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An investigation of undergraduate calculus students' conceptual understanding of the definite integral

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AN INVESTIGATION OF UNDERGRADUATE CALCULUS STUDENTS' CONCEPTUAL UNDERSTANDING OF THE DEFINITE INTEGRAL

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An Investigation of Undergraduate Calculus Students' Conceptual Understanding of the Definite Integral

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This study sought to investigate undergraduate calculus students' understanding of the definite integral after initial exposure to it in a first semester calculus course. The research question addressed was: What is an undergraduate calculus students' conceptualization of the definite integral? In addition, the study explored students' ability to view the definite integral from different viewpoints; that is, viewing the definite integral on the closed interval $[a,b]$ in the following ways: (a) as a computation, (b) as an area, (c) as an accumulation or summation, (d) as a total change over the closed interval $[a,b]$, (e) as a function, and (f) as an abstract object.

Five beginning second semester calculus students were selected to participate in the study. The 3 female and 2 male participants ranged in age from 19 to 25 years and consisted of both mathematics majors and non-majors. Each participant took part in a series of task-based interviews that used the talking aloud and the clinical interview methods to gather data. The participants' verbatim interview transcripts and written work were analyzed inductively in light of the above research question. A matrix was constructed from which the data could be analyzed to discover major themes.

Students regard the definite integral most commonly as an area, closely followed by as a computation. The total change viewpoint is the least used. Surprisingly, students with average or below average understanding of the definite integral may exhibit some understanding of it as a total change or as a limit of a summing process, which those with above average understanding may not exhibit. Also, students' abilities to find an appropriate viewpoint for a particular situation are not fully developed. Additional trends from the interviews are presented, any of which would make interesting hypotheses worthy of further study. Finally, implications for the teaching of the definite integral are discussed.

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CHAPTER 1

INTRODUCTION

Calculus teachers are realizing the importance of understanding how undergraduate students conceptualize the definite integral. Students' conceptualization of the definite integral affects their abilities to solve problems involving it. Many students can carry out straightforward computational integration procedures, but have trouble with many conceptually-oriented and application-oriented questions. Furthermore, these students have difficulty thinking about the definite integral from multiple viewpoints. Personal experiences in teaching the definite integral to undergraduates have raised my awareness of these issues. Reflecting upon calculus teaching, it appears that there should be better ways to help these students construct their knowledge of the definite integral. A desire to aid undergraduate students' learning of the definite integral has led to an interest in learning how they develop an understanding of the definite integral.

A quantitative assessment project at the University of Wisconsin-Madison described by Bauman and Martin (1995) in *The College Mathematics Journal* led to questions about undergraduates' understanding of the definite integral. The project focused on whether emerging college juniors possessed the quantitative skills necessary for upper division courses at the University of Wisconsin-Madison. The article contained two remarks concerning the definite integral that were particularly interesting. The first of these remarks stated "many more students could exactly evaluate a definite trigonometric integral symbolically than could accurately estimate its value from a graph" (p. 219). The other remark related that many students could perform routine computational problems, such as

integration by parts, but few students could relate to nonroutine problems, such as using tabular data or graphs to estimate an integral's value.

Need for the Study

Despite the importance of the definite integral to calculus and to many mathematical applications, a search of the literature revealed that relatively little is known about student understanding of it. Many studies (e.g., Eisenberg, 1992; Orton, 1983b; Tall & Vinner, 1981) have been conducted regarding the learning of calculus in general, and on the topics of functions, limits, continuity, and derivatives. Studies (e.g., Alibert & Thomas, 1991; Artigue, 1991; Dubinsky, 1991) have also addressed student understanding of topics such as mathematical induction, proof writing, and introductory analysis. Comparatively few investigations have targeted students' understanding of integration and the definite integral, in particular (e.g., Ferrini-Mundy & Graham, 1994; Foley, 1992; Orton, 1983b). Most of this research exists as small pieces of larger studies of calculus.

This study provides a more focused contribution to the body of knowledge about students' learning of integration. Specifically, this research concentrates on investigating and describing undergraduate calculus students' knowledge of the definite integral. Hence, this study contributes to mathematics education by expanding the knowledge base concerning the definite integral, and it also contributes to the relatively new area of research in collegiate mathematics education.

Statement of the Problem

This study focuses on answering the question: What is an undergraduate calculus student's conceptualization of the definite integral? In addressing this question, the study describes students' understandings of the definite integral after initial exposure to it in a first semester calculus course. This includes documenting areas where their understandings are incomplete or still developing, as well as where they have misconceptions concerning the

definite integral. One particular aspect of the question the study explores is students' ability to view the definite integral from different viewpoints; that is, viewing the definite integral, $\int_a^b f(x)\,dx$, in any of the following ways: (a) as a computation, (b) as an area, (c) as an accumulation or a summation, (d) as a total change between $x = a$ and $x = b$, (e) as a function, and (f) as an abstract object.

Importance of the Study

This study responds to a need for research into the understanding of the definite integral, and it could influence methodology for the teaching of the definite integral. A concentration on the definite integral provides a more focused contribution than the previous studies. The use of a case study provides a detailed description of a few cases in order to illuminate the question under study (Patton, 1990). Finally, by focusing on undergraduates, the study provides insight into the learning of mathematics at the collegiate level.

A long-term goal of the study is to contribute to the discussion about how students construct their understandings of the definite integral, so that better methods can be developed to aid their learning. As more research is done, a schema may emerge. Once this schema is understood and verified, it can be incorporated into the development of calculus curricula. Even more importantly, this schema can be used to improve classroom instruction.

CHAPTER 2

LITERATURE

This chapter is divided into two sections placing the study in context with relevant research and theory. The first section focuses on material addressing understanding of the definite integral, and the second focuses on the psychology of mathematical thought processes.

Definite Integral

The literature concerning the definite integral, $\int_a^b f(x)\,dx$, is reported in terms of the following viewpoints: (a) as a computation, (b) as an area, (c) as an accumulation or a summation, (d) as a total change between $x = a$ and $x = b$, (e) as a function, and (f) as an abstract object. These viewpoints are not mutually exclusive and often one viewpoint must be used in conjunction with others when solving certain problems. Central to most of these viewpoints is the concept of the definite integral as the limit of a summing process. In order to avoid duplication in reporting the literature, a choice will be made as to which representation makes most sense for a particular item.

Computation

Viewing the definite integral as a computation refers to evaluating it without reference to aspects such as area or summing processes. In most instances this involves applying The Fundamental Theorem of Calculus to a definite integral that contains an explicit function (Ostebee & Zorn, 1997, pp. 366 & 368). Computation will also encompass any algebraic techniques associated with using The Fundamental Theorem of

Calculus. In addition, this viewpoint includes situations where students use memorized techniques to set up a definite integral, but are unable to explain why it is appropriate. An example is knowing how to set up an integral to find a volume, but not making the connection to dissecting the object and summing over the pieces.

One consistent literature finding was that students can be very proficient at learning and carrying out required calculations and yet possess a minimal understanding of the conceptual basis for the calculations and the theory in general (Ferrini-Mundy & Graham, 1984; Ferrini-Mundy & Lauten, 1993; Orton, 1980, 1983a). Foley (1992) followed honors and engineering Calculus II students through a semester and reported that, by the end of the semester, growth in procedural knowledge was noticeable. However, she found that students who lack conceptual knowledge demonstrated misconceptions when asked to justify a calculation or procedure. Using the task shown in Figure 1, Foley illustrated her point with the following types of student responses: (a) just applied the appropriate formula, (b) the integral gives the area and multiplying by π gives the volume, and (c) integration adds a dimension.

2. The volume of the solid shown can be obtained by evaluating

$$\pi \int_0^2 x^4\,dx.$$

Explain why this is so in language that a Calculus I student can understand.

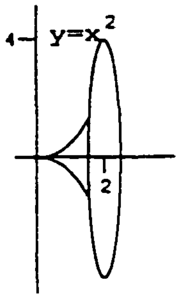


Figure 1. Task used by Foley (1992, p. 28) that elicits student responses showing the definite integral from the viewpoint of a computation.

Mundy (1985), studying calculus students' errors, noted that "students appeared to apply a rule or procedure in a problem setting where it does not apply" (p. 171). As an illustration of this misconception, she showed a definite integral involving absolute values,

where many of the students computed the integral by ignoring the absolute value symbols. In this case the problem was simplified to a known integration procedure.

Mundy (as cited in Ferrini-Mundy & Lauten 1994) "found that 90 percent of the students computed $\int_{-2}^{2} |x| dx$ as if the problem were $\int_{-2}^{2} x\,dx$" (p. 118), but used this to illustrate that students tended to view graphical and analytical representations of the definite integral as if they were separate, unconnected notions. Eisenberg (1991) noted the same phenomenon when students were evaluating the integral

$$\int_{-4}^{4} \left| x^2 + 5|x| + 6 \right| dx .$$

The students attempted to evaluate it analytically, ignoring the graphs that they had correctly drawn. Furthermore, in a different study, Eisenberg (1991) found that mathematics professors were reluctant to use graphical techniques. In particular, he showed that when a group of professors of mathematics were asked how they would solve the integral

$$\int_{-3}^{3} \sin(x)\left[\cos(x) + 3x^2 - x\sin(x)\right] dx ,$$

they "all started by saying that they would try integration by parts or substitution, but not one saw initially that the function is odd, and because of the limits of integration, had to be zero" (p. 147). This reluctance to use geometric interpretations to help with completing algebraic calculations relating to the definite integral was noted by Ferrini-Mundy and Graham (1994) as well.

Studies by Ferrini-Mundy and Graham (1994) and by Norman and Prichard (1994) concluded that for some students, the integral symbol is a sign to do something. Norman and Prichard said this was suggested by responses to the problem: Find $F(a)$ given that

$$F(x) = \int_a^x f(t)\,dt.$$

This problem evoked several declarations that it was impossible to do without knowing the function f. Upon being supplied with a simple function for f, students were able to compute a value for $F(a)$.

Finally, in reporting on one case study from Calculus I, Ferrini-Mundy and Graham (1994) stated that one student, Sandy, saw no significant differences between indefinite integrals and definite integrals. Sandy is quoted as saying "'They're both taking the antiderivative—but the second is more definite'" (p. 41). However, regarding Calculus II students, Foley (1992) reported that one of the strongest connections for students was the relationship between definite integrals and antiderivatives.

Area

The area viewpoint refers to thinking about the definite integral in terms of area. When the function is nonnegative over a closed interval, the definite integral is associated with the area of the region bounded by the graph of the function and the horizontal axis over the interval. This connection is sometimes referred to as finding the "area under the curve." When the function takes on negative values, the definite integral computes the "net or signed area." In this latter setting, bounded regions that are below the horizontal axis are said to have "negative area."

Several authors have reported that students seem to have a good understanding of the conceptual link between the definite integral and area (e.g., Ferrini-Mundy & Lauten, 1993; Foley, 1992; Orton 1980, 1983a). Sandy, the student described in Ferrini-Mundy and Graham (1994), is like many students who view the definite integral as defining the area bounded by the graph of the function and the x-axis, but she also connects the antiderivative of a function with the area between the graph of that function and the x-axis.

Students do have some misconceptions and shortcomings regarding the definite integral as related to area. For example, for some students "the more approximating rectangles one had, the greater the area" (Norman & Prichard, 1994, p. 75) and "if the area is small, then height must also be small" (Foley, p. 48). Many students do not seem to understand that the concept of limit is the key idea that connects the notions of approximating rectangles and area under the curve (Orton, 1980, 1983a). As noted before, many students do not make a connection between the algebraic representation and the geometrical representation while computing definite integrals (Eisenberg, 1991; Ferrini-Mundy & Graham, 1994; Ferrini-Mundy & Lauten, 1994).

A major obstacle to student understanding of the definite integral as area is the idea of "negative area" and how "negative area" relates to the notion of area (Norman & Prichard, 1994; Orton, 1980, 1983a). Orton describes how, when given a problem about computing area of a region containing "negative area," many students could separate the region into two regions in order to calculate the area of each, but several did not know why this method worked. A few of these students responded by saying that they were taught to do this type of computation in this way.

Accumulation or Summation

Accumulation or summation refers to the notion of the definite integral as a summing process. Typically, it is used when an object is dissected and a sum of the pieces is formed. In this framework, one is usually intending to approximate a definite integral. Employing a limiting process on the sum leads to the definite integral, under appropriate conditions, and an exact value for the quantity being measured or computed.

Many students have difficulty conceptualizing the definite integral as a limit of a sum. In particular, it is not part of many students' understanding that the definite integral is connected with the procedure of dissecting an object, summing over all the pieces, and then

making use of a limiting process (Ferrini-Mundy & Lauten, 1993; Foley, 1992; Orton, 1980, 1983a). Orton comments that, in the case of working with areas, the idea of a limit of a sum may be complicated by the algebraic manipulations required. In her dissertation, Foley reports two particularly disturbing misconceptions that were held by a few students. The first of these is that the definite integral "was described as 'the sum of all heights', without the width" (p. 48). The other misconception became apparent when Foley posed a question connecting the integral to the convergence or divergence of series; in this case, the integral was thought of as "a sum over all numbers in the interval whereas the series is a sum over the integers" (p. 48).

Total Change

Viewing the definite integral as a total change means understanding it as integrating a rate function over a closed interval $[a,b]$ to obtain a change in an amount function over the same interval. The change in an amount function corresponds to the change in an antiderivative of the rate function, and therefore can be found by using The Fundamental Theorem of Calculus. However, the notion of total change requires that The Fundamental Theorem of Calculus be seen as more than just a computational tool. The idea of total change is also the end result of taking the limit of a summing process in applied problems where the definite integral is the appropriate tool for the task.

Foley (1992) presented the graph of a piecewise-defined velocity function consisting of line segments, and asked students to determine the distance traveled in 5 hours. To compute the distance traveled most students tried to use the constant velocity distance formula, $d = rt$, or the area under the graph of the velocity function. However, a few students observed or used in some version the idea that the distance traveled is just the integral of the velocity function over the time interval. The latter solution path showed some understanding of the definite integral as a total change.

Function

When one views the definite integral as a function, one thinks of a function G defined as

$$G(x)=\int_a^x f(t)\,dt,$$

where a is some real number. Thus the definite integral is viewed as an object in which the upper limit of integration is allowed to vary. Here one initially focuses on evaluating the integral function and describing the behavior of the graph of this function. This view has been explored in studies focusing on student understanding of function, as well as in studies focusing on integration.

Some researchers noted that students have difficulty comprehending that, by varying the upper limit of integration, the definite integral is a function (e.g., Dubinsky, 1991; Harel & Kaput, 1991). This could be why students have difficulty employing the full power of The Fundamental Theorem of Calculus or understanding the natural logarithm when it is defined as an integral (Dubinsky). According to Harel and Kaput,

> to understand the meaning of:
>
> $$I(t)=\int^t f(x)\,dx$$
>
> as a function of t, it is necessary to think of $I(t)$ as an operator that acts on the process $x \rightarrow f(x)$ *as a whole* to produce a new process:
>
> $$t \rightarrow \int^t f(x)\,dx$$
>
> It is the awareness of acting on a process as a whole, as a totality—not point-by-point—that constitutes the conception of that process as an object. (1991, p. 85; also see Harel & Kaput, 1990)

Furthermore, it is their belief that students, who do not understand the integral as a function, have not conceived the process $t \rightarrow I(t)$ as an object yet.

Students have also demonstrated misconceptions about the definite integral in responding to across-time questions; questions that ask students to "describe patterns of change in one variable that are related to patterns of change in another variable" (Ferrini-Mundy & Lauten, 1993, p. 159). An example of this type of question used by Norman and Prichard (1994, p. 76) is illustrated in Figure 2. They reported that one student indicated

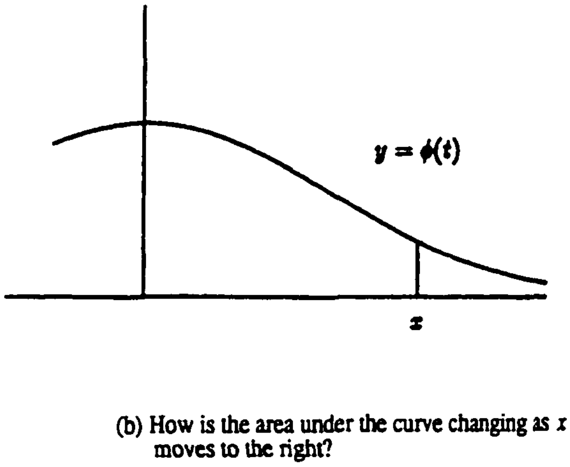


Figure 2. Question used by Norman and Prichard (1994, p. 76) to explore across-time understanding of the definite integral.

the area under the curve decreased as x moved to the right. The student's response showed confusion between the values of the function $\phi(t)$ and the values of the integral $\int_0^x \phi(t)\,dt$.

G. S. Monk (as cited in Ferrini-Mundy & Lauten, 1993) found similar confusions among students. Artigue and Szwed (Artigue, 1991) asked 89 first-year university mathematics students to describe the graph of the function g defined by

$$g(x) = \int_{-4}^{x} f(t)\,dt,$$

given the graph of the function f (see Figure 3). They summarized the results from this task as follows:

> The graph of the function g was only attempted by 35 students. The curves produced were extremely diverse and seemed to have only one property in common: the graph is a line segment on $[-2,2]$. Only 14 gave the correct slope and at least 14 graphs were discontinuous. Of the 35 graphs produced, only 13 considered the direction of variation in g, and only 3 could be considered acceptable solutions. (p. 179)

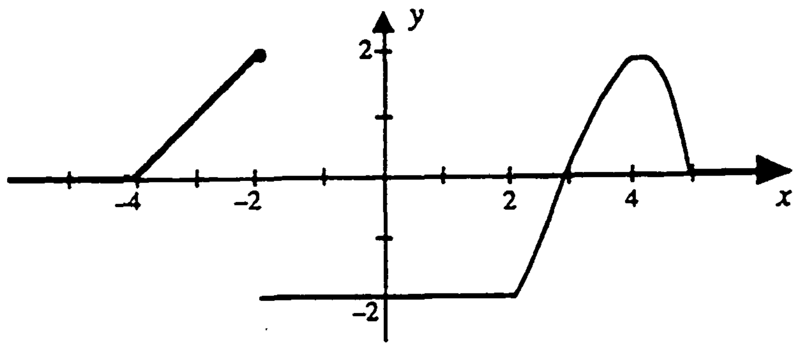


Figure 3. The graph of the integrand f used by Artigue and Szwed (Artigue, 1991, p. 179) in their across-time task.

Abstract Object

As an abstract object the definite integral is considered a formal object, independent of context. Thus one analyzes the properties of the definite integral and its relationships to other mathematical objects. Included in this representation is knowing, and being able to work with, a definition for the definite integral. Additionally, the notion of improper integrals could be included here.

Ferrini-Mundy and Lauten (1994), in investigating their student Sandy's understanding of Riemann sums and a definition for the definite integral, noted that at best this understanding was vague and "not necessary to her routine solution of calculus problems" (p. 42). When asked what the limit has to do with integration, Sandy admitted that she was not sure. Finally, within the discussion related to a definition of the definite

integral, Ferrini-Mundy and Lauten concluded that, despite Sandy's ingenuity in making sense of the material, whatever was done to motivate the notion of the definite integral in class seemed to be lost to her.

Orton (1980, 1983a) used two tasks that fit into this representation: (a) one asking students to explain why the definite integral of a sum is the sum of the definite integrals and (b) the other focusing on the relationship between integration and differentiation when looking at the rate of change of area under a curve. The students found these tasks to be very difficult. For the first, nearly all students were unable to begin an explanation, even though they all took the property for granted. The second task required that students realize the need to integrate first and then differentiate; however, many students could appreciate the need to either integrate or differentiate, but not both. Orton concluded that the students could not grasp that integration and differentiation are inverse processes.

Psychology of Mathematical Thought Processes

The basic concept of the constructivist view of learning is that the individual learner constructs knowledge by seeking meaning for ideas based upon personal experiences (Goldin, 1990; Noddings, 1990). In order to conduct research from such a perspective, one has to accept certain basic assumptions. The following are the assumptions that I hold pertaining to constructivism (Davis, Maher, & Noddings, 1990): (a) Students make sense of ideas based upon the connections that they establish between new ideas and their own experiences and knowledge. (b) Students' constructions are rational, and they can explain their understanding. (c) Errors and misconceptions are a natural part of the learning process and are to be expected. (d) Errors provide insight into the developmental process of gaining an understanding of a new idea.

From the broad range of constructivism, three perspectives in particular have influenced the direction of this study. These perspectives are the concept image–concept

definition perspective, the object–process perspective, and the human information processing perspective. The remainder of this chapter contains brief descriptions of these perspectives.

Concept Image–Concept Definition Perspective

An individual's accumulated mathematical knowledge concerning a particular topic can be described from a concept image–concept definition perspective (Tall & Vinner, 1981; Vinner, 1983). The *concept image* is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, p. 152). As the concept image evolves over time, different parts of it can be stimulated and altered, depending on the particular experience. Thus, the concept image may become compartmentalized in such a way that the individual can simultaneously hold differing notions for the same concept without the individual being aware of any conflicts. The part of an individual's concept image that is activated when that concept is needed, and that is the part of the concept image that is accessible to the observer, is the *evoked concept image*. Since an individual utilizes the part of the concept image that seems appropriate at a given moment, it is possible for that person to evoke seemingly conflicting images for similar situations. It is only when conflicting aspects of the concept image are evoked simultaneously that an individual may notice the discrepancy.

The *concept definition* is "a form of words used to specify that concept" (Tall & Vinner, p. 152). An individual may hold a personal concept definition, a formal concept definition, or both, with the formal concept definition being some mathematically acceptable concept definition. A concept definition is part of an individual's concept image, but may be void, or may be incorrectly or incompletely included in the concept image. Thus, a concept definition can be at variance with some part of the concept image. In addition, a

concept definition could remain inactive or even forgotten, as an individual may evoke a concept image to handle any given task or scenario.

Object–Process Perspective

The object–process perspective of Sfard (1991), informs this study. Here, a *process* (or *operational*) conception of a notion is one in which the notion is perceived as dynamic or as a detailed sequence of steps. An *object* (or *structural*) conception of a notion is one in which the notion is thought of as a static structure existing on its own, and one on which processes can be performed. Even though these two conceptions appear to be distinct and disjoint, Sfard argues that they are in fact "inseparable, though dramatically different, facets of the same thing" (p. 9). In other words, the process and object conceptions complement each other, and therefore are necessary for a deep understanding of mathematics.

Sfard (1991) hypothesizes that when concepts are developed, the operational or process conception comes first, and then the structural or object conception follows later. The transition from operational to structural occurs in three stages: interiorization, condensation, and reification. A learner at the stage of *interiorization* is becoming "acquainted with the processes which will eventually give rise to a new concept" (p. 18). In essence, the learner is exploring and becoming skilled at performing the process. The *condensation* stage is "a period of 'squeezing' lengthy sequences of operations into more manageable units" (p. 19). During the condensation phase, a learner is condensing a process into a meaningful whole and is able to use it with other processes. Condensation is analogous to forming a procedure in a computer program. *Reification* occurs when a learner experiences "an ontological shift–a sudden ability to see something familiar in a totally new light" (p. 19). Thus, reification occurs when a process can be seen as an object and, as such, can have other processes applied to it. Sfard does point out that, because

reification requires an ontological shift to occur, it can be extremely difficult to achieve, and it may take years for it to happen.

Human Information Processing Perspective

The perspective of human information processing (Davis, 1983: Davis, 1984; Davis & McKnight, 1979) works on the premise that, in order to understand how people think about mathematics, one must understand how they see and work with information. In building a framework from which to accomplish this task, Davis and his study group borrowed ideas from the fields of cognitive science and artificial intelligence, along with ideas drawn from observations of student performances on mathematical task-based interviews. In addition, they use the computer as a useful, although not perfect, metaphor for describing how humans work with information. In the remainder of this section, a brief overview of some basic concepts and mechanisms is given.

Davis and his study group (1983, 1984; Davis & McKnight, 1979) developed mechanisms to describe sequential processes. For this purpose, they defined the notions of procedures and sequences. A *procedure* consists of several steps or instructions carried out in a definite order to accomplish a particular goal. One of the instructions within a procedure may be a cue to another procedure to complete a particular task. Two distinctive types of procedures are visually-moderated sequences and integrated sequences. The *visually-moderated sequences* are such that a visual input cues the retrieval and execution of a procedure that in turn modifies the visual input. The modified visual input cues another procedure, and the process continues until a termination is achieved. Visually-moderated sequences that are practiced enough to become independent of the actual visual clues for carrying out the entire task are called *integrated sequences*. In addition to procedures and sequences, some other capabilities and ideas are associated with the sequential processing of information. Some of these capabilities are: (a) an ability to look ahead and plan for the

next step as various procedures are carried out, (b) an ability to keep a record of procedures actually used, (c) a mechanism for keeping track of which procedure has control over a particular sequence of steps that are being carried out, (d) a mechanism for detecting certain types of errors, and (e) a way to retrieve and copy procedures from passive memory to working memory. These capabilities need to be developed and debugged themselves before they can be effective.

Furthermore, Davis and his study group (1983, 1984; Davis & McKnight, 1979) developed a structure that serves as a collection of many pieces of connected information. They defined the notion of frames for this purpose. A *frame* is a structure that contains a sizable amount of interrelated knowledge in memory. A frame differs from a procedure in that a frame need not be sequential and therefore allows for multiple points of entry, which provides some flexibility in its use. Frames allow "top down" or "bottom up" processing. Another feature of frames is that they are hypothesized to include variables, with default procedures to assign values when there is no input data. As with procedures, there are mechanisms for working with frames. These mechanisms are responsible for things like: (a) retrieving appropriate frames, (b) sequencing frames together when more than one frame is needed, (c) sending appropriate input data to the variables of the frame, (d) deciding whether the input is a satisfactory fit for the variable, (e) inserting default data based on past experiences into variables for which there is no input data (the default data could be inappropriate), and (f) modifying and creating frames. Finally, just as with procedures, the frames and mechanisms must be debugged before they can become effective.

CHAPTER 3

METHODOLOGY

The purpose of this study is to investigate, describe, and understand undergraduate students' understandings of the definite integral after initial exposure to the notion in a first semester calculus course. This chapter, in five sections, describes the methodology used in this study: (a) the design of the study, (b) the issue of the trustworthiness of the study, (c) the instruments and tools used in the study, (d) the interview process, and (e) the method of analyzing the data.

Design of the Study

Students construct their knowledge based on their personal experiences and backgrounds, which in turn influence their formation and perceptions of mathematical concepts, as well as their problem-solving strategies. Furthermore, the way in which students communicate in oral and written forms impacts their abilities to demonstrate understanding of key ideas effectively. All of these issues combine to create a complex web from which a researcher must seek to extract meaning and understanding. Therefore, a qualitative design was selected to explore students' conceptual worlds in order to understand the meaning(s) they constructed (Bogdan & Biklen, 1992) for the definite integral. Also, the use of a qualitative approach leads to more attention "given to nuance, setting, interdependencies, complexities, idiosyncrasies, and context" (Patton, 1990, p. 51).

In particular, this study used the case study format for three reasons. First, case studies are a common format used in conducting qualitative investigations. Second, case studies are "particularly useful where one needs to understand some special people,

particular problems, or unique situation in great depth" (Patton, 1990, p. 54). Finally, the case study design allows a researcher to gather in-depth, comprehensive information in a manageable, systematic manner, and additionally, allows the researcher to focus on individual variations (Patton).

The data for the study were collected using the talking aloud and the clinical interview methods (Ginsburg, 1981; Ginsburg, Kossan, Schwartz, and Swanson, 1983). The talking aloud method allows a researcher to elicit information about the complex activities involved with problem solving and to describe these activities. The method also allows a researcher to specify the cognitive processes that take place in the mind of a participant (Ginsburg et al.). The clinical interview method permits probing and questioning to explore a participant's ideas and thoughts. The method seeks to achieve three aims in understanding cognitive processes: (a) discovery of such processes, (b) identification and description of such processes, and (c) establishment of competency with such processes (Ginsburg; Ginsburg et al.).

Trustworthiness of the Study

The initial ideas for this study were discussed with professors and colleagues in order to (a) test ideas for the study, (b) determine alternative approaches or ideas, and (c) expose biases in myself and the methods being considered. This exposure to questioning of methods and design plans allowed for clarification and revisions in the study. These discussions and questionings continued as the design of the study evolved and developed. Furthermore, as the interview tasks were developed, discussions with and questioning by professors and colleagues exposed various shortcomings in tasks and allowed for the development of richer tasks.

The participants in the study were selected purposefully in order to ensure that the study included individuals who would give the richest set of information about students'

understanding of the definite integral. Each participant was interviewed three times during the study, thus allowing them to become less self-conscious about being interviewed and more open to expressing their thoughts. In addition, having three interviews allowed for follow-up questions to resolve misunderstanding over what was previously said, clarify vague statements, and address conflicts in the explanations of individual participants. Addressing issues through more than one task or question assessed consistency in what was said by a participant during the interview process. Data was gathered using a tape player, a camcorder, and participants' written work, in order to provide multiple sources of data to compare and cross-check. Additionally, informal conversations with participants were used to gather additional data and to note outside influences that could affect the results of this study.

Data from the study were examined from several different points of view during the analysis stage of the research in order to develop a better understanding of the data. The perspectives used in this study were: (a) concept image—concept definition perspective, (b) object—process perspective, and (c) human information processing perspective. Discussions with members of the doctoral examining committee provided opportunities to expose any personal biases, as well as allowing possible alternative explanations to be brought forward for examination.

Instruments and Tools

This section provides a discussion of the instruments and tools used in the study for compiling data. These include the researcher, the tasks, recording devices, a background survey, participant selection tasks, and researcher notes.

Researcher

A summary of my education, teaching experiences, courses taught, and professional interests appear on the vita in Appendix A. My teaching philosophy evolves from a

constructivist point of view, in which students build their own understandings of mathematics based on personal experiences and backgrounds. Thus, to aid students in improving their problem-solving skills and to develop better understanding of concepts and content, students need to have numerous opportunities to demonstrate their understandings and to receive feedback. With this in mind, I teach from a student-centered view rather than a teacher-centered view. Furthermore, to enhance understanding and problem solving skills, students need to develop multiple representations for any mathematical topics encountered. Finally, technology is a tool used in my teaching and provides additional options for problem solving and content development.

Tasks

The tasks for the study were developed during the summer of 1998. The development was influenced by previous research concerning the definite integral (Artigue, 1991; Ferrini-Mundy & Graham, 1994; Ferrini-Mundy & Lauten, 1994; Norman & Pritchard, 1994; Orton 1980, 1983a) and by the text *Calculus from Graphical, Numerical, and Symbolic Points of View* (Vol. 1) (Ostebee & Zorn, 1997), used at the university where the study was undertaken. Adapting questions based on previous research provided opportunities to link this research and the literature. The Ostebee and Zorn calculus book was chosen because most participants in their Calculus I course studied it.

The initial set of tasks was piloted in the summer, 1998, in an Applied Calculus—Calculus for the Biological Sciences—class and a Calculus II—second semester calculus—class. The tasks were piloted as homework problems, extra problem sets, quiz questions, and in-class examples. In addition, 5 students from these two classes volunteered to be interviewed. They were interviewed using the initial set of tasks, as well as some rewritten tasks. After the tasks were piloted, all were revised to gain the richest set of data to describe undergraduate calculus students' concept images and understandings of the

definite integral. Additional tasks and follow-up questions were also developed to investigate interesting or unexpected participants' evoked concept images produced during the piloting of the tasks.

The set of interview tasks is presented in Appendix B. The tasks were chosen because of their potential for eliciting responses and demonstrating diverse ways of thinking about the definite integral. The tasks vary in level of difficulty from easy computations to novel tasks that require a solid understanding. Each task was designed to gain insight into participants' understanding and use of the following viewpoints of the definite integral, $\int_a^b f(x)dx$: (a) as a computation, (b) as an area, (c) as an accumulation or a summation, (d) as a total change between $x = a$ and $x = b$, (e) as a function, and (f) as an abstract object. Many of the tasks lead to responses using different viewpoints, and thus provide many opportunities to assess an individual participant's understanding and ability to relate to various viewpoints of the definite integral. These opportunities help determine whether a participant can apply different viewpoints to a given task, or if the individual can view the task in only one way. Overall, the tasks were designed to give each participant ample opportunity to show an understanding and any ideas about the definite integral.

The 20 tasks were used during the three interviews as follows: (a) Tasks 1 through 7 were the bases for the first interview, (b) Tasks 8 through 13 were the bases for the second interview, (c) Tasks 14 through 19 were the bases for the third interview, and (d) Task 20 was used during any of the three interviews when there was time or when appropriate. Furthermore, the tasks were broken down into core tasks presented to every participant, and supplemental tasks presented only to further explore a participant's concept image of the definite integral. For this study, Task 13, the second graph of Task 15, Task 19, and Task 20 were considered to be supplemental. The remaining tasks formed the core

of the study and provided opportunities for the participants to demonstrate their understandings of the definite integral and their abilities to use any of the six viewpoints listed previously.

Recording Devices

Each interview was recorded with a tape recorder and a camcorder. The primary data gathering method was tape-recording. The audiotapes were transcribed to make data readily available. The videotape provided a backup to the audiotape, provided supplemental and clarifying comments to be inserted into the transcripts of the interviews, and provided a dynamic way to re-create the written work of a participant during data analysis.

For each interview, the tape recorder was placed on the table in front of the participant. The camcorder and tripod were also on the table in front of the participant. In accordance with the agreement for anonymity (Appendix C), the camcorder focused on the writing area only.

Background Survey and Participant Selection Tasks

A background survey (Appendix D) and participant selection tasks (Appendix E) were administered to Calculus II students to aid in selecting individuals to be invited as participants in the study. The survey consisted of questions regarding both personal and mathematical backgrounds. The survey information gave an overview of each student and provided a basis for a participant profile. The selection tasks were designed to elicit information about student understanding of the definite integral. The tasks consisted of four problems relating to various aspects of the definite integral.

Researcher Notes

Researcher notes are a repository for procedures, observations, reflections, and insights. The notes include fieldnotes, interview notes, and analysis notes. Fieldnotes were kept to document the participant selection process, to reflect on the interview process as it

unfolded, and to record topics covered in the Calculus II classes during the interview process. Individual sets of interview notes were made after each interview to describe observations, thoughts, and initial insights, as well as to document relevant informal conversations occurring before and after each interview. Analysis notes were kept to aid in coding data and to record results as they emerged and developed. Finally, the notes provided a way to reflect on my subjectivity and biases during this study.

Interview Process

This section discusses how the interview process was conducted for this study. In particular, this section addresses participant selection, the materials provided for the interview, the interviews, and data gathered during the interviews.

Participant Selection

The sampling approach used in the study was purposeful sampling (Bogdan & Biklen, 1992; Patton, 1990). This sampling allowed the selection of individuals who could give the richest set of information about undergraduate calculus students' understanding of the definite integral. For selection purposes, stratified purposeful sampling (Patton, 1990) was used in the study. A goal of this sampling is to "capture major variations rather than to identify a common core, although the latter may also emerge in the analysis" (Patton, 1990, p. 174). The intent was to select participants with a range of understanding and ability concerning the definite integral in order to try to document several different stages of development in the construction of knowledge about the definite integral.

The study involved Calculus II—second semester calculus—students from a state university located in the northern Rocky Mountains of the United States. The university's enrollment at the time of this study was approximately 12,000 full-time students. The Ostebee and Zorn (1997) book was the adopted text for Calculus I and II for the semester that data were gathered, as well as the year prior to the study. Thus, the majority of the

students enrolled in Calculus II at the time of the study had used Ostebee and Zorn for Calculus I, and had studied only one chapter on integration.

During the first week of fall semester, the students in the two sections of Calculus II were asked to help in a study focusing on undergraduate calculus students' understanding of the definite integral. In addition, they were told that this was the start of a long-term project to search for better ways to help students learn about the definite integral. The students were given the background survey (Appendix D) and the participant selection tasks (Appendix E) to complete. Overall, 51 students completed the survey and selection tasks to become the pool of students from which to draw participants for the study.

The surveys were read to determine which students should be removed from the selection pool for the study. Three reasons for eliminating students at this stage were: (a) not a university student, (b) repeating Calculus II, or (c) two or more years had passed since having completed Calculus I. The students who were not university students were high school students. They were removed because the focus of the study was undergraduate calculus students. The students repeating Calculus II were removed from consideration because this investigation was created to determine the conceptual understanding of the definite integral that students possessed after having completed their initial exposure to the concept. Finally, those students who indicated it had been more than two years since they took Calculus I were removed so that not remembering key ideas about the definite integral would not be an issue in trying to learn about their understanding of it. In all, these three conditions eliminated 20 students from the selection pool.

The participant selection tasks of the remaining students were scored using the rubric presented in Table 1. The individual scores for the selection tasks were totaled. The total scores were examined, and these scores seemed to clump together naturally into three distinct groupings. These groupings were used to divide the selection pool into three

Table 1.
Rubric for Participant Selection Tasks

Score	Description
2 Demonstrated Competence	• correct response • correct idea(s) employed in solution, minor mistakes • understands basic concepts, but may lack a deep understanding
1.5	• response is borderline between demonstrated competence and satisfactory
1 Satisfactory	• good approach, but made mistakes • may not have finished the problem, but mostly complete and correct • shows some understanding of the basic concept(s)
0.5	• response is borderline between satisfactory and inadequate or superficial
0 Inadequate or Superficial	• no response • off-base or no idea of how to approach the problem • started only, but did not finish • no or little understanding

distinct groups (see Table 2): (a) below average conceptual understanding of the definite integral, (b) average conceptual understanding of the definite integral, and (c) above average conceptual understanding of the definite integral. Those scores on the borderlines were placed in groupings based on Calculus I grades.

Table 2.
Score Ranges Used to Form Three Groups for the Study

Scores	Description
0.0 - 2.5	below average conceptual understanding
2.5 - 4.5	average conceptual understanding
4.5 - 8.0[a]	above average conceptual understanding

Note. [a]No student scored above a 6.0.

Two students–one male and one female–from each group were contacted about participating in the study. The intent was to find a total of 6 students representing different

levels of understanding concerning the definite integral. Each of the contacted students was invited to an individual meeting to discuss participation in the study. During these meetings each student was told about the research project, about the interview process, about the expectations of them, and that participation would be completely voluntary. They were informed that each interview would be tape-recorded and videotaped, and were guaranteed that participant anonymity would be preserved. Each student was given the opportunity to ask questions about the study. Finally, each student signed a consent form (see Appendix C) before participation was allowed.

Five participants completed all the interviews. The pseudonyms for these 5 participants are Joan, Rob, Stan, Tina, and Lynn. Joan and Rob formed the above average group, Stan and Tina formed the average group, and Lynn formed the below average group. The sixth participant (from the below average group) dropped out of the study just before the first interview.

Materials

For each interview the participants were provided with graph paper, black pens, a ruler, and both lined and unlined paper. Starting with the first interview of the fourth participant, a red pen was provided as well. Participants were allowed to use their own graphics calculators during the interviews, and backups were available in case one malfunctioned.

Interviews

The interviews took place in two rooms on the campus where the study was conducted. One room was a conference room in the university's Mathematics Building. The other was the mathematics education classroom located in the university's Liberal Arts Building. Evening interviews were held in the mathematics education classroom and all

other interviews were held in the conference room during the day. During each interview, only the participant and myself were present in the room.

Each participant gave one interview per week over a period of three weeks early in the fall semester. The reason for this was to ascertain participants' understanding of the definite integral after their Calculus I experience, but before they were heavily involved with their Calculus II study of integration. The interviews were scheduled around the participants' schedules. A schedule of the interviews is given in Appendix F. The first and second interviews each lasted about 1 hour and 20 minutes, and the third interviews lasted about 1 hour each.

Each interview began with general, informal conversation while the camcorder was focused on the participant's writing area. The first interview for each participant proceeded immediately to the tasks. Each participant's second and third interviews began with follow-up questions from the first and second interviews, respectively, before proceeding to new tasks. The only exception to this was Tina's third interview, in which the follow-up questions were asked at the end of the interview because of a time constraint on her schedule. The problem-solving tasks were presented to participants on paper so that they could refer to the problem as needed. However, for verbal explanations, verbal tasks were presented and the tasks on paper given only if necessary. After initial responses, participants were questioned about their responses and work. This process was repeated for all core tasks and any extra tasks that were useful in understanding the participants' concept images of the definite integral. After each interview was completed, there was general, informal conversation to learn more about each participant.

Data

The data for the study came from participant interviews and written work. All interviews were collected on audiotape and videotape, and transcripts were made. Written work consisted of the participants' writings and drawings during the interviews.

Analysis

Analysis began immediately with the interview. Initial analyses occurred between interviews to determine whether follow-up questions were needed. After verbatim transcripts were made from audiotapes, the transcripts were double-checked and supplemented with the videotapes to include nonverbal actions. The transcripts were studied and analyzed inductively from a concept image–concept definition perspective, an object–process perspective, and a human information processing perspective. These allowed the data to "speak" for itself, and for a "story" to emerge. The data were further analyzed in terms of the six viewpoints of the definite integral, $\int_a^b f(x)\,dx$, presented in Chapter Two: (a) as a computation, (b) as an area, (c) as an accumulation or summation, (d) as a total change between $x = a$ and $x = b$, (e) as a function, and (f) as an abstract object. This information was used to construct a matrix (Appendix G) to summarize the use of each viewpoint within each task by each participant. The viewpoint usages on each task were recorded whether or not the participant was successful. An additional code was incorporated for those instances when the participant was unsuccessful and did not appear to use any of the six designated viewpoints. This matrix then served as a framework from which to analyze all the data together to discover major themes that emerged from the data.

