Holiday Bows.

By the time the fifth graders had completed the tasks described above, they had demonstrated an understanding of division of fractions. In summary, there was indication of each of the following:

- Solving division of fractions problems within a concrete context.

- Solving division of fractions problems using symbolic notation.

- Understanding of the inverse relationship resulting in a decreased quotient when the dividend is increased.

- Understanding that the quotient is a count of how many times the divisor can be measured along the dividend.

- Understanding that the operation of division for fractions has the same meaning as the

> operation of division in natural numbers so that one can fluidly move between these two forms to get the same solution.

- Even when a procedure for division of fractions is described it is rooted in the understanding of the conceptual basis for why this works. Not all students conceptualized the procedure in the same way.

- Even when students did not draw representations, their explanations provide clues to the internal models they used to solve problems.

Tuna Sandwiches and Applications – Grade 6

The classroom culture that had been established in the fifth grade was part of the sixth grade mathematics class from the onset. Therefore, when the students were asked to work on the task, Tuna Sandwiches, they knew they had to explore the problem on their own and with their peers without waiting for the teacher to "give" them an algorithm or procedure. Again, they were not told that this was a division problem. They were merely told to solve the problem and to use the letter that was required by the problem to justify their findings.

Not one of the thirteen sixth graders used a linear model to solve the Tuna Sandwich Problem. Ten of the thirteen students actually drew area models to represent their solutions and three of the thirteen explained their thinking without referring to any representation. It is interesting to note that all but one of the area models included discrete drawings for each pound of tuna. One would think that the problems involving the hero sandwiches, those which each required $2/3$ of a pound of tuna, would be more difficult to solve when using discrete representations. There was no mention, either verbally or in writing, of greater difficulty with the non-unit fractions, as had been the case in the fifth grade. In fact, several students stated that each one-pound of tuna would yield one and one-half hero sandwiches. It appeared that the shift in unit was made seamlessly by the sixth graders. One-third pound of tuna was recognized to be one half the quantity needed to make a hero sandwich, which required two-thirds pound of tuna. This kind of understanding was not demonstrated in the fifth grade.

Though they were not required to do so, most of the sixth graders spontaneously formed some kind of graphic organizer to structure their results (See Figure 6). Seven of the thirteen students formed a matrix indicating the amount of tuna required for each sandwich as one dimension and the different-sized cans of tuna as the other dimension. Four of the students indicated their solutions in an organized listing. One of these students had both an organized listing and a matrix.

	1 pound	2 pound	3 pound	5 pound	
Junior Sandwiches	4	8	12	20	
Regular Sandwiches	3	6	9	15	
Large Sandwiches	2	4	6	10	
Heroe Sandwiches	$1\frac{1}{2}$	3	$4\frac{1}{2}$	$7\frac{1}{2}$	

This is the chart of explanations.
for the Heroe Sandwiches, first you make $1\frac{1}{2}$ of a
Heroe Sandwiches from a 1 pound can of Tuna. add the
Half to the $2\frac{1}{2}$ Hero sandwiches you get 3, then $4\frac{1}{2}$
+ really you would get 6 if you used a 4 pound can but
Since you cant, you get 7 or a $\frac{1}{2}$

Junior Sandwiches = $\frac{1}{4}$
Regular Sandwiches = $\frac{1}{3}$
Large Sandwiches = $\frac{1}{2}$
Heroe Sandwiches = $\frac{2}{3}$ fits into 1 pound like so =
if you take a pound
$\frac{2}{3}$ of tuna you could make
Your Welcome; 3 peices of 1 each. you need
$\frac{2}{3}$ for a 3 heroe sandwiches,
$\frac{2}{3}$ So there for 1 or a half.