CHAPTER 4

PRESENTATION OF THE RESULTS

This chapter presents the data gathered during the interview process. Each participant's "story" has been written in the form of a case. Each case starts with an introduction to the participant, which is then followed by a description of the participant's work on each of the tasks completed. The statements of the tasks are given in Appendix B. The ordering of the cases within the chapter is Joan, Lynn, Rob, Stan, and Tina.

Before proceeding to the first case, two remarks about the material taken verbatim from the transcripts are in order. The ellipsis symbol, ..., is used in the transcript material to show long pauses or breaks in the participant's flow of thoughts. Insertions from the videotapes are enclosed in parentheses and written in italics to separate them from the audiotape text.

The Case of Joan

Introduction

Joan was a 21-year-old junior pursuing a pre-dental major at the time of the study. She also indicated that she was starting a mathematics major. Prior to Calculus II, she had taken Precalculus and Calculus I at the institution where the study took place. She took Calculus I during the semester prior to the interview. Joan earned an A in Calculus I.

At the end of the third interview, Joan was asked to respond to questions focusing on her thoughts about her understanding of the definite integral. This information was used to provide insight into how she approached the tasks themselves. In response to a question about how she thought about the definite integral, Joan responded "I do think of it

graphically. That's my first thought. What does this represent? Can I draw a picture? It makes it tangible." In a remark regarding the role of symbols in her thinking about the definite integral, her comment was "I prefer graphically over symbolically, but sometimes I can grasp the idea symbolically." Finally, regarding conceptual or abstract thinking in reference to the definite integral, she stated that this "would be the one way I don't think of it."

In response to the inquiry about what idea(s) she viewed as most difficult for her to grasp about the definite integral, she provided two comments. The first one related to her perceived understanding of the definite integral. The essence of this comment was that she had thought the definite integral was a fairly easy topic, prior to participating in this study. Her other comment related to her way of thinking about the definite integral. She admitted to having difficulty

> trying to represent it [the definite integral] as anything but the area under a curve. I don't see it any other way. I just see it as the area between the curve and the x-axis. That's what it is. I have a hard time looking at it any other way.

For Joan, some of the mechanical aspects of the definite integral were the easiest ideas for her to grasp. In particular, she mentioned using The Fundamental Theorem of Calculus and Riemann sums (see Ostebee & Zorn, 1997).

Response to Task 1

Joan found the first two parts of this task to be basic calculations. Neither of these subtasks was difficult for her, and she was able to solve them correctly using The Fundamental Theorem of Calculus. The following passage from the interview provides Joan's own description of her solution method:

> Um, at first, I was taking, you know, finding the antiderivative, and then, um, plugging back in the limits, upper limit first, subtract the lower limit, after they'd been plugged in.

In this instance, Joan viewed The Fundamental Theorem of Calculus as a mechanical process rather than a condensed whole. Joan's view of The Fundamental Theorem of Calculus had not necessarily been consolidated into a single procedure.

Joan was not familiar with the greatest integer function, but she picked up the idea easily when the definition was explained. When she felt comfortable with the definition, she was asked to consider the third part of this task. Her initial reaction suggested that the task's analytical format was prompting an analytical solution path: "Well, just looking at it [the integral], my first thought is just to integrate it like I would any other problem." She proceeded to produce the following calculation:

$$\int_0^5 \lfloor x \rfloor dx = x^2\Big|_0^5 = 5^2 - 0^2 = 25.$$

As she wrote out her work, she verbalized her thoughts:

> OK, then, integrating it for the greatest integer less than or equal to five, which would be, er, putting that in, which would be 5. So, I'm a little unsure of myself, but it just seems like it would be done like a regular problem, because you've given me integers for the limits.

The last sentence of this passage suggested that she operated under the notion that, since $\lfloor x \rfloor = x$ for all integers x, then

$$\int_0^5 \lfloor x \rfloor dx = \int_0^5 x\, dx.$$

However, no reason was ever given for dropping the necessary $1/2$ from the antiderivative. It should also be noted that at no time did she initiate any steps to resolve her discomfort with the chosen solution path.

At this point, Joan was challenged to check her work with her graphics calculator. After being given instructions for entering the greatest integer function into the calculator, she proceeded to compute the integral. The results of this check produced the following exchange:

Joan: It's 10!
Todd: OK. Any idea why?
Joan: None at all. (laughs)
Todd: OK. What would be another way that you could think about it [the function] in terms of the integral?
Joan: I'm drawing a blank. Um...*(pause)*...no ideas.
Todd: OK.
Joan: Um, (laughs)...I mean, I've never seen it before, so I've never worked with it. So, um, I'm sure that makes a big difference, but, not a clue.
Todd: OK.
Joan: Um, I mean, 'cause 2 times 5 is 10, so...
Todd: Right.
Joan: (laughs) So, again, that, um....
Todd: But no other thoughts how you could have possibly approached this problem other than trying to find an antiderivative?
Joan: The only other...um...the only other way I know to describe a definite integral is the area under the curve, so...
Todd: OK. Might that be a...is that a possibility?
Joan: It was, well, I wouldn't know how to graph it.

The results of this "answer-in-the-book" check produced disequilibrium within Joan. She tried to resolve the discrepancy between answers, but initially had no ideas. She attributed her lack of ideas to not having seen this function before. However, the 2 times 5 is 10 comment that followed showed she was still searching for a resolution. This continued search eventually led her to the notion of area under the curve, but she could not apply this idea because she could not graph the greatest integer function. At this point, she was told what to enter into the graphics calculator. Joan then graphed the function (see Figure 4). She applied the area model to conclude that the answer from the graphics calculator made sense.

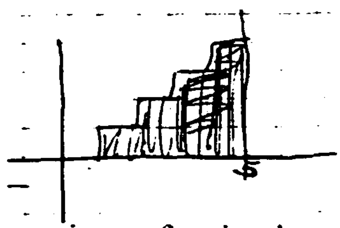
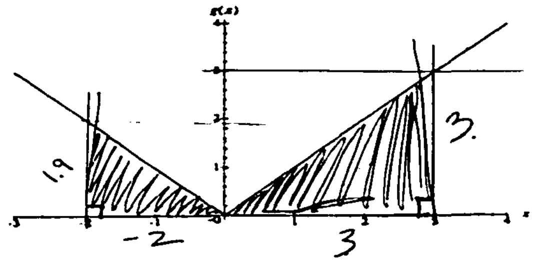


Figure 4. Joan's graph of the greatest integer function that was used to compute the definite integral $\int_0^5 \lfloor x \rfloor dx$.

Response to Tasks 2 and 3

In Task 2 Joan applied the area model to compute the requested definite integral (see Figure 5). In particular, she summed the areas of the two shaded triangular regions. Her choice of solution path appeared to be influenced by the geometrical nature of the task, as she did not consider an analytical solution to the task.



$$A = \tfrac{1}{2}bh$$
$$A_T = A_1 + A_2$$
$$A_T = \tfrac{1}{2}(-2)1.9 + \tfrac{1}{2}\,3\cdot3$$
$$-1.9 + \tfrac{9}{2} \approx 2.6\,.$$

Figure 5. Joan's evaluation of the definite integral $\int_{-2}^{3} g(x)\,dx$.

When questioned about her calculation for the area in Quadrant II, she said, "Just from what I've learned, you subtract a negative area, and that would make that area negative because it's in the second quadrant." Her statement indicated that she was familiar with the idea of negative area, and that she had transferred this notion to Quadrant II, where there were negative coordinates involved. Thus the negative x-values acted as a signal for the negative area image from memory.

When Joan was presented with Task 3, she disregarded Task 2. Initially, she did not recognize that the two tasks were identical. She used her graphics calculator to carry out the integration because she has "always had a phobia of absolute values," and the graphics

calculator allowed her to address this phobia. Even though the graph for $y = |x|$ was in plain view, she employed a non-geometric solution. Thus the analytical statement of the task seemed to influence how she approached the task.

When the calculator produced the answer 6.5, she reacted by stating, "so it looks like they added these areas, rather than subtracted it," as she referenced her work in Task 2 concerning the triangle in Quadrant II. This reaction was like that of a student checking an answer in the back of the book only to discover a disagreement. For Joan, there was immediate disequilibrium, and she tried to resolve the conflict. It was at this point that she said, "I'm assuming this is a graph of the $|x|$ *(pointing to the graph in Task 2)*. It looks familiar." Therefore, she must have realized that, since the two tasks asked the same question, the answers should be the same. It was possible that Joan may have noticed, but not stated, that the sum of the values for the areas of the triangles in Task 2 was the same as the calculator's answer. Joan provided another possible connection when she said,

> Um, I haven't looked at definite integrals this in-depth for a while, so, um, I was kind of confusing it with the area under the x-axis is negative...considered a negative area, and that would be subtracted. So I was just kind of mixing those two up; but, above the x-axis it's positive, below the x-axis it's negative.

This statement revealed that she had not looked at the definite integral in detail since the previous spring semester when she took Calculus I, and she had confused the situation in Quadrant II with the situation in which a curve is below the horizontal axis. After she found the proper image, she concluded that the areas of the two triangles should be added to attain the proper result. Joan also took an opportunity to clarify the issue of when negative area makes sense and when it does not make sense.

Response to Task 4

Joan started by finding the areas of the triangular region and the non-triangular region individually using geometric area formulas. She then summed the results to obtain

the total area of the region (see Figure 6). In her work, A_T was for the area of the region above the x-axis, A_1 was for the area of the triangular region associated with the closed

$$A_T = A_1 + A_2$$
$$\tfrac{1}{2}3(1.5) + \tfrac{1}{4}\pi 3^2$$
$$\tfrac{3}{2}(1.5) + \tfrac{9}{4}\pi$$
$$\tfrac{3}{2}\cdot\tfrac{3}{2} + \tfrac{9\pi}{4}$$
$$\tfrac{9}{4} + \tfrac{9\pi}{4} \approx 9.3185$$

Figure 6. Joan's area approximation using geometrical area formulas.

interval $[1,4]$, and A_2 was for the area of the non-triangular region associated with the closed interval $[4,7]$. In her work she assumed the latter region was a quarter of a circle with radius 3. When questioned about her assumption, the following exchange took place:

Joan: 'Cause it's...it's easy to assume it's a circle. We've got a formula for that.
Todd: But, what's telling you that that [the second region] should be a circle? Or a quarter circle?
Joan: Actually, nothing. It's because...um, the radius isn't equal. (laughs) It's not...you've got a radius of 3, whereas the height of it here is, um, 1.5. So it's not a true circle, but the reason, the...by assuming it's a circle, you can find...you can estimate the area. A second approach would be to use the Riemann sum.

The first part of this exchange was interpreted to mean that she decided the non-triangular region was a quarter circle because there is an area formula for a circle. Also, she might have assumed that this region was a quarter circle because of a perceived likeness to a circle. The second part showed that she recognized that this region was not a quarter circle because the "radii" were not equal. The remainder of her second statement showed how she tried to resolve the disequilibrium that had arisen. The first attempt was to use the area formula for a circle to estimate the amount of area enclosed by the region. The second attempt,

using the Riemann sum, could be the result of a continued search for a better approach and further resolution to the internal conflict that her initial idea produced.

The Riemann sum approach was pursued briefly in order to gauge what she was thinking at this point. Her approach could be summarized in the following way (see Figure 7 for graph): She partitioned the interval into 12 subintervals of length $\Delta x = 0.5$, and then

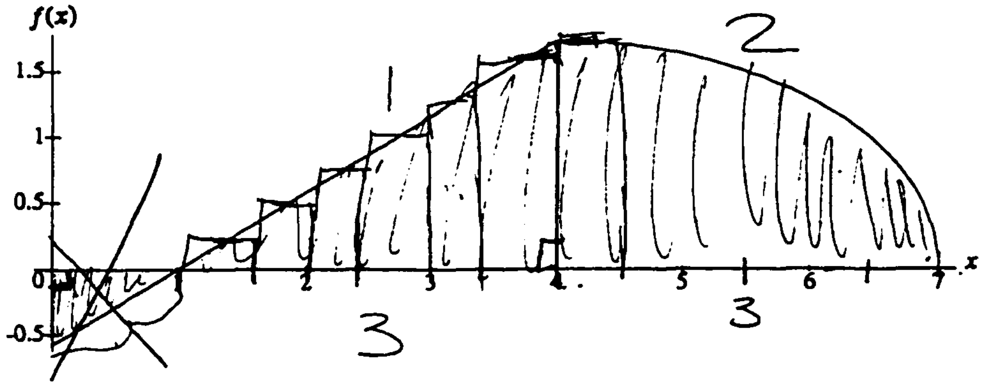


Figure 7. The diagram showing Joan's set-up for her Riemann sum calculation.

selected the midpoint of each subinterval to determine the height of each of the rectangles that she had drawn onto the diagram. Once some sample rectangles were drawn, she continued her summary by saying, "you find the area of each rectangle, and, um, you add those together. And that gives you an estimate of the area under the curve." This apparently provided a resolution to her earlier disequilibrium. However, nowhere in the description of her Riemann sum process did the idea emerge that taking the limit of her summation would yield the actual area. She appeared to be content carrying out the approximation.

When asked what she would choose, if she could have any missing piece of information, she replied that she would like to have "the formula for the function." She went on to say that if she had the formula, she "would, um, find an antiderivative and, um,

and then evaluate that from 1 to 7." At this point she appeared to believe that the function could be expressed as a single expression, and she would just apply The Fundamental Theorem of Calculus to it. After being shown the piecewise definition for the function *f*, she was able to set up the necessary integrals (see Figure 8) and apply the summation property

$$A_1(x) = \int_1^4 \frac{\sqrt{3}}{3}(x-1) \qquad A_2(x) = \int_4^7 \sqrt{\frac{9-(x-4)^2}{3}}$$

$$A_T = A_1 + A_2$$

Figure 8. Joan's sum of definite integrals to compute the area of the region.

of the definite integral to represent the area of the region. However, she failed to include the differential dx in her two integrals. It was possible that she simply forgot to include it. Because she evaluated the definite integrals using her graphics calculator, the absence of the differential did not influence her response. Joan evaluated the integral, using her graphics calculator, to arrive at an approximate area value of 6.671 for the region. She commented that by using a circle for the nonlinear region, she had made the "area larger than it actually was."

Response to Task 5

Joan took an interesting approach to solving this task (see Figure 9). She integrated and expressed the function G as a nonintegral function of x. She then substituted the value 5 for x just before she computed the total change occurring in the antiderivative, even though she acknowledged that she "would be finding it [the definite integral] with respect from 0 to 5." When asked why she took this approach, she replied by saying, "I don't know…it just helped me to keep it that way and take it step by step, and then, you know, the final step would be letting $x = 5$." This was followed a little later by the comment, "I just saw it as

$$G(x) = \int_0^x (\lambda-1)^2\,d\lambda \qquad u = \lambda - 1$$

$$du = d\lambda$$

$$\int_0^x u^2\,du$$

$$\frac{1}{3}u^3\Big|_0^x$$

$$G(x) = \frac{1}{3}(\lambda-1)^3\Big|_0^x$$

$$\frac{1}{3}(\lambda-1)^3\Big|_0^5$$

$$\frac{1}{3}(5-1)^3 - \frac{1}{3}(0-1)^3$$

$$\frac{1}{3}(4)^3 - \frac{1}{3}(-1)^3$$

$$\frac{1}{3}\,64 + \frac{1}{3}$$

$$\frac{64}{3} + \frac{1}{3} = \frac{65}{3}$$

Figure 9. Joan's evaluation of the integral function G.

one more step, so I put it off as long as possible, and then I did it when I felt like I needed it." These statements indicated that the definition for the function G evoked her Fundamental Theorem of Calculus procedure, and therefore she followed through by integrating. When the time came to evaluate the antiderivative at the upper limit of integration, she was able to replace the x with the value of 5. The substitution could not occur earlier because it would have interfered with her perceived need to integrate first; she could not deviate from her established procedure until she began the evaluation stage of The Fundamental Theorem of Calculus.

Joan decided to integrate with the substitution method because this technique was on her mind. The week prior to the interview, her Calculus II section had covered finding antiderivatives using substitution. Thus, her Calculus II course influenced her solution to

the task. However, she did not carry out the substitution method properly, as she did not switch the limits on integration to align with her new variable.

After Joan had finished evaluating the integral function, she was asked to describe her reaction to having both x and λ in the definition of the function G. She provided the following response:

> Well, my first reaction as I was looking at the λ going, oh, what the heck is that, you know, just kind of an initial thought. Um, my second one was the integral is...seems to be the function with respect to x, but then we've got an integral with respect to λ, so it kind of confused me. (laughs) And it's like, why would anyone do that? Then when I started the computation, it just...um...I'm looking at that we have the limits. The only place to put them in would be for λ. So I'm still not sure why it was in...originally in two variables. I guess so that you can change the upper limit if you needed to?

The first part of the passage indicated she was initially perplexed by the situation. This was followed by a struggle to sort out the roles of each variable. She inferred that she saw x and λ as having different roles to play in the definition of the function. From her perspective, x was the variable for the function G and allowed for varying of the upper limit of integration. Responding to a follow-up question, she indicated that, since x was representing the upper limit of integration, it would be a number. She saw λ as being the variable over which the integration took place and as the variable for which the limits of integration would be substituted. However, she was not extremely confident about her response.

Finally, Joan was asked why it made sense to view G as a function of x. She responded in the following manner: "It's one of those things that I've always done. I mean, um, that's notation representing functions, that was my understanding, it's what...it's an assumption, I guess." She assumed that G was a function because function notation was used in its definition. When challenged to speculate about how one could show that G was actually a function, she replied, "the easy way is to graph it." However, she did not know of any way to actually carry out her idea.

Response to Task 6

Joan had to re-read part a) several times in order to absorb and interpret the given information. Once she finished this, she provided the following initial observation: "Now, as x moves to the right, the area is going to increase, because we're still adding on more area, looking at it geometrically." When asked to elaborate, she used both pointwise and across-time notions to convey her thoughts and ideas. She said that "if x can equal 0, then your area would be equal to 0." As x moved to the right along the t-axis, she tried to express the idea that more and more area was being accumulated under the curve, as shown by the following exchange:

Joan: We're...we're constantly adding on area. I mean, we can't possibly lose area. As long as we're moving to the right, we've still got more and more...
Todd: OK.
Joan: ...stuff under the curve.

The phrase "we can't possibly lose area" meant that, since the graph of the function was above the t-axis, there was no negative area to take away from the total value. This agreed with a remark she made while first looking at the task, when she observed that there was no negative area involved in the task.

When asked what would ultimately happen if x was allowed to continue to move to the right, Joan summed up her thoughts as follows:

Joan: Um, as x goes to infinity, the area goes to infinity.
Todd: OK.
Joan: It's constantly increasing. Um, it's going to grow slower, because we've got...I don't how to say this. Um, this distance between the graph and the x-axis is, um, smaller than it was here *(points to a part of the graph closer to the y-axis)*, so we're gonna have a slower growth rate, but it's constantly gonna grow, as long as x moves to the right.

She appeared to be trying to express the idea that the area under the graph of $y = f(t)$ and above the t-axis was increasing at a decreasing rate as x goes to infinity because the distance between the graph and the t-axis was getting smaller as x continued to the right. In order to see how firmly she held the belief that as x goes to infinity, the area goes to infinity, Joan

was asked to determine the area of the region bounded by the t-axis and a curve $y = f(t)$ from $t = 0$ to $t = x$ for various values of x with two different functions.

The first of these functions was the function

$$f(t) = \frac{1}{t+1}.$$

Joan used the integration feature of her graphics calculator to construct the following table of x-values and corresponding area values (see Table 3). As she was finishing the table, she

Table 3.

Joan's Table for Area Under the Curve $f(t) = \frac{1}{t+1}$

x	Area
0	0
10	2.39
100	4.615
100,000	11.513
10,000,000,000	23.0259

made the comment, "it's constantly growing, like...I assume." This appeared to refer to the fact that the area values were continuing to grow as the value of x continued to increase. She believed that in this case the area would grow to infinity as x went to infinity.

She then considered the function

$$f(t) = \frac{1}{(t+1)^2}.$$

She again used the integration feature of her graphics calculator to construct another table (see Table 4). Joan had difficulty explaining what was happening in this example, but eventually made several statements of the following sort: as x moved farther and farther to the right the area is "still increasing but at a much slower rate than, um, our area was growing with the function $1/(t+1)$." After further efforts, she finally made this connection to what was happening to the area values in this example: "It's [area] approaching 1."

Table 4.

Joan's Table for Area Under the Curve $f(t) = \frac{1}{(t+1)^2}$

x	Area
0	0
10	0.9090
100	0.990099
100,000	0.99999
10,000,000	0.99999900

One possible explanation for why she struggled so much in reaching this conclusion was that she was focusing so exclusively on the increasing area values, and the expectation that these values should be going to infinity, that she was not seeing what really was taking place. She had to override her expectations before she could expand her concept image to include the possibility that the area could approach a finite value.

It was shortly after she discovered that the area approached the value 1 in the second example that she began to realize the significance of what had just happened. She was in disequilibrium over what had taken place, and tried to find an explanation. In an attempt to find a resolution, she turned to her graphics calculator and graphed the two functions (see Figure 10). She made several attempts to explain why the two functions produced such

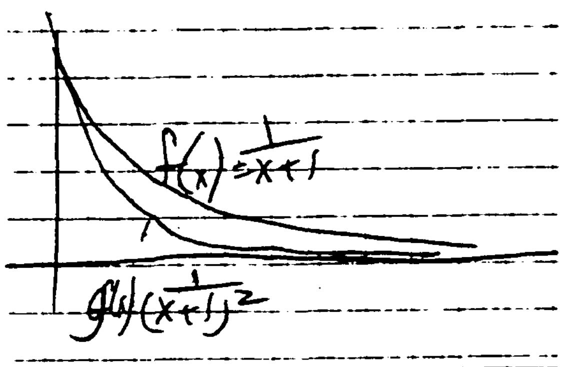


Figure 10. Joan's graphs for comparing the functions $f_1(t) = \frac{1}{t+1}$ and $f_2(t) = \frac{1}{(t+1)^2}$.

different results. Below is her best attempt at such an explanation:

> Well, the first one [f_1] does, um...*(pause)*...because it doesn't, um...what I'm seeing is because the graph doesn't approach zero as quickly, so you can actually get values above one, and then the second one [f_2] does not, I guess, because it's hugging the x-axis. That's not anything solid, but that's what I'm seeing.

Even though her explanation focused on the rate of decay in the two functions and the possible effect on the areas, she was not convinced that this was reasonable. The whole idea seemed strange to her, but she did seem to be less sure about her original conjecture. However, no real resolution took place. The second example appeared as an exception to the rule.

The following passage summarized Joan's thoughts concerning the second part of this task:

Joan: OK. *(Reads the task out loud first.)* And, right now, I'm, again, just rereading the questions to make sure I know what I'm looking for. *(long pause)* Isn't this the same question just worded a different way? (laughs)
Todd: Why are you saying it's the same question?
Joan: Um, well, the way I'm reading this, um, according to, um, the area function, by using a definite integral, this $g(x)$ [*sic*] is just the, uh, formula for the area under the curve $f(t)$, and bounded by these limits, 0 to x. So the way I'm reading it, it looks like I'm looking for the same thing. Um, as x increases, what's happening to the area under the curve. That's how I'm reading it. So, um, I would answer it the same way, just by looking at the graph. I would still, just by looking at the graph, with no numbers on it, I still would assume that the area goes to infinity, so therefore, $g(x)$ [*sic*] would go to infinity as x moves, uh, as x moves to the right.

Note first that Joan realized that the two parts were asking the same question: the first part in terms of area and the second part in terms of the definite integral. In other words, the two notions, area and the definite integral, were equivalent in this setting.

Two additional notes were particularly interesting from the above passage. The first of these was the beginning of Joan's second statement. This section indicated that she associated the function G from this task with the area function A_f defined in Ostebee and Zorn (1997, p. 357) as "$A_f(x) = \int_a^x f(t)\,dt =$ signed area defined by f, from a to x." This

connection was most likely made because of the similarities between the functions G and A_f. The latter part of her second statement illustrated that she was still not ready to modify her original conjecture that the area will go to infinity as x continues to move to the right. However, it also revealed that she was not as sure of herself as she had been earlier.

Response to Task 7

Joan's original response to part a) of this task can best be summarized by the following quote she gave while clarifying her initial response:

> If you start at 0, and you go...well, if it equals 0, the area is 0. OK, then as you move to this point, to here *(points to the t-intercept)*, it becomes...the area's positive. Then as you move beyond this point, um, we start subtracting this area, because it's below the x-axis [*sic*]. Then it becomes smaller, and then at some point I assume it'd be...it would be negative, and then become more and more negative as x goes to infinity.

Based upon this passage and other comments that she had made, she viewed the area of the region as beginning at a value of 0 and increasing in value until x reached the t-intercept. Once at the t-intercept, the area would start to decrease, because the region was now below the t-axis. She viewed the area of the region below the t-axis as eventually being greater than the area above the t-axis, and so the area of the region would then be negative. Finally, as x continued to move to the right, the area of the region would become more and more negative. Thus, she viewed the first part of the task as asking for the signed area.

When Joan was asked about how area was viewed geometrically, she replied that "areas are supposed to be positive." However, the following passage suggested that, when the coordinate system is involved, the situation had to be treated differently from how it would be treated in a geometrical setting.

> This is calling upon knowledge that, again, is somewhere in the back of my mind. Below the curve we have negative values for the function, so we consider it a negative area, a term used by an instructor of mine, and we subtract that from anything above the x-axis [*sic*].

Her usage of the phrase "negative area" indicated that she knew how the idea worked and that it was an important idea. However, it was reasonable to conclude that she had not made a distinction between the notions of area and signed area. In the end, she stayed with her original answer, even though she knew areas should be positive. Apparently, the connection between the coordinate system and the notion of negative area was too strong for her geometric knowledge to introduce a cognitive conflict.

When Joan considered part b) of the task, she saw the function G as "just a formula for finding the area under the...bounded by the curve and the x-axis [*sic*], with a lower limit 0 and the upper limit, um, start...beginning at 0 and increasing." She then gave a description about what would happen to the area as x moved to the right that was similar to the response she had given in part a) of this task. When specifically asked, she confirmed that she did not perceive a difference in what part a) and part b) were asking her to do.

Some follow-up questions regarding this task were asked at the beginning of the second interview. Joan confirmed that she still believed her original answer for part a) was true. When asked what would happen if the coordinate system was ignored, she had to stop and think. She was not sure, but said,

> You'd add the two together, and you'd come up with a total area. Take...find the area of this region *(points to region above the t-axis)*, find the area in this region *(points to the region below the t-axis)*, and then add the two together, so you'd get a total area. Um, and then as...x or t or...yeah, this is x as you have it written...um, as it continued to the right, it would continue to get bigger.

Thus, in this case the areas of the two subregions would both be treated as positive, and the sum would be positive and increase in value as x moved to the right. When the coordinate system was returned, Joan saw two possibilities: her new idea that all the areas were positive and her original idea that the region below the t-axis was to be treated as having negative area. She had great difficulty deciding which of these two ideas fit part a) because

both ideas made sense to her. She finally decided to stay with her original idea that part a) was about signed area.

Joan was then asked what would happen in part a) if the phrase "area of the region" was replaced by the phrase "total area of the region." This question produced the following exchange:

Joan: Oh, then you would add the two together, rather than subtract the lower area from the upper, you would add the two regions together, and then you'd have the total area.
Todd: When you're adding them together, would this *(pointing to region below the t-axis)* be considered positive? Negative?
Joan: Oh, positive.
Todd: OK.
Joan: It'd be like...like geometrically speaking or graphically, you'd find this area *(referring to region above the t-axis)* as if it were a geometric figure, and find this area *(referring to the region below the t-axis)* as if it were a geometric figure, you know, and then just put...add the two together.

Joan's notion of total area corresponded with the actual usage of the word area, whereas her usage of area corresponded to the notion of signed area. She viewed total area and area as being two different issues within the context of this task.

Joan was then asked if there was a difference between parts a) and b) as originally stated in the task itself. She responded by saying: "I don't believe there's any difference. I mean, um, graphically speaking, I understood that the definite integral was represented by the area under, bound by the graph and the x-axis, so, um, I think they're the same thing." She saw the two parts of the original task as asking the same question. The reason for her belief might stem from her understanding of the definite integral as the area bounded by the graph and the t-axis, in this case, even though she had associated signed area with the calculation of the definite integral. It should also be noted that the beginning of her response indicated that she might be starting to question her original ideas about the connection between the two parts as stated in the task.

Finally, she was asked if there was a difference between part a) if the phrase total area was used and the original part b). This produced the following passage:

Joan: That's different, because, um, total area, you're not taking into account negative values or values below the axis. Um, you're just assuming everything is positive, and taking this area...the area of this region *(region above the t-axis)* plus the area of this region *(region below the t-axis)*. And again, you're not taking into account the coordinate plane.
Todd: And with the definite integral...?
Joan: You are. It's, uh, it's a coordinate system, so you've got positive values, negative values, and that affects your answer.

In this setting, the two parts were asking different questions. Part a) asked for the geometrical area, and thus all the areas were viewed as being positive. Part b) was still viewed as finding signed area, and thus there was a need to account for the locations of the two regions relative to the t-axis. On this point, she was very certain about the distinction between the two parts.

Response to Task 8

Joan immediately concluded that

$$\int_5^5 h(x)\,dx = 0$$

because "you start there, you stop there, it's the same place, so there's no width." She then proceeded to use the jump discontinuities to divide the whole region into three subregions corresponding to the closed intervals $[0,5]$, $[5,8]$, and $[8,11]$ (see Figure 11). Once this was completed, she rationalized that

$$\int_0^5 h(x)\,dx < 0.$$

Although she did not provide a reason, it was assumed that she drew this conclusion based upon the location of the graph on the closed interval $[0,5]$. Joan also said that

$$\int_5^{11} h(x)\,dx > 0$$

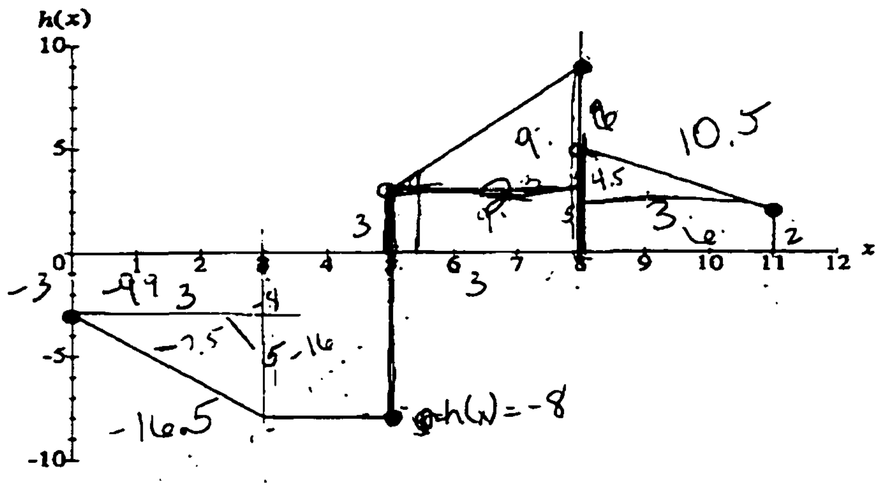


Figure 11. Joan's graphical work to determine values for the definite integrals $\int_0^{11} h(x)dx$ and $\int_3^8 h(x)dx$.

because "we see that both regions are positive." She decided to approximate values for the remaining two integrals, since both integrals involved regions where the graph was above the x-axis as well as below the x-axis. She accomplished this by dividing each subregion into triangles and rectangles and then applying the respective area formulas (see Figure 11). Joan determined that

$$\int_0^{11} h(x)dx = -4 \text{ and } \int_3^8 h(x)dx = 2.$$

She then proceeded to order the integrals from smallest to largest with the following rationale:

> So from smallest to largest...again, we can add these up *(referring to the two parts of subregion 1)*. We know that...just by looking at the values they're obviously, um, more negative than this one *(points to part a))*, so this one's the smallest *(part c))*. So c) is smaller...smallest, then, a), because it's also negative...d), because it's 0...e), and then b). So then from smallest to largest is c), a), d), e), b).

The integral associated with part b) was placed as the largest because "it's all positive." This was interpreted as referring to the fact that the region associated with this integral was completely above the x-axis.

At this point, some comments are in order regarding Joan's work on this task. She worked the entire task in terms of signed area. She did not find any of the expressions for the line segments in order to integrate symbolically. The approach she used to approximate values for those integrals she evaluated illustrated a geometric understanding of the summation property of the definite integral. She was able to apply the summation property to $\int_5^{11} h(x)\,dx$ to obtain:

$$\int_5^{11} h(x)\,dx = \int_5^{8} h(x)\,dx + \int_8^{11} h(x)\,dx.$$

It was presumed that she obtained this statement from the graph itself. However, when asked to explain why this was true, all she could give for a reason was that "it's a theorem or something."

During the investigation of Joan's understanding of the summation property, the role of the jump discontinuity at $x = 8$ arose. For Joan, this jump discontinuity would have "little numerical effect on the definite integral" over the closed interval $[5,11]$. Her explanation for this conveyed the idea that the height of the second trapezoid on the closed interval $[5,11]$ was infinitely close to the height it would have if the open circle at $x = 8$ was changed to a closed circle. Therefore, the difference between these two heights would be negligible. Joan appeared to be viewing the left side of the second trapezoid as being located infinitesimally close to the vertical line $x = 8$. Evidently, she did not see the vertical line $x = 8$ as a boundary line for both of the trapezoids on the closed interval $[5,11]$ and that

therefore the height for the second trapezoid would be determined by the y-coordinate of the open circle at $x = 8$.

Response to Task 9

This task was easy for Joan to complete. Based on the rapidity, in which she completed her sketch, she easily interpreted the situation in terms of area. Also, the reversal of the usual integral calculations did not seem to distract her from completing the task. She used the notion of signed area to complete her graph (see Figure 12). Her scaling on both

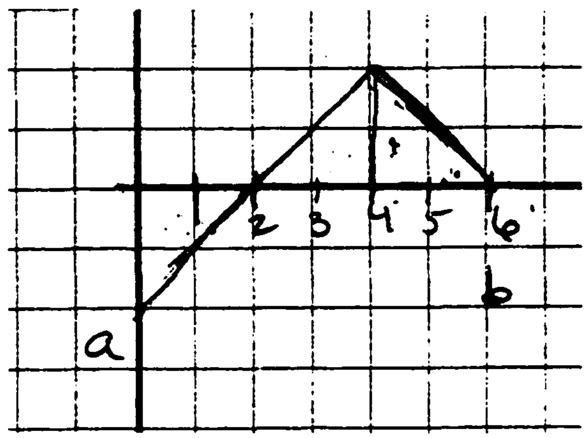


Figure 12. Joan's graph of a function satisfying the integral equation $\int_a^b f(x)\,dx = 2$.

axes was one, and she integrated over the closed interval $[0,6]$. When asked why she decided upon this particular graph, she said:

> 'Cause I have a graph of my choice, so I chose. Um, uh, just to be interesting. I mean, I could have done anything. I could have just done an area like this *(points to the last part of her graph)*. Or I could have tried to do a curve. Or I could have just done that *(one unit long segment at $y = 2$)*, and then the area equals 2. But I chose that one just to kind of experiment with a negative and a positive area.

Within her response she acknowledged that there was more than one answer to this task. She also illustrated that she could view the definite integral as more than just area, but as signed area.

When asked what her thoughts were while she constructed her graph, Joan said:

> I still have two little squares, total, that, um, I can consider into my area. Well, I have more than two squares, but because this is negative and this is equal to that, then they cancel each other out, so I have a total of two squares.

Her usage of "total" in this response was more consistent with the notion of signed area than with area. Thus her entire goal was to create a graph in which there was a net of 2 units of area, where a square was her unit of area.

This task appeared on the participant selection tasks (see Appendix E). Joan's response (see Figure 13) at that time also employed the area model. However, she did not

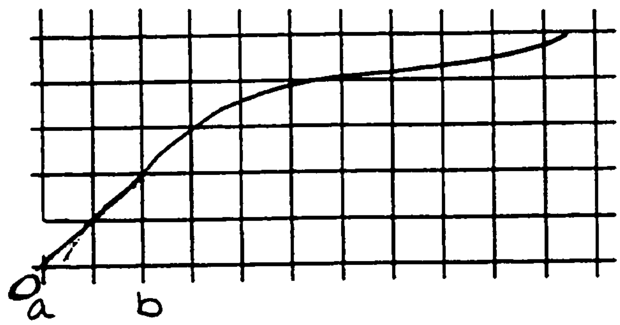


Figure 13. The graph that Joan sketched during the administration of the participant selection tasks.

evoke the idea of signed area in that sketch. Thus, she consistently viewed this task in terms of the area model. Usage of signed area might be considered indicative of growth in her understanding, or that a different part of her concept image was evoked during the interview process.

Response to Task 10

This was the first task that Joan was not able to complete. She made seven attempts at solving this task before she admitted to being stuck. During these attempts she used both continuous and discontinuous piece-wise defined functions. Her discontinuous functions

involved jump discontinuities. She admitted that piece-wise defined functions might be useful for this task, and for her they are "easier just to see geometrically than like a curve."

Joan knew what the task was asking her to do and was able to describe what each of the integrals was asking. She saw the first integral as requiring that the "integral of the function itself equals 2." The second integral required that "if you take the absolute value of the function it [the second integral] equals to 4," by which she appeared to mean the integral of the absolute value of the function. She also said, "if you take the absolute value of the area which would be, in other words the sum of the two parts, the total area, it would be 4." Thus she viewed the first integral as asking for a signed area of 2 and the second integral as asking for an area of 4, although she continued to use "area" in place of "signed area" and "total area" in place of "area" as she talked about her work and thoughts.

Joan's sixth attempt at drawing a graph to satisfy the given equations (see Figure 14) was her best attempt. This attempt was a modification of her fifth attempt. The fifth

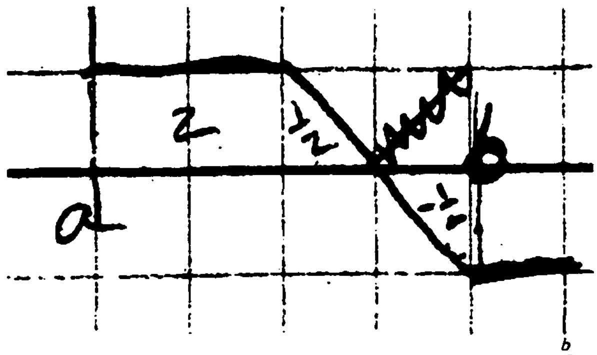


Figure 14. Joan's best attempt at satisfying the integral equations $\int_a^b f(x)\,dx = 2$ and $\int_a^b |f(x)|\,dx = 4$.

attempt was on the closed interval $[a,b]$ as indicated on the graph, and the scaling on both axes was 1. The crossed-out segment of the graph was part of her demonstration for the $|f(x)|$ in the fifth attempt. Once she realized that the fifth attempt did not work, she added the horizontal segment below the x-axis to create her sixth graph. What she did not realize was that, if she had used her additional horizontal segment below the x-axis along with the crossed-out segment instead of the corresponding reflected segment below the x-axis, she would have satisfied the two conditions. She might have failed to consider this possibility because she associated the crossed-out segment with the $|f(x)|$.

In six out of the seven tries to solve this task, Joan was able to satisfy one of the two criteria. In the remaining attempt, she believed that she had satisfied one of the criteria, but she actually had miscalculated the value for the second integral. In all of her attempts she used area ideas to guide her creation of a function. However, it became apparent that she was focusing on satisfying one of the criteria and then hoping the second one would be satisfied in the process. It did not occur to her to consider the two criteria simultaneously. This might be the reason that she was unsuccessful with this task.

Response to Task 11

Joan started this task by sketching a graph (see Figure 15) for f using the two given

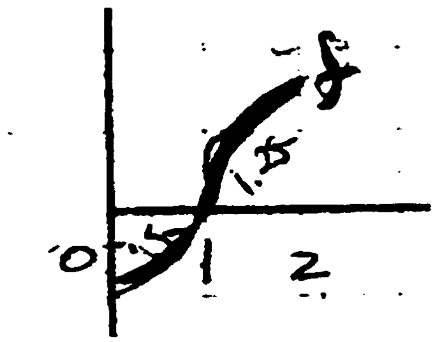


Figure 15. The graph Joan sketched to find the value of the definite integral $\int_0^1 f(t)\,dt$.

integrals. She admitted to creating a graphical representation to help her make sense of the task and to generate ideas. In the process of completing her graph, she used a graphical version of the summation property of the definite integral. It was during her construction of a graph for the function f that she announced "I've found this part, the $\int_0^1 f(t)dt$ has to equal -0.75." However, given the opportunity to explain her graphical approach, Joan wrote out the summation property of the definite integral for the integral $\int_0^2 f(t)dt$. She then solved algebraically for the integral $\int_0^1 f(t)dt$. In the process, she realized that she had made an error and corrected it to obtain

$$\int_0^1 f(t)dt = -0.5.$$

She was unable to make any progress on finding values for the antiderivative F evaluated at 0 or at 2. She commented while searching for ideas that:

> I have the worst time with, like, imaging an antiderivative without having some fun[ction], something to look at that, you know, like f equals x^2. Without having an equation. I kind of don't know where to go with this right now.

This was interpreted as an admission of having difficulty working with a general antiderivative at this stage of her learning. Thus, the lack of a particular antiderivative could be a cognitive obstacle for Joan at this stage. After she had considered the situation a while longer, she admitted, "I don't know really... Really don't have a clue." At this point we moved on to the next task.

At the beginning of the third interview, Joan was asked about computing values for $F(0)$ and $F(2)$ in order to see if she might evoke a different image or provide more insight into why she had trouble with this part of the task. She was unable to make any progress in

computing the requested values, but she amplified an idea that she had mentioned in passing during the second interview. The basis for her idea revolved around having a more accurate version of the graph shown in Figure 15 and that f was the derivative of F. She described her thought in the following manner:

> I could...um...find, like, the slope at 1 of this *(points to the graph of f)*, or the value of 1 at this *(points to the graph of f)* would be the slope here *(refers to the F)*, and then approach it that way. Then I'd maybe be able to find the values for $f(0)$ and $f(2)$ [*sic*] *(meaning* $F(0)$ *and* $F(2)$*)*, but there's more than one possibility of having these values for the integrals, so I couldn't be absolutely sure of my values for $f(0)$ and $f(2)$ [*sic*] *(meaning* $F(0)$ *and* $F(2)$*)*, so...

She seemed to be indicating that she wanted to create a slope field for F using the values from the graph of f as the slope-values. The latter part was interpreted as saying that she perceived that there was more than one integrand f that could have produced the integral values used in this task, so the values for $F(0)$ and $F(2)$ would not be unique. She focused on the relationship between f and F in terms of differentiation rather than in terms of the definite integral as total change. Once she focused upon the differentiation aspect, she was unable to see the connection to the definite integral. This suggests that her knowledge was compartmentalized and there were no links between the compartments. Thus, once she focused on a particular feature and was drawn into its compartment, she was unable to see out into other compartments in order to find other connections.

Response to Task 12

After she had read the task, she plotted the data points with time on the horizontal axis and harvest rate on the vertical axis (see Figure 16). She completed her graph by connecting the data points with a smooth curve. On her graph, she did not allow herself enough room to plot the point (30,14), and so left the gap at the top of the graph to represent this point. Also, she initially plotted the points (40,12) and (50,12) incorrectly.

She drew the curve out of habit, and generated ideas for a plan from it. Later, she acknowledged that her curve was not the only possible curve for the data.

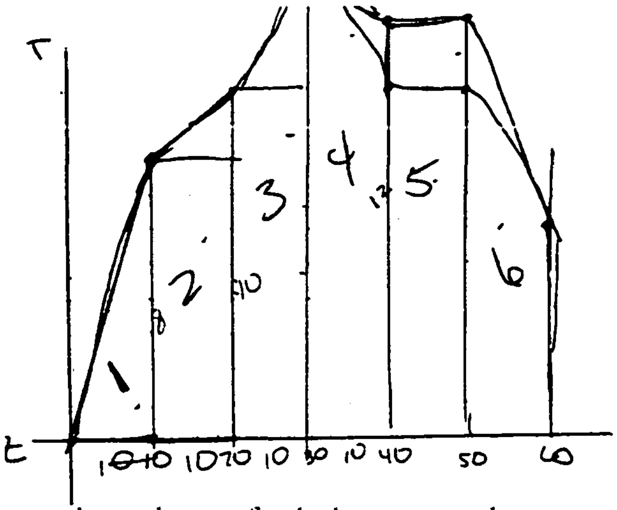


Figure 16. Joan's scatter plot and curve for the harvest rate data.

Once Joan completed her graph, she said, "this gives us the rate, and I'm thinking... I think that we can find the, uh, total number of bushels by finding the area under this rate curve." She then went on to associate the harvest rate with velocity, and the total number of bushels harvested with position. Her ideas were summed up in the following statement:

> So if we consider, like, his...how many bushels he has, his position, then, um, we can say that the area underneath the curve, because this is his speed or velocity of picking bushels, would be this total number of bushels.

It appeared that she was trying to relate the situation presented in the task to the idea that the area of the region bounded by the horizontal axis and a curve representing a velocity over some interval represents the displacement that takes place over that interval. It could also have been the case that she was recalling the following fact from Ostebee and Zorn (1997): "Integrating f' (the rate function) over $[a,b]$ gives the change in f (the amount function) over the same interval" (p. 371). In fact, at a later point during this task, she came back to

this notion and mentioned that she "could apply a polynomial to the curve, and then integrate it that way from 0 to 60" to find the actual area under the curve. Thus, she extrapolated the idea of area under the curve from her graphical representation. However, she never tried to carry out this idea as she went on to "estimate it [the bushels harvested] by geometric estimation" techniques.

The technique she employed (see Figure 17) involved using area formulas for

region $A = \frac{1}{2}(10)(8) = 40$

r_2 $A = \frac{1}{2}(10)(8+10) = 5(18) = 90$

r_3 $A = \frac{1}{2}(10)(10+14) = 5(24) = 120$

r_4 $A = \frac{1}{2}(10)(14+12) = 5(26) = 130$

r_5 $A = 120$

r_6 $A = \frac{1}{2}\, 10(12+6) = 5(18) = 90$

590 bushels

Figure 17. Joan's area calculations to estimate the total harvest.

triangles, rectangles, and trapezoids to estimate the total number of bushels harvested. Upon closer inspection, her approach was the Trapezoid Rule, a technique shown in Ostebee and Zorn (1997) for approximating definite integrals. Once she finished finding the approximate total number of bushels harvested, she outlined another approach for estimating the harvest. For this approach, she viewed the graph as "more like a step function", and then found the number of bushels harvested by carrying out a left sum approximation. She illustrated part of her step function on the graph presented in Figure 16. Overall, her approach to the task involved summing up areas.

Finally, Joan was asked what could be done to improve her approximation. She responded that "if I kept it like a curve, then I could make my intervals smaller," to form trapezoids with narrower bases. She expanded on her response by saying, "the more

intervals I have, the closer to the shape of the curve I've got...of like my little geometric figures, the closer to the exact number of bushels he picked, I get." This meant she would use finer partitions to more closely approximate the area under her harvest curve with trapezoids. The area represented the number of bushels harvested by the farmer. She alluded to the notion that including more trapezoids into the region would make her approximation for the total harvest more accurate. Thus she hinted at the definite integral as a summation process, but she did not provide any insight into whether she saw the total harvest as the limit of a summing process.

Response to Task 13

Joan admitted to having a hard time imagining the context for this task. She also had a hard time accepting the notion that the density could vary with the length of the rod. She did draw a picture of a rod, but never referred to it afterwards. Her initial idea revolved around her view that the density was constant and, as such, the mass should then be $\rho(100)\cdot 100$. She used a formula of the form rate times length to obtain the mass.

Once Joan had finished with her response and appeared to be finished with the task, she was asked about the role of the definite integral in this situation. She decided to reconsider her response and, in particular, her idea that the density was constant. She drew a generic graph for $y=\rho(x)$ (see Figure 18) to aid her thinking. After a very long pause,

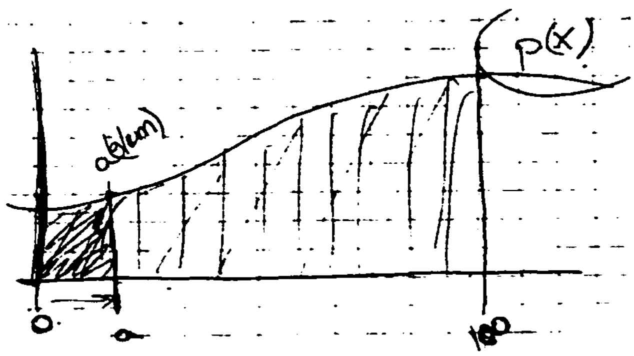


Figure 18. The generic graph Joan sketched to represent the density function $\rho(x)$.

she said "for some reason I need to find the area underneath here *(shades in underneath her curve)*, and that should give me the mass. I'm just trying to see how that relates to the units." The units comment was relevant to the way Joan was thinking as she was using units in order to make sense of the task. However, she was focusing only on the density in relation to her area comment, but the units were not working out the way she thought they should, namely that $g = \left({}^{g}\!/_{cm}\right) \cdot cm$. Thus, her two ideas caused disequilibrium, and she struggled to resolve this conflict.

After another long pause, the following exchange about how to compute the mass of the rod took place.

Joan: I think so. I think you can find the mass that way.
Todd: What way?
Joan: If you integrated $p(x)$ from 0 to 100, you'd find the mass.
Todd: Why?
Joan: Now, the why part, I'm still working on. But I'm pretty sure. *(pause)* I'm just still trying to figure out how it affects the units. Because my frame of thought is that makes a big difference in whether or not you can figure out, if it would give you the mass or not.

Joan was certain that the mass could be found by integrating the density function between 0 and 100, but she struggled with why this was the correct formulation of the task. She was still trying to determine how the units should fit into this process because, to her, the units were the key to understanding why integration should work. While she struggled with the units, she was still trying to comprehend the idea that the density could vary. This was a challenge for Joan, and it was not resolved when the interview was finished. However, she did try one more time just after the camcorder was turned off to resolve the conflicting ideas. In a lengthy sequence of disjointed statements, Joan attempted to express how this situation was analogous to the relationships between position, velocity, and acceleration, but she was unsuccessful in her attempt to arrange her ideas into coherent statements.

This task was revisited at the beginning of the third interview to find out whether Joan could provide any more insight into why she believed that integrating the density function between 0 and 100 would give the mass. She returned to her position, velocity, and acceleration thought in concert with her units thought from interview two. This time she was able to express her ideas more coherently. She thought of density as velocity, as the units of both are rate of change units; so if mass were like position, then "if we take the antiderivative of it [grams per centimeter], we end up with just grams." However, she still had not resolved the issue of how the units actually work. Although she continued to believe that this was the correct approach since "units change in position, velocity and acceleration, so it has to be comparable in this problem" when integrating. Her thoughts could be expressed as the act of deducing amount information from rate information (Ostebee & Zorn, 1997, p. 371).

Nowhere in any of her work did Joan indicate that she considered partitioning the rod in order to find the mass of little sections that could be summed to obtain the total mass. Her Calculus II section had been working on volume and other applications of the definite integral prior to the third interview. Despite this recent course work, Joan failed to recognize that she was raising many issues that could best be addressed by approaching the definite integral through a partitioning process.

Response to Task 14

Joan's response was that the definite integral represented "the area between the function and the x-axis from some value a to value b." Since she did not mention either signed area or net area, her statement was interpreted to mean that she associated the definite integral only with the notion of area. This was ironic because she had demonstrated that she was aware of the idea of signed area and had been able to successfully apply it to some of the previous tasks. There are two possible explanations worth considering here. On the

one hand it may have been the case that she had not expanded her personal representation of the definite integral to include the notion of signed area. On the other hand, she may have collected these ideas under the single umbrella of "area" without having yet made the determination as to which should really be the representation. It was unclear at the time which of these possibilities best reflects Joan's thought processes.

This task appeared on the participant selection tasks (Appendix E). At the time it was administered, Joan wrote the following: "If F is an antiderivative of f then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This would also be the area between the graph of f and the x-axis from $x = a$ to $x = b$." Even though she included The Fundamental Theorem of Calculus idea at this time, she still showed that she viewed the definite integral in terms of area. There was no indication that her representation underwent any alteration during the time between the administration of the Participant Selection Tasks and the third interview. Therefore, her representation for definite integral appeared to be firmly established in her mind.

Response to Task 15

Joan connected the integral $\int_a^b f(x)\,dx$ with the quantity $F(b) - F(a)$. While she labeled her graph (see Figure 19), she said, "Looking at it that way, this would be $f(b)$ *(means* $F(b)$*)*, this would be $f(a)$ *(means* $F(a)$*)*. And I guess it would be this distance right here *(marks vertical displacement on graph)*. I have no idea." She then offered a second possibility for what the integral $\int_a^b f(x)\,dx$ represented on the diagram. Her second thought was that the integral was "the length of f between b and a." She represented this by darkening the section of the curve over the closed interval $[a, b]$. Thus she postulated

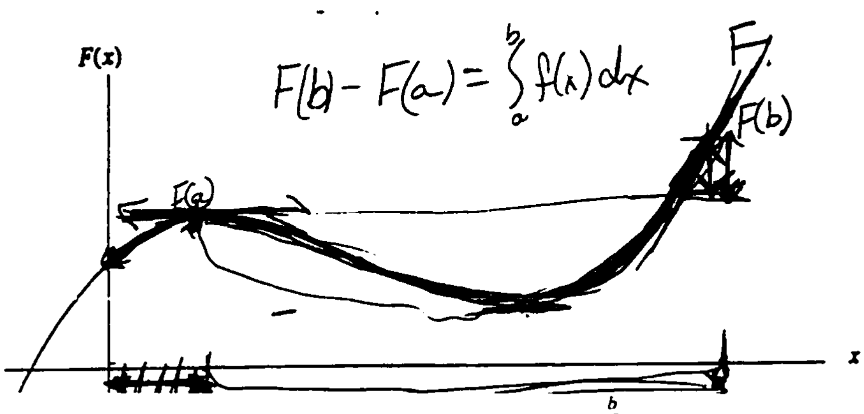


Figure 19. Joan's attempts to represent the definite integral $\int_a^b f(x)dx$ on the diagram containing the graph of an antiderivative F.

use of either vertical displacement between the points $(b,F(b))$ and $(a,F(a))$ or arc length along the closed interval $[a,b]$ as possible ways to signify the value of the definite integral on the given diagram. She was unsure as to which of her two thoughts was correct, but stated that "it's going to be one or the other." However, she went on to say that vertical displacement "kind of makes more sense to me, because they're y-values [$F(b)$and $F(a)$], and so it [the definite integral] would be the difference in y-values."

Evidently, the graph for the function F may have played a deciding role in Joan's decision that vertical displacement was more sensible to her. This was revealed by her use of phrases like "y-values," "difference in y-values," and "height between the two [points]." However, the idea of arc length was a legitimate option for her because her Calculus II section had just studied arc length in class. Thus, the notion of length between the two points on the graph was a familiar idea to her, and therefore she may have been trying to apply it to this setting.

Finally, when asked whether the quantity $F(b)-F(a)$ was going to be positive or negative, she said:

> I would say it's positive. 'Cause $f(b)$ *(pointing to her* $F(b)$*)* is bigger than $f(a)$ *(pointing to her* $F(a)$*)*. So, looks like it'd be positive to me. I mean, anyway, whatever it represents, it's *(points to* $F(b)$*)* bigger than $f(a)$ *(pointing to her* $F(a)$*)*, so your going to get a positive outcome.

Even though she responded to the question as if she were thinking of the quantity $F(b)-F(a)$ as representing the vertical displacement, the phrase "whatever it represents" was interpreted to mean that she still believed arc length was a possible way to view the definite integral in terms of the graph of F. She remained uncertain as to what the definite integral actually represented in this context.

Response to Task 16

Joan began by saying she would tell the student "the best representation is to draw a picture. Once you have a picture, you can kind of get a grasp of the idea" (see Figure 20). Once she had her graph drawn, she continued her explanation:

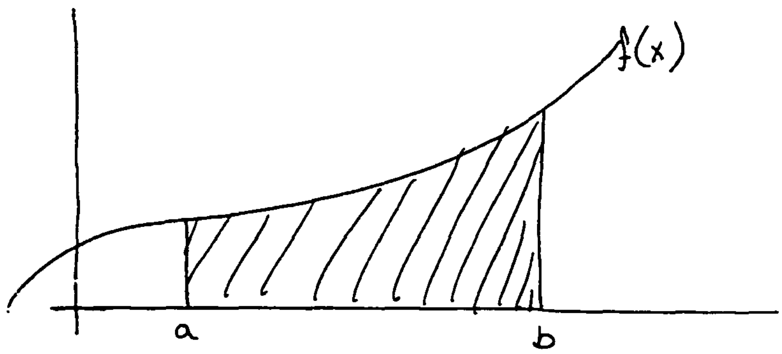


Figure 20. The graph Joan used to illustrate her definition of the definite integral.

> OK, so, you have this function $f(x)$ on some interval, say like, *a* to *b*, where their *a* and *b* are just some numbers. And the definite integral written like this *(writes out* $\int_a^b f(x)\,dx$*)*, represents the shaded area. The shaded area is the area between the function and the x-axis. And if you wanted to, um, give it a value, then if *f*, big *f*, is an antiderivative [mumble] of *f*, then the integral from *a* to *b* of *f* equals $F(b)-F(a)$.

Thus, it was clear that for Joan the definite integral was defined in terms of area. Furthermore, the value was computed by using The Fundamental Theorem of Calculus.

When asked to respond to the situation in which $f(x) < 0$, she gave the following explanation:

> Then, draw them another picture [see Figure 21]. It always represents the area between the curve and the x-axis. Always represents that. So then it would be this area. So, um, since this area is below the x- axis, it has a negative value, because you've got some...looking at it this way, it's got some...because its...the absolute value for $f(x)$ is negative, less than zero. So, um, then you would take...we'll call this c *(the x-intercept)*. You would take the area from a to c, or the...the area from b to c and subtract the area from a to c. Should say c to b. And that would be your area.

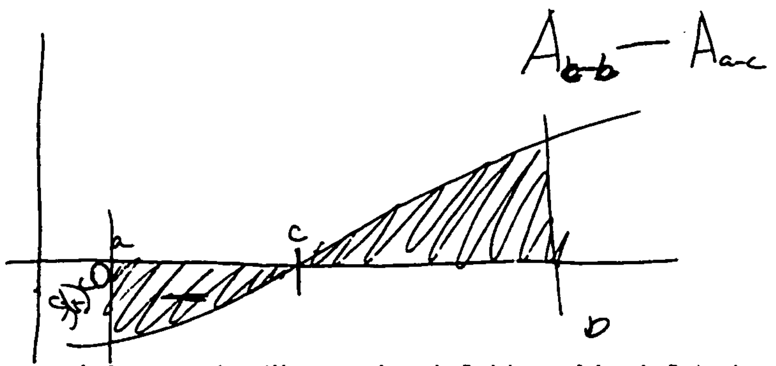


Figure 21. The graph Joan used to illustrate her definition of the definite integral in terms of signed area.

This passage illustrated that for Joan the definite integral represents the area of the region bounded by the x-axis and the curve, but she then proceeded to describe a signed area calculation. This was consistent with previous episodes in which she had made no distinction between area and signed area.

Her definition revolved around the notion of area. This was consistent with her response earlier in the third interview to the task regarding what the definite integral represented, namely, that the definite integral represented the area between the x-axis and the curve. Just prior to moving on to another task, she offered the following comment: "It's

the only way I can grasp it." This was interpreted to mean that the only way Joan could understand the definite integral was in terms of area. Thus her personal definition was in terms of area. Finally, even though Joan was able to use Riemann sum calculations, this notion never entered into her concept definition of the definite integral.

Response to Task 17

Joan believed that the two questions relating to the picture given in this task were essentially identical. When asked about the area of the region, she responded by saying, "the sum of the areas of the rectangles, an estimate of the definite integral from *a* to *b*." When questioned about her statement, she provided the following additional comment: "the area is the definite integral of *f* from *a* to *b*." In response to the question regarding a definition of the definite integral, she provided this thought:

> The definition I gave, um, relied on graphing and the area under the curve. So, again, it would relate in such a way that we took the areas of each rectangle and summed them up, we'd have an approximation to the area. That's how it'd relate to my definition.

This was not surprising, considering that she had previously said the definite integral represented "the area between the function and the *x*-axis from some value *a* to value *b*." However, her responses indicated that she was able to relate a Riemann sum with a nonregular partition to her understanding of area and the definite integral. Therefore, she was not bound to the idea that the partition must be regular in order to compute an approximate area or a definite integral.

The topic of improving the approximation was broached within this task. For Joan, this meant "making more rectangles" within the region. In accordance with her standing instructions to "think aloud," Joan attempted to verbalize her thought processes. The main idea from the resulting disjointed phrases was that she wished to repartition the interval so that all the subintervals were equal and the left-endpoint, midpoint, or right-endpoint on each subinterval was also used each time. When questioned further, she indicated that following

the above scheme would "make life easier" because "then you can use a Riemann sum." It initially appeared that she viewed Riemann sums only in this narrow setting. However, further probing on the length of the subintervals provided the following exchange, where the phrase "sum of the rectangles" was interpreted to mean " the sum of the areas of the rectangles," in order to be consistent with her past usage of the phrase:

> No, I don't think...the way it's defined I don't think so, but...it's easier. (laughs) Um, I believe that the Riemann sum was just an approximation by using geometric shapes underneath the curve. Meaning we could use trapezoids, as well, and, uh, they didn't have to be the same width. It's just the sum of the rectangles where some point, on the top of them, lies along the curve.

This response demonstrated that her image of the definite integral included a general notion of a Riemann sum, even though her preference seemed to be a regular partition with a consistent choice of the left-endpoint, midpoint, or right-endpoint of each subinterval. However, her inclusion of trapezoids in the discussion indicated that she saw the general notion of approximating sums as being equivalent to Riemann sums, rather than Riemann sums being a particular type of approximating sum.

Finally, Joan was asked if there was any connection between a definition for the definite integral and a Riemann sum. She responded, "A definition of the definite integral, as I defined it, is the area underneath the curve from *a* to *b*. Then, um...Riemann sums are approximations of that value...of that area." Thus Joan viewed the Riemann sum as a summing process which allowed her to compute an approximation for the definite integral. Nowhere did she evoke the notion of the limit as a way to bridge the gap between an approximation for the definite integral and the definite integral itself. However, this did provide a natural way in which to pose the next task to her.

Response to Task 18

Joan's initial response was to clarify that the task was not referring to the limits of integration. Once this was clarified, all she was able to recall at first was "a brief memory

of mine, um...somewhere in the back of my mind, says it does, but I can't remember what." This could signify that she was aware that limits were connected with the idea of the definite integral but were not a part of her regularly evoked concept image of the definite integral. After a few moments, she recalled a connection to the Riemann sum, as illustrated by the following:

> Well, I can, OK. If you've got a Riemann sum, then...then your limit...oh, I just thought of something. OK, the limit as, uh, Δx goes to 0...[mumble] wrote that wrong. But, anyway, as that goes to 0...no...(laughs)...yeah, the change...then the number of rectangles increases. So, um, you get closer and closer and closer approximation of that area. So, if you have, like, an infinite number of rectangles, then you actually have that area. 'Cause your Δx has gone to 0, and your number of rectangles has gone to infinity.

In interpreting this passage, it was assumed that she was referring to a regular partition of the closed interval $[a,b]$, in order to be consistent with her previous discussions about approximating sums. Thus the limit provided Joan with a method to obtain a better approximation of the area, with the eventual result being the area under the curve, which would occur when the approximating sum contained infinitely many rectangles. However, the realization of the definite integral as a limit of a summing process did not seem to be intrinsic to Joan's concept image of the definite integral. She did not mention any connection between the limit and the definite integral until asked, and struggled to express the idea once she recalled it. Finally, even though she was able to connect the limiting process to her approximating sums, her definition for the definite integral did not seem to be altered, as she continued to speak in terms of area.

Response to Task 19

The statement of this task initially overwhelmed Joan. It took her several minutes to fully grasp what the task was saying, and even then she needed to have the definition of the sequence explained to her. After she had written out the first three terms of the sequence, she experimented with two speculations:

$$S_n = \int_0^n f(x)\,dx \text{ and } \int_0^n f(x)\,dx = e - n.$$

However, upon checking these values numerically, she discovered that neither of her speculations was true. At this point she was ready to give up on the task as she had "no other thoughts."

In order to ascertain whether she could make any further connections, she was asked to consider the alternative version of S_n. After seeing the rewritten S_n, she stated: "I somehow want it connected to the Riemann sum." It turned out that this idea had occurred to her while looking at the original version of S_n because "it's a sum." The alternative version of S_n seemed to reinforce this idea because Joan associated the $1/n$ as being associated with Δx. However, she still believed "there's a connection with Riemann sum; I just can't put my finger on it." Joan revealed later that the fractional exponents stymied her, and she was unable to incorporate them into her Riemann sum idea. Shortly afterwards she concluded, "I'm not seeing it," and could make no further progress.

Joan never considered approaching this task graphically. She considered the task only numerically and by looking for a pattern in the definition of the sequence itself. When it was suggested that the setting was the closed interval $[0,1]$, she did not think to examine the situation geometrically.

Response to Task 20

After a period of time during which Joan absorbed the information presented in the task, she proceeded to sketch a graph of the function f (see Figure 22) which she had concluded was a step function. Once she had finished her graph of f, she announced, "just off the top of my head, I'm thinking that, um, I'm sketching the graph of the area under the function." The function that she referred to was interpreted to be the original function f. Thus she was viewing the function F as an area function over the closed interval $[0,2]$.

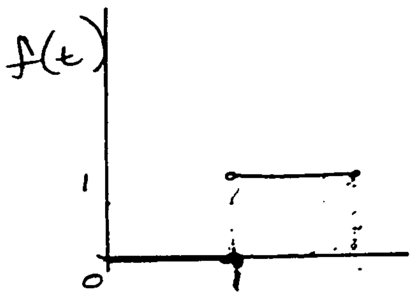


Figure 22. Joan's graph for the piecewise-defined function *f*.

In completing the first part of the graph of *F* (see Figure 23), Joan described her work by saying that if "$x = 0$, then the area's 0, and the area stays 0 all the way to 1." This was meant that $F(0) = 0$, which allowed her to uniquely identify the function *F*, and that $F(x) = 0$ on the closed interval $[0,1]$. When asked to explain herself, she provided the following rationale: "Um, well, think of it in terms of area again. There's nothing under it, because it's sitting on the *t*-axis, and, um, because the value is...it equals 0, so there's nothing." The second and third "it" in the previous quote referred to the function *f*, and the last "it" in the same quote referred to the area which was then connected to the function *F*.

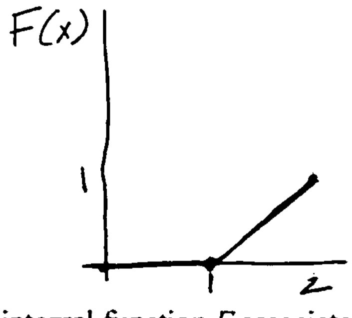


Figure 23. Joan's graph for the integral function *F* associated with the piecewise-defined function *f*.

For the second half of her graph of the function *F* (see Figure 23), Joan completed the sketch while saying that "as *x* continues to move across there *(interval* $(1,2]$ *on the*

graph of f), then the area goes to 1. So the area would increase to 1." Further probing provided more insight into her solution method. At $x = 2$ the value of F was 1 because:

> Um, and this area here *(referring to the interval* $[0,1]$ *on the graph of f)*, which was nothing. Then we've just got one big block *referring to the interval* $[0,1]$ *on the graph of f)*. So the area is 1. So we marked off...put a point at 1 *(meaning the y-coordinate)*.

This last passage showed that, in essence, she was thinking in terms of the summation property of the definite integral exclusively from a geometrical point of view. She then went on to provide a rationale for the line connecting the point $(1,0)$ and the point $(2,1)$ on the graph of F as follows:

> Then, um, I figured that it's a line, because as you move across it's...like if you were...just kind of taking something and moving across it *(illustrates by moving a ruler along the t-axis of the graph of f)*. Then you're just steadily increasing, like at a constant rate. So I assumed that it was a line.

Her use of the ruler to move across the second half of the graph of f indicated that she was operating under an across-time perspective when thinking about the area under the graph of f on the interval $(1,2]$, and the area was accumulating at a constant rate. Thus, the area values on the graph of F were linear. In this setting, the function $f(t) = 1$ is the rate at which the area was increasing.

Finally, Joan was asked what influence the jump discontinuity at $t = 1$ had on her approach. She responded by saying:

> I thought about it, and, um, again, this is one of my phobias, er, step-wise functions. Um, at 1, it's for sure 0, because, um, $f(t)$ still equals 0 at 1. And then...then right after 1, infinitely close to 1, it begins increasing.

The last part of this said that, right after $t = 1$ on the graph of f, the area under the curve begins to increase, and so the values of F will begin to increase right after $x = 1$. With her description of what took place, she was convinced that the jump in the graph of f was not affecting what took place at $x = 1$ on the graph of F.

The Case of Lynn

Introduction

Lynn, a 19-year-old sophomore pursuing general studies at the time of the study, revealed during the interview process that she was contemplating a math minor. Prior to Calculus II, she had taken Precalculus and Calculus I at the institution where the study took place. She took Calculus I during the semester prior to the interview. Lynn earned a C in Calculus I.

At the end of the third interview, Lynn was asked to respond to questions focusing on her thoughts about her understanding of the definite integral to provide insight into how she approached the tasks themselves. In response to a question about how she thought about the definite integral, Lynn responded "the first thing I do is I just try to find the antiderivative. I just totally use the equation." This response indicated that she tended to think on a more symbolic level when working with the definite integral. She also indicated that she neither thought "about it [the definite integral] real hard," nor about what she was trying to find when working on a problem. She did reveal that, since starting integration, she had begun to look at things more graphically. She graphed the integrand "to see what it looks like. To see if, like, I can estimate it [definite integral], and then whether the answer matches." Thus graphing was used as a way to check her answers. She indicated that she did not think about the definite integral from an abstract or conceptual point of view.

In response to the inquiry about what aspect(s) of the definite integral she viewed as most difficult, she provided two comments. The first comment related to her ability to find antiderivatives. She said, "finding the antiderivative is usually the only thing that gives me a problem." In particular, finding antiderivatives for the trigonometric and logarithmic functions was hard for her. Lynn's other comment focused on her perceived difficulty with understanding and using the notation associated with the definite integral. For Lynn, the

completion of The Fundamental Theorem of Calculus by substituting the limits of integration into the antiderivative and then finding the difference was one of the easiest aspects. The other was the idea that the definite integral was "the area underneath the graph, and that you're studying it from a certain interval."

Response to Task 1

Lynn solved the first two parts of this task correctly by applying The Fundamental Theorem of Calculus. Neither of these two subtasks was difficult for her to complete. She followed a very mechanical step-by-step process in carrying out the calculations. Lynn's description of her solution method was very similar to Joan's description, and revealed that she viewed The Fundamental Theorem of Calculus as a mechanical process rather than a condensed whole. She had not condensed the theorem into a single procedure that she could use to address a variety of tasks of this nature.

Lynn was not familiar with the greatest integer function, and after the definition was explained she initially struggled with the idea. After many examples, she started to become comfortable with the definition, and felt ready to consider the third part of this task. Lynn's initial reaction was "you just take the antiderivative again wouldn't you? Just put $x^2/2$." With this statement she also wrote down $\left|\frac{x^2}{2}\right|\Big|_0^5$, where the absolute value bars were interpreted to be the symbol for the greatest integer function. As with Joan, Lynn indicated that the task's analytical format suggested an analytical solution path to her. Also, Lynn indicated that she focused on the variable x and finding its antiderivative. She focused on what was familiar to her as she struggled with an unfamiliar function. Lynn concluded her work on the integral with the string of calculations shown in Figure 24. Due to lack of notation, it was unclear how she used the symbol $\lfloor\ \rfloor$ in her calculations, or whether she used it at all until the end. In any case, she arrived at an answer of 12. Even though she was

$$\left|\frac{x^2}{2}\right|\Bigg|_0^5 \qquad \frac{5^2}{2} - \cancel{\frac{0^2}{2}} \qquad \frac{25}{2} \qquad 12.5 \qquad \boxed{12}$$

Figure 24. Lynn's integration calculation for the greatest integer function.

uncertain how to proceed with this integral at the beginning, she did not validate her answer. In fact, when asked how confident she was of her answer, she replied, "Very confident."

Lynn was challenged to check her work by carrying out the calculation on her graphics calculator. After receiving instructions for entering the greatest integer function into the calculator, she carried out the requested calculation. Her reaction to the calculator's result of 10 was to say, "I was wrong." She tried to resolve the discrepancy by rechecking her work. When asked how else she could resolve the disequilibrium she experienced, she indicated, after a long pause, that she could look at a graph "to decide whether the area looked like 10 or 12 underneath the graph of the function." Once Lynn had the graph of the greatest integer function, she applied the area model, just as Joan had done, to conclude that the value for this particular integral should be 10. She also concluded that her initial calculation was wrong, and when questioned about what she thought was incorrect, she said, "I don't know. I really don't," after she re-examined her work.

Response to Tasks 2 and 3

Lynn applied the area model to Task 2 and computed the area of the region in Quadrant II as a negative number. However, when questioned about this, she said, "I was just looking at that," and admitted that the base of the triangle should be 2 instead of -2. She corrected the rest of her calculations (see Figure 25). She explained that the value in the question was 2 "because this is the positive area, because it's above the x-axis." She

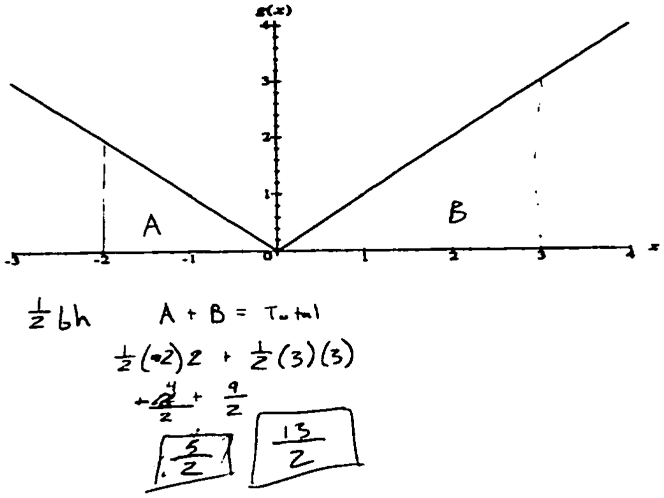


Figure 25. Lynn's approach to evaluating the definite integral $\int_{-2}^{3} g(x)\,dx$.

indicated that, since the region in Quadrant II was above the x-axis, the area of the region should be positive, and in order for this to happen, the base of the triangular region must be 2. The impression left by her comments and work was that, even though she started with an invalid image for the task, something caused her to re-assess her work and locate the correct image.

When Lynn was presented with Task 3, she disregarded Task 2, even though it was in plain view. She applied The Fundamental Theorem of Calculus to this task (see Figure 26) without any acknowledgment of having just completed the same task. This seemed to indicate that the analytical form of the task was drawing Lynn towards an analytical solution. Her work focused on what was familiar, the antiderivative of x, and she kept the absolute value symbols only because they were needed "to keep this *(points to the numerical value in line two of Figure 25)* positive." She expected that the answer should be positive because she started with $|x|$. She debated over the placement of the absolute

$$\left|\frac{x^2}{2}\right| \Big|_{-2}^{3}$$

$$\left|\frac{3^2}{2} - \frac{(-2)^2}{2}\right|$$

$$\frac{9}{2} - \frac{4}{2} = \frac{5}{2}$$

Figure 26. Lynn's evaluation of the definite integral $\int_{-2}^{3} |x|\,dx$.

value symbols during her work. In effect, Lynn computed the value for the definite integral $\int_{-2}^{3} x\,dx$, but the significance of this never occurred to her.

Lynn noted no connection between Tasks 2 and 3, so she was asked if there was one. Her responses indicated only superficial connections, because the tasks involved the fractions $9/2$ and $4/2$. She tried to form a connection based upon the appearance of these two numbers and the fact that, by adding them in Task 3 instead of subtracting them, she would get the same answer as in Task 2. However, careful consideration of the following two pieces of evidence indicated further knowledge. The first evidence was the following passage, in which she tried to make a numerical connection between the graph in Task 2 and her computation in Task 3.

> I said this *(points to triangle A in Figure 25)* was positive area, so if this *(points to triangle A in Figure 25)* was positive area, this *(points to the subtraction sign in the expression in the second line of Task 3)* would have to be plus, both would have to remain positive. So this *(points to* $3^2/2$ *and encloses it in absolute value symbols)* would have to be positive and this *(points to* $(-2)^2/2$*)* would have to be positive, right? And then it would have equaled $13/2$. So you have to treat them individually...with the absolute value?

The second evidence was when asked, she realized that the graph in Task 2 was the graph of the absolute value function. She confirmed her belief by graphing the function $y = |x|$. She concluded that the answers to the two tasks should be the same. Thus, she may have been at the verge of understanding what needed to be done in order to compute $\int_{-2}^{3} |x|dx$ analytically. In particular, her comment "you have to treat them individually," indicates she thought that

$$\int_{-2}^{3} |x|dx = \int_{-2}^{0} |x|dx + \int_{0}^{3} |x|dx,$$

and that she could find the values over each subregion individually, as in her geometric solution to Task 2. This was never stated outright nor implemented in any of her work during the interview, but this conclusion was strengthened by an event that took place at the conclusion of the interview. At that time, Lynn took the initiative to return to Task 3 to resolve her disequilibrium over these two tasks. When the definition of the absolute value as a piecewise-defined function was mentioned, she wrote on a whiteboard the following sum for the integral in Task 3:

$$\int_{-2}^{0} -x\,dx + \int_{0}^{3} x\,dx.$$

Furthermore, she believed that it made sense to evaluate the two definite integrals separately and sum the results. Thus, Lynn only needed a way to address the absolute value in the definite integral in Task 3 in order to compute it analytically.

Continuing with what transpired during the interview, Lynn was asked which of the two approaches she felt was correct. She said, "I feel this one is *(pointing to Task 2)* because I am unsure of how to treat the absolute value signs." Furthermore, she was more confident about the work completed in Task 2 "because it looks easier, to see it and to use the graph to figure it out than to figure, like, the integral and everything." Thus, she was

more confident about the graphical approach because it was easy to see and to use, whereas she was very unsure of the role of the absolute value bars in the definite integral of Task 3. At the end of the task, she was still confused about the role of the absolute value when it was connected with an integral; this was never resolved.

Response to Task 4

Lynn's initial approach was to use the vertical line $x = 4$ to separate the region into two subregions: a triangle and a quarter circle. From here she proceeded, as Joan did, to use area formulas to approximate the area of the whole region. When she tried to explain why she viewed the subregion on the closed interval $[4,7]$ as a quarter circle by showing various radii of the circle, she switched to viewing it as an oval. Her reason for this was "this radius is smaller *(vertical segment at* $x = 4$*)* than this one *(segment constituting the closed interval* $[4,7]$*)*." However, at this point she was uncertain how to proceed.

Lynn was asked how else she could find the area of the requested region. She indicated that she would still use the area formula for a triangle on the closed interval $[1,4]$, but would use what she called "the approximation, the squares like the left" or "the right-hand squares" to find the area of the oval. Lynn was unsure that she could explain her idea because "I don't really understand it very much. I don't know if I can explain it to you right." Even though she was worried about not being able to explain her idea, she described what turned out to be an approximating sum notion. It turned out that her "left-hand square" involved forming rectangles using the left endpoint of each subinterval on a partition, and that her "right-hand square" involved forming rectangles using the right endpoint of each subinterval on a partition. In both cases the base of her rectangles was constant. She illustrated these ideas by drawing "left-hand squares" on the left side of the oval, and by drawing "right-hand squares" on the right side of the oval (see Figure 27). The following passage indicated what she would do once all her "squares" were drawn:

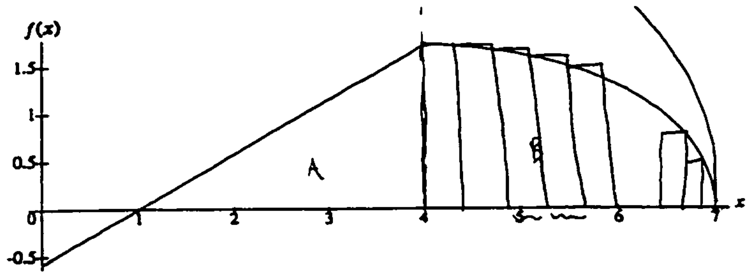


Figure 27. Lynn's left- and right-hand squares for approximating an area.

Lynn: I'm not sure if I know what the word is...is it...
Todd: What would the idea be that you're trying to communicate to me?
Lynn: That you're just trying to approximate the amount by finding the area of each of these and adding them together.

Though she was unsure of the correct terminology and unsure whether she really understood the method, she described an approximating sum technique. What might have made her feel that she did not understand the method was that she did not know the correct terminology.

When asked what she would choose, if she could have any missing piece of information, she replied "a simple equation to find an oval." By this, she meant a simple area formula analogous to πr^2 for a circle. At that point it was evident that she was still thinking in terms of geometry, and was not thinking in terms of calculus or the definite integral.

When asked what she would require to apply calculus ideas to this task, she was fixated on not having an equation to describe her oval. Because of this, she could not move beyond using the phrase "the integral from 4 to 7." If she had a formula for the quarter oval, she said, "I would find the antiderivative, and I would plug in these numbers...plug in 7 to the antiderivative and subtract and plug in 4 to the antiderivative, and that would be the answer." In other words, she would apply The Fundamental Theorem of Calculus to the

expression for the quarter oval in order to arrive at an exact value for the area. If she had a function, she would have used a definite integral to compute the area.

When presented with the piecewise-defined function for the function *f*, she struggled to comprehend the function, and eventually made connections between the function and the graph. She then explained how she would use the given function to find the area by saying:

> You find the, um, definite integral of this one using this function *(points to linear part in the function f)*, and you'd find the definite integral of this *(points to the radical expression in the function f)*, and you could add them together.

Lynn was asked to write what she had just said, and she wrote the mathematical statement for computing the area in terms of the summation property of the definite integral (see Figure 28).

$$\int_1^4 \frac{\sqrt{3}}{3}(x-1)\,dx \quad + \quad \int_4^7 \sqrt{\frac{9-(x-4)^2}{3}}\,dx$$

Figure 28. Lynn's use of the summation property of the definite integral.

Through both a verbal and a written description, Lynn demonstrated a firm grasp of the summation property of the definite integral in terms of this particular task. Finally, concerning the first integral shown in Figure 28, she said, "...it's just easier doing it graphically because you have a formula...an easy formula (*pointing to the triangular region)*." Essentially, she viewed the first integral as being easier to evaluate using the area formula for a triangle, mixing geometrical methods with analytical methods.