Michelle   ②

Figure 6. Michelle's Graphic Organizer

Since the students specified their solutions using *reasoning involving fraction knowledge* by looking first at how many sandwiches of each type could be made from a one pound can of tuna, it is interesting to note that very few used proportional reasoning approaches (ie., they did not

use multiplicative structures to arrive at solutions involving multiple cans of tuna). Rather, most used additive structures. Stephanie begins by alluding to proportional reasoning when she writes the following as she explains her solutions for finding out how many regular sandwiches, those requiring $^1/_3$ pound of tuna, could be made from each of the various sized cans.

> You can only make 3 sand. With one lb of tuna because 3 thirds make 1. (3/3=1) With one more lb of tuna (2lb) you can make twice as many sand. So you have 6 sand. With 3 lb of tuna you can make 3 more sand. (9 altogether) because you have one more lb of tuna which make 3 sand. Because 3 thirds (3/3) =1. Now with 5 lb. you add not 3 sand. But 6 because it is not 4 lb, but 5 lb of tuna.

Stephanie appears to be going back and forth between multiplicative and additive approaches, adding on multiples of three sandwiches. When Stephanie explains her solution to the hero sandwich problem, the one involving division by a non-unit fraction, she states the following.

> So with a 1 lb can you can make 1 sand. and a $^1/_2$ of another because it is $^2/_3$ of a lb of tuna [required for each hero sandwich] so you have $^2/_3$ left which is $^1/_3$ left which is $^1/_2$ of $^2/_3$. A 2 lb can of tuna you can make 3 sand. easily and the excess is 1/3 from both so that makes 3… Now for a 5 lb. can you can make 6 $^1/_2$ sand. because you can make 5 easily and 2 $^1/_2$ more with the extra of each lb.

Though Stephanie's solution of 6 $^1/_2$ sandwiches is not consistent with her explanation, she has evidently demonstrated an understanding that $^1/_3$ of a pound of tuna represents $^1/_2$ of a hero sandwich, an idea that students had more difficulty understanding the previous year when they worked with the linear model suggested by the Holiday Bows problem. This change in the unit is a very difficult one in general. It would appear that Stephanie is first counting the complete sandwiches that can be made from each pound, the ones she refers to as being made "easily", and then is gathering up the remaining $^1/_3$ pounds from each can to combine them in order to make additional sandwiches. While doing so, she made the error of recording 6 $^1/_2$ sandwiches as her final answer, though she says "…you can make 5 easily and 2 $^1/_2$ more with the extra of each lb." This kind of thinking was also observed in the representations of other students, such as Gabriella, Lynn, Amy, Sarah and Bea, who drew connecting lines to the "leftover" one-third pound of tuna in each representation of a one-pound can. (See Figure 7.)

Bea

then by 1/3 a regular sandwitch
I made one pie
then by two pounds
I made 2 I cut it in to 3
I got 6 sandwiches Then I added 3 more
and got 9 sandwiches
Then I added 3 and 3 because it's
5 pound that made 15

Then for a large sandwich is 1/2
so for the first one I made one pie
That is 2 sandwichs
then I added 2 more for 2 LB and
got 4 then I added two more and
got 6 then by 5 LB I added 2, and 2
and got 10

then by the huge sandwich its 2/3 so
I made 1/3 one pie

Then by 2 whole pounds I made 3 Sandw

Then by 3 LB I made 4 1/2

Then by 5 LB I made 5 pies
I cut it in to 3rds and I got 7 1/2 Sand

Figure 7. Bea's solution showing lines connecting fractional parts to make a complete sandwich.

After completing a lengthy explanation and justification of her solutions, Eve wrote the following reflection on her work.

P.S. When I was figuring this out for you I noticed something interesting. I noticed That by the junior sandwich (1/4 lb.) you added 4 by every can of tuna. This is because every time the can get bigger by 1 lb (from which you can make 4 sandwiches) so you just add another 4 and the 5 lb., it is 2 more lbs. So you add 8 instead of 4.