Response to Task 5

When solving this task, Lynn mentioned that she would either have to multiply $(\lambda - 1)^2$ or use the substitution method. The latter probably occurred to her because her

Calculus II section had covered the substitution method prior to her interview. She initially chose to use substitution "because it's just easier than multiplying it out," but changed her mind shortly after identifying u as $\lambda - 1$. She decided it would be easier to multiply the elements of the integrand, then apply The Fundamental Theorem of Calculus. From here, she proceeded mechanically through all the details (see Figure 29).

$$\int_0^5 (\lambda-1)^2 \, d\lambda \qquad \int_0^5 \lambda^2 - 2\lambda + 1 \, d\lambda$$

$$\frac{\lambda^3}{3} - \frac{2\lambda^2}{2} + \lambda \Big|_0^5$$

$$\frac{5^3}{3} - \frac{2(5^2)}{2} + 5 - \cancel{\frac{0^3}{3}} - \cancel{\frac{2(0)^2}{2}} + \cancel{0}$$

$$\frac{125}{3} - \frac{50}{2} + 5$$

$$\frac{250}{6} - \frac{150}{6} + \frac{30}{6}$$

$$\frac{130}{6}$$

Figure 29. Lynn's evaluation of the integral function G.

Lynn did not indicate any confusion over the two variables, x and λ, that were part of the integral function. Regarding the variable x, she said:

> The reason I decided where the 5 went was because this was x *(points to the x in $G(x)$)* so I matched it with the x *(in the upper limit of integration)*, and I didn't think it would have anything to do with the λ, that that was another variable.

In addition, by carrying out this substitution, she could identify "the range that I evaluate it for," meaning she would know the interval over which she was to carry out the integration. Thus she dealt with x prior to actually integrating, and therefore eliminated x from the integral for the rest of the calculation. Lynn saw the role of λ as being where "you will put

the 5 in, but you have to find the antiderivative before you put it in and evaluate it at 5." This led to the conclusion that Lynn viewed λ as the variable over which the integration took place, and x as being associated solely with determining the upper limit of integration.

From Lynn's work, it was evident that she could think of the function G in two different ways, but at no time was she able to address G as a function itself. The first view of G presented itself earlier in her work, when she made two interesting comments while talking about the variables x and λ and the meaning of $G(5)$. The first comment was: "'Cause this is pretty much saying, isn't this *(pointing to* $G(x)$*)* saying that, um, antiderivative defined at x. And this *(pointing to* $G(5)$*)* is asking for the antiderivative defined at 5." The other comment was part of a discussion about the meaning of $G(5)$, "That you want to find what this function *(points to* $G(x)$*)* is at 5? What the definite integral, I can't say that, the antiderivative of this definite integral is at 5." Both of these comments suggested that Lynn viewed the function G as being an antiderivative of the integrand. This would be in accordance with the formal version of The Fundamental Theorem of Calculus. The interest lies in the fact that, even though she saw G as an antiderivative, a function in its own right, she was unable to consider G as a function while working on this task.

Lynn's second way of thinking about the function G was in terms of evaluating the function itself. This was evident from her descriptions of how to compute $G(3)$, $G(10)$, or G evaluated at any number. In all of these explanations, she referred to her previous work for computing $G(5)$. As an illustration of her descriptions, here is what she had to say about computing $G(10)$: "I'd put 10 in for x. Find the antiderivative. Then I'd put 10 in here, and I'd put 0 in here *(pointing to her work using The Fundamental Theorem of Calculus)*." From these examples, it was evident that Lynn used a pointwise understanding

of the function G, even though she was unable to actually talk about it as a function. Evaluating or working with a function one point at a time will be referred to as showing a pointwise understanding of the function.

Lynn was questioned regarding her ability to talk about G being a function, but very little information was gleaned from her. She was unable to connect the idea of function to the function G itself. This point was not pursued exhaustively, because it was evident that she was becoming frustrated by the probing and would be unable to provide any useful information.

Response to Task 6

Lynn had to read the first part of this task a couple of times to absorb all the information, but once she did, she gave a response that was similar to the initial response given by Joan. In Lynn's own words, "The area will increase as x moves right [*sic*]." Her descriptions for what was taking place were analogous to Joan's, so these are not reported here. However, Lynn never predicted whether the area would approach a finite number or would become infinite. Her final statement summarized her view regarding this part of the task: "Eventually, it's just gonna increase and increase until you decide to stop or it's going to find the entire area of the whole...until this reaches *(referring to the curve)* the x-axis if it ever does or the t-axis, I mean."

After reading through part b) of this task, Lynn stated that the values of the function G "increase because each time you put in a greater value into x; so you're evaluating a larger integral; so you'd be finding a larger area each time." Based upon further clarifications, the phrase "a larger integral" was interpreted as meaning that the outcomes from the integral would increase as the value of x increased. In addition, she had connected the integral function to the notion of area, and so demonstrated that she viewed the two parts of this task as asking the same type of question. Finally, based upon follow-up questions,

she provided some indications that, with the aid of the graph, she possessed some across-time understanding of the integral function.

Response to Task 7

Lynn's initial response to this task was the same as the response given by Joan, namely that "the further you went out, the more negative it [area] would become." She described what would happen as x continued to move to the right in the same way as Joan did. However, when asked why the area would be negative, she said, "Because, this is called signed area, and when it's above the t-axis, it's positive, and when it's below, it's negative, and there's a greater amount that's negative than positive." Lynn clearly indicated she used signed area in responding to this task.

When Lynn was asked about how area was viewed geometrically, she acknowledged that the value should be positive; but because this region was on a coordinate system, the value for an area could be negative. She was challenged on this point to determine whether she actually believed that it made sense for the area to be negative here, and her response was as follows:

> I mean...I guess it asks...I guess it depends on if it asks whether what the total area is or what the signed area is, you know. I don't have a problem with it because it's on a coordinate system. I don't...it was just what I was taught to find it on a coordinate system.

This passage was interpreted to mean that there were two ways of viewing this situation, and how the task was worded would dictate her response. Since this task asked for "area," she believed she should calculate the signed area, because that was what she was taught to do when working with a coordinate system. To her, the coordinate system was a cue to find the signed area. However, if the task asked for the "total area," then she believed she should calculate the geometrical area of the region, without taking into consideration the coordinate system. Throughout this entire discussion Lynn continued to believe that, since the coordinate system was involved, she should compute the signed area for the area of the

region. This led to the conclusion that, for Lynn, the coordinate system was acting as a cognitive obstacle when addressing the notion of area.

After reading part b), Lynn took a long time to present her answer. When she did, she said, "OK, if you have x and you increase the values, it's going to increase the value of the function." Further questioning revealed that she meant that as x moved to the right along the t-axis, the values for the function G would increase. In describing what the values of the function G would be at various points along the t-axis, she evoked the notion of positive area. Lynn summarized what she thought the task was asking with this statement: "With definite integrals, the t-axis, um, doesn't matter. It doesn't matter where the...it doesn't matter where...whether it's located above or below it's all...it just wants total area, and it's gonna be positive." Thus, she saw the definite integral as being independent of the coordinate system and, as such, giving the total area of the region.

At the beginning of the second interview, this task was revisited for follow-up questions. She did confirm her original answer that part a) was measuring the signed area of the region. Lynn was then asked what would happen in part a) if the phrase "area of the region" was changed to the phrase "total area of the region." Her response was:

> That would mean it would remain positive and just keep increasing as you go along, because you'd want to know what the total area of this *(points to the region above the t-axis)* plus this *(points to the region below the t-axis)* is. The total region bounded by the graph...by the function.

The inclusion of the word "total" evoked the actual notion of area for Lynn, whereas without the word "total", she believed the task was asking for the signed area.

Moving on to part b), Lynn reconfirmed her original answer that the definite integral measured the total area. When asked to evaluate the integral $\int_0^5 -3t+4\,dt$, she correctly applied The Fundamental Theorem of Calculus and arrived at an answer of $-35/2$. This answer produced the following exchange:

Lynn: Negative. So you do have negative...*(pause)*
Todd: Why do you think that might be?
Lynn: Because, hmm...*(pause)* This −3 makes a negative, but I don't know if it would be because I was wrong in saying that integrals are always positive, or if this is just finding the negative of the integral rather than the positive. You know what I mean? Like, I don't understand whether... OK. *(For the remainder of this utterance "this" refers to the region below the t-axis on the graph of the task.)* Integrals do measure this area as negative. I don't...either that, or else, um, you're finding the negative area of this, rather than finding the total, you're finding the opposite. I don't know why. Other than that, I don't...it must measure this as negative.

At this point, Lynn struggled to understand what had just taken place and to formulate an explanation for what had happened. She suggested several ideas about why the answer was negative, but it appeared that she leaned toward the definite integral measuring the area of region under the horizontal axis as negative. On a suggestion, she graphed the function $y = -3t + 4$. After having drawn the graph, she struggled some more with the results, but finally produced this response:

> Oh, I know why it's negative. Yeah, because you have...it would have to be because the, um, it's dealing with signed area. Because this area right here, underneath the function *(refers to the region above the horizontal axis)*, is smaller than this area that's underneath the x-axis. It's the only thing I can think of.

Thus Lynn concluded that the definite integral dealt with "signed area," and therefore the value from a definite integral could be negative. However, her final statement suggested that the issue was not resolved yet. When asked about her original answer to part b), she said, "I think I'm wrong now. It measures signed area." This provided further credence to the idea that she had changed her thinking about what the definite integral measured in this task, but the issue was not fully resolved. It was evident that she viewed both parts a) and b) as asking for the signed area of the region in question.

Finally, Lynn was asked why G was a function. Her initial response was, "I just expected it to be because it was written in the problem." She was asked if the word "function" had not been included in the statement of the task, would she believe that G was

a function. She responded, "Um, yeah, I probably would have, actually." When asked why, she said:

> 'Cause I just figured that, whatever the answer to this was *(right side of the function)* would equal this *(left side of the function)*, and I just thought it was a function because of that, um...*(long pause)* I don't know.

Her response indicated that she believed G was a function due to the assignment nature of the equal sign in the statement of part b) of the task. However, in the end she admitted that she did not know.

Response to Task 8

Lynn began by considering the definite integral $\int_{0}^{11} h(x)dx$, and pointed out that this integral involved three distinct regions, which were defined by the jump discontinuities. However, she then struggled to determine whether this integral represented total area or signed area. This struggle was a continuation of the difficulties she experienced during the follow-up questions for Task 7 part b) regarding whether the integral represented signed area or total area. Although she never expressed it aloud, once she started to work with the integral again, she used the notion of signed area to do her calculations.

She subdivided each of the three subregions into rectangles and triangles (see Figure 30). She then applied the respective area formulas to compute signed area values for

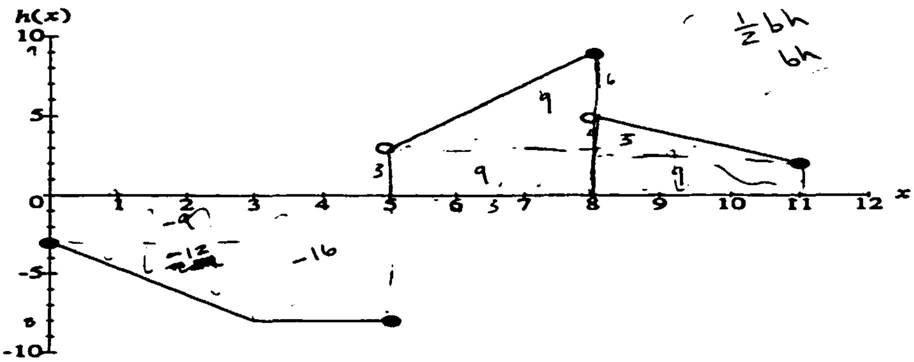


Figure 30. Lynn's diagram for ordering the five definite integrals.

each of the geometric shapes she had created on the graph. She made an error in computing the signed area of the triangle on the closed interval $[0,3]$ because she used 8 for the height instead of the actual height of 5. She also estimated the height of the rectangle on the closed interval $[8,11]$ as 3 instead of 2. Once the signed areas for the individual rectangles and triangles had been computed, she added the appropriate signed areas together in order to compute a value for each of the integrals in the task. Lynn then correctly ordered the integrals: "From the smallest to largest, you got c, a, d, e, and b."

Based upon her calculations for the integrals in this task, Lynn possessed an understanding of the summation property of the definite integral from a geometric standpoint. However, when she was asked to express her geometric work for the definite integral $\int_5^{11} h(x)\,dx$ in terms of integrals, she experienced great difficulty with the idea. Her initial response was, "Wouldn't I need the function to do this?" She was asked what she would do if there were an actual function to use, since it appeared that the lack of a function was an obstacle for her. Lynn was still unable to respond. A little later in the interview, when she was asked to comment on the effect of the jump discontinuity at $x = 5$, the following exchange demonstrated that she could express the summation property of the definite integral in a symbolic form:

Lynn: You'd have to evaluate each of these as separate functions, like, you'd have the integral of...you'd have the integral from 0 to 5, and then you'd have a function plus the integral of 5 to 8, and you'd have a function. You know what I mean?
Todd: Could you write it out please.
Lynn: The integral from 0 to 5 and then you'd have the function. And then you'd have the integral from 5 to 8, and you'd would have a function. Then you'd have the integral from 8 to 11; you'd have a function.
Todd: And all that together would be giving you...?
Lynn: This total. The total area of all 3 of these. I mean, it'd give you the total area of this *(shades in the three regions)*.

For the written work that she was asked to produce, she wrote:

$$\int_0^5 \_\_\_\_ dx + \int_5^8 \_\_\_\_ dx + \int_8^{11} \_\_\_\_ dx.$$

The blanks in her mathematical work above were for the functions she mentioned in the preceding passage from the interview. It was apparent that she thought a different function was needed for each of the blanks, and she provided support for this notion during follow-up questions concerning why she split up the integral at $x = 8$. Lynn said, "'Cause it's a different function, once again." It was unclear whether she meant that there were actually different functions involved or she was referring to the individual expressions that made up the piecewise-defined function h.

By the end of the preceding interview passage, Lynn had switched to viewing the definite integrals as computing total area. This indicated that she had slipped back into her original understanding, as expressed in Task 7 during interview 1, that the definite integral represented total area rather than signed area. Her usage of total area indicated that she was still in disequilibrium over how to interpret the definite integral when using the area viewpoint.

Response to Task 9

Lynn read the task twice, and then paused for several seconds before providing the following insight into her thoughts:

> Why am I making this so hard? It's easy. The area is just 2, so as long as underneath the graph, there's only an area of 2, then anything will fit it. Whether it starts down here *(indicates below the horizontal axis)* and moves up here *(indicates above the horizontal axis)* or not.

This passage indicated that she struggled with how to start the task, even though she believed that it was easy. Her description demonstrated that she was aware of what she needed to do to complete the task. One possible explanation for this particular statement could be that she was still dealing with the disequilibrium caused by the total area versus signed area dilemma introduced at the beginning of the second interview. Another possible

explanation would be that she was struggling with how to select an example from the many examples that would work.

She finally drew a graph of a function that satisfied the given conditions (see Figure 31). She used the function $y = 1$ over the closed interval $[1,3]$. When asked what she

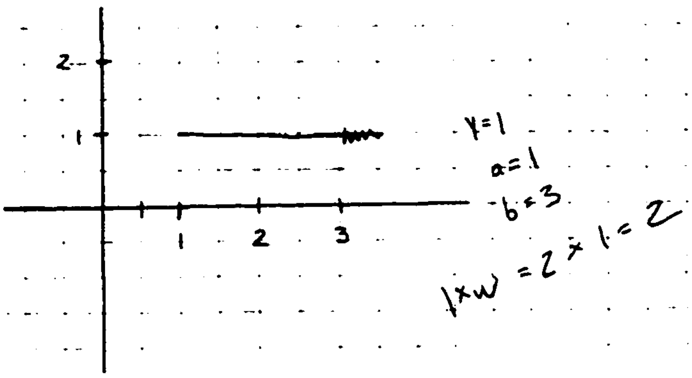


Figure 31. The graph Lynn created to satisfy the integral equation $\int_a^b f(x)\,dx = 2$.

thought about as she constructed her graph, Lynn stated that she was "trying to find a function that would, um, have an area of 2, like, beneath the function, between the axis and the function." This statement provided evidence that she used the viewpoint of area to sketch her graph. She confirmed this by sketching a graph of $y = x$ over the closed interval $[0,2]$ as a second example, but described her construction as a triangle made up of "just one little box being 1 unit, and then two half a boxes equaling 1 unit." As a final note, Lynn used a similar box and two half boxes construction to complete this task on the participant selection tasks (see Appendix E). This indicated consistency in the type of image she evoked when considering this task.

Response to Task 10

Lynn experienced extreme difficulty with this task, and in the end decided that she could not do it. She attempted the task twice. Each time, she started off with an area of 2 units above the x-axis, which forced her to have 2 units of area below the x-axis in order to satisfy the second integral condition. However, by doing this she always ended up with the computed value of the first integral being 0. She did mention having another unit of area above the x-axis, but she decided against that because "it'll make this *(first integral)* greater [than 2]."

Her interpretation of the integral conditions provided insight into how she approached the task. Lynn viewed the second definite integral as stating "it doesn't matter whether it's below or above the x-axis. You just want the total area to be 4." She also made the statement that the second definite integral was "determining the total area, rather than signed area." She then explained that the first definite integral was "saying that it needs to be above the x-axis, and it needs to be 2." This statement meant that the first definite integral dictated that there needed to be 2 units of area above the x-axis. This view explained why she started each attempt with 2 units of area above the x-axis, as well as why she would not consider having a greater amount of area above the x-axis. However, when she checked each of her attempts, she calculated the first definite integral as signed area, rather than simply pointing to the 2 units of area above the x-axis with which she started. This difference in how she used the first definite integral in her work seemed to be a likely source for her inability to sketch an appropriate graph for this task.

Response to Task 11

Lynn made a couple of false starts on this task before getting all of the information organized. Once organized, she found the value of the integral $\int_0^1 f(t)\,dt$. The following passage described how she carried out this calculation:

> Um, from 1 to 2 you have 1.25 and from 0 to 2 you have 0.75. So to determine the integral from 0 to 1, you could take 0.75 minus...it would be minus 1.25. Let's see, 1.25... I'll do it on this *(meaning the calculator)*. 0.75, which is the total from 0 to 2, minus 1.25. So you have -0.5 equals the integral from 0 to 1.

Follow-up probes revealed that she was thinking in terms of the summation property of the definite integral while carrying out her work, but she never showed her work in terms of the summation property. These probes also showed that she was using the word "total" to mean the final value over the closed interval $[0,2]$ obtained from combining the values of the definite integrals over the closed intervals $[0,1]$ and $[1,2]$.

Lynn then found the value for $F(0)$. She expressed and clarified her approach to computing this value in the following passage:

Lynn: OK, if...*(pause)* So I'll use this one *(underlines the integral from 0 to 1 that she previously wrote)*. OK, if F...if antiderivative at 1 equals 0.3, you'd have 0.3 minus a number equals -0.5.
Todd: And why are you saying that?
Lynn: Because, OK, you have...you're evaluating what the function would be at 1, 'Cause you're taking the anti[derivative]... OK, you're taking the antiderivative of a function, right?
Todd: OK
Lynn: And you're evaluating it at 1.
Todd: Mm-hmm.
Lynn: And you get 0.3.
Todd: OK.
Lynn: And then you subtract the antiderivative evaluated at 0, to get the total of the function.
Todd: And the total represents?
Lynn: The total area of the integral.

Based upon this passage, she used The Fundamental Theorem of Calculus as the guiding force behind her work. From a mathematical perspective, she used

$$\int_0^1 f(t)\,dt = F(1) - F(0)$$

to set up the calculation

$$-0.5 = 0.3 - F(0).$$

From this point, she solved for the value of $F(0)$ algebraically to determine that $F(0) = 0.8$.

The end of the preceding passage indicated that Lynn still associated the definite integral with the idea of total area, where total area referred to the geometrical area of a region. The instability of her view of the definite integral was further revealed a little later when she stated the integral $\int_0^1 f(t)\,dt$ measured the signed area. In particular, the negative value for this integral meant there was "more area below the x[-axis] than above the x-axis." Therefore, it was evident that the disequilibrium introduced at the beginning of the second interview persisted as the interview continued.

As Lynn completed her calculations to find $F(0)$, she moved directly into calculating $F(2)$. Again, the essence of her explanation revolved around The Fundamental Theorem of Calculus, and was very similar to the approach to find $F(0)$. In particular, she was using

$$\int_1^2 f(t)\,dt = F(2) - F(1)$$

to set up the calculation

$$1.25 = F(2) - 0.3.$$

She then solved the latter equation for $F(2)$, and found that $F(2) = 1.55$.

Response to Task 12

Lynn plotted the data points and drew in a polygonal curve that passed through each of the data points (see Figure 32). When asked why she drew in a curve, she admitted, "it

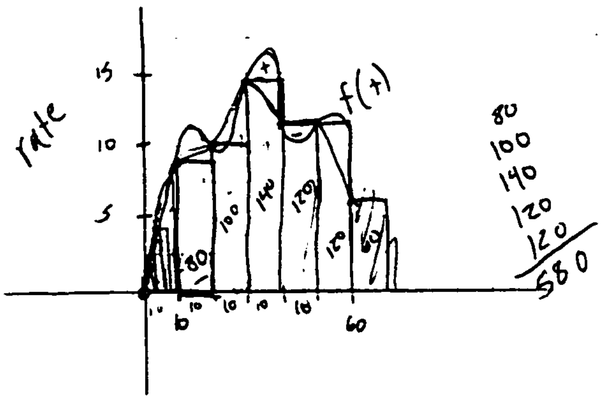


Figure 32. Lynn's graph and graphical work using the harvest rate data.

was easier for me to see, like, the increases and decreases in the rate during the time interval." The curve was used as an aid to help her relate to the data. She also acknowledged that there were other curves that could work, as long as they passed through the data points. Some other samples were sketched in on the graph in Figure 32.

Once she had completed her graph, Lynn said that she wanted "the integral from 0 to 60." Before this idea could be explored, however, she moved on to discuss estimating "the area underneath this *(lightly shades region under the curve)* by, um, you could do it by Riemann sum or however you wanted to estimate it." At this point, she reiterated that she did not understand Riemann sums very well, but set up a left-approximating sum as illustrated in Figure 32. In drawing the necessary rectangles for the left-approximating sum, she drew one too many rectangles at the right end of the graph. When asked to explain why this rectangle was needed, she realized that it did not belong there and marked it out. Lynn then computed the area of each of the five rectangles and summed up the areas to compute an approximate total harvest, although she did add incorrectly (see Figure 32).

At this point, the existence of different curves that would fit these data points arose again. When asked to comment on how the estimate would be affected by the use of different curves, she replied, "the estimation would be the same because the points are in the

same spot." In this sense, she demonstrated that she understood that the estimate would be independent of the curve. She also commented that using different curves "could affect the total amount," by which she meant the total harvest. Her explanation for this statement revolved around the notion of area under the curve and that different curves could bound different amounts of area.

Lynn was asked about improving her approximation of the total harvest. She said, "You'd have to make the, um, rectangles smaller and smaller until they get really, really close together." The following section of the transcript provided a clarification of what she meant by the phrase "smaller and smaller...:"

Lynn: What would have to be done? Just to make the...*(pause)*...to make the rectangles smaller. Just to, like, make them, like...instead of having them 10 *(referring to the base)* right here, just have them like 5 or smaller and smaller until they get to 0.
Todd: Till what gets to 0?
Lynn: Till the distance between them gets to 0. Like so that it's just a point after a point on this line *(refers to curve for harvest rate)*, so it's like, know what mean, like it's just the line, instead of worrying about having rectangles. It...they decrease So that they become...they just become (mumble)... (laughs) I don't know how to explain this.

It was concluded that Lynn was trying to describe the notion that the estimation could be improved by letting the mesh of the partition go to zero. Although she never stated it specifically, she had a sense of the idea of a limiting process on the approximating sums. She also indicated that the rectangles would eventually become lines, meaning height of the function. However, since she did not explain her thoughts, it could not be determined how firmly she held the latter image, or if it was simply an attempt to find an explanation that made sense. She was not able to provide additional insight to help clarify her original notion.

When Lynn began working on this task, she mentioned integrating from 0 to 60. Towards the end of her work on this task, she was asked to explain what she meant. Her response was:

Lynn: Um. If you could make a function for the line *(meaning the curve)*, you could evaluate from 0 to 60 just by finding the antiderivative and sticking in 60, subtract what you get from that amount, well... OK, stick 60 in the antiderivative and subtract what you...when you put 0 in the antiderivative. You get the total. You get the complete amount of area under the graph.
Todd: And that would represent?
Lynn: That would represent total number of bushels harvested over a 60-minute period.

This was interpreted to mean that if she had a function for the harvest rate, she would apply The Fundamental Theorem of Calculus in order to find the total number of bushels harvested over the 60-minute period. However, at no time did she make a connection between her approximating sums and the definite integral. Her work on this task indicated that the only connection between the two ideas was that the approximating sums provided a way to estimate the value of the definite integral.

Response to Task 13

Lynn had difficulty comprehending this task, and this was a source of frustration for her as she attempted to work on it. The notion of density itself contributed to her frustration, as she could not relate to it. She confused density with the notions of mass and area. As she spoke her thoughts, there were several times when it was evident that she held the image that density times length was like area, and this image persistently interfered with her thoughts. Another source of difficulty for her was that she wanted to view mass as area or volume. She had difficulty relating to the one-dimensional setting of the task, because it conflicted with her three-dimensional thoughts. Finally, the newness of the task bothered her; she had never seen a task like this before, so she did not know what to do with it.

After her initial struggle to understand the information given in the task, she formed the definite integral $\int_0^x 100p(x)\,dx$ to compute the mass of the rod. Lynn explained that the 0 in the lower limit of integration was for "the beginning of the rod." The x in the upper limit of integration denoted how far along the rod she had measured from the beginning.

The integrand of $100p(x)$ was the length of the rod multiplied by density and, as such, gave the mass of the rod. She tried to explain why integrating the mass would give the mass of the rod, but was unsuccessful. The best explanation that she could give came early in her work, at which time she said, "...this is the way it fits together for me." She indicated that it was a "gut reaction" on her part as to why the stated integral would compute the mass of the rod. One explanation for why integration made sense to her was that Lynn was confusing mass and density with area and volume. Area and volume were topics covered recently in her Calculus II section, so these ideas could have influenced her thinking towards this task.

Response to Task 14

Lynn's response to the question posed in this task was that the definite integral "represents the area underneath the graph of $f(x)$ from the interval of *a* to *b*." She followed this up with this additional comment: "you find the antiderivative of $f(x)$, and put *b* in antiderivative subtract *a* into the antiderivative." The latter comment indicated that The Fundamental Theorem of Calculus was part of her representation for the definite integral.

During the conversation concerning her view of the definite integral, she returned to her dilemma between signed area and total area, and the connection to the definite integral. She referred to her work on Task 7 from the follow-up section of the second interview by saying that she "found out that integrals are, can be positive or negative." Lynn struggled to remember what had taken place with Task 7, and by the end of her work with the current task concluded that the definite integral represents signed area. How firmly this view was fixed in her mind could not be determined.

This task appeared on the participant selection tasks (Appendix E). At the time that was administered, Lynn wrote the following: "The antiderivative *F* from a point *a* to a point

b." It was concluded that she probably had The Fundamental Theorem of Calculus in mind. In addition, she included a picture below her written statement that indicated a connection to area under the curve, but no connection between these two ideas was made on the participant selection tasks. A comparison of the two responses to this task led to the conclusion that her view of the definite integral was in flux. This conclusion was based on her use of the notion of area during the interview, which was only hinted at on the selection tasks, and the fact that the computational aspect of the definite integral had taken a secondary role in her response during the interview.

Response to Task 15

Lynn's initial response to this task was that the definite integral $\int_a^b f(x)\,dx$ would be the area under the curve of *F*. However, she did not appear to be comfortable with this response, as she kept re-reading the task and talking through The Fundamental Theorem of Calculus as applied to the function *f*. During this time, it was verified that she knew the graph she was looking at was the graph of *F*, but she demonstrated that she was so focused on the idea that the definite integral represented the area under the curve that initially she missed the significance of having the graph of *F*. When asked if her response made sense, she said "no." She continued to explain why her response did not make sense with this comment: "say this was $f(x)$, rather than the antiderivative, then that *(meaning the given integral)* would be the area." $\int F(x)\,dx$. on to say the area bounded by the graph of *F* and the *x*-axis would be given by $\int_a^b F(x)\,dx$.

After this, Lynn continued to struggle with this task. She continued to focus on The Fundamental Theorem of Calculus and the notion of area. She also began to express her frustration with being unable to do the task. This was evident through comments like,

"Why is this so confusing for me?" and, "This can't be all that hard of a question. I don't really know why I'm having such a problem with it." Her continued focus on the notion of area led to the conclusion that area was a cognitive obstacle to her ability to complete the task. This focus on area was so strong that she was unable to see any connection between her work with The Fundamental Theorem of Calculus and the graph of F.

In an effort to determine whether she could think beyond the notion of area, Lynn was asked where $F(b)$ and $F(a)$ would be located on the graph. She plotted the points on the graph, and soon after she had done this, she drew a line between these two points (see Figure 33). However, she was still focusing on area, because she went on to conclude that

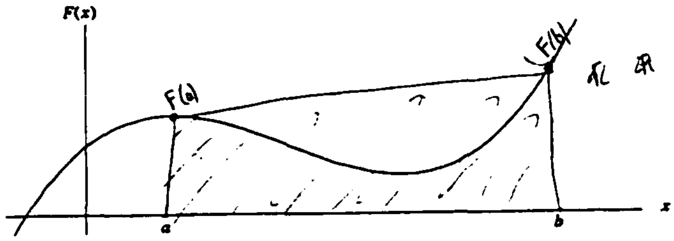


Figure 33. Lynn's use of area place the value of the definite integral $\int_a^b f(x)\,dx$ on the graph of an antiderivative F.

the area of the trapezoid she had just formed would be what the definite integral $\int_a^b f(x)\,dx$ represented on this diagram. Furthermore, she said, "it makes sense to me, but I don't know why." As she continued to try and explain why her last attempt made sense to her, she came the closest to the idea of the definite integral $\int_a^b f(x)\,dx$ being the vertical

displacement or the total change in F between $x = a$ and $x = b$. The following comment provided the basis for this conclusion: "OK, this point *(pointing to point labeled $F(b)$)* subtracting this point *(pointing to point labeled $F(a)$)*, and you're finding the total area *(shades the trapezoid's area)* between those two points, underneath of them." However, as the latter part of the comment indicated, she was still thinking in terms of area. As further evidence of the influence that the area obstacle held over her, she made her last comment about subtracting $F(a)$ from $F(b)$ with the statement $F(b) - F(a)$ written out on the paper from her earlier work. Therefore, it was evident that Lynn was not going to be able to overcome the impediment caused by her focus on area during the course of the interview.

Response to Task 16

Lynn's definition for the definite integral can be summarized by the following passage:

Lynn: I'd probably just tell them what it stood for, and then I'd probably just show him how to work a problem. 'Cause that would be the easiest way for me to explain what it is to them.

Todd: So, could you come up with a simple explanation? Pretend I'm this student.

Lynn: OK. I'd probably just use, like, the function $y = x$, and I'd let him graph it on a calculator if they needed to. And then I would say that this is the function. *(Writes $= f(x)$ after her $y = x$.)* x equals the function, and you're evaluating it from, well, say you're evaluating it from $x = 1$ to $x = 3$. All right, so, x equals, if you're evaluating it from $x = 1$ to $x = 3$, then a would be the starting point of the evaluation and b would be the ending point of the evaluation, and you always start from the less number and go to the greatest. And you evaluate your function, which is x. *(Has been writing out $\int_1^3 x\,dx$ as she has been talking.)* The first thing that you would do is you take your antiderivative of x. That's the first step in determining a definite integral. And that would be $x^2/2$. And then you just evaluate that on the interval that was chosen. And you'd just stick it in, you have $\frac{3^2}{2} - \frac{1^2}{2}$. And you just do that, and that's your answer. That would be the easiest way for me.

It was concluded that Lynn's definition of the definite integral revolved around the mechanical aspects of working with the definite integral and, in particular, The Fundamental

Theorem of Calculus. Even though the majority of her explanation focused around the computation of her example, she also mentioned telling the student what the definite integral stood for and graphing the function. However, she never returned to expound upon these ideas as she worked on this task.

Even though she stated that the definite integral represented "area underneath the graph of $f(x)$ from the interval of a to b" in Task14, this idea never was mentioned during her description of a definition of the definite integral, unless this was what she meant by her phrase "I'd probably just tell them what it stood for" from the above passage. Finally, even though Lynn could use the notion of Riemann sums to work various tasks, the idea of a Riemann sum never surfaced during her work on this task.

Response to Task 17

Lynn was asked how the picture given in this task might be related to a definition of the definite integral. She responded, "Well, this picture, by using these rectangles is, um, estimating the area below the function, which is what a definite integral is finding, is the area below the function, um, on the interval from a to b." This was interpreted to mean that the definite integral computed the area under the curve, but the picture would provide a way to estimate this area. In addition, she connected this task with Task 16 when she said, "Well, when you're finding an integral, you're finding the exact area, but show them how to estimate it first would be best. So they can understand what you're doing with the definite integral before they actually do it." This indicated that Lynn viewed this task, through its use of estimations, as a good way to begin the process of comprehending the definite integral. However, when asked whether a definition for the definite integral had more to do with the picture from this task or the type of computation illustrated in Task 16. She replied by saying:

> I think the picture would be a lot more helpful for somebody that's just beginning to learn it. Although I find it a lot easier for me, now that I know what I'm doing, to

> work with this form *(pointing to the calculation from Task 16)*. Rather than a graph, estimating the [area] with squares or boxes.

This passage was interpreted to indicate that Lynn saw the benefit in estimating to begin learning about the definite integral, but her own definition still revolved around The Fundamental Theorem of Calculus.

From here, the focus changed to approximating sums. When questioned regarding the fact that the rectangles in the picture had different widths, Lynn commented that "you can still estimate the area with them." This indicated that a summation process over a non-regular partition of an interval was a possibility to her. However, she continued by saying, "it's not like using a Riemann sum where all the rectangles are the same." The phrase "the same" was interpreted to imply the rectangles all had the same width. In fact, it turned out that Lynn did not view the summing of the areas of the rectangles in the picture as being a Riemann sum for two reasons: the widths of the rectangles were not the same and the sum was not a midpoint, left, or right approximating sum.

Lynn continued by saying that the given picture "could be the first step" to estimating the "area under the curve." However,

> when you're actually finding it *(referring to area)*, like with Riemann sum, you do use the same size rectangles, and they do become closer and closer to 0 as you get closer and closer to the correct, I mean, to the complete answer, rather than approximating.

This passage was interpreted as her attempt to explain the notion of letting the widths of the rectangles approach zero so that the approximation of the area would approach the actual area of the region. Further questioning confirmed this interpretation. In addition, her comments indicated that she was trying to describe a limiting process without using the formal notion of "limit." This conclusion was supported by the following passage: "the closer and the closer that you come together, that disappears *(refers to missed area between rectangles and graph)*. And it's all, eventually, it's completely, on the line." The ideas

expressed by Lynn during the latter part of this task indicated that there was a limiting process connected with her image of a Riemann sum. This provided a natural lead-in to the next task, which focused on the connection between the limit and the definite integral.

Response to Task 18

When asked about the connection between the limit and the definite integral, Lynn's initial response was "I know it's involved, but I don't...I don't know how to explain." She then indicated that the source of her inability to explain her ideas was that she did not understand the notation that was used with the Riemann sum; however, she claimed to understand what the limit does. Therefore Lynn was asked to explain how the limit was involved with the definite integral without worrying about the formal notation. She struggled with verbalizing her thoughts, but eventually she provided this response:

> *(pause)* The limit...*(long pause)* Would the limit approached 0? If the limit would approach 0 or infinity, but I don't know whether the limit's describing , like, the number of infinite lines that you're having drawn down after you get them all put together, you know what I mean, or if it's just 0, because they're all together and there is none, you know what I mean?

This passage indicated that Lynn saw the limit as a part of the definite integral. However, she was not sure about the exact nature of the connection. It was evident that she was unable to decide whether the limit involved the idea of the number of rectangles going to infinity, or the idea that the mesh of the partition was approaching zero. Within her explanation she used the phrase "number of infinite lines," but it was unclear what she meant. If she continued to follow her past usage of the phrase, then it was likely that she meant the number of rectangles would approach infinity. Lynn did provide some clarification as to what she meant by the latter part of her response when she said, "I mean, if they *(referring to rectangles)* become closer and closer together, pretty soon you're going to have just this area underneath here *(points to the curve in Task 17)*." With the addition of this last statement, it appeared that she possessed an understanding of the

rudiments of the limiting process involved with the definite integral, but she was not sure how to quantify the idea that the rectangles became "narrower and narrower." She eventually gave up in frustration, without ever tying the limit notion into her definition of the definite integral.

Response to Task 20

Lynn struggled to comprehend the definition for the function *f*, but after she had evaluated *f* at several *t*-values, she began to understand it. She correctly sketched a graph for the function (see Figure 34).

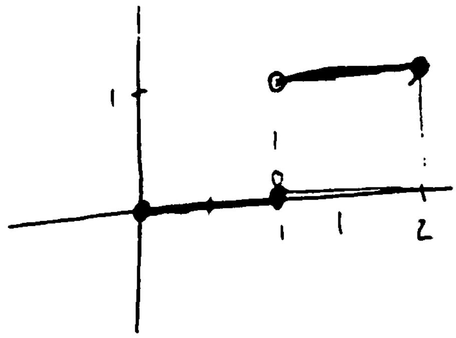


Figure 34. Lynn's graph of the piecewise-defined function *f*.

Once she had her graph of *f* sketched, she was asked how she would sketch a graph for the function *F*. Lynn started by focusing on computing $F(2)$. She explained her computation by saying "it'd just be this area *(forms a rectangle on the interval from 1 to 2, and lightly shades it in)*. This area right here is what you're evaluating." This indicated that she was using the area of the region bounded by the graph of *f* and the *t*-axis over the closed interval $[0,2]$. She was then asked about the value of $F(1)$. She replied that it would be 0, because "it's just this *(points to* $y = 0$*)*, there's no area" while referring to the graph of *f*. Based upon these two computations, it was evident that she viewed the integral

function F as determining area. It was also apparent that she used only a pointwise understanding of the function F to approach the task.

After her two computations, she was again asked about a sketch for the graph of F. She struggled initially, and expressed some frustration, but she eventually produced the graph shown in Figure 35. When asked to explain why this graph worked, she provided the

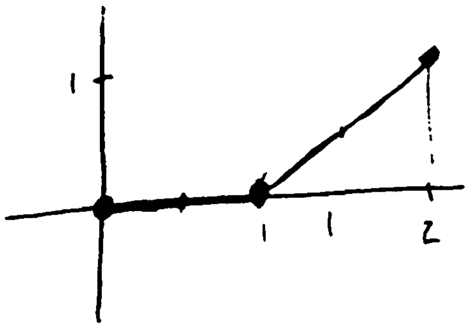


Figure 35. Lynn's graph of the integral function F associated with the piecewise-defined function f.

following explanation: "OK, from 0 to 1, I said it was 0, so it's 0 *(points to the point* $(1,0)$*)*, and then from 0 to 2, I said it was 1 *(points to the point* $(1,1)$*)*." Based upon her work, the intervals she referred to in her explanation intervals she was computing the area over to find values for $F(1)$ and $F(2)$. Pressed for more details about the graph of $y = x - 1$ on the closed interval $[1,2]$, she began an explanation, but ended up saying, "I don't know. I was guessing." Further probing did not reveal any additional insight into why the graph she drew for F made sense to her.

The Case of Rob

Introduction

Rob was a 19-year-old second semester freshman pursuing a Japanese major at the time of the study. Prior to Calculus II, he had taken Calculus in high school and Calculus I at the institution where the study took place. He took Calculus I during the academic year prior to the interview. Rob earned a B in Calculus I.

At the end of the third interview, Rob was asked to respond to questions focusing on his thoughts regarding his understanding of the definite integral to provide insight into how he approached the tasks. In response to a question about how he thought about the definite integral, Rob responded "probably graphical or symbolical, 'cause I usually either figure it *(referring to the definite integral)* out using the function or using a graph." When asked if he viewed the definite integral abstractly or conceptually, he responded "not usual[ly], no."

In response to the inquiry about what idea(s) he perceived as most difficult for him to understand about the definite integral, he said that prior to the techniques of integration he was then learning in Calculus II, nothing was really difficult about the definite integral. For Rob, the application of The Fundamental Theorem of Calculus to compute the area under the curve was the easiest aspect of the definite integral for him to understand.

Response to Task 1

Rob's first response to part a) was to try the substitution method. He let $u = 3x^2$, which was inappropriate for this task. However, before finishing the task, he said, "actually, I don't even need to do substitution. I can probably just work the problem out without the substitution, because it's all addition and subtraction, so we'll try it like that." His attempt to use the substitution method seemed to have been influenced by the fact that this had recently been a topic in his Calculus II section. After he realized that the substitution method was not needed to evaluate this integral, he applied The Fundamental Theorem of

Calculus. He correctly found the antiderivative, but substituted 1 for x instead of -1. Rob's work demonstrated that he knew how to apply The Fundamental Theorem of Calculus, but he did not pay sufficient attention to the details of the calculations.

Rob then moved on to part b) of the task. He easily evaluated this definite integral using The Fundamental Theorem of Calculus. These two subtasks showed that Rob had no difficulty finding antiderivatives for elementary functions and applying The Fundamental Theorem of Calculus.

He was not familiar with the greatest integer function, but Rob picked up on the idea quickly once the definition was explained to him. After a few examples involving the greatest integer function, Rob was ready to look at part c). He applied The Fundamental Theorem of Calculus to the integral in much the same way that Lynn computed this integral. Since no additional information was gained from Rob on this aspect of this subtask, his work will not be reported here. However, when asked about how he could check his answer, he replied "I'm not sure of one, I'm sure there is, but...could we do it by differentiation? Um, I'm not really sure." At this time, he gave no indication of viewing the integral graphically. Rob gave no indication to use the graphics calculator as an alternative method to evaluate the definite integral.