Though Eve used *reasoning involving fractional knowledge*, she applied additive reasoning to get the solutions, adding the number of sandwiches that can be made from each pound of tuna. It is interesting to note that Eve and other students who did this did not recognize the repeated addition of the same addend as multiplication.

Sarah used multiplicative reasoning in finding the solutions. She wrote the following.

Out of 3 pound you can make 12 junior. There is 4 in each [pound] and 4 x 3 = 12.

Sarah included a diagram of 3 circles divided into four sections or fourths. She numbered the sections from one to twelve. She used this representational structure for all of her solutions.

Gabriella also used multiplicative reasoning. She drew five circles, divided them in half vertically and stated the following.

How much large sandwiches can you make from 5 pounds. Let's try those imaginary pounds [her drawings]. Well 2 in each of the 5 pounds 5x2 = 10!

In the summative class discussion of the Tuna Sandwiches Problem, students talked about the problem and how it was like the problem they had done the previous year, called Holiday Bows. Those who did not recognize it at first agreed when their peers noted the structural similarity in the problems. They recognized that the problem required division of fractions and easily explained their solutions using symbolic notation. For example, when summarizing that three hero sandwiches could be made from two pounds of tuna, they were able to create the number sentence, $2 \div \frac{2}{3} = 3$. Some of the number sentences that the students provided were recorded on an overhead projector transparency. These number sentences are seen as solutions representing conceptual understanding derived from the use of student-generated representations and internal models, rather than as algorithmic answers. The students agreed that they had solved these problems using division of fractions.

Approximately six weeks after the students began working on the Tuna Sandwiches problems, they were assigned division of fractions problems using only symbolic notation, one at a time. During those six weeks, the class went on to explore other unrelated topics in the sixth grade curriculum such as practice in rounding decimals and surface area. There were a significant number of religious holidays (6 days plus a full week) for which the school was closed during this time creating both a lack of consistency and an aura of festivity. What is interesting to note when examining the students' work done with the decontextualized problems is that when drawing representations, students invariably went back to linear representations. Many referred specifically to Cuisenaire Rods® when they discussed their linear models. Thus components of the conceptual models they had built early in the fifth grade had endured, which is consistent with the conclusion drawn by Lesh, Lester and Hjalmarson (2003) that elements of initial models are retained.

The first of the two problems involving symbolic notation was to find the value of $2 \div \frac{3}{4}$. The students were told to build a model, to solve the problem and to explain how the model could be used to find the solution. Some (Michelle, Amy and Rose for example) wrote the problem as "How many $\frac{3}{4}$'s are in 2?" This would indicate an understanding of the meaning of division.

All of the students used linear representations and these representations were all continuous. The students referred to Cuisenaire Rods® in their explanations and descriptions. Even though Amy used a continuous linear model, she referred to sandwiches in her explanation. This would seem to indicate that she is comfortably moving back and forth between linear and area models cognitively to represent the fractions. She writes the following.

> I made a train[5] of 2 Brown's wich is <u>1</u>. I wanted to make sandwiches with 3 scoops of margarine so I took 6 scoops then I had 2 extra wich was 2. and I had 2 $^2/_3$.

R	R	R	R	R	R	R	R
Br				Br			

R	R
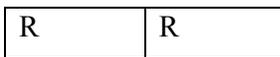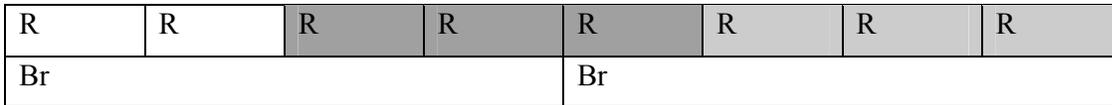
$^2/_3$

Figure 8. Amy's representation of her solution for $2 \div ^3/_4$.