Response to Tasks 2 and 3

When Rob was presented with Task 2, he immediately recognized the function g in the graph as the absolute value function. He then rewrote the task by replacing $g(x)$ with $|x|$ and by so doing, addressed Task 3. With the rewritten integral, he tried to apply The Fundamental Theorem of Calculus by finding an antiderivative. He was uncertain how to determine an antiderivative of the absolute value function. After a pause of several seconds, the following exchange took place:

Rob: If there's absolute value we couldn't just have $(1/2)x^2$.

Todd: Why do you say that?
Rob: Um, just doesn't seem right, um, 'cause the absolute value has to fit in there somewhere, with the problem, and you can't just forget about that. Um...*(pause)* Why don't [I] just try evaluating it that way. Of course, uh, with it being squared, it's gonna, not...we're not gonna have to worry about the absolute value anyway. 'Cause all the values'll be positive, 'cause of they're squared. *(Works through task.)* So it will be 9...so 4.5 minus...4, so 2...so it's equal to 2.5.

Rob did not believe $(1/2)x^2$ was the proper antiderivative because he did not know what to do about the absolute value bars. However, he decided to proceed anyway, since squaring x alleviated the need for the absolute value bars. His actions indicated that he felt the need to do something here, even if it was not a correct approach. Even though the graph from the original task was still on the table, he did not consider using it to evaluate the integral. This led to the conclusion that Rob was unable to disengage from his analytical approach to consider other possible methods. His thoughts were rigidly compartmentalized and there were no escapes from a compartment once he had accessed it.

After he had a value for the integral, he said, "And that's the area under the curve from 3 to -2. And we can check that graphically...." He then turned to a graphical approach for purposes of checking his answer. His desire to check his answer may have resulted from the uneasiness he felt over his antiderivative. He drew a graph and divided the region bounded by the graph of the absolute value function and the x-axis on the requested closed interval into squares and triangles (see Figure 36). After he completed his drawing, he said, "Actually, [it's] not gonna come out to the right value, 'cause you'll have 1, 2, 3, 4, 4.5, 5, 6, 6.5. *(Counts blocks under the graph.)* So the total area would be 6.5, instead of 2.5." Even though the original task was presented to him as a graph, and that graph was on the table in plain view while he tried his analytical approach, he used a graphical approach only to check his work. His graphical solution provided a resolution to the disequilibrium he felt during his attempt to find an antiderivative for the absolute value function.

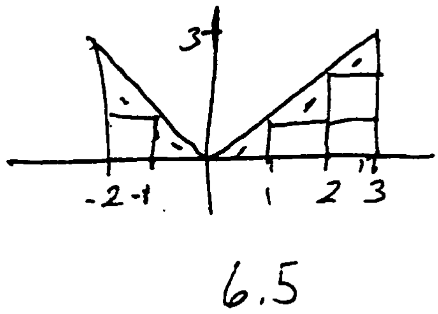


Figure 36. Rob's graphical check of his analytic work to evaluate the definite integral $\int_{-2}^{3} |x| dx$.

Response to Task 4

After reading the task, Rob said, "first of all, you're going to have to find what the function $f(x)$ is." He then indicated that he would compute the area of the region by using the summation property of the definite integral by writing out:

$$\int_{1}^{4} f(x) dx + \int_{4}^{7} f(x) dx.$$

The reason that he gave for using a sum of two integrals was, "since they're not continuous, you're gonna have to add those areas separately." When asked what he meant by "not continuous", Rob answered, "$f(x)$ is not continuous, um, because there's a break in the function right there where it switches from a curve to a straight line." Although he used the wrong terminology, his remarks indicated that he recognized that two integrals were needed to find the area using integration techniques because the function contained both a linear expression and a non-linear expression. When asked to explain how he would find the area, he changed to a graphical approach. His reason for the change was "I think that might be an easier solution than trying to find the function without having the function." Later in the interview, Rob was asked what he would do if he had found the

function for the curve. His response indicated that he would apply The Fundamental Theorem of Calculus to the two integrals separately, and then sum the individual results to find the area. At that time he was presented with the function. He successfully set up the necessary integrals, and then unsuccessfully tried to apply The Fundamental Theorem of Calculus.

Rob was asked to expand upon his graphical approach. He subdivided the closed interval $[1,7]$ into 12 equal-sized subintervals, then constructed a polygonal curve to approximate the function f (see Figure 37). The reason he gave for his approach was, "then

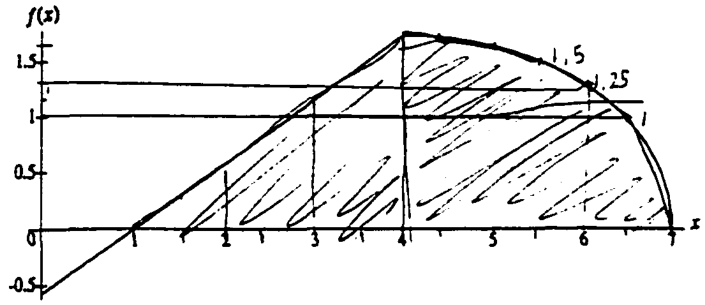


Figure 37. Diagram for Rob's area approximation using a polygonal curve to approximate the function f.

you can approximate the area under the curve with the straight lines." After he had completed sketching the polygonal curve, he focused upon the part of the graph associated with the closed interval $[4,7]$. He approximated the y-coordinates for the endpoints for the six subintervals of length $\Delta x = 1/2$ within the closed interval $[4,7]$. He then formed the following two sums where *rh* represented right-hand and *lh* represented left-hand:

$$rh = 1.75 + 17. + 1.5 + 1.25 + 1 + 0$$
$$lh = 1.78 + 1.75 + 17. + 1.5 + 1.25 + 1.$$

The values for *rh* and *lh* were then averaged to obtain a value for the area of the subregion associated with the closed interval $[4,7]$. The value Rob reported for this area was 8.09. Upon careful inspection, it was concluded that Rob was indirectly trying to apply the Trapezoid Rule. However, he did not include the height of each trapezoid, that is, he left $\Delta x = 1/2$ out of his calculations. This caused his approximation to be twice the value that it should have been, but he did not realize this while he carried out his work. He applied the same technique to the closed interval $[1,4]$, only this time he used $\Delta x = 1$. He approximated the necessary *y*-values, and computed the following two sums:

$$rh = 0.5 + 1.15 + 1.8$$
$$lh = 0 + 0.5 + 1.15.$$

Again, he computed the average of the two sums to determine an area of 2.55 for the subregion associated with the closed interval $[1,4]$. This time his sum was not affected by his having left out the value $\Delta x = 1$ in his calculation. He never acknowledged that he was dealing with a triangular subregion.

As soon as Rob had completed his calculation of 2.55, he said, "there's quite a bit of discrepancy in there, so I'm a little unsure about 8.09 now." Asked to elaborate, he replied, "the discrepancy is 8.09 is quite a bit bigger than 2.55, and the areas really aren't that much different in size." This indicated that he used the two area values to validate his work, but since the two area values were not similar in size, something was wrong. He struggled to find a reason for the discrepancy. His best idea was that the areas were so different because he had used three subintervals to compute one of the areas and six subintervals to compute the other area. It never occurred to him that the discrepancy was the result of not including Δx in his calculations. His final thought was "I might have even done it entirely wrong. I'm kind of at a loss right now." At this point he was quite perplexed.

Response to Task 5

After Rob read the task, he substituted the value 5 into the upper limit of integration. He decided to use the substitution method to compute the integral. This probably occurred because his Calculus II section had studied the substitution method prior to his interview. The main difference between Rob's work and Joan's work on this task was that Rob replaced the x with 5 right away. He initially computed the antiderivative incorrectly, and he made an arithmetic mistake while he finished evaluating of the integral. Rob did not provide any new insights into students' understanding of the calculations associated with this task.

Rob did have an interesting explanation for the roles of x and λ. He said:

> Lambda is the independent variable, um, it takes the place of the, um, x-axis…the x in this, uh, equation just represents the upper limits of the function. So λ is just the independent variable.

Thus, Rob viewed the function G as a part of the $\lambda G(x)$-coordinate system, and therefore x was just a value along the λ-axis. Apparently, he focused on the integrand instead of the integral function when he decided the roles of the variables, and tried to place everything into one coordinate system.

He demonstrated that he had a pointwise understanding of the function G, since he was able to explain how the evaluation would work for other values of x. He indicated that the evaluation could be carried out for any value of x. However, he did have an unusual comment about what would happen if $x < 0$. When $x < 0$:

> You'd have to switch the upper bound with the lower bound, because 0 would be the upper bound if it was a negative number, unless you were…'cause that would reverse the, uh, area, so you'd have negative area instead of positive area.

It was evident that Rob wanted to preserve the order relation of the real line within the limits of integration. However, what he meant by the latter part of the explanation was unclear.

Response to Task 6

After reading part a), Rob paused for several seconds, then provided his initial observation about this task. He said, "the area's gonna get bigger, but it'll get bigger slower as you move to the right more." Based upon additional information gained through follow-up questions, Rob meant that as x continued to move to the right, the amount of area in the specified region would continue to increase, but at a slower rate. In other words, as x increased in value, the amount of area would continue to accumulate, but smaller amounts of area would be added on as x continued to move to the right. Rob was asked to speculate what would happen if x continued to move to the right without ever stopping. He initially indicated that the amount of area would become infinite, but when asked if there was a chance that the area would approach a finite number, the following exchange took place:

Rob: There is a chance.
Todd: How so?
Rob: Because as the values get smaller and smaller, um, you can keep adding less and less and it's gonna eventually get to a point where it's only gonna be reaching a certain number, um, 'cause you'll be adding such small amounts to that number that it'll get closer and closer but never actually reach it.

His explanation indicated that he had switched to believing that the area would approach a finite number because the graph was approaching the t-axis, and therefore the amount of area being added on was becoming smaller. He also indicated that his "certain number" would depend on the behavior of the function's graph and on the value of the y-intercept. Further probing demonstrated that he believed the area would stay finite. Asked if the area could ever become infinite, he said, "not if it *(indicates the curve itself)* keeps going to 0, because it'll eventually get so small that it can never get higher than this number." This was interpreted to mean that the area would not become infinite, because the function values approached 0 as x moved to the right, and so the area values would never exceed the particular finite number that represented the area of the region.

In order to ascertain how firmly Rob held the belief that as x went to infinity, the area approached a finite number, his certain number, he was asked to determine the area of the region bounded by the t-axis and the function

$$f(t) = \frac{1}{t+1}$$

for various values of x. He used the integration feature of his graphics calculator to construct a table of area values (see Table 5). When asked what he thought the area would

Table 5.

Rob's Table for Area Under the Curve $f(t) = \frac{1}{t+1}$

x	Area
10	2.398
100	4.615
1000	6.909
100,000	11.513
100,000,000	18.421
100,000,000,000	25.328

do if x continued to increase, his initial response was "you'd get to infinity, but it would be such a huge number." This was interpreted to mean that the area would approach infinity, although he saw infinity as being "a huge number". However, as he continued to explain his thoughts, he talked himself out of this idea after he considered the function again. In his final explanation, he said the area will, "keep getting bigger, but at a very small comparison towards, uh, in comparison with t. So it'll keep getting bigger, but never infinity, because of the function $1/(t+1)$, because it'd be 1 over a really large number plus one." The essence of what he seemed to say was that the area would continue to increase in value, but would not increase fast enough to approach infinity, because the function values were approaching

zero. Therefore, he was staying with his conclusion that the area would approach a finite value.

When he considered part b) of the task, he pointed out that what he had been computing in part a) using the graphics calculator was an example of part b). He also concluded that the values of G would get larger, "but at a much slower rate than x." By this, he seemed to mean that the values of G would continue to grow, but at a much slower rate than the values of x would grow. His response was essentially the same as in part a), only this time he had not taken a stand on whether the values of G would approach a finite number or approach infinity. Further questioning revealed that he believed the two parts of this task were "asking the same question." This connection was further established by his comment that part a) was looking for the area of the bounded region, and the integral in part b) "finds the area under the curve." His belief that the two parts asked the same question indicated that he still held to the notion that the area under the curve would continue to grow and would approach a finite value as x approached infinity.

Response to Task 7

Rob's original response to part a) can best be summarized by the following quote, in tandem with the graph presented in Figure 38:

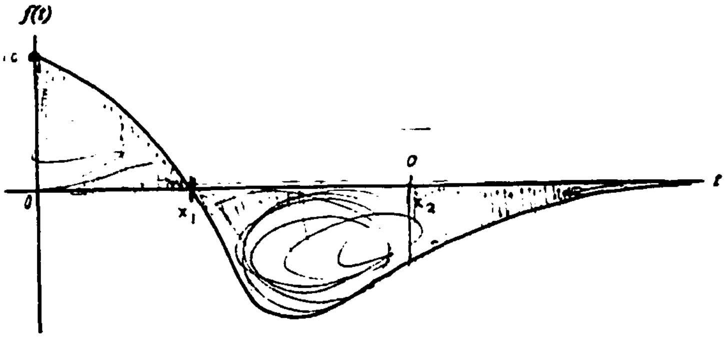


Figure 38. Graph Rob used to explain his notion of area for a function with some negative values.

> The area is going to, um, until this point *(marks the t-intercept)*, be positive. And then at a certain point it'll reach 0, um, maybe here *(marks location with a vertical line)*. And then, it'll get negative, um, kind of around about way of the opposite of the other function, because it's slowly going to go back towards…it's gonna have less and less negative numbers but it'll never actually get positive again.

This meant that, initially, area would increase, but once x passed the t-intercept, the area would start to decrease and eventually would become negative. The last sentence of the quote indicated that the area would continue to become more and more negative, only at a slower rate, because the graph approached the t-axis. Therefore, Rob viewed part a) as asking what the signed area would do as x moved to the right.

During the follow-up questioning to clarify his signed area views, Rob provided an alternative view of the task. This idea is presented in the passage below, along with the graph presented in Figure 39:

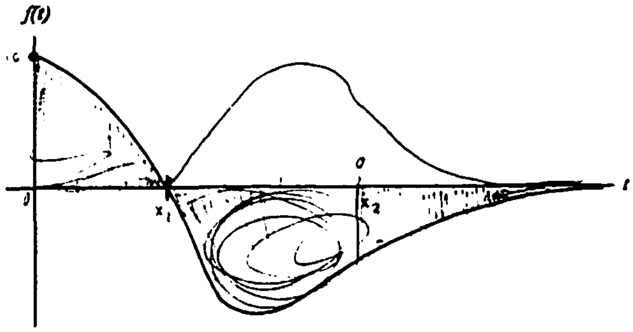


Figure 39. Graph Rob used to explain his alternative view of area for a function with some negative values.

Rob: If you don't take into account that this *(pointing to the region under the t-axis)* is negative, then it'll keep adding area. Well…
Todd: Explain.
Rob: Well, this is positive because it's above… *(Indicates the region above the t-axis.)*
Todd: I see that.
Rob: …the t-axis. This is negative because it's below the t-axis. *(Points to the region below the t-axis.)*
Todd: Right.

Rob: But it's still area, um, between the curves, and so if, kind of like if you took the absolute value of this *(region below the t-axis)* and said, oh, it's gonna go, like that then *(attempts to draw the reflection over the t-axis)*, since it's still area no matter if the value is negative, you might want to just consider it positive area 'cause it's still, the area between the curves. So you might continue to add the area.

This idea was presented as a way to consider the area of the region below the t-axis as being positive. In this case, the area of the region would be positive and would continue to increase in value as x moved to the right. Thus, he considered the area of the region below the t-axis in two different ways: his original notion of treating that area as a negative quantity, and his alternative notion of treating that area as a positive quantity. He referred to this latter approach as the "total area."

Further questioning revealed that the two notions expressed above were different ideas to Rob. He was asked which of the two notions was requested in part a). His response indicated that, when he was finding the area, the region below the t-axis had negative area. Thus, the area of the bounded region would become more and more negative as x increased in value. However, if the task had asked for the total area, he would have treated the region below the t-axis as having positive area. The area of the bounded region would then increase in value as x increased in value.

When Rob considered part b), he saw the function G in terms of area. He went on to give a description that was similar to the response he had given in part a). This description, also, demonstrated some across-time understanding of the integral function G. Finally, he indicated that he viewed parts a) and b), as worded, as being the same question.

Response to Task 8

Soon after Rob began to work on this task, it became apparent that the jump discontinuities caused him some difficulty. The following quote in reference to the definite integral $\int_3^8 h(x)\,dx$ illustrated this:

> Um, well, we have an open circle here *(points to the one associated with $x = 5$)*, and then, so it's not continuous function from, like, 3 to 8. And I don't know if...how to evaluate that with an integral. A non-continuous function, I'm not sure how to evaluate that one.

At this point, he was struggling unsuccessfully to achieve a resolution to his disequilibrium. He was so focused upon the discontinuities that other ideas, such as a geometric solution, were not being referenced.

As the interview continued, Rob was asked what he might do to compute the integrals. His thoughts regarding a plausible way to deal with the discontinuities were illustrated in this passage:

Rob: You'd have to break it up into separate parts. Like, your integrals from 3 to 8, you'd have to go from 3 to 5 and then from 5 to 8.
Todd: Why does that seem reasonable to do?
Rob: Um, well, you can't do 3 to 8, because it's not continuous, but, um, taking from 3 to 5...3 to 5 is continuous, and then from 5 to 8 is continuous, so.... That breaks it up into two continuous functions, then you add those together.
Todd: Wouldn't that give me then, what happens from 3 to 8?
Rob: Yeah, but, since it's not continuous at 5, then I don't think that you can evaluate the integral from 3 to 8 without doing them separately.

This passage indicated that Rob would employ the summation property of the definite integral to evaluate the definite integral $\int_3^8 h(x)\,dx$ indirectly by separating it into two definite integrals at $x = 5$. This would then allow him to work with continuous functions that he could integrate. Rob was asked to comment on how the jump at $x = 5$ would affect the definite integral $\int_5^8 h(x)\,dx$. He responded by saying: "Um, it shouldn't affect it at all, because the curve immediately after 5 is continuous on through 8." This indicated that Rob might comprehend that a jump discontinuity like the one at $x = 5$ does not affect the integrability of h over the closed interval $[5,8]$. Therefore, although the jump discontinuities

were a source of discomfort for Rob, he was able to devise a plausible way to handle the situation. However, he never put his idea into practice.

Rob was asked if there was a way in which he could order the integrals. The following showed his thoughts about this:

Rob: Um, just by breaking them into separate parts, and then adding them up to be a whole. Um, if I had to guess, I'd say that c) is the smallest, then d), then a), and then e), and then b).
Todd: And, why are you saying that?
Rob: Um, c) is 0 to 5, and 0 to 5 is all in the negative area. d) is just a point, um, but it's negative, but, since it doesn't cover as much area, I don't think that it would be as negative as c), or as small as c). Um, then a) is from 0 to 11, so it's gonna have negative area and positive, so, that will be the next smallest. And then 3 to 8 is gonna have some negative, and then some positive, too, so it's gonna be a little bit larger than a), and then b) is all positive area, so....

Even though he again seemed to suggest using the summation property of the definite integral at the beginning of the passage, his explanation of his ordering from smallest value to largest value, namely c), d), a), e), and b), indicated that signed area and visual estimates influenced his work. He never formally computed any values for the integrals, and actually said that he was guessing.

There were two noteworthy issues in the above passage. The first of these was that he in he indicated that the definite integral $\int_5^5 h(x)\,dx$ had a negative value because the point associated with $x = 5$ had a negative y-coordinate. The second issue was that he used area ideas to carry out his ordering. When he was asked about his use of area instead of evaluating the definite integrals, his response was "that's basically asking the same thing, 'cause integrals are the area between the x-axis and the curve." He was asked why he could not find the definite integrals, since he had indicated he could find the areas of the regions bounded by the curve and the x-axis. Rob replied, "I guess you could. Um, I just thought that the integrals had to, er, had to be a continuous function for you to evaluate the integral." Based upon his previous work on this task and Task 4, this was interpreted to mean that for

a function to be integrable, the function had to be continuous and had to consist of a single expression. Even though he indicated that he could compute values for the definite integrals using areas, he did not to go back and reconsider his work. In addition, he again indicated that the jump discontinuities made him uncomfortable.

Response to Task 9

Rob initially misread the task and tried to work it based upon this misunderstanding. He perceived the task as saying $f(x) = 2$, and believed he was supposed to find and graph an antiderivative. Once he understood the task, he said that the task was "saying that the integral is equal to 2, so that means that the area under the curve is equal to 2, between *a* and *b*." However, he hesitated over how to proceed, because he could think of several functions that would work. At this point, he did not realize that the task was asking for the graph of any function that would satisfy the integral condition. Once it was pointed out that he could choose any function he wanted to choose, he graphed what appeared to be the function $y = x$ and created a triangular region consisting of the equivalent of two blocks of area bounded by the function and the horizontal axis (see Figure 40). He illustrated that more

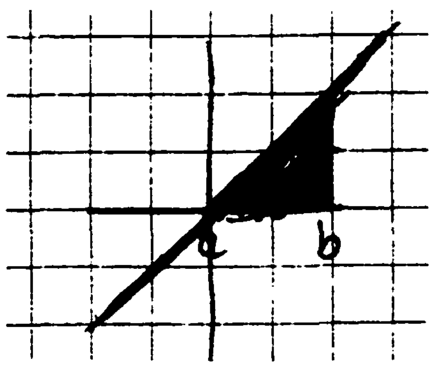


Figure 40. First graph Rob created to satisfy the integral equation $\int_a^b f(x)\,dx = 2$.

than one function could satisfy the integral condition by drawing three other examples. Further probing revealed that, in all of his examples, he was using area to construct his functions. In particular, he was using the horizontal axis and the graph of the function to construct regions consisting of 2 units of area. Finally, reversal of the usual integral calculation created no difficulty for him, since he viewed the task as easy, once he understood what it asked him to do.

This task appeared on the participant selection tasks (see Appendix E). Rob's response at that time (see Figure 41) employed the area model. However, he used the idea of signed area to construct his graph. Thus his approach to the task each time was based upon area. When this task appeared on the selection tasks, Rob did not seem to experience the difficulties that he did during the interview.

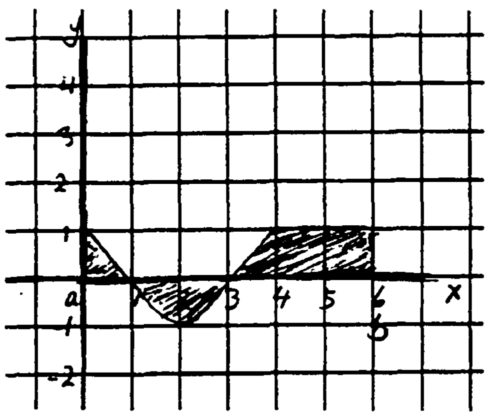


Figure 41. The graph Rob sketched for the participant selection tasks.

Response to Task 10

Rob was stumped by this task and initially was not sure what the second definite integral meant. After a long pause, he said, "I think what it's saying here is with *a* to *b* is...one part's gonna be in the negative or it's gonna have negative area." This was

interpreted to mean that he had decided the two integral equations together forced him to have some negative area under the curve. He tried to explain further what the two definite integrals meant to him, and his thoughts were summarized in the following passage:

Rob: A function that, um, between the *a* and *b*, has area 2 under the curve, but the absolute value of that function, um, from *a* to *b*, has a value of 4.
Todd: And the value of 4 is what?
Rob: Um, the total area, um, added under the curve, no man...like, just, not worrying about the negative values.

Taking into account Rob's usual usage of the term "area," he viewed the first integral equation as asking for a function f on a closed interval $[a,b]$ such that the signed area was 2. He viewed the second integral equation as requiring the function f on the same closed interval to have total area of 4. This indicated that he was able to interpret the equations, and understood what was being asked of him. Even though he was using area as a guide, he was unable to think of any function that would satisfy both equations. His final comment about this task was: "I don't know. Maybe I'm having trouble with reading it, and it's not registering right, but..."

Response to Task 11

After Rob had read the task, he paused for a long time. When asked what his initial thoughts were, he replied, "Um, the area under the curve for 0 to 2 is equal to 0.75, and 2 to 1 is 1.25, which means that the curve is, um, going up." This response indicated that he viewed the task in terms of area. In addition, he believed that the given information indicated that the function was increasing. Rob continued to refine his thoughts regarding the meaning of the two given integrals. He eventually demonstrated that he was thinking in terms of the summation property of the definite integral, as is shown in the following passage:

> The whole area from 0 to 2 is equal to 0.75. Um, but the area from 1 to 2 is equal to 1.25. Um...hmm...that means there's a part from 0 to 1, where it's negative. So from 0 to 1, *(Draws graph.)* that area is gonna be 0.5 area.

The graph referred to in this passage is shown in Figure 42. It should be noted that the

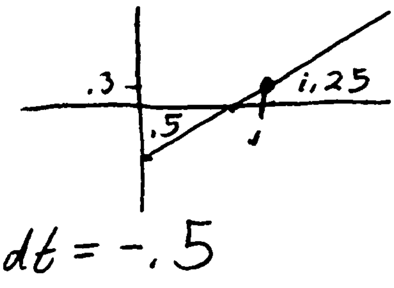


Figure 42. The graph Rob drew to compute the value for the definite integral $\int_0^1 f(t)\,dt$.

point associated with the coordinates $(1, 0.3)$ was added to the graph later. Rob's usage of the phase "whole area" in reference to the closed interval $[0,2]$ was understood to mean the sum of the integrals over the closed intervals $[0,1]$ and $[1,2]$. A follow-up question revealed that the value of the definite integral $\int_0^1 f(t)\,dt$ was -0.5.

Rob was unable to make any progress on finding values for the antiderivative F evaluated at 0 and at 2. In fact, he admitted that he did not know how to find these values. He was asked what he would need in order to find the requested values, and he said that he would need to know the function f. Further probing revealed that if he knew the function f, he would find the antiderivative and then evaluate it to find $F(0)$ and $F(2)$. This led to the conclusion that he considered only the evaluation aspect of the antiderivative and did not see a connection to the integrals that were present. His thoughts were compartmentalized, and therefore he was not able to think beyond the evaluation aspect suggested by the notation. Thus the notation seemed to be an obstacle to his perceiving the connection to the integrals.

Rob was asked if $F(1) = 0.3$ had any bearing on the task. He said, "I'm guessing it does. I just don't know what to do with it."

This task was revisited at the beginning of the third interview to determine whether Rob had any further ideas about finding the values of the antiderivative at 0 and at 2. He was unable to make any further progress, and his comments re-iterated his previous explanations. Therefore, he evoked the same image as he had during the second interview.

Response to Task 12

After Rob had read the task, he plotted the data points with time on the horizontal axis and harvest rate on the vertical axis (see Figure 43) and drew in vertical lines from the

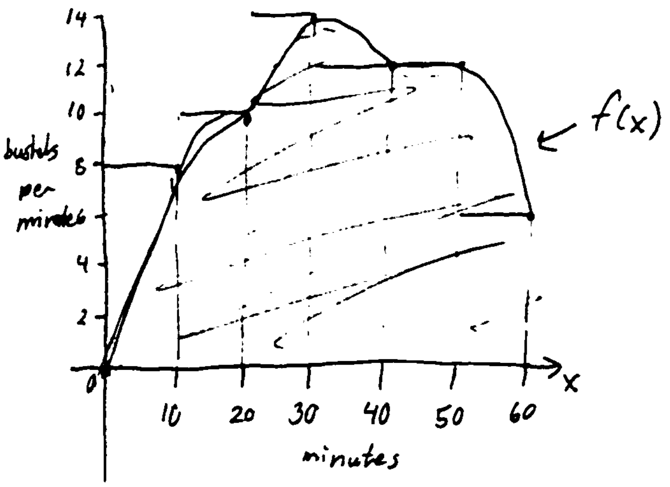


Figure 43. Rob's plot of the data, subsequent curve, and graphical work using the harvest rate data.

horizontal axis to each data point. He then said he was going to use his approximating sum method to approximate the total harvest, and prepared for this by drawing in the horizontal lines that would be associated with a right approximating sum. The curve connecting all the

data points was not added until, towards the end of his work on this task, he was asked about a connection to the definite integral.

After Rob had drawn in his horizontal segments, he formed the two sums that were used to complete his version of the Trapezoid Rule. Using *rh* for right-hand and *lh* for left-hand, he formed the following two sums using the values given in the table:

$$rh = 8 + 10 + 14 + 12 + 12 + 6 = 62$$
$$lh = 0 + 8 + 10 + 14 + 12 + 12 = 56.$$

The values for *rh* and *lh* were averaged to obtain an initial value of "59 bushels over the 60-minute period." His work implied that he was using $\Delta x = 1$ for the width of his trapezoids, as he had done when he worked on Task 4 during Interview 1. Rob was asked whether his answer made sense for a 60 minute time period. His response was:

Rob: No. (laughs quietly) Um, 'cause that would be... You have to take the sum times 10? So 590.
Todd: Why is that?
Rob: Because, for a 10 minute period, he did 8 bushels per minute, and then, for another 10 minutes, he did 10 bushels per minute, and then another 10, 14 bushels...12 bushels a minute. So, for 10 minutes, he would do 80 bushels here, 100 bushels there, 140 bushels there, 120 bushels, 120 bushels, and 60 bushels. So, that, um, is gonna add up to be 590, average between the left and right sum.

The passage reveal that Rob realized the importance of taking into account the width of each trapezoid in his calculations. This conclusion was strengthened by his reply to a question about how the above realization was different from his past usage of his approximating sum. His reply was: "'Cause I didn't take into account having to multiply." He then tried to explain why multiplying by 10 was necessary. His best explanation was:

> And, this is the harvest rate in bushels per minute, but it only gives the rate every 10 minutes. So, there's a whole space *(pointing to the first interval)* in here where he could have been doing 8 bushels a minute, and so, you have to take...account for each minute.

Further questioning using different regular partitions of the 60-minute interval illustrated that multiplying by the length of the subintervals was part of his approximating sum image at that time.

Rob was asked why he chose to use an approximating sum to estimate the total harvest. His response indicated that, to him, it was a logical approach to find an estimate:

> Just made sense. Um, I...you didn't have the function, all you had to do...all you had was the information to draw the graph, and from the graph, you can figure out the Riemann sums, like that *(points to his sums)*.

He did not explicitly state he was using the area model, but his work indicated that area was the basis for his work. His use of the plot he sketched and his formation of his approximating sums indicated that he was reducing the task to the simpler problem of a constant rate over a time increment, and then summing over all of the time increments.

As a final question, Rob was asked if this task had any connection to the definite integral. After he replied in the affirmative, Rob was asked to expand on this. He provided the following thoughts:

Rob: Well, if you had the function for...say this is *(sketches a curve through the data points)* a curve, like that, and you had the function for the curve. You can find out, um, the integral $\int_0^{60} f(x)\,dx$...and that's this, $f(x)$ *(labels the graph)*. And that will give you the same...well, it will be more accurate. This is an approximation *(points to the 590 bushels)*. And if you take that definite integral, you can find out the area under the curve *(Shades in the area under the curve he drew in.)*

Todd: And that would give us?

Rob: The harvest rate...or the total bushels done in an hour.

The curve that he sketched can be seen in Figure 43. During follow-up questions, he acknowledged that his curve was "not necessarily what the graph has to look like." He was just trying to "make it look more smooth, like it was an actual curve, instead of separate points." His reasons indicated that he used the curve as an aid in making a connection between the definite integral as area and the discrete data. Furthermore, he indicated that he viewed the approximating sum as only an estimate of the total harvest, and the definite

integral as yielding the actual total harvest. No connection between the approximating sum and the definite integral was given or indicated, so it was deemed unlikely that he perceived the definite integral as being a limit of a summing process. What he might have recalled, instead, was the idea that the area under a rate curve can be interpreted as a measurement of an amount.

Response to Task 13

Rob admitted to being confused by this task after he had read it. He also seemed overwhelmed by it. After he had received some help in understanding the task and the density function, in particular, he was able to progress. His next difficulty was how to work into his solution the notion that mass was given by density times length and, in particular, the length factor. Rob proposed three possible definite integrals, in the following order:

$$\text{(a) } \int_0^{100} \rho(x)\,dx, \text{ (b) } \int_0^{100} \rho(x)x\,dx, \text{ and (c) } 100\int_0^{100} \rho(x)\,dx.$$

When he tried to explain the formation of his first definite integral, the issue of the proper way to incorporate length into the unit analysis arose. The second definite integral was proposed as a way to incorporate length. The factor x in the second integral "would be the length at each spot." However, he immediately had second thoughts, and wrote out the third integral. His explanation for this integral was:

> I don't know if you'd want to have multiplied by the length every time. In which case, here's the density *(circles $\rho(x)$ in the integrand)* and there's the length *(circles the 100 outside the integral)*. So you find out the density for the whole integral [*sic*] and then you multiply by 100, 'cause that's the length.

He believed the third integral gave the density of the whole rod, and then multiplying by 100 would yield the mass of the rod. When asked to explain his limits of integration, he replied, "'Cause that's how long the rod is." He then had second thoughts about his third integral and went full circle back to his first integral. Rob thought that the length was "already figured into the equation when it's 0 to 100," referring to the limits of integration.

When asked why integration made sense, Rob pointed to the definite integral listed in (a) and said, "'Cause that, um, the integral's going to add up the mass from 0 to 100." This indicated a possible connection to the notion of a summing process. Asked to expound upon this last thought, he said:

> Well, the...like in the last problem [Task 12] where we were talking about how it was bushels per minute. It added up the total number of bushel done in that time period, and this *(points to current task)* is going to give us the whole density [*sic*] from 0 to 100 in the rod. And that's the whole length of the rod. So it'll give us the density [*sic*] through the whole length of the rod.

His use of density in this explanation was later shown to be mass. In addition, Rob connected the current task back to Task 12. The connection appeared to be that density was the harvest rate, the mass was the total harvest, and the rod's length was the time interval. Within this context, Rob then connected the mass and the definite integral through the notion of area under the curve. Thus Rob illustrated a possible connection to the notion of summing up the masses, but replaced it with the idea that the mass was the area under the density curve, where the density curve was viewed as a rate curve. At this point, Rob was unable to elaborate further why the definite integral made sense for this task.

Even though Rob had mentioned the idea of summing up the masses, he never elaborated upon this idea. There was no indication that he had considered the notion of partitioning the rod into sections so that summing the masses made sense. Rob failed to recognize the usefulness of his summing idea in understanding this task, despite the fact that his Calculus II section had been working on applications of the definite integral prior to the third interview.

Response to Task 14

Rob's response to the question posed in this task was that the definite integral represented "the area between a and b on the x-axis of the, uh, function, uh, $f(x)$." He did not mention signed area in his response. Therefore, his response associated the definite

integral only with the notion of area, even though he had demonstrated an awareness of the notion of signed area. One possible explanation for this was that he had not expanded his personal representation for the definite integral to include the notion of signed area. A second possible explanation was that he might have collected all these ideas under the umbrella of "area" without having yet made the determination of the proper representation for the definite integral. At the time of the interviews, it was clear that Rob had not brought the notion of signed area into his personal representation of the definite integral.

Rob was asked if area was all that the definite integral represented to him. He replied, "It's actually just a number, too. After you figure it all out, it's just gonna be a single number, so it's a value." Therefore, Rob could see beyond the various interpretations of the definite integral to view it solely as a number.

This task appeared on the participant selection tasks (Appendix E). At the time it was administered, Rob wrote the following: "$\int_a^b f(x)dx$ represents the area between $f(x)$ and the x-axis from a to b." This showed that he viewed the definite integral in terms of area. This was consistent with his first view of the definite integral during the third interview. Although he demonstrated in the third interview that he understood the definite integral to be just a number, his initial expressed view of the definite integral had not changed during the time between the participant selection tasks and the third interview. Therefore, the area viewpoint of the definite integral was firmly established, and his comment about the definite integral being "just a number" indicated that he was beginning to think of the definite integral in more abstract terms.

Response to Task 15

Rob was unable to make any progress on this task. It was evident that he found it difficult to understand what was being asked in the task, even after having it explained to

him. His focus was on pointwise aspects of the graph of the function *F*, as illustrated by the only coherent connection he was able to provide between the functions *f* and *F*. This connection, which was stated as a question, was that "each value along here *(points to the graph of F)* [is] going to be what the, um, area under the curve of $f(x)\,dx$ [*sic*] is?" However, he never provided a base point from which the area under the curve was to be computed. Although this was a legitimate connection between the antiderivative of the function *f* and the function *f* itself, it did not help Rob to identify what the definite integral $\int_a^b f(x)\,dx$ represented on the graph of the function *F*. In the end, Rob admitted that he did not know how to connect the integral to the given graph of the function *F*.

Response to Task 16

Rob's definition for the definite integral centered on the indefinite integral, the notion of area under the curve, and The Fundamental Theorem of Calculus. The first two of these appeared in his definition for the definite integral. He had difficulty arranging his thoughts, but he eventually summarized them: "The definite integral is just an indefinite integral evaluated between two points on the *x*-axis that will give you the area under the curve from those two point [*sic*] *a* and *b*." The connection with The Fundamental Theorem of Calculus was shown during his initial attempt to formulate his definition when he wrote the following:

$$\int_a^b f(x)\,dx = A$$

$$F(x)\Big|_a^b = A.$$

Rob used A in his written work to represent area. Therefore, the notion of area formed the basis for his definition for the definite integral.

The overarching theme of Rob's definition revolved around the notion of area. This was consistent with his response earlier in the third interview to questions regarding what the definite integral represented, namely, that the definite integral represented the area between the curve and the x-axis. In addition, the notion of the Riemann sum did not enter into his definition for the definite integral, even though he demonstrated he was familiar with the notion.

Response to Task 17

Rob responded to the question about how the picture given in this task related to a definition of the definite integral by saying the definite integral would give the exact value for the area under the curve. This response was consistent with his representation for the definite integral as "the area between a and b on the x-axis of the, uh, function, uh, $f(x)$." Asked about the rectangles that were drawn on the picture he said, "These are going to, uh, be approximations, like, the left- and right-hand sums." Further questioning revealed: "The area of the rectangles added up is gonna be an approximate value of the area under the curve." These comments indicated that the rectangles would provide only an estimate for the area under the curve, and thus for the value of the definite integral. Furthermore, his responses indicated he viewed the estimation process as separate from the definite integral.

Rob viewed the given partition of the closed interval $[a,b]$ as providing a way to estimate the area under the curve, but he was concerned by the fact that the heights and bases of the rectangles were not chosen consistently. He was particularly concerned by the lack of a pattern for the choice of the heights. He searched for a pattern, but finally said, "Um, I don't know how they decided on the height of the rectangles." Furthermore, he indicated that in order for the given setting to be like his left-and right-hand sums, he would have to repartition the closed interval "to, um, make all the rectangles equal in width."

Therefore, it was evident that he was not comfortable working with a general formulation of a Riemann sum.

Response to Task 18

Rob's first response was to determine whether the task was referring to the limits of integration. Once this was clarified, he attempted the task. The only notion of limit that he was able to evoke was that of the limit of a function. His only reference to limits and the definite integral was "the definite integral is just a one value...a single value anyway, so it really can't have a limit." His statement was in line with his view of the definite integral as "just a number." With this understanding, he did not view the notion of limit as relevant. He finally admitted, "I'm not seeing it," in reference to a connection between the notions of the definite integral and the limit.

Response to Task 19

Rob had difficulty comprehending this task and, in particular, understanding the definition of the sequence. As he worked to understand the sequence itself, he made his only real connection between the sequence and ideas related to the definite integral. While referring to the sequence itself, he said:

> It's kind of what I do when I do the Riemann sums or, try to figure out that. It's adding up all the values at specific points along the graph, which would be the rectangle values, for instance, and then dividing them by the number of, um, steps along the way.

The phrase "rectangle values" was interpreted to mean the heights of the rectangles used to approximate the area under the curve. His usage of the phrase "dividing them by the number of steps along the way" seemed to correspond to dividing by the length of the subintervals in a regular partition. Additional probing supported these interpretations. Therefore, Rob linked the sequence with a Riemann-sum-like notion. Although he alluded to graphical and geometrical ideas, he never translated his thoughts into a graphical setting.

Once he understood the definition of the sequence, Rob was unable to make any further connections between the sequence and the definite integral. Before he gave up on the task, he was presented with the alternative version of the sequence to ascertain whether he could make any further connections. After looking at this version, he announced, "I don't see the relationship between the first one, and I don't see it between the second one, either." The relationship that he referred to was the one between the definite integral and the sequence. Thus the alternative version of the sequence did not provide any additional insight into his ability to relate the sequence to the definite integral.

Response to Task 20

After a period of time in which Rob tried to absorb the information given in the task, he sketched a graph of the function *f* (see Figure 44). He then questioned the accuracy of this graph because he was still tying to understand the piecewise-defined function *f* itself. After he talked through the definition for *f*, he concluded that his graph was correct.

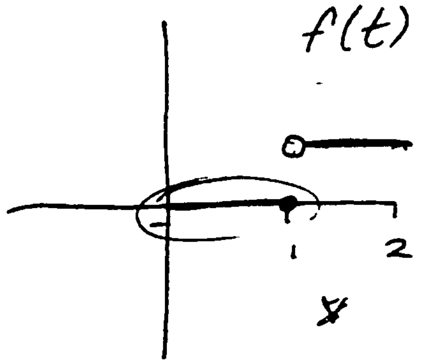


Figure 44. Rob's graph for the piecewise-defined function *f*.