In order to create this representation, Amy needed to be able to select the appropriate length Cuisenaire Rods®, the ones that enable her to find a suitable solution. In this case, Amy chose the brown rod, which is eight centimeters in length to call "one". She showed that each red rod, which is two centimeters in length, is therefore one fourth. She clearly indicated that three red rods (which form the number three – fourths) is now defined as one, indicating that each time this length is achieved, it represents one time that $^3/_4$ goes into 2. Amy's representation shows that she has seamlessly been able to move between the changes of unit necessary to interpret her representation. She darkened the outline of each set of 3 red rods (shaded in the diagram) to indicate a count of $^3/_4$ that goes into 2.

Subsequent to providing solutions for the problem above, students worked on the problem of finding the value of the expression $^5/_8 \div 2 \, ^1/_2$. Though this problem is considerably more difficult, involving a quotative division model, every student in the class found the correct solution. Again, students used continuous linear models and referred to Cuisenaire Rods® in their explanations and their representations.

Most of the representations and explanations involved the reasoning that if $4 \times ^5/_8 = 2 \, ^1/_2$, then $^5/_8 \div 2 \, ^1/_2 = ^1/_4$. Rearranging the equation in this way is a very complex notion. All of the representations that accompanied the explanations indicated the use of Cuisenaire® rods, using the brown rod to represent 1.

Olivia and Eve were the only students to submit a joint solution. They wrote the following, which is based upon the idea that $^1/_4$ of $2 \, ^1/_2$ is $^5/_8$.

> Brown = 1. 8 whites go into brown so, each white = $^1/_8$. 5 whites are equal to one yellow, so

each yellow is $^5/_8$. 2 purples equal 1 brown therefore each purple equals $^1/_2$. According to what we just wrote, 2 browns and one purple would be 2 $^1/_2$. Yellow goes into 2 $^1/_2$ four times. Therefor, the answer is $^1/_4$ because $^1/_4$ of 2 $^1/_2$ is $^5/_8$.

Their representation also indicates the understanding that $^5/_8$ is ¼ of 2 ½ . They bracket each yellow rod and indicate that it is ¼ of 2 ½.

½		½	
1			
$^1/_8$ $^1/_8$ $^1/_8$ $^1/_8$ $^1/_8$ $^1/_8$ $^1/_8$ $^1/_8$			
$^5/_8$			

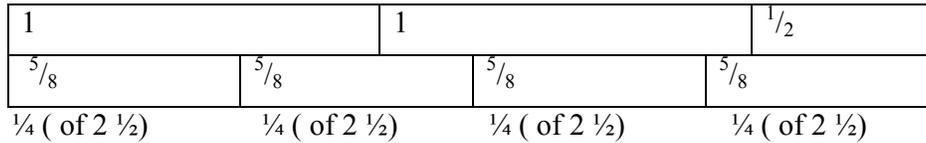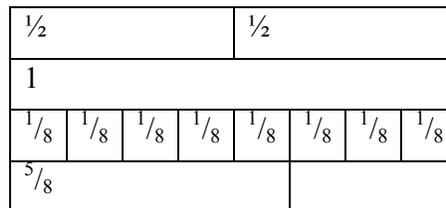| 1	1	$^1/_2$
$^5/_8$	$^5/_8$	$^5/_8$	$^5/_8$
¼ (of 2 ½)	¼ (of 2 ½)	¼ (of 2 ½)	¼ (of 2 ½)

Figure 9. Olivia and Eve's representations for $^5/_8 \div$ 2 ½.

This solution, like the others, indicates conceptual understanding of division of fractions.

A summary of the representations created by the students over time is shown in the chart below.

Tasks:	Ordering Fractions	Holiday Bows	$5 \div {}^1/_3$ $12 \div {}^3/_4$ $7 \div {}^2/_3$	Tuna Sandwiches	$2 \div {}^3/_4$ ${}^5/_8 \div 2\ {}^1/_2$
Grade:	5	5	5	6	6
Lynn	Linear	None	None	Discrete Area****	Continuous Linear
Gabriella	Linear	None	Continuous Linear	Discrete Area****	Absent
Samantha	Linear	None	None	Continuous Area	Continuous Linear
Eve	Linear	Continuous Linear	None***	Discrete Area****	Continuous Linear
Sarah	Linear	Discrete Linear	Discrete Area	Discrete Area****	Continuous Linear
Amy	Linear	Continuous Linear	None	Discrete Area	Continuous Linear
Stephanie	Linear	None	None***	None	Continuous Linear
Nicole	Linear	Continuous Linear	None	Discrete Area****	Continuous Linear
Linda	Linear	**	Absent	-	-
Bea	Linear	Discrete Linear	None	Discrete Area****	Continuous Linear
Brooke	Linear	None	None	Discrete Area	Continuous Linear
Olivia	Linear	Discrete Linear*	None***	None	Continuous Linear
Rose	Linear	Continuous Linear	None	Discrete Area	Continuous Linear
Michelle	-	-	-	Graphic Organizer	Continuous Linear

*After having used fractional knowledge to solve the problem, Olivia used reasoning involving natural numbers. She was the only student in the 5th grade to come up with this form of solution.

**Linda was absent on the first day that the problem was assigned and therefore submitted only her completed chart. (See Figure 1.) Because she is a child with special needs, completion of the written explanations was not required.

***Though these students did not draw representations, they referred to ribbons and/or bows in their explanations. This would lead to the conclusion that the cognitive representations they had created were linear.

****Also included a graphic organizer.

Figure 10. Summary of the representations used by students.

Conclusions and Implications

There are several main points that can be gleaned from this investigation into the nature of the representations that students built and used as they developed an understanding of division of fractions in the absence of being taught the formal algorithms or procedures. The first, and perhaps the most significant, is that students can, under certain conditions, create and link representations that can help them to make sense of problems involving division of fractions, and in the process produce solutions that are mathematically correct. Furthermore, this can be done within the context of regular classroom practice. This study documents that the ideas that are built are robust and can be spontaneously retrieved even after long periods of time have elapsed. The students in this study were able to competently and easily recognize, retrieve, and use ideas that they had formulated several months earlier, and these ideas could be used to solve a variety of symbolic and decontextualized problems—all in the absence of formal instruction on the use of algorithms. Lesh, Lester and Hjalmarson (2003) indicate that when model eliciting problems are assigned, models are developed for specific purposes, just as the students in this study created their initial models as representations to solve particular problems. Moreover, they suggest that as the models become generalizable and transferable, they retain some characteristics of the original situated context. We see this in the students' references to ribbons and bows and Cuisenaire® rods as they construct solutions for the decontextualized problems.

A second significant point is that like the students cited in Bulgar, 2002; 2003a; and 2003b, the students in this study made use of three main methods to solve the problems, even though they had not been taught a procedure or algorithm to solve problems involving division of fractions. These methods are *reasoning involving natural numbers, reasoning involving measurement* and *reasoning involving fraction knowledge.* In *reasoning involving natural numbers,* students converted the units to other units, changing the division from a fraction problem to a natural number problem. For example students might convert fractional parts of meters to centimeters. In *reasoning involving measurement,* students created a measurement tool the size of the divisor and counted how many times they could place the tool along the object that is the dividend. In *reasoning involving fraction knowledge,* the students made use of their knowledge of the number of unit fractions in each "one". Only one of the students in this group of subjects used *reasoning involving natural numbers.* This student claimed to have originally solved the problem differently, using *reasoning involving fraction knowledge.* Some students began by using

reasoning involving fraction knowledge, but then applied *reasoning involving measurement.* This was seen when students worked with non-unit fractions or when they applied additive structures to their solutions. The predominant method that students used when solving these problems, in this study, was *reasoning involving fractions.* These solutions evolved from the representations that the students created. All three of these fraction solution methods provide students with knowledge frames based upon counting schemes, which as stated by Speiser and Walter (2000) are easily interchanged between integers and fractions, making a counting frame for fractions a natural extension of the one for integers. Because of the contextualized problems, the students were not limited in their ability to extend their knowledge of division to division of fractions (Tirosh, 2000) and were able to construct meaningful solution strategies.