Rob's attention then turned to graphing the function *F*. As he worked on this graph (see Figure 45), two distinct aspects of the function *F* emerged. He viewed the function *F* as graphing the area under the curve given by the function *f*. However, when he actually

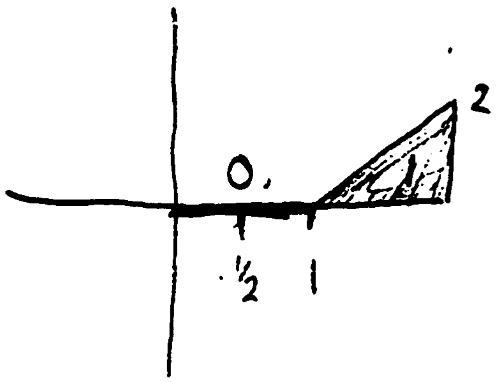


Figure 45. Rob's graph for the integral function F associated with the piecewise-defined function f.

graphed F, it became evident that he was not using the area values as y-coordinates for his graph. Instead, he used the area values to create regions under the graph of F, with the corresponding amount of area as found under the graph of f. The following passage provided the initial insight into his solution path:

Rob: It would...*(pause)* Well, if it's giving the area under the curve from 0 to 1, it's gonna be 0. *(Starts sketching graph.)* And then it's gonna go up *(sketches diagonal part)*. And that'll be 1, the area *(puts a 1 in triangular region of graph)*.
Todd: OK. Can you explain how you came up with that?
Rob: Well, I don't know if it'll actually be one area. But from...from 0 to 1, there's no area under the curve. *(Points to graph of f.)* It's 0. So there's not gonna be any area from 0 to 1, so $F(x)$ is equal to 0 from there to there *(meaning from* 0 *to* 1*)*. Um, from 1 to 2, $f(t) = 1$, so, um, the area is going to start getting bigger, and it'll, um, have, from 0 to 1 [*sic*], it'll have an area of 1 under the curve.

Further probing about what would happen at $x = 2$ produced the following exchange:

Rob: It's gonna be 1.
Todd: And, the reason for that?
Rob: Well...(sigh) *(pause)* Actually, it would be 2. Because from 1 to 2, it has to have an area equal to 1, which means that it's gotta go from 0 to 2 to get that area equal to 1. 'Cause if it was 0 to 1 it'd just be 1/2 *(referring to the area under the graph of F)*.

This meant that, in order to have 1 unit of area under the graph of F over the closed interval $[1,2]$, the values for the function F must range over the closed interval $[0,2]$. Otherwise, if

the function values for *F* ranged only over the closed interval $[0,1]$, the area value would be $1/2$. In addition, the previous passage further supported the conclusion that he created his graph of *F* so that the area under the curve of *F* was the same as the area under the curve of *f*. Also, he used his area values to determine the corresponding *y*-coordinate for the function *F* at $x = 2$. Based upon his work, it was evident that, even though he knew the function *F* represented the area of the region bounded by the *t*-axis and the graph of *f*, he was unable to disengage from the area under the curve concept image in order to comprehend these values as function values for *F*.

Additional probing revealed that Rob viewed the function *F* as an antiderivative of the function *f*. Furthermore, he provided evidence that he was thinking in terms of polynomials while working with *F* as an antiderivative. However, in the end, he remained firmly set on his solution method for this task and his use of area to sketch a graph for the function *F*.

The Case of Stan

Introduction

Stan was a 20-year-old sophomore pursuing a geology major at the time of the study. Prior to Calculus II, he had taken Precalculus and Calculus I at the institution where the study took place. He took Calculus I during the semester prior to the interviews. Stan earned a C in Calculus I.

At the end of the third interview, Stan was asked to respond to questions focusing on his thoughts regarding his understanding of the definite integral to provide insight into how he approached the tasks themselves. In response to a question about how he thought about the definite integral, Stan responded, "I think about it graphically." He continued by saying, "I mean, it's easier to picture it graphically, definitely. And I try to relate it symbolically, but symbolically is where I get messed up." Regarding conceptual or abstract thinking in reference to the definite integral, he revealed that, for him, conceptual thinking revolved around trying to relate what happened symbolically on paper to what was happening with the graph, and, in particular, to his belief "that the antiderivative of $f(x)$ is the area under that graph." During the interviews, this belief was his way of thinking about the area function $A_f(x) = \int_a^x f(x)\,dx$ that was presented in his Calculus I textbook (Ostebee & Zorn, 1997). His final response to this question was, "So I guess I would use all of them, a little bit, but mostly graphically."

In response to the inquiry concerning what aspect(s) of the definite integral he viewed as most difficult for him, he noted that he found the notation difficult to understand, and that "taking the antiderivative of the function is hard for me." For Stan, his notion "that the area under the curve is the antiderivative of the curve" was the easiest aspect of the

definite integral. Again, he seemed to be thinking in terms of the area function mentioned above.

Response to Task 1

In working on parts a) and b) of this task, Stan recognized that he needed to apply The Fundamental Theorem of Calculus to both parts, but he had difficulty finding the antiderivatives in both cases. In part a), he correctly found antiderivatives for the first two terms, but incorrectly computed the derivative of the constant term. After being challenged to check his work, he struggled with the roles of antidifferentiation and differentiation before he finally realized that he had confused the two notions. He was then able to find the error in his original antiderivative and correct it, but then made some arithmetic mistakes. In part b), he again had difficulty with the antiderivative of e^t because he felt that he needed to use the chain rule. However, he was able to correctly compute the definite integral. Some of his difficulty with the two parts of this task might be attributable to the very noticeable nervousness he displayed when he arrived for the interview. However, a larger part of his difficulty was probably related to his admitted difficulty with finding antiderivatives. This point was made clear later in the first interview.

Stan admitted to being familiar with the greatest integer function and viewed it as a step function. He was questioned to verify that he did understand this function. Except for the function values at the integers, he was able to correctly work with the greatest integer function. After his definition was corrected, he correctly sketched a graph for the function. However, as soon as his attention was turned to part c) of the task, he set his graph aside.

When Stan first looked at part c), he acknowledged a connection to the area under the graph he had previously drawn. When asked to demonstrate his plan, he applied The Fundamental Theorem of Calculus, in the same manner as Rob, to arrive at an answer of 12. Since this approach was reported in Rob's case study, it will not be covered here. In

addition, it was concluded that the analytical nature of the task led him to use an analytical solution method, even though he indicated that there was a connection to the graph he had previously drawn. The simpler geometrical solution appeared to be overruled by the analytical nature of the task.

Stan was asked whether he could relate his answer of 12 to the graph he had previously drawn. He said, "I would think if I did it right that that would be the area under this function, from 0 to 5." Because he showed no initiative to check his answer with the graph, he was challenged to do so. At this point, he sketched in the graph of $y = x$ (see Figure 46) and counted up the number of squares under this line to be 12. He did this

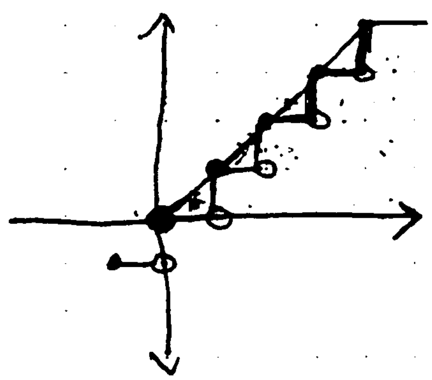


Figure 46. Stan's graph for the greatest integer function along with the line $y = x$.

"just to connect them *(meaning the solid dots)*," since he initially did not view the graph of the greatest integer function as a curve because it was disconnected. However, when asked to point out the graph of the greatest integer function, he realized that this line did not belong on the graph and that he should have included only the area below the graph of the greatest integer function. He then drew in the staircase outline of the area under the graph (see Figure 46) and counted up the number of squares under the graph, arriving at an

answer of 10. When asked, he indicated that he believed his graphical solution was correct. His reason was summed up in the following passage:

> Because it's, um, it's right there, I mean, it's evidence. I mean that *(points to the original problem)* is something that I've never really done before, and I don't know if I antidifferentiated that right or, if there was something else I was supposed to do.

He was more confident in his concrete approach to the subtask using area than he was of his analytical work, because of its newness. Even though he already had a graph present on which he could carry out a graphical solution, Stan was still more comfortable approaching the task analytically.

Response to Tasks 2 and 3

Stan responded to Task 2 by saying, "Um, I mean I could do it, like, approximating the sum, I think." When asked to demonstrate his idea, he subdivided the closed interval $[-2,3]$ into 5 equal-sized subintervals, and formed a Riemann sum using the left endpoint of each subinterval (see Figure 47). In his initial formulation of the sum, he included an

$$1\left(g(-2) + (g(-1) + g(\cancel{g}0) + g(1) + g(2) + \text{[illegible]}\right)$$
$$1(2 + 1 + 0 + 1 + 2)$$
$$= 6$$

Figure 47. Stan's Riemann sum calculation for the definite integral $\int_{-2}^{3} g(x)\,dx$.

extra value, but he caught that error when he was asked to explain his work. He completed his approximating sum by approximating the function values from the graph, and then carried out the necessary arithmetic to arrive at an answer of 6. His approach was interesting because he overlooked the simpler geometric solution path.

Because Stan had only approximated a value for the requested integral, he was challenged to provide a second solution. The following exchange contained the outline for a geometrical approach based upon the area model:

Stan: I could just draw...draw 'em in...draw some lines in.
Todd: And then what would you do?
Stan: Count the number of blocks.

He illustrated what he meant by forming a grid system on the graph given in the task and counting the number of squares enclosed by the graph and the x-axis on the closed interval $[-2,3]$ (see Figure 48). With this approach, he counted 6.5 squares. Thus, rather than using the area formula for a triangle, Stan used elementary counting techniques to find the area of the enclosed region. Finally, Stan indicated that he felt better about this second approach because it was more accurate than the approximating sum he first used.

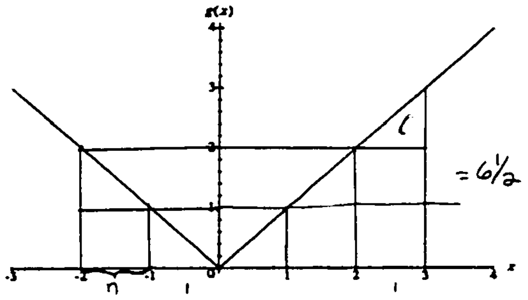


Figure 48. Stan's use of his "counting blocks" technique to evaluate the definite integral $\int_{-2}^{3} g(x)\,dx$.

When Stan was presented with Task 3, he applied The Fundamental Theorem of Calculus, just as Rob and Lynn had when evaluating the requested integral, and computed an answer of $\frac{5}{2}$. However, he was uncertain about his work and seemed to be searching for a resolution. After briefly considering checking his antiderivative, he suggested he could

check his work graphically by drawing the graph for the absolute value of x. It was at this point that he connected the two tasks.

Stan acknowledged that the graph presented in Task 2 was the graph of $y = |x|$, and that Tasks 2 and 3 were really the same. After comparing the two tasks, he decided that his answer to Task 3 was wrong and that his answer to Task 2 was correct. Asked why his work in Task 3 was incorrect, he replied, "I think I'm antidifferentiating the $|x|$ wrong." He provided two reasons for why his antiderivative was incorrect. The first reason was that his work on Task 3 did not coincide with his graphical solution in Task 2. He seemed much more inclined to believe his graphical solution, possibly because it was so concrete to him. The second reason was that he did not believe that the antiderivative for the $|x|$ would be as easy as he had made it.

Response to Task 4

After Stan finished reading the task, he said, "OK, um, since I have no function, I would...I mean, the easy thing for me to do myself would be to draw lines in and just count the boxes." He then laid out a grid over the region, as shown in Figure 49. Once the grid

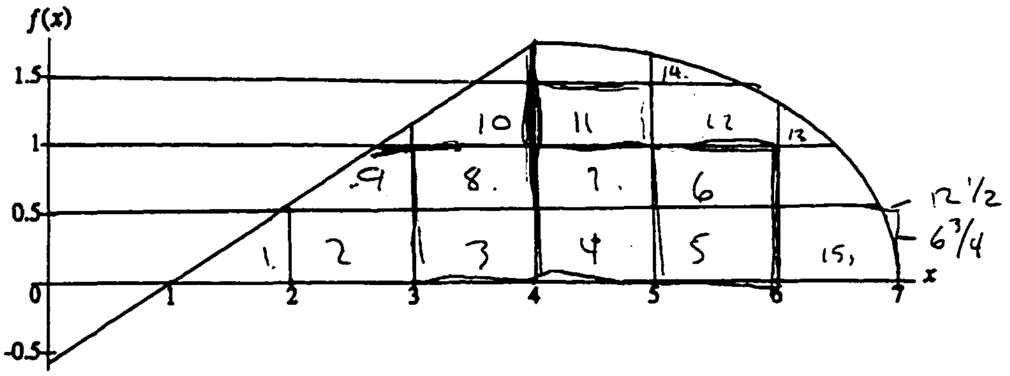


Figure 49. Stan's set-up for "counting boxes" to approximate the area of the given region.

was completed, he started to count his "boxes," which were actually 1 unit by 0.5 unit rectangles. He counted the number of whole rectangles, then pieced together incomplete rectangles to approximate whole rectangles. When he had finished counting, he stated the area would be "12.5, whatever the units will be, I guess, squared." At this point, he had not noticed that his "boxes" were not unit squares, so he was asked about the size of an individual "box." In responding to the question, he realized his error and started to pair off the rectangles to form unit squares. However, rather than divide his first answer by 2 to obtain the approximate number of unit squares, he counted the number of unit squares. This proved to be slightly more difficult for Stan to do, and he acknowledged this by saying, "Yeah, this doesn't work very well with the different scales." After counting the number of unit squares to the best of his ability, he guessed the answer was "6.75 or something."

Based upon Stan's approach above and on previous tasks, he was quite content to use counting techniques to compute areas. In fact, he never expressed any recognition that there was a right triangle available to use in the given figure, probably because he was so intent on counting "boxes." It was evident that counting provided a concrete way for Stan to relate to the area aspects of the definite integral.

When asked what piece of information he would choose to have, if he could have any additional piece of information he wanted, he replied, "The function, I guess." Further questioning revealed that if he had the function to work with, he would "take the antiderivative" and then "use The Fundamental Theorem of Calculus." However, he was hesitant to examine the function because it "might be a…a wild lookin' function. I don't know if I could take the antiderivative because of the way it behaves here." This meant that he was not confident enough in his ability to find antiderivatives to warrant working with the function itself.

Response to Task 5

When Stan first looked at this task, he thought that it was very difficult because of the λ in the integrand. Upon being encouraged to take a second look, he did, and said, "I suppose I could have a *u* there," in reference to λ. This was interpreted to mean that he was able to view λ as simply another variable. Stan tried to complete the task, but he was unable to compute an antiderivative for $(\lambda-1)^2$, and admitted, "I don't really know where to go here." Furthermore, he said that finding antiderivatives was difficult for him. He claimed he knew what the task was asking him to do. In particular, the task "wants the function of the area. Which is the antiderivative of this *(points to the integrand)*. And then it wants me to put the 5 in for *x*."

As questioning continued, it became apparent that Stan saw two different, yet related, relationships packed into the statement of the task:

$$G(x)=\int_0^x(\lambda-1)^2\,d\lambda \text{ and } \int_0^x(\lambda-1)^2\,d\lambda.$$

Even though $\int_0^x(\lambda-1)^2\,d\lambda$ was the right-hand side of the integral function *G*, Stan treated it as a separate entity. When he did this, the integral was used "to find the area under the function *(points to the integrand)* on the interval 0 to *x*." Further probing revealed that he perceived the form $\int_0^x(\lambda-1)^2\,d\lambda$ as requiring him merely to evaluate the integral. On the other hand,

$$G(x)=\int_0^x(\lambda-1)^2\,d\lambda$$

was "a function of the area underneath the curve of $(\lambda-1)^2$." He also used

$$G(x)=\int_0^x(\lambda-1)^2\,d\lambda$$

for the antiderivative of $(\lambda-1)^2$. It was evident that he was thinking in terms of the area function A_f from his Calculus I textbook, defined by

" $A_f(x)=\int_a^x f(t)\,dt=$ signed area defined by f, from a to x" (Ostebee & Zorn, 1997, p. 357),

and shown to be an antiderivative of $f(t)$. Thus, when Stan saw

$$G(x)=\int_0^x(\lambda-1)^2\,d\lambda,$$

he believed he was being asked to find the antiderivative as a function of x. On the surface, it seemed that Stan was saying the same thing from two different points of view, but careful consideration of what took place over the course of the interview revealed that, to Stan, this was not the case.

The distinctions were more fully revealed when he discussed finding $G(5)$. For Stan, finding $G(5)$ involved finding the antiderivative first as a function of x, and then substituting 5 for x. This process was not the same as computing $\int_0^5(\lambda-1)^2\,d\lambda$, because evaluating the integral involved finding the area under the function $(\lambda-1)^2$, and therefore had nothing to do with evaluating the antiderivative as a function. He illustrated his method for finding $G(5)$ during a follow-up question posed in the second interview, in which he was asked to consider the simpler task:

$$\text{If } G(x)=\int_0^x(\lambda-1)\,d\lambda\text{, then } G(5)=?$$

He was able to carry out the necessary computations at that time (see Figure 50). As can be seen, he found the antiderivative first, and then evaluated that function at $x = 5$. His method was very similar to that used by Joan when she solved the original task.

$$G(x) = \int_0^x \lambda - 1 \, d\lambda$$

$$= \frac{\lambda^2}{2} - \lambda$$

$$= \left(\frac{x^2}{2} - x\right) - 0$$

$$G(x) = \frac{x^2}{2} - x$$

$$G(5) = \frac{25}{2} - 5$$

$$= 7\tfrac{1}{2}$$

Figure 50. Stan's evaluation of a simplified integral function.

In addition, Stan viewed the variable x in the upper limit of integration as setting the interval to integrate over. However, the x in $G(x)$ was for evaluating functions of x. To him, the two x's in the integral function were different, but he never was able to explain why.

Although Stan indicated that

$$G(x) = \int_0^x (\lambda - 1)^2 \, d\lambda \text{ and } \int_0^x (\lambda - 1)^2 \, d\lambda$$

were different entities, there were indications that his views were in flux. For example, he once wrote:

$$G(5) = \int_0^5 (\lambda - 1)^2 \, d\lambda,$$

and again later talked as if he were carrying out the above calculation, but in both instances he talked himself out of these ideas and returned to viewing the two entities as different. This indicated that the separation between the two notions was not absolute, and he was in the process of refining the relationship between the two ideas.

Response to Task 6

Like Lynn, it took Stan a while to absorb all the information given in part a) of this task. His answer was similar to answers already given by the previous 3 participants in that the area would continue to grow, but at a slower rate, as x continued to move to the right. In addition, he said that the area would grow forever "as long as the function goes on." This to meant that the area would eventually approach infinity.

After studying part b) for some time, Stan concluded that the values for the function G would grow. He also indicated that part b) was the same as part a). In addition, he demonstrated some across-time understanding of the given integral function. This part of his response was similar to responses given by the previous 3 participants.

What was interesting about Stan's response to this subtask was the change in his views from the previous task, concerning the integral function itself. He still viewed G as an area function. However, when asked what the function G represented, he replied, "The area under the curve." Further evidence of this change was given as he described the placement of $x = 5$ in the integral function:

Stan: Well, it would just go here then, right? The 5 would. *(Writes a 5 as the upper limit of integration.)*
Todd: OK. So I'd have the...
Stan: In this...but I don't think, I mean, it wouldn't go here, right?
Todd: It wouldn't go where?
Stan: Right here *(points to the G side of the equal sign).*
Todd: Why not?
Stan: Because that's a different function. That's...that's...well, I guess it's the same as this *(points to the integral side of the equal sign)*, actually, from 0 to 5. We're evaluating this function *(points to* $f(t)$*)*...or the area under the curve from 0 to 5.

In addition to viewing the function G as both an area function and a way to compute area under a curve, he no longer saw the x in $G(x)$ and the x in the upper limit of integration as being different. This change concerning the function G led to two possible conclusions. The first was that he evoked a different image for the integral function. The second was that, over the course of this task and the previous task, some learning took place. Although it was clear that his view of the function G had changed, it was not apparent which possibility best explained why.

Response to Task 7

Like the 3 previous participants, Stan viewed part a) of this task as asking about signed area. He believed that the area would become negative and continue to decrease as x continued to the right. He, also, acknowledged that if the coordinate system was removed, the area would be positive and increase in value as x moved to the right. However, when the coordinate system was included, then the region under the t-axis was to be treated as having negative area. Like the previous 3 participants, the coordinate system played a strong role in how Stan interpreted area in part a). Similarly, if the phrase "area of the region" was replaced by the phrase "total area of the region," then the total area would continue to grow as x moved to the right.

Stan did contribute one interesting new comment. As the questioning related to the role of the coordinate system was being finished, the following exchange occurred:

Todd: OK. But yet you're telling me when I put the coordinate system on this, the area of that same region then would take on a negative [value], right?
Stan: Yeah. Just because it's below the x-axis [*sic*].
Todd: Just because it's below the x-axis [*sic*]?
Stan: (laughs) Yeah. I mean, that's as far along as I am, I think.

Stan's last sentence contained the interesting comment. It was interpreted as an admission that he was still learning, and that this was as far as his understanding had progressed. It also indicated that he expected to understand more as his studies continued.

When part b) was presented, there was a long pause before Stan said, "$G(x)$ will decrease." Further questioning clarified this statement to mean that as x continued to the right, the values of $G(x)$ approached negative infinity. Stan's description of what the values of $G(x)$ did as x continued to the right indicated that he had some sense of across-time understanding of the integral function. In addition, he, also, viewed parts a) and b) as asking the same question.

Stan was asked why G was a function. His response was, "I guess because, it is the antiderivative of $f(t)$, and $f(t)$ is a function...and so the antiderivative of a function, would be a function, I suppose." He was unable to elaborate on his answer any further but, based upon his previous work, it was concluded that he was thinking of the function G as his area function. As such, G was an antiderivative of the function $f(t)$. Therefore, G was a function, although he was not certain about this. This uncertainty was expressed by his use of the phrase "I suppose."

Response to Task 8

Stan began this task by outlining the boundary of the entire region. After considering the region for several seconds, he began to order the integrals by visually comparing the areas of the subregions, rather than computing values for most of the definite integrals. As he ordered the integrals, it was evident that he was using higher-level reasoning to visually relate the various regions within the context of signed area. This approach was similar to the one used by Rob, but Stan provided a detailed rationale for his method.

He decided that the integral in part c) would have the smallest value "'cause its got total negative an area." In addition, he commented that the only integral that could be less than the one in c) was the integral associated with a), and then ruled this out by saying, "but I know that 0 to 11 is not less, because it's got the same area from 0 to 5, but it has positive

area as well." He next concluded that the value for the integral in d) would be 0, and placed it in the middle by saying, "so, that would probably be, like, my middle one, somewhere in there, I'm thinking. 'Cause I'm probably going to have another one with negative area." He then moved on to the integral in part b). For this definite integral he said, "let's see 5 to 11, is going to be, all positive so that will probably be, so b) will be the largest one. Because the whole area is positive." At this point, his ordering for the integrals was c), followed by d), and followed by b).

Next he turned to the integral in part e). The following passage provided the rationale for his placement of e) between parts d) and b):

Stan: And so I got c) is the smallest, d) is probably the middle, and b) is the largest. So...*(Marks the ones he has used.)* OK, now I just have to determine...now I know that, 3 to 8...from 3 to 8 is going to be larger than 0 to 11 because there's more negative area involved *(referring to the integral from 0 to 11)*. It's got this much more negative area *(outlines the region associated with the interval from 0 to 3)* than 3 to 8. You see that.

Todd: OK.

Stan: See what I am saying. And I know that it *(points to integral from 3 to 8)* going to be greater than 0 because this...all this area *(points to region above the x-axis)* is greater than this *(points to the region below the x-axis on the interval from 3 to 5)*, I mean, I can just see that.

It was interesting to note that, in deciding whether the integral in part a) or the integral in part e) was larger, he focused on only the closed interval $[0,8]$. He did not consider the closed interval $[8,11]$. Furthermore, Stan concluded that

$$\left|\int_0^3 h(x)\,dx\right| > \int_5^8 h(x)\,dx,$$

and that

$$\left|\int_3^5 h(x)\,dx\right| < \int_5^8 h(x)\,dx.$$

These last two inequalities were visual estimates that he made from the graph.

Stan could not decide whether the value for the integral in part a) was positive or negative. This was interesting in itself, because in similar situations, he just looked at the graph and made a decision. In order to decide this time, he returned to his "block counting" method. He drew in a grid of unit squares, and then counted the number of blocks above and below the x-axis. When he had completed his counting, he said that the integral in part a) was, "less than 0, because there's a little bit more negative area than there is positive area." This led to his final ordering of the integrals, from smallest to largest: c), a), d), e), and b).

Once he had finished explaining his ordering of the integrals, Stan was asked whether the value of the definite integral over the closed interval $[5,11]$ would be affected by the jump discontinuity at $x = 8$. He answered by saying:

Stan: I don't think so. Um, 'cause this is all included under the curve, straight down from that point *(the solid dot with x-coordinate 8)*.
Todd: OK.
Stan: And from 8.1 or 8.001, or whatever you want to call it. Right here *(points to the open dot with x-coordinate 8)*, it's just straight down as, under the curve as well. So, it's just like they go from 8 *(points to solid dot)* to 8.001 *(points to open dot)*....

Stan was saying that the definite integral over the closed interval $[5,11]$ was the same as

$$\int_{5}^{8} h(x)\,dx + \int_{8.001}^{11} h(x)\,dx,$$

and therefore, he believed the jump discontinuity was not going to affect the value of the definite integral. However, he did not give any indication as to what happened over the closed interval $[8, 8.001]$; he apparently did not recognize that he had created a hole within the closed interval $[5,11]$. It further appeared that the positions of the line segments that made up the function, rather than the endpoints, were determining what happened with the definite integral.

Finally, Stan was questioned about his use of "counting blocks" to compute areas, since his work on this and previous tasks indicated this was a well-established procedure for him. When asked if this technique worked well for him, he responded:

> It's the easiest way for me to comprehend. 'Cause it's, like, right there in front of me. I can see what is going on, you know, I got this area and I got this negative area and I got… I can just see it more clearly, I guess.

Although the areas he was referring to were unclear, it was suspected that he was thinking in terms of the calculation he had completed for part a). His comment indicated that "counting blocks" was simply a concrete way for him to understand the definite integral in terms of the area model.

Response to Task 9

This was an easy task for Stan to complete. He reversed his "block counting" technique to construct the graph of a nonlinear function over the closed interval $[0,2]$ with approximately 2 "blocks" of area bounded by the horizontal axis and his curve (see Figure 51). This indicated that he was able to reverse the usual integration calculation to determine

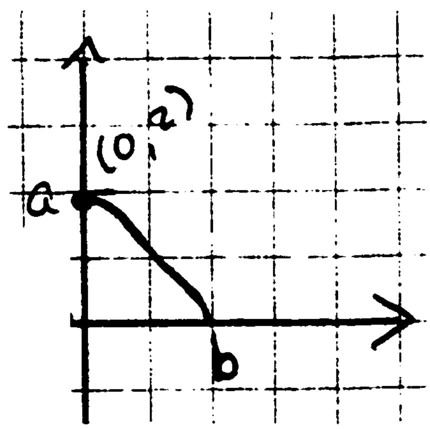


Figure 51. The graph that Stan sketched for a function satisfying the integral equation $\int_a^b f(x)\,dx = 2$.

a graph of a function that satisfied the given integral equation. When asked to describe his strategy for sketching his graph, the following exchange took place:

Stan: So, my main priority was getting 2. *(Points to the 2 in the integral equation.)*
Todd: OK.
Stan: Two blocks...2 positive blocks.
Todd: So you're viewing the problem as asking you to...?
Stan: To graph an area.

This meant that he was trying to create a region that consisted of 2 units of area bounded by the horizontal axis and the curve he was sketching. In reality, he over-approximated the area by using a nonlinear function. It was not clear why he chose to use a nonlinear function instead of a linear one. As a note of interest, he sketched a similar graph when he completed the same task on the participant selection tasks (Appendix E). This showed that he was consistent in how he approached this particular task and in the type of image he evoked when solving the task.

Response to Task 10

Stan had difficulty getting started on this task. After a couple of long pauses while he appeared to be thinking, he was able to interpret the two integral equations. Stan understood the first integral equation as, "looking for an area of 2 with the area, any area underneath the x-axis still being negative." Therefore, Stan viewed the first integral equation in terms of "signed area." Stan's understanding of the second integral equation is given in the following passage:

Stan: That the total area would be 4. That's what I'm thinking.
Todd: OK. And...
Stan: And not...not...so like, when you count this area under the x-axis it's not considered a negative area but a positive area as well.

Therefore, he saw the second integral equation in terms of the mathematical notion of area. His initial idea for the graph had only 2 units of area above the x-axis, just as Joan, Lynn, and Rob had, but then Stan had a different idea.

Stan's idea used a different combination of the area above the x-axis and the area below the x-axis. His resulting graph is shown in Figure 52, and his explanation of how

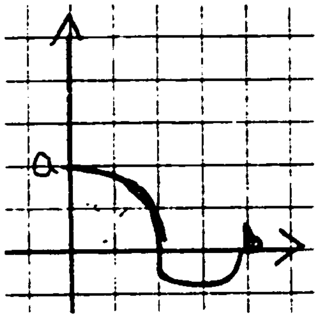


Figure 52. The graph that Stan sketched for a function satisfying the integral equations $\int_a^b f(x)\,dx = 2$ and $\int_a^b |f(x)|\,dx = 4$.

this graph satisfied the stated conditions appeared in the following exchange:

Stan: So this is b, or this is *a (the point* $(0,2)$*).* That's *b (the point* $(4,0)$*).* I've got 3 there *(above the x-axis)* and 1 there *(below the x-axis).* Which is a total...which is an area of 2, on this integral without the absolute value.
Todd: Because?
Stan: Because you got 3 there *(above the x-axis)* minus one block *(from below the x-axis)...*
Todd: OK.
Stan: ...which equals 2.
Todd: OK.
Stan: And a total area of 4 blocks because this *(points to block below the x-axis)* is considered an absolute value of -1, block would be 1.

Stan used a nonlinear function, but it was unclear why he did this. One possible reason was that he was trying to use only familiar types of graphs, namely, the graphs of smooth functions. He also over-approximated his area, but he did acknowledge that he had done so. Closer inspection showed that his graph was not the graph of a function, as it failed the Vertical Line Test, but it did effectively convey that he had the right idea in mind for solving the task itself.

The key to Stan's success on this task appeared to be his ability to use his "block counting" technique to configure different arrangements of blocks as he searched for a solution. It was possible that his "block counting" technique was what allowed him to find a solution where others had not. In addition, he was able to characterize all graphs that would satisfy the two integral conditions: graphs that "have 3 blocks above the x-axis and 1 below [the x-axis]" in some configuration.

Response to Task 11

There were several long pauses as Stan tried to "take an overview of all the information" that was presented in the task. When he was finally ready to work on the task, he started by finding the value for the integral $\int_0^1 f(t)\,dt$. As illustrated in the excerpt below, he used the notion of area and the geometrical version of the summation property of the definite integral to find his value.

Stan: The area under the curve, from interval 1 to 2 *(pause while writing)* is equal to 1.25.
Todd: OK.
Stan: Now the area under the curve from 0 to 2 *(pause while writing)* is 0.75.
Todd: OK.
Stan: *(pause)* So that would mean that the area under the curve from 0 to 1 *(pause while writing)* equals a -0.5.
Todd: OK, why are you saying that?
Stan: Well, from 1 to 2, it's at a positive 1.25.
Todd: OK.
Stan: Then you throw in the 0 to 2 here *(points to the appropriate integral)*, and it drops a 0.75.
Todd: OK.
Stan: So that means that from 0 to 1, it's gotta be a negative area of 0.5.

During the pauses Stan wrote down the words that he had spoken.

Stan then considered $F(0)$ and $F(2)$. He experienced great difficulty finding an idea for how to proceed. In fact, he commented that this was a hard problem and "kind of tricky," as well. In the end, he returned to viewing the antiderivative F as being an "area function." In particular, it became apparent that he was viewing F as being defined by:

$$F(x) = \int_0^x f(t)\,dt,$$

although he never stated it in this fashion. Therefore, for $F(2)$, he said:

Stan: That at $f(2)$ [*sic*], they're looking for the area under the curve, so I think that f of...well, no... I think $f(2)$ [*sic*] would be 0.75.
Todd: Why?
Stan: Because 0.75 is the area under the curve from 0 to 2. And $f(2)$ [*sic*]...*(pause)* $f(2)$ [*sic*] is the area under the curve? I think. At 2, when $x = 2$.

Note that the last part of the passage showed that he was not confident of his solution. For $F(0)$, he said, "$F(0)$ is the area under the curve when $x = 0$. And at 0, it hasn't gone anywhere. Assuming that we're starting at 0." It was at this point that it became clear that, in his "area function" framework, he was thinking of the lower limit of integration as being 0. At this point, he had not noticed the conflict between $F(1) = 0.3$ and

$$F(1) = \int_0^1 f(t)\,dt = -0.5$$

that his interpretation of the antiderivative *F* had caused.

This conflict was investigated at the beginning of Stan's third interview. Stan was greatly perplexed by this, and appeared to be in disequilibrium over it. He verified his original calculation for

$$\int_0^1 f(t)\,dt = -0.5,$$

and concluded that it was correct. After some prompting, he considered his view of the antiderivative *F*, but this only added to his confusion. The only idea that he was able to generate was that different antiderivatives were being used. In addition, Stan was asked why $F(2)$ could not be given by

$$\int_1^2 f(t)\,dt = 1.25.$$

This intensified his confusion and disequilibrium. He finally admitted that he did not have an answer and was "kind of stumped" by all of this.

As a final follow-up question during the third interview, Stan was asked what additional piece of information he needed to know to do this task. His response was, "the function." This was further clarified to show he was wanting the function *f*. Further, he indicated that he would: (a) graph *f*; (b) antidifferentiate *f* to find *F*; and (c) use The Fundamental Theorem of Calculus, although he was not certain how he would go about using The Fundamental Theorem of Calculus. After having mentioned these ideas, he was unable to make any further progress on this task.

Response to Task 12

It took Stan a few minutes to digest the given information and to decide what he was going to do with the instantaneous rates of change. Without plotting the data, he produced two estimates during his work on this task (see Figure 53). In both cases, he worked

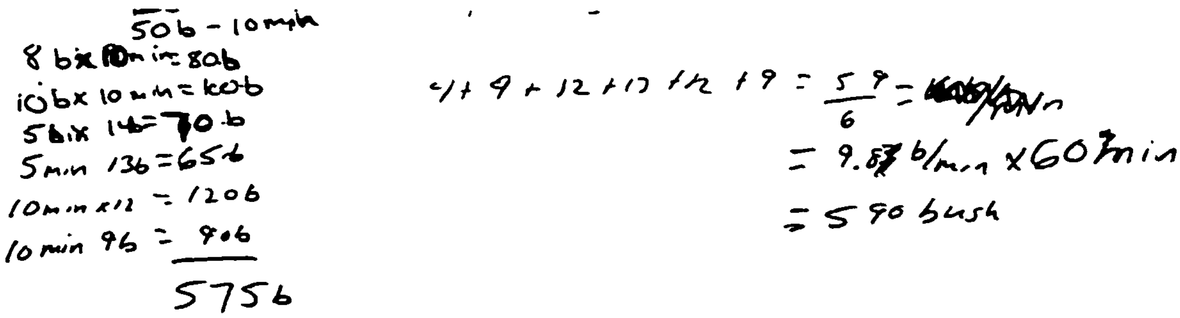


Figure 53. The two estimations Stan formed for the total harvest.

directly from the table of data. It was evident that he was thinking that the number of bushels was equal to the harvest rate multiplied by the length of a time interval. His first estimate (see left side of Figure 53) was essentially to sum:

$$5*10+8*10+10*10+14*5+13*5+12*10+9*10=575 \text{ bushels}.$$

As he carried out his estimate, there appeared to be no discernible pattern in the formulation of his products. When asked to summarize his calculations, he decided that he could do a better job estimating the total harvest. This was when he produced his second approximation (see right side of Figure 53). This approximation can be summarized as follows:

$$\frac{1}{6}\left[\frac{1}{2}(0+8)+\frac{1}{2}(8+10)+\frac{1}{2}(10+14)+\frac{1}{2}(14+12)+\frac{1}{2}(12+12)+\frac{1}{2}(12+6)\right]*60=590 \text{ bushels}$$

Judging by his written work, he first averaged the harvest rates that formed the endpoints of each of the subintervals given in the table of data. He then averaged the resulting values over the number of subintervals, and finally took his average harvest rate and multiplied by 60. Careful inspection revealed that this was, in essence, an application the Trapezoid Rule. When asked why this last method made sense, he was unable to form a coherent response. The essence of his response was that the harvest rate was probably going to change more gradually than the table showed, so he was trying to compensate for this by averaging the harvest rates.

Stan was able to view this task in terms of a summation process or an accumulation process, but he was unable to communicate the idea when asked to explain his work. Furthermore, he was unable to relate this process to the definite integral when asked whether there was any connection between his work and the definite integral. His initial response was, "I can't really think of a way to relate it." He did plot the data points and connect them with a curve. When he finished, he said, "so I suppose that, area under that curve could be the total number of bushels harvested." However, this was only a speculation, as he was unable to provide any reason for his idea when asked to explain it. This led to the conclusion that he did not perceive a connection between the summing process he had just completed and the definite integral.

Response to Task 13

Stan was unable to make any progress on this task. He had to re-read the task several times before he appeared to have comprehended all of the information, and even then the density function had to be explained and clarified for him. The only connection that he was able to make was that the mass was equal to the density times the length, where the length was the distance from the beginning of the rod. He was unable to make any connections between this task and the definite integral. After some probing, it became clear that he was becoming very frustrated with the task.

Response to Task 14

Stan's response to the question posed in this task was that the definite integral "represents the area under the curve $f(x)$ from a to b." He did not provide any more insight into his representation than the above comment. This task also appeared on the participant selection tasks (Appendix E). At the time the selection tasks were administered, Stan wrote that the definite integral represented "the area under the curve on the interval a to b." This showed that Stan's representation of the definite integral had not changed during the time between the administration of the selection tasks and the third interview. Therefore, the area representation of the definite integral was firmly established for him.

Response to Task 15

Stan drew in vertical lines at $x = a$ and $x = b$ soon after reading the task, as if he was going to indicate the area under the curve, but stopped when he realized that he was looking at the graph of the antiderivative F. He then indicated that:

$$\int_a^b f(x)dx = F(b) - F(a).$$

This was interesting, because while working on Task 11 during the second interview and revisiting it at the beginning of the third interview, he did not make this association. It is

unclear what caused him to make the connection at this time. When asked whether he could relate the quantity $F(b)-F(a)$ to the graph of the function *F*, he marked the point $(b,F(b))$and the point $(a,F(a))$ on the graph. His first thought after plotting the points was given in the following exchange:

Stan: Yeah. I don't know how it would relate...I don't think it relates to this area under this curve *(the given curve)*.
Todd: OK.
Stan: 'Cause that's...if it did...if this was $f(x)$, then that would.

In essence, he rejected the notion that the integral represented area under the curve because the graph he was looking at was the graph of the antiderivative *F*, not the graph of the function *f*. When asked if he had any other thoughts, he eventually produced of the idea of taking the differences of the *y*-coordinates, as shown in the following passage:

Stan: Um, I don't know. *(long pause)* I don't know. I mean, this coordinate right here *(the point $(b,F(b))$)*, this *y*-coordinate...
Todd: Mm-hmm.
Stan: ...minus this *y*-coordinate *(the point $(a,F(a))$)* would give you the answer for this, for integral $\int_a^b f(x)dx$.
Todd: OK.
Stan: I mean, for me, that's what $F(x)$ is, is what this is *(points to the curve for F)* telling me.

The first part of this passage meant that he was unsure whether he was supposed to give the obvious answer. He apparently thought the answer should be more complicated than the obvious answer that the integral represented a difference between the *y*-coordinates of the function *F*, with the *y*-coordinate for $x=b$ coming first. However, he did not indicate the vertical displacement on the graph of *F* itself. Further questioning revealed that, because "the *y*-coordinate of *b* is greater than the *y*-coordinate of *a*," that is, $F(b)>F(a)$, the integral would have a positive value.

Stan was given the opportunity to look at the second graph for this task in order to further investigate his understanding of the situation presented in the task. Asked how the integral $\int_a^b f(x)\,dx$ related to this graph, the following exchange took place:

Stan: That it's negative.
Todd: Why is it negative?
Stan: Because the y-coordinate on this one *(refers to the point* $(b, F(b))$*)* is smaller than the y-coordinate on *a (refers to the point* $(a, F(a))$*)*.

Again, he never indicated this on the second graph. However, it was evident that he had come to view the integral $\int_a^b f(x)\,dx$ as representing the difference in the y-coordinates of the points $(b, F(b))$ and $(a, F(a))$ on the graph of the antiderivative F.

Response to Task 16

Once Stan understood the question, he drew a graph for a function f because a "picture helps me think about it" (see Figure 54). He then said, "I would tell him that

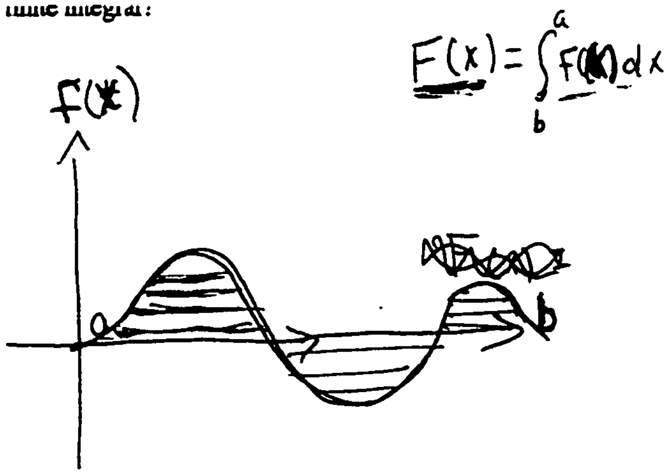


Figure 54. Graph Stan used to explain his definition of the definite integral.

$F(x)$ is the antiderivative of $f(x)$." He then gave his definition of the definite integral, "I would tell him, that this is $f(x)$ *(referring to graph)*, and the area under this curve is the function of $F(x)$. And that is defined as the function for $F(x)$, I would tell him." Therefore, his definition was based upon two themes that appeared previously. The first of these was the notion of area under the curve. This was consistent with the representation of the definite integral that he gave in Task 14, namely, that the definite integral "represents the area under the curve $f(x)$ from *a* to *b*." However, the notion of signed area would have been more appropriate for the graph he had drawn. The other theme was the area function, which was simply the antiderivative *F*. Here the antiderivative *F* was a function of the area underneath the curve $f(x)$. Although his definition was fragmented, for Stan, a definition of the definite integral centered on the area model.

Response to Task 17

Stan had great difficulty with this task. Early in the task, he thought that the definite integral was an approximation for the area under the curve. This issue was resolved once he realized that The Fundamental Theorem of Calculus would give an exact value for the definite integral, as well as the area under the curve. Another idea that he had difficulty with was working from the graph. In particular, he had difficulty determining the heights of the pictured rectangles and finding the corresponding x-value for these heights. The lack of scaling on the axes concerned him. Finally, it was possible that he was unsure about what to do in this situation, since his "block counting" technique was not applicable.

Although Stan rambled and had disconnected speech, he eventually pulled together some thoughts. He acknowledged that the diagram could be used to form an approximating sum that would yield an approximation for the area under the curve. At first he wanted to repartition the x-axis to form a regular partition. When asked why he could not work directly from the given diagram, he replied, "those are different size rectangles. I mean, you

could, but you'd still have to rescale it so you knew where this x-coordinate was or where this x-coordinate was." The x-coordinates referred to were associated with the vertical lines that intersected the x-axis. This suggested Stan was concerned by the lack of explicit x-coordinates on the x-axis, but he acknowledged that it was possible to form an approximating sum from the given diagram. Further questioning revealed that he would do this by computing the area of each rectangle individually and then summing these areas over the number of rectangles. The end result would be an approximation for the area of the region bounded by the graph of f and the x-axis on the closed interval $[a,b]$. All of this led to two conclusions: the connection between the diagram given in the task and his definition was in terms of area; and that he was able to work with the idea of an approximating sum over an uneven partition of the closed interval. However, the latter was a notion with which he was uncomfortable.

In order to further gauge Stan's understanding of the relationship between his definition for the definite integral and the approximating sum suggested in the diagram, he was asked about a way to connect the two ideas. He indicated that the situation pictured in the task was an approximation of the area under the curve, and that the definite integral would yield the exact value for the area under the curve. In addition, he admitted, "I don't see where I want to be going with it." However, he also said, "and the closer you get these together, the more subdivisions you have under the curve, the closer to exact you're going to be." This meant that the more subdivisions the closed interval was partitioned into, the closer the approximation would be to the area of the region bounded by the graph of f and the x-axis on the closed interval $[a,b]$. Furthermore, there were the rudiments of a limiting process, based upon the use of the phrase "the closer you get these together", but he never made this connection. This provided a natural lead into the next task, in which the focus was the connection between the limit and the definite integral.

Response to Task 18

When asked about the connection between the limit and the definite integral, Stan was unable to provide any answer that connected the limit to the definite integral. His use of the phrase "the closer you get these together" in the previous task did not suggest the notion of limit to him. He did mention limits in connection with functions and with derivatives, but he did not make a connection to the definite integral. He finally admitted, "I don't know."

Response to Task 20

Stan had great difficulty comprehending the definition of the function f. After evaluating the function at several t-values, he began to understand the function f. He correctly sketched a graph for the function (see Figure 55).

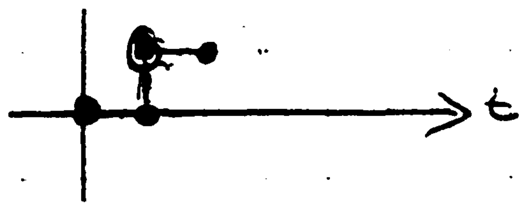


Figure 55. Stan's graph for the piecewise-defined function f.

Once he had the graph of f sketched, he focused on the function F. However, he focused on the fact that the function F was an antiderivative for the function f, rather than on the integral function. The connection between f and F was revealed in the following:

Stan: 'Cause this *(points to* $F(x)$*)* is the antiderivative of this *(points to* $f(t)$*)*, so this *(points to* $f(t)$*)* is the derivative of that *(points to* $F(x)$*)*.
Todd: OK.
Stan: And so, I'm trying to look at how the slope of this *(underlines* $F(x)$ *on graph)* would act *(circles graph of* $f(t)$*)* with that. I mean, that *(points to the graph of* $f(t)$*)* is the slope of this function *(points to* $F(x)$*)*.

Thus, he understood the graph of f as giving the slope values for the function F. His view at this point was consistent with his past views of the integral function, namely, as a function which is the antiderivative of the integrand. He then sketched a graph of F based upon the slope values given by the function f (see the top graph in Figure 56). When questioned

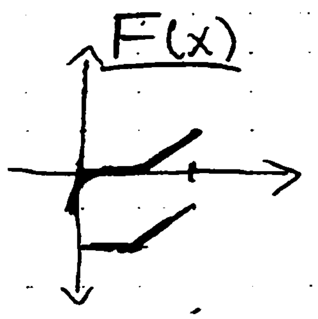


Figure 56. Set of graphs Stan sketched for the integral function F associated with the piecewise-defined function f.

about the placement of his graph for F, he admitted, "I guess it could be anywhere really." This demonstrated that he recognized the antiderivative was not unique. He continued to try to explain the placement of his graph, but finally said, "yeah, I don't why it would go there. It's gotta me befuddled." At this point, he was presented with a graph that was similar but shifted down 2 units (see bottom graph in Figure 56), to see if it would induce any further thoughts. Stan said that this graph would "probably" work, and explained:

> 'Cause it's just the slope of this *(points to new curve)*. That's all this graphing *(points to graph of* $f(t)$*)*. It's just the slope, I mean, the slope of this one is the same as the slope of this one. *(Points to the two graphs drawn for the* $F(x)$*.)*

This indicated that, to Stan, the additional graph would work, because it had the same slopes as his graph, and these slopes corresponded with the function values of f. This led Stan to conclude that the graph for the function F was not unique. It was interesting to note that Stan never considered the integral side of the integral function in any of his work. It was

evident that he focused only on the antiderivative aspect of the integral function, and by doing this he was blind to other ways of thinking about the integral function.

At the beginning of the third interview, Stan was asked about the integral in the definition of the function *F*. He had difficulty finding words to express his thoughts, but eventually he said, "you're looking at the area under the curve only from 0 to 2." Stan was then asked how "area" fit into his work in determining a graph for the function *F*. At this point, he focused on both slope and area and sought a connection between the two ideas. Eventually, he presented his thoughts on a connection between slope and area. His solution was given and explained in the following passage:

Stan: See it's 0 from 0 to 1 *(upper curve on the graph of F)*, and the area under the curve from 0 to 1 is 0 *(points to the graph of f)*, as well.
Todd: OK.
Stan: And, from 0, or, from 1 to 2, the slope *(points to the upper curve on the graph of F)* is equal to the area under the curve *(points to the graph of f)*, as well, which is 1.
Todd: OK.
Stan: So, I guess the area under the curve and the slope could be the same, maybe. On this example, it looks like they are.

In formulating his resolution, he focused on intervals, and not on what happened at individual points. He then used this criteria to determine that the upper curve graphed in Figure 56 would be correct because "the slope and the area under the curve are equal." Furthermore, the lower curve graphed in Figure 56 is not the graph of *F* because "the area under this curve *(points to the lower curve on the graph of F)* is negative, and the slope is not negative, the slope's positive." However, he concluded his thoughts by saying that he was not certain that his idea was correct. He was able to connect the integral function with area under the curve once he focused his thoughts on the integral portion of the integral equation, but only after being forced to examine the idea. However, it was apparent that the area concept was being overshadowed by his view of the function *F* as an antiderivative and the notion of slope that this evoked. Stan was unable to resolve this issue within the confines of the interview.

The Case of Tina

Introduction

Tina was a 25-year-old senior pursuing a geology major at the time of the study. Prior to Calculus II, she had taken Calculus I at the institution where the study took place. High school algebra was the last mathematics class Tina had taken before enrolling in Calculus I. She took Calculus I for the second time during the semester prior to the interview. Tina earned a D in Calculus I the second time she enrolled in the course, and said that her grade was the result of not having devoted the time required to grasp the concepts.

At the end of the third interview, Tina was asked to respond to questions focusing on her thoughts about her understanding of the definite integral to provide insight into how she approached the tasks themselves. In response to a question about how she thought about the definite integral, Tina said:

> Graphical first. That's the first thing I think of, I think. Because I learn much better by seeing things. I can grasp things much more easily, if I can actually see. That way in my head, it's easier for me to, um, figure out exactly what I'm doing, or what it means, I guess. So, I need some sort of application. In math, I cannot think of things conceptually. My brain doesn't work that way.

This indicated that her first thoughts would be graphical because she was a visually-oriented person. In addition, she admitted being unable to envision mathematics from a conceptual perspective. She made no comment concerning thinking symbolically.

In response to an inquiry regarding aspect(s) about the definite integral she perceived as most difficult for her, she said:

> The whole process. (laughs) Um...*(pause)* Uh, I don't think I could really nail down one specific thing. I mean, pretty much the majority of it eludes me. (laughs)

For her, nothing about the definite integral was simple. She also said, "In my head something may seem easy, but it's simply because I'm oversimplifying what it really means, 'cause I don't understand it."

Response to Task 1

In working on parts a) and b) of this task, Tina recognized that she needed to apply The Fundamental Theorem of Calculus to both parts; however, when she actually carried out the calculations, it was evident that she performed from rote memorization. Her usage of phrases like "I do believe" and "I can't really remember, if..." provided evidence for this conclusion. In addition, phrases like "as far as I understand" indicated that she might be still learning how to apply The Fundamental Theorem of Calculus. Despite some difficulty caused by arithmetic errors in part a), she was able to find the antiderivative correctly for both parts. In both cases, she concluded that she was not comfortable with her work. This was interpreted as evidence that Tina was in the early stages of understanding The Fundamental Theorem of Calculus.

As Tina worked on parts a) and b), two items of interest arose from her work. In part a), she included a "$+C$" with her answer. When asked about this, her first response was, "as far as I understand it is, um, it's just an arbitrary constant." She then had great difficulty formulating her thoughts concerning why it should be present. After she stopped and collected her thoughts, she provided her best explanation for using the "$+C$" in her answer:

> Um, it's just, I think, basically, whatever number that is *(points to the* 1 *in the integrand)* when you get rid of it when you do...when you take the derivative. It [$+C$] just represents any number that can be there *(points to the* 1 *in the integrand)*.

Her response indicated that "$+C$" was a holdover from finding antiderivatives. She was familiar with the need for the constant of integration when finding antiderivatives, and she transferred this idea to the current subtask because it involved finding an antiderivative. However, when she worked on part b), she did not include "$+C$" with her answer.

The other item of interest occurred in part b). When she first looked at the subtask, the use of the variable t in the integral caused disequilibrium because she was unsure how to

evaluate a definite integral with a variable other than x. However, she resolved the conflict by concluding, "um, I'm assuming that regardless of the variable, it's still the same." This statement was interpreted to mean that she would compute the definite integral using The Fundamental Theorem of Calculus whether she was using x or t for the variable. Further probing revealed that she was accustomed to using x as the variable of integration in the definite integral, but

> because there is only one variable in there, I guess, I mean, it doesn't matter what variable it is. Like, if it was $e^t\,dt$, like, having to take the derivative of e^t with respect to t, I would be really confused. But, uh, I guess it's just a single variable, it doesn't matter.

This passage indicated Tina had determined that, when the definite integral involved only a single variable, the variable name did not matter when computing the definite integral. Furthermore, it was evident that some learning had taken place on Tina's part, and that her concept image of The Fundamental Theorem of Calculus had been expanded to include the notion that the value of a definite integral does not depend on the name of the variable.

Tina was not familiar with the greatest integer function. After it was explained, and she had worked some example calculations involving the greatest integer function, she appeared to have a basic understanding of the function. When part c) of this task was presented to her, she had difficulty finding a solution method to try. Eventually, she admitted, "I really have absolutely no clue how to approach that," where "that" referred to the integral involving the greatest integer function. Tina was unable to provide any solution ideas for this subtask at the time of the first interview, apparently because this definite integral did not fit into any of her learned rules.

This subtask was revisited at the beginning of the second interview to determine whether she had any additional insights for evaluating this definite integral. After the greatest integer function was explained to her again, she guessed that the value would be 5 and provided an explanation that indicated she was computing the value of the integral as

$\lfloor 5 \rfloor - \lfloor 0 \rfloor$, but was not finding an antiderivative. Further probing concerning an antiderivative yielded this response:

> I wouldn't have a clue how to antidifferentiate, the greatest integer function at x. I mean, I guess if I were to antidifferentiate x inside the brackets, and then keep it inside the brackets, and then evaluate it, just with what I know how. But I don't know if that's the way things work with this function.

This passage indicated that she was searching for a rule for finding an antiderivative, but she was unable to find an antiderivative since the antiderivative of the greatest integer function conforms to none of the rules she had been taught. Tina was asked if there were other ways in which she could evaluate an integral. After a long pause, she responded, "I guess I could graph it, but I don't know how to graph the greatest integer function."

Tina was given instructions for entering the greatest integer function into her graphics calculator. The graph that she produced is shown in Figure 57. After the graph

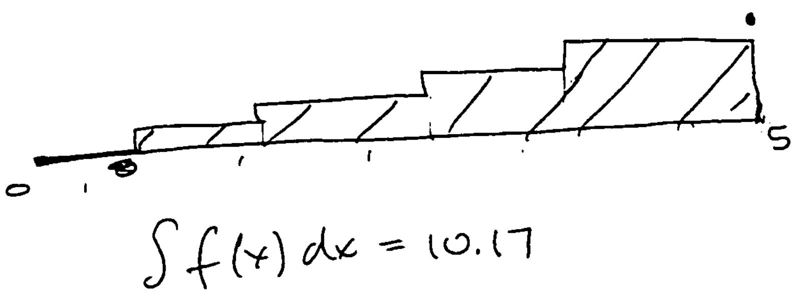


Figure 57. Tina's graph for the greatest integer function and her approximation for the definite integral $\int_0^5 \lfloor x \rfloor dx$.

was drawn, she approximated the integral from her graph, using the numerical integral application on her calculator. She reported an answer of 10.17 for the value of the integral. She was asked if this made sense to her. While she attempted to make sense of the graph

itself, she indicated that she was thinking about the area under the curve. In particular, she was trying to make sense of what was happening on the vertical axis. She offered this speculation: "if we've gone 5 units on the x-axis, and 5 on the y-axis. You could add those together is 10." While she tried to explain her rationale for this, she made a connection to Riemann sums, as illustrated by her comment, "Um, I guess it's, like, Riemann sums, basically, where you're taking the area of each one of these little rectangles and, adding 'em up." As she spoke, she lightly drew in rectangles on her graph (see Figure 57). However, when she tried to compute the areas of the rectangles, she had difficulty reading the heights off the graph. After reviewing the graph, she said:

> Um, let's see. *(pause)* It does make sense, because, now if you're taking the area, or the...if you were to take the areas of each one of these little rectangles, um...*(pause)*...the area of this *(points to the fourth rectangle, and then moves to the left)* would be 4, this would be 3, this would be 2, this would be 1. And if you add those all up, it would be 10. So, I guess it does make sense. (chuckles) Or, maybe, I don't...it appears to. I don't know.

Therefore, she was able to make sense of her calculator approximation by relating it to the area under the graph of the greatest integer function on the closed interval $[0,5]$. However, she appeared to talk herself out of her understanding by the end of the passage. An explanation for this was that her unfamiliarity with the function caused her to doubt her work. In the end, she was unable to make any further progress towards comprehending her calculator approximation for the value of the definite integral.

Response to Tasks 2 and 3

When presented with Task 2, Tina approached the task in the same manner as Joan. Tina found the area of the two triangles that she created, but subtracted the area of the triangle in Quadrant II from the area of the triangle in Quadrant I (see Figure 58). Furthermore, she subtracted the areas because the area in Quadrant II represented negative area. Since Tina evoked similar methods and explanations as Joan, no further comments will be made about Tina's work on this task.

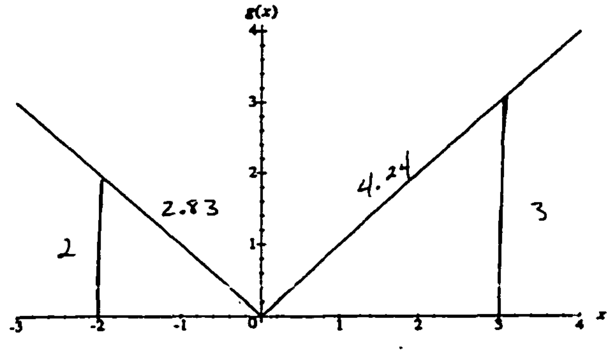


Figure 58. Set-up for Tina's graphical approach to evaluating the definite integral $\int_{-2}^{3} g(x)\,dx$.

When Task 3 was presented to Tina, she disregarded her graph from Task 2, even though it was in plain view. Like Lynn and Stan, Tina worked the problem as if it was $\int_{-2}^{3} x\,dx$. Her answer to this integral was the same as she had obtained in Task 2. When asked why she had evaluated the definite integral the way she did, she replied, "well, the rules of antidifferentiation." She then went on to explain the Power Rule for Antidifferentiation. This indicated that she was working from rote memorization and was applying a familiar rule to an unfamiliar situation.

Tina made no attempt to form any connection between Tasks 2 and 3, so she was asked if there was a reason why she had the same answer for both tasks. After a long pause, she said, "'cause I messed up. I don't know." After another look at the two tasks, she concluded, "I don't know. No, I don't have a clue why." After she said this, it was apparent that she was not going to make any connections between the two tasks.

Response to Task 4

Tina's initial approach was to separate the region into two subregions: a triangle and a quarter ellipse (see Figure 59). Her reason for using a quarter ellipse was:

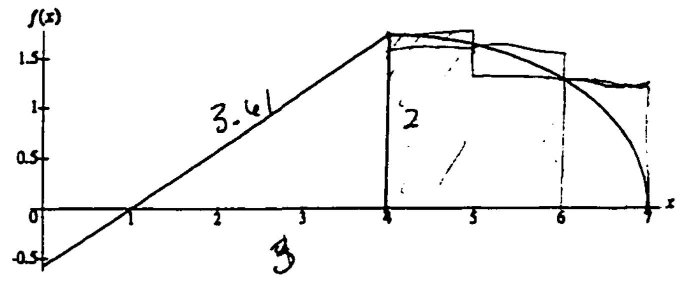


Figure 59. Tina's graphical work for approximating the area of the region associated with the closed interval $[1,7]$.

> 'Cause it's not, curved like a circle. There's a steeper curve here *(points to the part closer to the x-axis)*, then there is up here *(points to near the maximum)*. At least that's what it looks like to me.

After she had partitioned the region into the two subregions, she indicated she would "find the area of this *(points to the triangular region)*, and find the area of this *(points to the other region)*, and add them together." She used the area formula for a triangle to approximate the area of the triangular region as 3; however, she was stymied by not knowing an area formula for an ellipse.

Tina was asked if there were other ways she could find the area of the quarter ellipse. She replied:

> Um. Yeah, I guess I could use the Riemann sums thing. Um, and break it into trapezoids. *(Sets up 3 rectangles using the right endpoint of each subinterval.)* Hmm, this doesn't work out very well, from the right side. *(Sets up 3 rectangles using the left endpoint of each subinterval.)* And find the area of each one of these *(shades in each rectangle)* rectangles that I just... And it would be an approximation.

As she spoke, she drew on her graph (see Figure 59). A possible reason for why she rejected use of the right approximating sum was that it would not capture any of the area over the closed interval $[6,7]$. By using the left approximating sum, she was able to include area from each of the three subintervals of the closed interval $[4,7]$. However, she did not actually compute the left approximating sum.

When asked what additional piece of information she needed, she said, "the area equation for an ellipse." At that point, it was evident that she was still thinking in terms of geometry, and was not thinking in terms of calculus or the definite integral.

She was asked if she had any other thoughts about how she might find the area of the given region. After a long pause, she said:

> I guess if I actually knew the function, I could, um, evaluate it from [a] lower bound of 1 and an upper bound of 7. If I knew the actual $f(x)$. Then I'd just antiderivative and get F or whatever, and, um, plug in 1 and 7, and subtract that value *(points to* $x = 1$*)* from that value *(points to* $x = 7$*)*.

This indicated that, if she knew the function f, she would apply The Fundamental Theorem of Calculus over the closed interval $[1, 7]$. It was noted that even though she had broken the region into two regions previously, she now indicated that only a single antiderivative was needed for the entire interval. She did not evoke the idea that the summation property of the definite integral was a necessary tool at this time.

Tina was given the opportunity to view the piecewise-defined function. The function was not what she expected to see, but she was able to work with it, as shown in the following excerpt:

Tina: Well, assuming I did this correctly *(points to her calculation for the area of the triangle)*, which is a huge assumption, I could evaluate this piece of the function *(points to the second expression of the piecewise-defined function)*. Um, and I guess I'd have to antiderivate that, and again plug in these two values *(points to the inequality containing the* 4 *and* 7*)* for my upper and lower bounds for this part of the function.
Todd: And that would give you what?
Tina: The area of this part right here *(points to the region to the right of* $x = 4$*)*.
Todd: OK.
Tina: And then, to check this *(points to value for the area of triangle)*, I guess I could do the same with that *(points to the first expression of the piecewise-defined function)*. And then add them together.
Todd: OK.
Tina: For the overall area.

What this passage showed was that Tina could work with the summation property of the definite integral. In addition, she was able to correctly write out the necessary definite integrals:

$$\int_1^4 \frac{\sqrt{3}}{3}(x-1)\,dx \text{ and } \int_4^7 \sqrt{\frac{9-(x-4)^2}{3}}\,dx.$$

Response to Task 5

Tina had great difficulty with this task. After perusing the task for several seconds, she commented that she would substitute the 5 for the x in the upper limit of integration, and that she would "have to first antiderivate the expression with relation to λ." She made an unsuccessful attempt to use the substitution method to find the antiderivative, possibly because her Calculus II section had studied the substitution method the week prior to this interview. However, after her attempt, she became very confused. Further probing revealed that she was confused by the use of the two variables λ and x in the integral. She still understood the task as asking her to evaluate the function G when $x = 5$, but she was not seeing that she could substitute 5 for the x in the upper limit of integration to eliminate x from the integral. She was very confused and extremely frustrated, so it was decided to move on to a new task.

At the end of the first interview, this task was revisited to determine whether she had any additional thoughts. She admitted that the task was less confusing than before, but she was unable to make any additional progress towards completing the task. She indicated that Tasks 6 and 7 helped her, but she was unable to articulate why this was the case.

This task was revisited at the beginning of the second interview to ascertain whether she could provide any more information about her difficulties with this task. At that time, she was presented with this following simpler task:

$$\text{If } G(x) = \int_0^x (\lambda - 1)\,d\lambda, \text{ then } G(5) = ?$$

This simplification of the task did not help her evaluate the integral function, but it gave her an opportunity to further clarify the source of her confusion:

> I think where my confusion lies is, I'm not exactly sure, how λ applies to $g(x)$ [*sic*], in the sense that, if I were to find $g(5)$ [*sic*], what do I do with the variable λ? That's, I think, where my biggest confusion is.

Tina believed the variable λ was somehow involved in the substitution process of evaluating the integral function. She may have evoked the familiar concept image that, when a value is substituted for a variable in a function, all occurrences of the variable are replaced by the value. However, in this particular example, the presence of two variables caused her to experience disequilibrium. She appeared to be trying to relate λ to the substitution aspect of evaluating the integral function instead of the integration aspect of it.

Tina was then asked to consider this problem:

$$\text{Evaluate } \int_0^x (\lambda - 1)\,d\lambda \text{ when } x = 5.$$

She placed $x = 5$ into the upper limit of integration, but expressed concern over what to do with the 5 after she antidifferentiated. After several seconds of silence, she said:

> Well, let's see, maybe, this would be $\lambda^2/2 - \lambda + C$. I mean, is it that I'm overlooking the most obvious answer? That it would be $5^2/2 - 5 + C$. I mean, I don't know. Because, I guess if x were [a] point on the graph of this function *(points to the integrand of the new function)*, x were point on the λ-axis, and $x = 5$, then indeed, plugging in 5 into here *(referring to the upper limit of integration)* would get me, the overall value of the integral.

At this point, Tina apparently experienced a revelation and decided to pursue the new idea. Her question about "overlooking the most obvious answer" was interpreted to mean that she wondered whether she was reading too much difficulty into the task. Her graphically-oriented comments were interpreted as her way of deciding whether her idea made sense.

Since $x = 5$ could be viewed as a point on the λ-axis of the graph of the integrand, the integral made sense to compute. Mathematically, she applied The Fundamental Theorem of Calculus to obtain a correct value for the integral, except that she included the "$+C$" in her answer again.

Tina was then asked what the difference was between the two statements:

$$\text{If } G(x) = \int_0^x (\lambda - 1)\,d\lambda, \text{ then } G(5) = ?,$$

and

$$\text{Evaluate } \int_0^x (\lambda - 1)\,d\lambda \text{ when } x = 5.$$

Her response was:

> I guess nothing. *(pause)* I think where I get confused is when, I see the $g(x)$ [*sic*] here *(points to the original integral function)*. I think, OK, this is the function, it's a function of x, and then when I'm evaluating this *(points to the integrand in the original problem)* with respect to the λ. Um, *(pause)* I'm not exactly sure how to tie the λ-expression *(points to the integrand)* in with my x-expression *(points to $G(x)$)*.

Thus she thought the two statements that she was being asked to compare were the same. The latter part of the response focused on further clarifying her confusion with the task. Her explanation indicated her confusion might center upon not understanding the assignment aspect of the equal sign in the definition of the integral function. However, any connections that she had made between the above two mathematical statements appeared to be lost by the end of the task, as she was again experiencing difficulty understanding the roles of the variables x and λ.

At one point Tina commented that G was a function of x, so she was asked to explain why this made sense. She replied, "because of the notation." She continued with this graphical explanation:

> And, I guess, graphically, like, if you were to graph some function $g(x)$ [*sic*], you're graphing the x-values as opposed to...the x-values when evaluated as this function *(circles the* $G(x)$ *in the original problem)*. So, basically, like, the evaluation of the function gives your y-axis, and then your x-values are on your x-axis. And once the function is graphed, it's the values of x evaluated at this function. If that made any sense. (laughs)

Her understanding of G as a function of x appeared to be based upon her past experiences with functions; in particular, with function notation and the input-output aspect of functions. In this sense, she indicated that she had the beginnings of a pointwise understanding of G; however, she was never able to explain why G can be called a function.

Response to Task 6

Tina needed some help understanding the statement of the task. In particular, she had difficulty understanding how the x fit into it. Once this was clear, she concluded, like the other 4 participants, that the area of the region would increase as x moved to the right. What was interesting about Tina's explanation was that she used a pen as the boundary line $t = x$, and then slid the pen along the t-axis to illustrate that the area of the region would increase as x, or the pen, in Tina's case, moved to the right. This action demonstrated that Tina possessed some across-time understanding of area. Tina was asked to speculate about what would happen to the area if x was allowed to continue to increase. She provided the following answer and explanation:

Tina: Well, I can't say this for sure, but $f(t)$ looks like it might be hitting an asymptote at the t-axis. So it can continue on indefinitely, if that's the case.
Todd: What would happen to the area, then?
Tina: It would be infinite. I'm assuming. (laughs)
Todd: OK.
Tina: Well, if this can continue on forever *(points to the curve)*, so could the area.

Therefore, Tina believed that the area would increase toward infinity because the t-axis was acting as a horizontal asymptote for the function. However, she also indicated that she was not sure what would actually happen.

After reading the second part of the task, Tina, like the rest of the participants, said that the values of the integral function would increase. What was interesting, in light of her initial experience with the integral function in Task 5, was the explanation for her answer:

Tina: I guess, if x is continually moving to the right, its values are getting larger and larger and larger and larger.
Todd: OK.
Tina: And if you were to evaluate this function, given actual, like the, a numerical value, and plug in x, the larger the x-value, the greater that numerical value would be. Therefore, making g [*sic*] greater.

At this point, she had no difficulty working with the integral function, whereas she was unable to work with the integral function presented in Task 5. Tina appeared to be focusing only on the variable x. Further insight was gained when she was asked whether there was a connection between the two parts of the task. Tina's understanding of the situation was revealed in the following passage:

Tina: *(pause)* Yeah, I think it's just asking me...this is asking me in words *(points to part a))*, this is just asking me using, um, symbols *(points to part b))*.
Todd: Asking what?
Tina: What's gonna happen to the overall area. Because the integral signifies area under this curve, or so...I'm trying to remember. Um, so therefore, this being the curve *(points to the integrand)* and this being the value on the t-axis *(points to the upper limit of integration)*, it's asking me as this value *(points to the upper limit of integration)* increases, what's gonna happen to the area under this curve *(circles the integrand)*, which is $g(x)$ [*sic*]. I guess, well, the overall value... Auggg, I don't have a clue. I...I have a really hard time understanding how the variables x and t are related. I'm not understanding the relationship. I have a hard time articulating what I think might be going on.
Todd: So is that the same problem you're having back on H [Task 5]?
Tina: Yep.

Thus, the two parts of this task were asking Tina the same question. Her reason was based upon the definite integral representing area under the curve where $t = x$ was the right boundary, and thus the upper limit of integration, for the region bounded by the t-axis and the graph of the function f. Part b) simply asked her to find the area under the curve using only symbols. From this, it was evident that she was working from her remembrance of what the definite integral represented. However, the latter part of her second vocalization

revealed that her understanding of the definite integral was rather superficial and tenuous, as she was unable to continue her explanation. She attributed her inability to complete her explanation to insufficient understanding of the relationship between the variables x and t. While this might be part of the reason, it was also likely that she did not fully understand the connection between the definite integral and area. She also provided information about her confusion over the variables x and λ from Task 5; namely, not being able to understand how the two variables were related.

Response to Task 7

Tina's response to the first part of this task centered around signed area, and was very much like the answers given by the previous 4 participants. From a geometric perspective, she knew that area was a positive-valued quantity, but it could be negative in this situation "because that's what they told me in math class." Tina produced no other new insights into how the participants viewed part a) of this task. When Tina was asked how part a) would change if the phrase "area of the region" were replaced by the phrase "total area of the region," she answered, "no, I don't think it should change it, but I could be totally wrong." Further probing only confirmed that she believed the two phrases meant the same thing. Therefore, Tina was unaware of the distinction between the mathematical notion of area and the notion of signed area.

When Tina considered part b) of this task, she viewed it as asking the same question as part a), only "using symbols." Therefore, as x continued to move to the right, the values for the function G "would be in actuality decreasing." Furthermore, these values would be negative. The difficulties that Tina experienced with previous tasks involving an integral function did not arise this time. There were two possible reasons why this difficulty did not occur in this instance. The first was that her response to part b) focused on the graph itself

and signed area. The second reason was that she did not consider very closely the function G itself.

Response to Task 8

Tina used visual reasoning along with signed area to order the definite integrals given in this task. In completing her ordering, she computed a value for the definite integral in part d) only. The ordering that she obtained, from smallest to largest, was c), d), a), e), and b). However, she placed a) and d) in the wrong order.

She was asked to justify her ordering of the integrals. The following provided her rationale for the ordering, along with the graph shown in Figure 60:

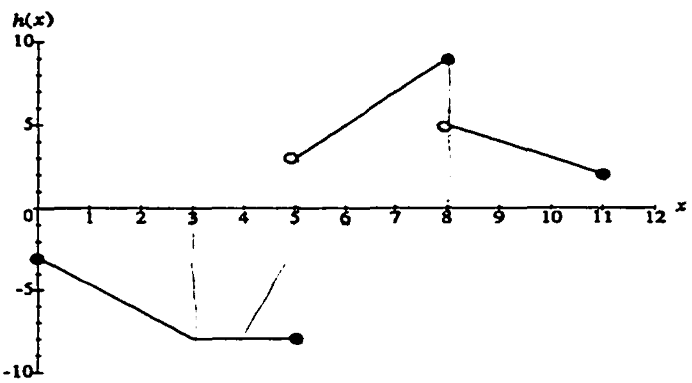


Figure 60. Graphical work Tina used to order the five definite integrals.

Tina: Um, c) is from 0 to 5, and the function is below the x-axis, so it's negative, and, um, so it's area is all this in here *(shades region under the x-axis)*. So, out of the two negative areas, it's the largest negative value, so therefore it's the smallest numerical value. d) is the next negative value. It's just simply a point, because I'm evaluating it from 5 to 5. And so this is the only point *(points to the solid dot)*, which would be, um...oh, maybe that would turn out to be 0. *(pause)* 'Cause you would subtract it from itself. Hmm. I guess it still would put it in the same position. Um, then a) is from 0 to 11, and, um, so that would be including all of this area here *(refers to the area below the x-axis)*...and this part *(shades region above the x-axis)*. And so, this *(points to the region below the x-axis)* subtracting from this *(points to the region above the x-axis)*, probably would yield a very small area.

Todd: Smaller...

Tina: Smaller, smaller...larger than this *(referring to d))* but smaller, say, than the next one, which would be e). Which is from 3 to 8. And um, that would be this area here *(shades region below the x-axis from* 3 *to* 5*)* plus this area here *(shades region above the x-axis from* 5 *to* 8*)*. And um, I guess it looks like it'd be a close race, but I think I'd have to evaluate it numerically, but it looks like it would yield a greater area than 0 to 11, but not as much as, b), from 5 to 11 yields all this area here *(shades region above the x-axis)*.

Although she used the term area, it was clear that she was using signed area, along with some guessing, to order the integrals. Her rationale for the placement of the integral $\int_5^5 h(x)\,dx$ was interesting. Tina initially associated its value with the point on the graph at $x = 5$, but then changed her mind to say the value was 0. The meaning of the reason she gave, "'cause you would subtract it from itself," was unclear, but it might have been a reference to The Fundamental Theorem of Calculus. The manner in which she decided the placement of the integrals in parts a) and e) was interesting because she appeared to guess, even though she made a comment about having "to evaluate it [integral in part e)] numerically" to compare to the integral in part a). Tina was asked what she meant by "evaluate numerically." She responded:

> Um, if I were to actually know the function $h(x)$ and, antidifferentiate it and then plug in each one of these values *(refers to the limits of integration)* for all of these. I would evaluate them, and see actually which numerical value...if they did indeed correspond with the order that I have chosen.

This indicated that if she knew the particular function $h(x)$, she would use The Fundamental Theorem of Calculus to evaluate each of the definite integrals, and then compare the numerical values. There was no realization that she could evaluate the definite integrals using simple geometric area formulas, especially in light of the fact that she had originally used area concepts to do her ordering.

Finally, Tina was asked if the jump at $x = 8$ would affect the integral over the closed interval $[5,11]$. She said, "Um, I would imagine it does, but I don't really how, um,

integration works with, um, non-continuous functions. So, I don't really know how it would, but perhaps, I mean, it would not surprise me if it did." She believed the value of the definite integral would be affected by the jump discontinuity, but since she did not know how integration worked with discontinuous functions, she was unsure of herself.

Response to Task 9

Tina interpreted this task as requiring her to find a function whose area under the curve on the closed interval $[a,b]$ was 2, but she had difficulty with the actual act of finding such a function. As she attempted to find a graph of a function that would satisfy the given conditions, she summarized her approach to this task as follows:

> Um, outside of just choosing a function and hoping that [laughs] I can get a and b to be some numbers that work. I mean, for it all to come out as 2. I mean, I don't really know how else I would do it.

Secondly, she illustrated her approach by saying, "it would just be an insane guessing game for me to try and think that I can come up with some function and have it all work out." These two thoughts indicated that Tina wanted first to choose a function and then to find limits of integration that would make the integral have the value 2. Later questioning revealed that, while she would have used the notion of area to determine the limits of integration, she still would have chosen a function prior to using the area model. This indicated that Tina was not using area to determine the graph itself.

Immediately after Tina's comment that finding a function would be "an insane guessing game," she made the following statement:

> Um, I guess I can just choose some linear function and say $f(x)$ is equal to...you know, 5. Or, I don't even know, I could even make easier than that, it's equal to 1, and a is 0 and b is 2.

She then drew her graph to complete the task (see Figure 61). While explaining her solution, Tina said, "I was trying to think of some great function, but obviously that wasn't gonna work." This indicated that part of the reason for her difficulty was that she was

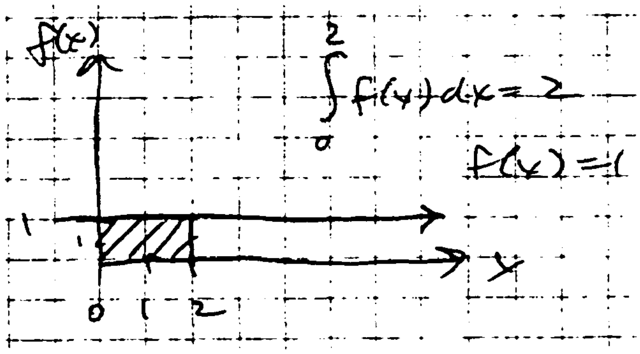


Figure 61. Work and graph Tina used to satisfy the integral equation $\int_a^b f(x)\,dx = 2$.

trying to find a special function rather than simply finding one that would work. In addition, she produced a second example, namely, "$f(x) = 2$, and take my interval from 0 to 1." When asked how she chose these functions, she said, "knowing how my function was going to behave graphically, and then keeping that mind, choosing my upper and lower bounds such that it *(refers to the integral)* would equal 2." This was consistent with her aforementioned solution strategy of choosing a function and then using area to determine the limits of integration.

As a final note, Tina used a linear function on the participant selection tasks (see Appendix E) to compete the task at that time. In particular, she used the function $f(x) = -x + 2$ on the closed interval $[0,2]$. This demonstrated consistency in the type of image she evoked to complete the task, namely, the area model.

Response to Task 10

Although Tina could not complete the task, she demonstrated a basic understanding of what the task required. Her understanding was illustrated by her disjointed explanation of the meaning of the integral equations: the first integral equation involved the amount of signed area defined by the graph of the function, and the second involved the amount of area

bounded by the graph of the function and x-axis on the closed interval $[a,b]$, because of the absolute value bars around the function. Tina made only one attempt to satisfy the requirements of the task. She used the function $f(x)=x-2$ on the closed interval $[0,4]$. Furthermore, she concluded that this function would not work because the integral $\int_0^4 x-2\,dx=0$, which did not satisfy the first integral equation.

However, Tina provided some insight into the nature of a function that would meet the requirements stated in the task. She indicated that "there's obviously gonna have to be negative values for x [*sic*]," because

$$\int_a^b f(x)\,dx < \int_a^b |f(x)|\,dx.$$

Although she said x in the above statement, it was confirmed that she actually meant there needed to be some negative y-values. Using the graph that she had sketched for the function $f(x)=x-2$, she demonstrated that a function meeting the requirements outlined in the task could not be symmetric with respect to a point on the x-axis. Finally, she indicated that the function "wouldn't be something that's linear."

Response to Task 11

After Tina read this task, she paused for a very long time before saying, "I don't know really how to attack this one." She had great difficulty sorting out the information given in the task. When asked what was making the task difficult for her, she replied, "what's making it difficult is that I don't know how to correlate what's been given with the values I need to determine." However, she did admit that if she had a graph for the function f, she could estimate the value for the definite integral $\int_0^1 f(t)\,dt$. Tina was asked whether she

could draw a picture that could help her with the task. After some thought, she sketched the graph shown in Figure 62. Asked why she chose to draw this sketch, she said:

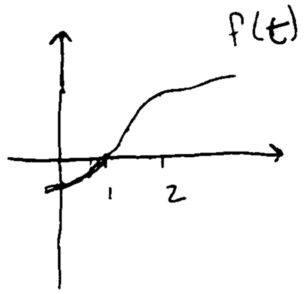


Figure 62. The graph Tina sketched to make sense of the given information and to compute the definite integral $\int_0^1 f(t)\,dt$.

> Well, between 1 and 2, uh, the area is greater than between 0 and 2, so therefore there has to be some negative area in there to make that reasonable *(shades in the two regions)*. Um, and, let's see. So I guess the value for that would be negative *(points to the integral to be evaluated)*.

This was interpreted to mean that having

$$\int_0^1 f(t)\,dt < 0$$

made sense to Tina because

$$\int_0^2 f(t)\,dt < \int_1^2 f(t)\,dt.$$

However, this was as much as she was able to do at that time. It was interesting to note that, even though she had all the pieces in place to determine the definite integral $\int_0^1 f(t)\,dt$, she was unable to perceive the connections.

At the end of Tina's third interview, this task was revisited to determine whether she could make any additional progress towards completing the task and, in particular, to discover whether she could put the pieces together to evaluate the definite integr$\int_0^1 f(t)\,dt$ dt.

Her attention was directed first toward the definite integral $\int_0^1 f(t)\,dt$, and she was asked whether she could determine a value for this integral. She was able to make the necessary connections in order to compute the value this time, as illustrated below:

Tina: Um...*(pause)* Maybe if I subtracted the *(pause)* that *(points to the 1.25)* from that *(points to the 0.75)*, I would get the value for that region.
Todd: Why are you saying that?
Tina: Because, um, this region here is from 0 to 1. And you've given me that from 0 to 2, the area is equal to 0.75. And, so this whole thing is equal to 0.75 *(points to the region under the curve from 0 to 2)*. And I know that this part right here *(points to the region under the curve from 1 to 2)*, by itself, is equal to 1.25. So, given that I know the whole area *(from 0 to 2)* and I also know this little area here *(from 1 to 2)*, I can subtract this *(from 1 to 2)* from the whole area *(from 0 to 2)*, and it would give me that *(from 0 to 1)*.