Finally, as the cited research suggests (Fosnot & Dolk, 2001; Warner, Alcock, Coppolo & Davis, 2003; Warner & Schorr, in progress), the ability to move among and between different representations for the same concept, indicated for these students a deeper understanding of ideas relating to fractions. The students in this study seamlessly moved among continuous and discrete linear representations and area representations when solving problems involving division of fractions, indicating a very meaningful grasp of division of fractions, often thought to be the most difficult topic in elementary school mathematics (Ma, 1999). They were able to generalize their ideas and apply them to problems using symbolic notation, thereby using their conceptual knowledge rather than algorithmic procedures to solve problems. The knowledge that they built about division of a natural number by a fraction was so robust, durable and flexible that they were able to extend their understanding to solve problems with fractional dividends and fractional quotients. Their solutions were rooted in their interpretation, extensions and revisions of the representations that they had created.

These findings have specific significance for the teaching of mathematics. They underscore the need for teachers to build a deep understanding of students' representations in order to choose and design appropriate tasks that become concrete contexts for the development of abstract ideas about division of fractions. Contextualization has been identified as one of the complex notions surrounding teacher knowledge (Doerr & Lesh, 2003). Teachers also need to understand how to interact with students as they encourage them to develop, use and build meaning for the ideas associated with division of fractions (Davis & Maher, 1997). Such interactions would include the creation of a classroom environment in which justification, sense making and meaningful discourse are encouraged. The research reported in this paper substantiates the importance of encouraging discourse as an important means of strengthening students' ideas (Bezuk & Cramer, 1989; Post, Ipke, Lesh & Behr, 1985). This study confirms the findings of other researchers (Bezuk & Cramer, 1989; Post, Ipke, Lesh & Behr, 1985), wherein discourse was also shown to be critical to the development of concrete models, which were then used as cognitive representations of fractions. Additionally the findings presented herein substantiate the need to create the appropriate learning environments to provide the building blocks that will assist students in constructing meaningful representations (Davis & Maher, 1997).

In conclusion, this research has important implications for both teachers and researchers. As indicated above, the students in this study were able to avoid, for all practical purposes, the main difficulties typically associated with the study of fractions and division of fractions in particular. By considering the classroom context, the problem situations posed, and the trajectory of ideas formulated by these students, teachers and researchers can gain insight into ways in which to help students make meaningful sense of this material. In closing, the author wishes to

underscore the fact that the research contained herein was done in these students' regular mathematics class, facilitated by their regular mathematics teacher, and not by a team of researchers under highly idealized circumstances. The type of activities described here, and the culture of this classroom were typical of how these students regularly received mathematics instruction. This should lend a note of encouragement to teachers who are in search of practices that may help their students build a deeper understanding of this very complex topic.

Endnotes

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[2] Traditional, in this case refers to a more didactic environment of the type described in Cuban, 1993.

[3] This task was originally developed by Alice Alston of Rutgers University. It has been studied extensively reported upon by Bellisio (1999), Bulgar (2002, 2003a, 2003b) and Bulgar, Schorr & Maher (2002).

[4] The abbreviations used by the students in this model had been adopted by the class and represented the colors of the Cuisenaire Rods®. P represents purple, which is 4 cm in length; Dg represents dark green, which is 6cm in length; Lg represents light green, which is 3cm in length and W represents white, which is 1cm in length.

[5] A train is the class-accepted term to denote two or more Cuisenaire Rods® that have been placed side by side along their shorter end. The resulting combined length is now treated as if it were one longer Cuisenaire Rod®.

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