This passage indicated that Tina was thinking about the areas of the various regions and about the summation property of the definite integral. She eventually completed the necessary arithmetic to conclude that

$$\int_0^1 f(t)\,dt = -0.5.$$

Tina was asked whether she had any ideas concerning how to compute $F(0)$ and $F(2)$. She evoked The Fundamental Theorem of Calculus, but from the viewpoint of total change. She initially concluded that $F(2) = 0.75$. She indicated that, mathematically, she was completing the following calculation:

$$\int_0^2 f(t)\,dt = F(2) - F(0) = F(2) = 0.75$$

because she assumed that $F(0) = 0$.

Tina immediately computed the value for $F(0)$ and produced the work shown in Figure 63 to conclude correctly that $F(0) = 0.8$. She did not assume that $F(0) = 0$, as she had done previously. One possible explanation for this was that her attention was focused upon calculating the value, so her previous assumption did not occur to her.

$$\int_0^1 f(t)\,dt = F(t)\Big]_0^1 \quad \underset{+0.3}{\underset{\uparrow}{F(1)}} - F(0) = -0.5$$

$$0.3 - F(0) = -0.5$$

$$F(0) = +0.8$$

Figure 63. Tina's written work for computing $F(0)$.

Since Tina's original answer for $F(2)$ was spoken only, she was asked to write out her work for the computation. This was done to investigate why she had thought $F(0) = 0$ during this calculation. After she started writing out her work (see Figure 64), the following sequence of comments explained what she found:

Tina: Maybe I got ahead of myself on that one.
Todd: Ahead of yourself in what way?
Tina: Um, I just assumed that if 0 was plugged in that the whole thing would equal 0. So therefore, 2 would be the only thing to consider.
Todd: Well, what happens if 0's plugged in?
Tina: I get, um, 0.8. (chuckles)

It was here she admitted she had simply assumed $F(0) = 0$. She then completed the calculation using the fact that $F(0) = 0.8$. However, she initially found that $F(2) = 0.55$, but corrected herself and said that $F(2) = 1.55$.

$$\int_0^2 f(t)\,dt = F(t)\Big]_0^2 = 0.75$$

$$F(t)\Big]_0^2 = F(2) - F(0) = 0.75$$

$$F(2) = 0.75 + .8$$

$$= 1.55$$

Figure 64. Tina's written work for computing $F(2)$.

When asked what had enabled her to complete this task, when she could not do so the first time she saw it, Tina provided the following explanation:

> Um, I think, to be totally honest, it wasn't that I necessarily, um, all of a sudden something came into my head. I think it's just...it happens quite often that if I look at a problem and get stumped, and I leave it for a good period of time, and I go back to it, nine times out of ten, there'll be something that is a lot more clear to me. For no reason. I mean, there isn't necessarily...I don't look it up. I don't try and figure it out in the interim. Maybe it's just that, um, I'm not as intimidated by it, 'cause I've already seen it. And so, I'm more apt to be open minded to what's in front of me then...(chuckles)...to be intimidated by it, I think, maybe. I don't know.

Response to Task 12

It took Tina some time to read over the task and to gather her thoughts. She gave indications that she was having difficulties related to the discrete nature of the task, as well as what she referred to as the vagueness of the task. Her comments evidenced that she was having difficulty producing an approximation from the limited data. Eventually, she indicated that she could give a "rough estimate" for the harvest. By this, she meant that she would find a minimum harvest and a maximum harvest, with the understanding that the actual harvest would lie between the minimum and the maximum. The essence of her approach was that, for each ten-minute period, she would use the smaller harvest rate to compute a minimum harvest, and the larger harvest rate to compute a maximum harvest.

Once she had computed a minimum harvest for each ten-minute period, she would sum them up to obtain a total minimum harvest. Likewise, she would compute a total maximum harvest by summing up all the maximum harvests for the ten-minute periods. She never actually carried out her plan, yet it sounded as though she intended to compute upper and lower sums for approximating a definite integral.

Tina was asked how the exact harvest might be obtained. She replied, "Well, I think you'd have to know the rate of change over, (sighs) a very small time interval, much less than 10." This indicated that she would need the 60-minute time period subdivided into more subintervals than the six subintervals generated from the data set. In addition, she said, "I don't know if you ever get the exact number though. You'd get close to it, but I don't know if you can get the exact [value]." Based on her statements, she recognized that using a finer partition of the 60-minute interval would improve the approximation, but she was unsure about ever obtaining the actual harvest amount. This indicated that she could view the task in terms of a summation process, but she did not perceive the connection to the limiting process.

Tina was asked whether there was a possible connection between this task and the definite integral. She indicated that a connection existed, and provided this explanation: "Well, roughly, very roughly, I think, we're dealing with rates of change, with respect to time. And any time you deal with rates of change, you're dealing with differentiation, and integration is the opposite of differentiation." Her link appeared to be based on the relationship between differentiation and antidifferentiation, but never made a connection from these directly to the definite integral or the harvest amount. Additional probing did not reveal any new insight. Eventually, after trying to expand on her ideas, she said, "I don't know, I guess, I might be shooting in the dark there." This meant that she was not certain about the connection to the definite integral. Another interpretation was that she had a

glimmer of understanding, but the questioning caused her to think that perhaps she was off track. Either way, she was not sure of herself.

Response to Task 14

Tina gave two responses to the question posed in this task. Her first response was that the definite integral represented "area under a curve on the interval *a* to *b*." She followed with her second response, "Um, I guess it also represents, let's see, I guess, a change over time, I guess, with respect to application to the real world." When asked to explain her second response more fully, she said:

> Kind of like that last problem that we had towards the end of the last meeting. Um, where it was the number of bushels per minute type problem where, um, you can use a definite integral to, um, evaluate, like a certain value I guess for, um, a function of something like of...if the bushels were a function of time. How many bushels were being produced in a certain amount of time, and, uh, to be able to evaluate it. Numerically you could, um, use integration.

The task that she referred to was Task 12, which dealt with the harvest rate data. Based on Tina's second response, she had a rudimentary understanding of the definite integral $\int_a^b f(x)\,dx$ as representing a total change between $x = a$ and $x = b$, but this was not a firmly established image for the definite integral as evidenced by the disjointedness of her explanation.

This task appeared on the participant selection tasks (Appendix E). At the time it was administered, Tina wrote the following: "definite integral or area under curve or antiderivative." A comparison of the responses given on the selection tasks and during the interview process led to the conclusion that her understanding of the definite integral had matured during the time between the administration of the selection tasks and her last interview. Based on her use of Task 12 to explain her second response, it was evident that part of this growth was due to the interviewing process itself. Another contributing factor to her growth could have been her Calculus II course.

Response to Task 15

Tina had great difficulty formulating a response to this task. The only response she was able to produce was, "I guess, this *(points to the integral)* would indicate the area between a and b under this curve *(points to graph of F)*." Tina's response was similar to Lynn's initial response to this task in that both thought that $\int_a^b f(x)\,dx$ represented the area under the curve of the function F. Whereas Lynn was able to move beyond her initial response, Tina was not. Further probing produced no additional insight from Tina.

Response to Task 16

Tina's first response to this task was to say, "Wow, kind of hard to explain something you don't quite understand." When asked to explain as much as she could, she began by describing the relationship between a function and its derivative, and concluded with, "the definite integral would be, kind of undoing differentiation." At this point, Tina was challenged to provide more information about the definite integral. She then provided this additional explanation:

> I think the easiest thing to do would be to do it, with a picture, like, showing a definite integral, and then working through the evaluation of it and at the same time, having a graph showing what it means graphically.

She expanded on her explanation by writing out The Fundamental Theorem of Calculus with the function f as the integrand and the function F as an antiderivative, and drawing a graph of the function F (see Figure 65). Furthermore, she indicated that the result of applying The Fundamental Theorem of Calculus would give the area of the shaded region bounded by her graph, which was the graph of the function F. At this point her thoughts were probably influenced by her views from Task 15.

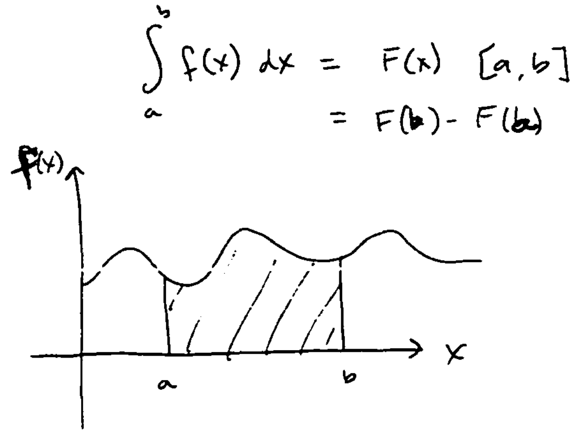


Figure 65. Tina's work concerning a definition of the definite integral.

Tina was asked about her use of the functions f and F. In response she said, "I'm totally all turned around, so I'm trying to untangle it." She indicated that she was confused about the role all the letters played in her written work of The Fundamental Theorem of Calculus. In an effort to sort out her thoughts, she worked on an example problem with the function $f(x) = x^2$. After she had started to apply The Fundamental Theorem of Calculus to $f(x) = x^2$, she concluded that her graph (see Figure 65) should be the graph of the function f, and said "OK, I think I have it untangled." At this point, she was asked to reexplain her definition. She decided to start her explanation at the beginning for the benefit of her "poor confused student." She then proceeded with her definition and referred to her previous work (see Figure 65):

> I would graph $f(x)$. And find, my upper and lower bounds on the graph. And then, evaluate it over here *(points to Fundamental Theorem of Calculus calculation)*, and get my, like, theoretical numerical value through calculations. And then, once I was on the graph, I would, um, find my upper and lower bounds. And, um, this numerical value *(points $F(b) - F(a)$)* evaluated here would be equal to the area under the curve.

From Tina's explanations, it was concluded that her definition of the definite integral centered on the area model and The Fundamental Theorem of Calculus. The use of area in

her explanation was consistent with the representation of the definite integral she gave in Task 14, namely, that the definite integral represents "area under a curve on the interval a to b." Finally, it was noted that, even though Tina had demonstrated she could work with Riemann sums, the notion of using a Riemann sum never surfaced in any of her work on this task.

Response to Task 17

Tina was asked how the picture given in this task might be related to a definition of the definite integral. She responded:

> Um. *(pause)* Oh, gosh, I don't know how to put that in words. Um...*(pause)* It's related, in the sense, that the definite integral provides, a numerical value for what the area should be between a and b under this curve *(points to the given graph)*. So this graph of this function $f(x)$ is using these rectangles to approximate, this area here by adding up the sums of all of these...the area of all these rectangles between a and b. So, it's related in the sense that it's physically working out the numerical evaluation, as opposed to calculating it out with the definite integral.

This passage indicated that Tina saw this picture as relating a definition of the definite integral to the notion of area under the curve. This was consistent with her version of a definition for the definite integral in Task 16. Furthermore, Tina understood the definite integral to compute the area under the curve between $x = a$ and $x = b$. She saw the role of the rectangles as providing her with a way to approximate the area under the curve through a summation of the areas of the individual rectangles. Her phrase "it's physically working out the numerical evaluation" referred to the approximation of the area by geometrical means; in this case, through the use of rectangles as the approximating means. Finally, she was able to work with a Riemann sum that was formed over a nonregular partition of the closed interval $[a,b]$.

In order to assess her understanding of the connection between the summing process and the definite integral, Tina was asked whether she could bridge the gap between the approximation of the definite integral and the definite integral itself and, in particular,

move closer to her definition for the definite integral. Tina was unable to make any additional connections. In her opinion, the explanation that she gave previously "makes the connection just fine and clear." However, she did admit that this might not be the case.

Response to Task 18

Tina's initial response indicated that the word "definite" in definite integral suggested that a limit is involved. She then connected the notion of limit to the derivative. She said, "being that the definite integral, or the integral, and derivations [*sic*] are extremely linked, um, limits are involved." However, she was not sure how and why they were connected. In fact, she admitted, "I don't fully understand how it would be connected." Therefore, it was evident that she believed there was a connection between the notion of limit and the definite integral, but she did not understand how the connection actually worked.

CHAPTER 5

DISCUSSION OF THE DATA

The data discussion is organized by the viewpoints of the definite integral from Chapter 2. Six viewpoints of the definite integral, $\int_a^b f(x)\,dx$, are presented; the definite integral as: (a) a computation, (b) an area, (c) an accumulation or summation, (d) a total change between $x = a$ and $x = b$, (e) a function, and (f) an abstract object. The analysis matrix that summarizes the participants' usage of these viewpoints appears in Appendix G. In addition, the analyses revealed results that do not fit any of these six viewpoints. These results are discussed in a separate section of this chapter.

Computation

The participants' responses demonstrate a strong tendency to view the definite integral as a computation. In fact, this viewpoint of the definite integral is the second most used viewpoint, and only the area viewpoint is used more frequently. Nearly all of the uses of this viewpoint involve applying The Fundamental Theorem of Calculus to a definite integral or using a rote application of the definite integral. There are instances where a participant knows how the definite integral is used in a task but not why it can be used there. Most of these issues can be attributed to an incomplete understanding of the foundation on which the definite integral is based.

The Fundamental Theorem of Calculus is a substantial part of the participants' viewpoint of the definite integral as a computation. The ability to apply The Fundamental Theorem of Calculus to various definite integrals ranges from competent, but mechanical, to rote. At one end of the spectrum, Joan, Lynn, and Rob prove to be competent, but

demonstrate a step-by-step process indicating that they have not yet condensed the idea into a comprehensive whole. On the other end, Stan and Tina work by rote, and appear to be searching for rules to apply during each step of the computation. All the participants can compute the definite integral, though not always easily, when it involves elementary functions (see Tasks 1a and 1b), but composite functions are an obstacle for some (see Task 5).

In addition to comprehending The Fundamental Theorem of Calculus as a computational device, 4 participants include it as part of their definition for the definite integral. For Joan, Rob, and Tina, The Fundamental Theorem of Calculus plays a secondary role to area in their definitions. It is used to compute the area of the region bounded by the graph of the function and the horizontal axis over the given interval. On the other hand, Lynn's definition of the definite integral is The Fundamental Theorem of Calculus.

There are a few instances where a misguided connection is made to the indefinite integral or notions related to antiderivatives. The most striking example is Tina's incorporation of a constant of integration in some of her evaluations of definite integrals (see Tasks 1a and 5). In these instances, she does not disengage from the antiderivative subprocedure after finding an antiderivative, and she is influenced by that subprocedure to include the necessary constant of integration. Joan (Task 11) and Stan (Task 20) provide another example when they try to evaluate an antiderivative using the definite integral of a function, but they see the integrand as determining slope values, and thus being the derivative of some function. They then try to "undo" the derivative to gain information about an antiderivative, but are unsuccessful because they have evoked a derivative–antiderivative image for the situation. The final two examples of misguided connections to the indefinite integrals occur when Rob and Tina describe their definitions of

the definite integral. Tina expresses the belief that the definite integral "undoes" differentiation, although her expanded explanations deal with using The Fundamental Theorem of Calculus to find area. In expressing his definition, focusing on area, Rob says that the definite integral "is just an indefinite integral evaluated between two points on the x-axis that will give you the area under the curve." Thus Rob sees the definite integral as a special case of the indefinite integral.

There is a noticeable reluctance on the part of the participants to use geometric techniques to aid in computing algebraically-presented definite integrals. This is particularly noticeable in the approaches participants use to compute definite integrals with the greatest integer function (Task 1c) and the absolute value function (Task 3). The participants try to evaluate both of these definite integrals algebraically. In the case of the absolute value function, 4 participants complete the equivalent graphical task (Task 2) and ignore the solution and graph that remain in plain sight while working on the algebraic version of the task. Rob turns the graphical task into the algebraic equivalent task and proceeds to work algebraically. In all cases, they ignore the simpler graphical solution, and use algebraic methods to attempt the definite integral with the absolute value function. With the integral of the greatest integer function (Task 1c), only Stan sketches a graph for the function; yet he ignores his graph as he works on the integration problem. In nearly all instances, a graphical solution is demonstrated only as an outcome of outside intervention. This indicates that if an integral is presented algebraically, algebraic methods will be used for computation instead of graphical methods.

Another theme arising from Tasks 1c and 3 is that, when the participants are presented with an unfamiliar definite integral to evaluate, they simplify the problem into a familiar, but not necessarily equivalent, integral that they know how to compute. In particular, for purposes of finding an antiderivative, they treat the unfamiliar integrals

$\int \lfloor x \rfloor dx$ and $\int |x| dx$ as if they are the integral $\int x\,dx$. In addition, several of the participants continue to use $\lfloor\ \rfloor$ and $|\ |$ with their antiderivatives, as if these symbols are important to the problem. In situations involving unfamiliar definite integrals, the integral sign is interpreted as a prompt to do something. This prompt may cause them to appeal to a familiar situation, as it allows them to get an answer, even though the technique chosen is not appropriate for the given problem.

Another interesting theme that the data reveal is that the lack of an explicit function for the integrand sometimes can be a cognitive obstacle, as is shown by Joan, Rob, and Stan in Task 11. They are unable to compute the values for $F(0)$ and $F(2)$ when the integrand is not explicitly given. In fact, Rob and Stan need the function in order to use The Fundamental Theorem of Calculus. However, not knowing the explicit form of the integrand is not an obstacle in other tasks because participants are able to apply other viewpoints, such as area. The abstract nature of Task 11 is not conducive to alternative approaches in the minds of some participants.

Some participants try to determine amount information from rate information, but they do not understand why this approach works. These participants are using the following from their textbook: "Integrating f' (the rate function) over $[a,b]$ gives the change in f (the amount function) over the same interval" (Ostebee & Zorn, 1997, p. 371). Participants provide several examples of finding either the total harvest (Task 12) or the mass of the rod (Task 13) by integrating the harvest rate or the density over the appropriate interval. Joan clearly refers to a relationship between velocity and "position," when she sees the harvest rate or the density analogous to velocity, and the total harvest or mass analogous to "position." In Task 13, Rob and Joan apply unit analysis techniques to the situation, but are unable to relate the length of the rod to the differential. None of the

participants provide an acceptable explanation for obtaining amount information from rate information.

Area

The area viewpoint is the one most used. This is consistent with the responses to the question concerning how participants think about the definite integral. All but one tend to think about the definite integral graphically, and the remaining one is starting to think graphically. Participants' personal representations and definitions of the definite integral are founded on the idea of the area of the region bounded by the graph of the function and the horizontal axis (see Tasks 14 and 16). Therefore, it is logical that participants tend to use the area viewpoint to understand or to solve many tasks related to the definite integral.

By using area in most of their work on the tasks, the participants demonstrate that they are fairly competent at understanding problems from this viewpoint. In addition, they are comfortable using graphical techniques to explore problems and to make sense of problem statements as they solve tasks. Even so, they still exhibit some shortcomings in their understandings of the definite integral from an area perspective.

When the function is presented as a graph, the participants demonstrate they can compute values or approximations for the corresponding definite integral. Some of the participants comment about the lack of an explicit expression for the function, but still complete the task with graphical techniques. Participants, unable to produce an exact value for an integral, are able to find an approximate value using either geometric area formulas, summation techniques involving rectangles or trapezoids, or counting techniques (see Stan's work in Task 4). In a few instances, more than one technique is used. Overall, these individuals are comfortable using graphical techniques to compute values for definite integrals without explicit expressions for the function. However, on Task 4, when participants are afforded the opportunity to view the piecewise-defined function, only Lynn

and Tina comment that evaluating the integral corresponding to the linear part of the function is easier to do from the graph. Rob and Joan, on the other hand, disregard the graph and proceed to integrate. These two examples reinforce the assertion that the frames or concept images for algebraic and geometrical solutions are only tenuously connected.

The participants are familiar with the concept of negative area, and are generally able to work with the idea successfully, as seen in tasks such as Tasks 7 and 8. However, there are indications that this concept image is not yet fully developed. In Task 2, 3 participants initially transfer the idea of negative area to Quadrant II. The negative x-coordinates used in that task may be a signal for the retrieval of the negative area image from memory. In particular, the association with negative area may result from seeing a negative x-coordinate. They eventually realize that negative area is relevant only when the y-coordinate is negative, and proceed to make that necessary correction.

The notion of negative area appears when the participants confuse the distinction between the definite integral and area. As is shown in Task 6, where the function has positive values for all $t \geq 0$, the participants correctly perceive no difference between the area of the region bounded by the graph of the function and the t-axis from $t = 0$ to $t = x$ and the definite integral $\int_0^x f(t)\,dt$. However, in Task 7, where the function takes on negative values, they also perceive no difference between the area of the region bounded by the function and the t-axis from $t = 0$ to $t = x$ and the definite integral $\int_0^x f(t)\,dt$. In the latter setting, area and the definite integral are both equated with signed area. To the participants, both parts of Task 7 are asking for the signed area, the first part in words and the second part in symbols. Thus, finding the area of a bounded region is equivalent to computing the signed area of the region when part of the function takes on negative values. They do not

equate computing area with computing the geometrical area of the bounded region unless the phrase "total area of the region" replaces the phrase "area of the region," or the coordinate system is ignored. The participants know that an area is supposed to be a positive-valued quantity, but with the coordinate system in place, they all believe that the area of a bounded region is found by subtracting the area below the horizontal axis from the area above the horizontal axis. This leads to the conclusion that the coordinate system is a cognitive obstacle to understanding that the definite integral is not necessarily the same as the area of a bounded region.

The participants also demonstrate an incomplete understanding of area and signed area in Tasks 9 and 10. In Task 9, all the participants draw a graph of a function such that

$$\int_a^b f(x)\,dx = 2$$

by finding a function that bounds 2 units of area. Only Stan is able to complete Task 10, to sketch a graph of a function such that

$$\int_a^b f(x)\,dx = 2 \text{ and } \int_a^b |f(x)|\,dx = 4.$$

The participants all perceive the first integral in terms of area or signed area and the second integral in terms of total area, where total area is the geometrical area of the region. With the exception of Stan, the participants appear to be thinking about satisfying the integrals sequentially as opposed to simultaneously; that is, they try to satisfy one of the conditions first, and then hope that the second one is satisfied as a result. Only Stan approaches the two conditions together. His block counting techniques for computing aids him in completing the task. Thus, the participants know what needs to be done to complete Task 10, but they are unable to actually carry out the necessary work. Therefore, the

interrelationship between area and signed area is a weak link in their understanding of the definite integral.

The participants show that they have some across-time understanding of the definite integral in terms of area, as well as a pointwise understanding of it. Pointwise understanding is demonstrated in Tasks 6 and 7 when participants describe the results of placing x at various places along the t-axis. Lynn and Joan show this type of understanding in Task 20. Since participants see the area of a bounded region continuing to grow as the value of the independent variable increases, they demonstrate across-time understanding (see Tasks 6, 7, and 20). Tina in Task 6a and Joan in Task 20 best demonstrate across-time understanding by using a pen or ruler as a sliding right boundary of the region to illustrate their thoughts.

Most of the participants do not perceive the connection between approximating rectangles or trapezoids and the area of a region bounded by the graph of the function and the horizontal axis over a closed interval $[a,b]$. In Tasks 4, 12, and 17, the participants approximate the area of the bounded region using either Riemann sums or the Trapezoid Rule, which they refer to as approximating sums. Most of them indicate that approximations can be improved by repartitioning the interval into smaller subintervals. They understand that the exact value for the area of the bounded region or the area under the curve is given by definite integral. Without being asked directly, only Lynn connects the limit of the approximating sums to the area under the curve. Her description is not precise, but the elements present indicate that this notion is part of her concept image. Joan makes this connection when asked about the role of the limit in Task 18, but her response is not immediate. In general, the concept of the limit as the connector between approximating sums and the area of a region is not part of the participants' concept image of the definite integral.

While the area model is a useful viewpoint for understanding the definite integral in many of the tasks in this study, it is a cognitive obstacle in Task 15 for Lynn and Tina and Task 20 for Rob. Here, these participants relate a definite integral of a function f to a graph of an antiderivative F of f, that is,

$$\int_a^b f(x)\,dx = \int_a^b F(x)\,dx.$$

They are fixed on the notion of area under the curve of f and are unable to disengage from this image as they move to the graph of F. The end result is a transfer of this image to the graph of the antiderivative F and the connection of the definite integral of f to the notion of the area under the curve of F. Only Lynn partially disengages from this connection, and then only with further probing. She ends her work on Task 15 with the definite integral of the function f representing the area of the region bounded by the line segment connecting the points $(a, F(a))$ and $(b, F(b))$ on the graph of an antiderivative F and the horizontal axis. She still associates the definite integral of f to an area on the graph of an antiderivative F.

Accumulation or Summation

The notion of accumulation or summation is one of the least used viewpoints. Nearly all occurrences are in combination with the area viewpoint when approximating a definite integral or an area. Only Stan and Tina apply summation ideas without evoking the notion of area in Task 12, where they form their approximation using only a table of values.

All of the participants improve an approximation by using a finer partition of the appropriate interval under consideration. However, there are only three occurrences connecting the summing process and the limiting process. Two of these are by Lynn (Tasks 12 and 17) and one by Joan (Task 18). Lynn does not use the word "limit" in her response, but the idea of a limit is the foundation from which she is working. Joan's only

connection to limit comes when she is asked about the connection between that limit and the definite integral. Her eventual answer is framed within the area context, but she is able to describe the limiting process on a sum. What is most surprising is that Lynn's conceptual understanding of the definite integral was below average according to the participant selection tasks (see Appendix E), and yet she demonstrates the most understanding of the limit of a summing process. This concept is not intrinsic to the other participants' understanding of the definite integral.

Outside of Task 4, no participant demonstrates any use of the dissecting and summing technique. In Task 4 they dissect a region to approximate the area of the region by summing up areas of rectangles. However, in the other tasks where dissecting and summing is necessary, they are not able to do it. Task 13 shows this when participants try to determine the mass of a one-dimensional rod given its density function. No one thinks of dissecting the rod into pieces. Even those who see the definite integral as the proper tool to use cannot explain why, because they do not comprehend the applicability of dissecting the rod. Although they relate the dissecting notion to the context of area, they do not apply it to scenarios that do not involve area.

The participants have a narrow view of what constitutes a Riemann sum. They see it as formed over a regular partition of an interval by consistently using the left endpoint, the right endpoint, or the midpoint to determine the approximating rectangles. For example, in Task 17, when they are shown a diagram depicting a general Riemann sum, the general response is that this diagram depicts a way to approximate the area of the region, but the participants indicate that this diagram does not depict a Riemann sum. Joan eventually sees this as a Riemann sum, but thinks it is because geometrical shapes are being used to carry out the approximating. Others indicate that they would have to draw different approximating rectangles, using a regular partition and heights determined by the left

endpoint, the right endpoint, or the midpoint, to have a Riemann sum. This shows that the general Riemann sum is not a part of the participants' concept image of the definite integral.

Total Change

The least used of all six viewpoints is total change. One participant never uses it, one participant uses it twice, and all others use it once. Tina, who evokes this viewpoint twice, does not use it confidently and should have been the least likely participant to use it. When she is asked, she is the only one who gives any indication of viewing the definite integral as representing total change (see Task 14). Her rudimentary understanding of the definite integral in this manner appears to result from her work on Task 12 involving the harvesting data, but this understanding does not appear to be a firmly established part of her concept image of the definite integral.

Task 11 is one of two tasks in which the participants used total change. Only Tina and Lynn are able to compute the values for an antiderivative evaluated at 0 and 2 using The Fundamental Theorem of Calculus in the more abstract setting of this task. Others say that they need an explicit function before they can compute the requested values using The Fundamental Theorem of Calculus. Lynn and Tina tend to view the definite integral symbolically or from rote, and this task was symbolically oriented. The participants who are graphical thinkers did not have success here.

Task 15 is the other task where the participants used total change. Only Joan and Stan are able to connect the definite integral with the vertical displacement on the graph of an antiderivative. Others unsuccessfully try to use the area model, although Lynn did come close to using total change. Joan and Stan indicate and demonstrate that they tend to view the definite integral graphically, and this task was graphically oriented. The graphical nature of this task lends itself to success by graphical thinkers, whereas symbolic thinkers did not have success here.

Function

The notion of the definite integral as a function is used only slightly more often than the accumulation or summation viewpoint of the definite integral. Many times, it is used in connection with the area viewpoint. Then, there is a graph present (see Tasks 6 and 7) or the statement of the task makes it natural to use a graph (see Task 20). Area is not evoked by most of the participants when evaluating the integral function in Task 5, only analytical techniques are used to evaluate the function. Stan is the exception when he talks about the integral function being an "area function," but he does not use area to work on the task analytically.

All exhibit a pointwise understanding of the definite integral as a function. Participants are gaining an across-time understanding of it. Pointwise understanding is demonstrated by their ability to evaluate an integral function (see Task 5), to describe what will take place when evaluating an integral function, or to discuss what happens to the value of the integral function for values of the independent variable. Tina demonstrates only possible pointwise understanding. The development of across-time understanding of the integral function is shown in descriptions of the change in integral-function values as the independent variable varies (see Task 6b and 7b) and by Joan's use of a pen as a slider on the graph of the integrand to justify her graph for the corresponding integral function (see Task 20).

Some of the participants refer to the integral function as an "area function," while others infer this through their work. This is probably a reference to the area function defined in Ostebee and Zorn:

" $A_f(x) = \int_a^x f(t)\,dt =$ signed area defined by f, from a to x" (1997, p. 357).

Some participants, Lynn and Stan in particular, talk about the integral function as an antiderivative of the integrand. This is particularly noticeable in Task 5, where Lynn and Stan see

$$G(x) = \int_0^x (\lambda - 1)^2 \, d\lambda$$

as an antiderivative of $(\lambda - 1)^2$ defined at x, and $G(5)$ as the antiderivative evaluated at $x = 5$. Stan illustrates this belief by using The Fundamental Theorem of Calculus to determine $G(x)$ as a non-integral function, and then using that to compute $G(5)$. Thus, there is evidence that some of the participants are developing an understanding of the full power of The Fundamental Theorem of Calculus.

All but one of the participants make some sense of the roles played by the two variables in the definition of an integral function. Tina is confused by the use of two variables and has difficulty with several of the tasks involving integral functions. She did experience a small revelation during one of the follow-up questions, when she realized that the variable upper limit of integration represents a value on the horizontal axis of the graph of the integrand. Rob indicates this understanding, as well. Others indicate that the variable upper limit of integration represents the independent variable for the integral function or allows one to vary the upper limit of integration. They understand the variable of integration to be the independent variable for the integrand, as well as the variable into which the limits of integration are substituted.

Participants have ideas regarding why an integral function is a function. The most common reason is that it is a function because of the notation used to define the relationship. Other ideas presented for why it is a function are: because the problem statement says it is a function, and because output values can be generated by input values. The most sophisticated reason for why an integral function really is a function is because

the antiderivative of a function is a function. Thus, participants do not really understand why an integral function is actually a function. However, the relationship between participants' concept of function and understanding of the integral function as a function has not been sufficiently addressed, and was beyond the scope of this study.

Abstract Object

The definite integral as an abstract object is the third-most-used viewpoint. However, in about half of the occasions in which it is used, it is only as the summation property of the definite integral. Contributing to the low number of occurrences of the abstract-object view of the definite integral is that it was not present in the participants' definitions for the definite integral. Nor is it present in the work of Task 19. Also, the participants believe they do not view the definite integral conceptually.

Participants use the summation property of the definite integral in both analytical settings and graphical settings (see Tasks 4 and 8). They also use the summation property to compute missing information in Task 11. However, some participants have difficulty using the summation property symbolically when an explicit function is not present (see Lynn's Task 8).

Most participants believe the jump discontinuities will not affect the value of the definite integral, or that any affect would be insignificant. The essence of the explanations for this is that "infinitely close to" or "right after" the x-value in question, the function is again defined, so they would be able to compute a definite integral. In practice, they approximate the improper integral with a proper integral, but they do not recognize the role of taking limits. In addition, the explanations indicate that they view the number line as discrete rather than continuous. This view is supported by the introduction of "little holes" in the number line around where a jump discontinuity occurs.

Only 2 participants take a stand in Task 6 about the convergence of the improper integral with an upper limit of infinity; the rest will say only that the value continues to grow. Rob believes that the value converges to a finite value because the amount of area being accumulated is becoming smaller. He also believes that the area cannot be infinite. Considering an improper integral that diverges did not alter his view. Joan believes that the value of the improper integral presented in the task approaches infinity. After considering an example that converges, Joan acknowledges that this is possible, but she is uncomfortable with the idea and believes that it is an exception to her currently held beliefs. Most of the participants are at the beginning stages of exploring this notion.

Participant connections between the limit and the definite integral are vague and tenuous. Some participants show that an understanding of how the two are connected is absent from their concept image. Examples that illustrate this are: (a) limit refers to the limits of integration: (b) limit is involved because of the word "definite" in definite integral; (c) limit is involved because limit is involved with the derivative, and derivatives and integrals "are extremely linked;" (d) limit is a meaningless idea because a definite integral is just a number so it cannot have a limit; and (e) not knowing whether there is a connection between limit and the definite integral. These participants do approximate a definite integral using a summing process, and improve the approximation using a finer partition of the interval. In most cases, however, the definite integral as the limit of a summing process is not part of their concept image. At the other end of the spectrum, Joan and Lynn do allude to limit in connection with a summing process. Joan, who demonstrated above average conceptual understanding of the definite integral, evokes the idea of a limit of a summing process only when specifically asked in Task 18. She describes the limiting process, but the idea is not an intrinsic part of her concept image of the definite integral. Ironically, Lynn, who entered the study with a below average conceptual understanding of the definite integral, possesses

some rudimentary understanding of the connection between the limit and the summing process. She shows her understanding on three different tasks (Tasks 12, 17, and 18) when she tries to describe the idea of the limit of a summing process where the widths of the rectangles approach 0. However, when asked about the concept of limit, she cannot decide whether the widths of the rectangles approach 0 or the number of rectangles approaches infinity; she does not see the connection between these ideas.

Additional Observations

For most of the participants, the formal language and notation of mathematics is a source of difficulty. They do not fully understand the language or notation or cannot use the notation and language correctly. However, if given the opportunity, they can convey in their own words the essential ideas for many concepts. In addition, they can describe and carry out many of the formal calculus procedures related to the definite integral, although the language and notation they use is based upon their personal development. Therefore, the lack of formal language and notation may be a cognitive obstacle to participants trying to demonstrate their understanding of the definite integral.

In many cases participants exhibited compartmentalization of knowledge. This compartmentalization prevented them from "seeing" alternative viewpoints of the definite integral for a particular task. It effectively blinds them from seeing beyond the viewpoint that they are focusing upon.

There is evidence that the participants' concept images of the definite integral expanded through participation in the study. The expansions are noticeable over the course of the three interviews. Examples of this are: (a) Tina's realization that the name of the variable does not matter when applying The Fundamental Theorem of Calculus, and her rudimentary awareness of the definite integral as a total change; (b) Rob's realization that the increment Δx is a crucial part of the Riemann sum; (c) Lynn's discovery of the proper

way to analytically compute the value of the definite integral $\int_{-2}^{3} |x|\,dx$; and (d) Joan's discovery that improper integrals with infinity for the upper limit of integration can diverge or converge.

CHAPTER 6

CONCLUSIONS

This chapter answers the research question, summarizes the primary themes emerging from the interviews, presents implications for teaching the definite integral, highlights the limitations of the study, and presents questions and issues for future research concerning student understanding of the definite integral.

Answer to the Research Question

This study addresses the question: What is an undergraduate calculus student's conceptualization of the definite integral? In addressing this question, the study focuses on undergraduate calculus students after an initial exposure to the definite integral in a first semester calculus course. One particular aspect of interest is how undergraduate students understand the definite integral, $\int_a^b f(x)\,dx$, from various viewpoints: (a) as a computation, (b) as an area, (c) as an accumulation or a summation, (d) as a total change between $x = a$ and $x = b$, (e) as a function, and (f) as an abstract object.

Students regard the definite integral most commonly as an area, closely followed by as a computation. The predominant views are that the definite integral represents the area of a bounded region or the area under a curve, and a computational viewpoint centered on The Fundamental Theorem of Calculus. Of the four remaining viewpoints, the definite integral is most viewed as an abstract object. Many of the uses are applications of the summation property of the definite integral. The accumulation or summation viewpoint and the function viewpoint are evoked almost equally after the abstract object viewpoint, and the total

change viewpoint of the definite integral is the least used viewpoint. Those with above average or average understanding of the definite integral are more likely to use the abstract object, function, and summation viewpoints of the definite integral. Surprisingly, those with average or below average understanding of the definite integral may exhibit some understandings of it as a total change or as a limit of a summing process, which those with above average understanding may not exhibit. This is surprising because these two notions require a deeper understanding of the definite integral. However, for all the things undergraduate students can do regarding the definite integral, their understanding of it is still incomplete.

Undergraduate students' abilities to find an appropriate viewpoint for a particular situation are not yet fully developed. In many cases, when they are unsuccessful or do not know why they cannot make any progress, it is because they are locked into one viewpoint or image and are unable to access other viewpoints or images that may be more useful. Occasionally, they may change to another viewpoint or image because of the direction the questions lead. Generally, once an inappropriate viewpoint or image is evoked, students require direction to find a more appropriate viewpoint or image.

The following is a summary of other trends emerging from the interviews with undergraduate calculus students after they have had an initial exposure to the definite integral in a first semester calculus course. All of these trends would make interesting hypotheses worthy of further research in large-scale studies.

- If integration is the proper technique for a task, students may not know why it is the proper technique.
- Due to similarities between the definite integral and the indefinite integral, notions from indefinite integrals are erroneously transferred to the definite integral.

- When an unfamiliar definite integral is presented, there is a strong tendency to simplify it into a familiar, but not necessarily equivalent, integral.
- The lack of an explicit function for the integrand sometimes can be a cognitive obstacle to completing a task.
- Students' definitions for the definite integral are mostly in terms of area or The Fundamental Theorem of Calculus.
- Although students are able to improve an approximation of a definite integral by using a finer partition of the interval under consideration, connections between limit and the definite integral are weak, vague, and rudimentary. Specifically, the definite integral as the limit of a summing process is not part of their concept image of the definite integral.
- Students are reluctant to use geometric techniques to aid in computing algebraically presented definite integrals. In particular, if a definite integral is presented algebraically, algebraic methods will be used for computation instead of graphical or geometrical methods.
- When a function is presented as a graph, students demonstrate they are comfortable computing or approximating values for a definite integral using area.
- The notion of negative area is familiar, but the understanding of the concept is incomplete.
- There is knowledge of the notions of area and signed area, but the distinction between the two is not understood.
- The coordinate system is a cognitive obstacle to understanding that the definite integral is not necessarily the same as the area of a bounded region.
- The idea of dissecting "an object" and summing its parts is noticeably missing in scenarios outside the area of a bounded region.

- The view of what constitutes a Riemann sum is very narrow. In particular, it is seen as being formed over a regular partition of an interval by consistently using the left endpoint, the right endpoint, or the midpoint to determine the approximating rectangles.
- Students who think symbolically tend to have more success applying The Fundamental Theorem of Calculus in an abstract, symbolic setting. Those who are graphical thinkers are less successful with this type of task.
- Only the graphical thinkers see the definite integral as a vertical displacement on the graph of an antiderivative of the integrand.
- Students develop pointwise understanding of the definite integral, or the area of a bounded region, as a function first. Across-time understanding of it as a function comes afterwards.
- In some instances the integral function is viewed as an "area function." Furthermore, there are indications that the integral function is seen as an antiderivative of the integrand.
- The integral function is seen as a function, but the reason why it is regarded as a function is not understood. Furthermore, most students do not have difficulty with the two variables in an integral function, and to various degrees are able to indicate the role of each of the variables.
- Students are comfortable using the summation property of the definite integral both symbolically and geometrically.
- The general view is that jump discontinuities do not affect the value of a definite integral or, if there is an effect, it is insignificant. However, there is no recognition of the role of limits in this process.

- The lack of formal language and notation may be a cognitive obstacle for students trying to demonstrate their understanding of the definite integral.

Implications for Teaching

When students are studying the definite integral for the first time, they need to be exposed to all six of the viewpoints presented at the beginning of this chapter in order to develop a well-rounded understanding of the definite integral. This study shows that students have a reasonable understanding of the definite integral as an area and as a computation, but are weaker in their understanding of it from the other four viewpoints. This is particularly true of the total change viewpoint. Students need to realize that the definite integral represents more than one or two of the viewpoints and that some viewpoints are more applicable than others when considering particular problems. Therefore, problems need to be developed that encourage students to consider the definite integral from multiple viewpoints and, in particular, problems that cannot be solved with computation or area viewpoints need to be developed and introduced into the students' initial exposure to the definite integral.

Students need multiple opportunities to demonstrate what they understand about the definite integral. For students who do poorly in a symbolic or computational setting, having alternative ways to demonstrate understanding is important to success in calculus. There are students whose understanding of the definite integral may never be demonstrated unless given an opportunity to work on problems requiring nonsymbolic or noncomputational responses. In addition, by employing a variety of assessment techniques, instructors create more opportunities to uncover misunderstandings and to help students expand their understanding of the definite integral.

Although many of the specific results are too preliminary to dictate action, two items do stand out as needing attention from a curriculum standpoint: the reluctance to use

geometric techniques to aid in computing definite integrals presented algebraically, and students' misunderstanding of the distinction between area and signed area. Regarding the use of geometrical techniques, students need more opportunities to work with unfamiliar definite integrals, such as the integral of the greatest integer function or absolute value function, to gain an appreciation for using geometric techniques. Moreover, to encourage visual thinking regarding evaluating definite integrals, geometric techniques need to be modeled by instructors and textbooks. This will help students to see the value of thinking geometrically when computing definite integrals presented algebraically. With regard to student misunderstandings about area and signed area, students need opportunities to learn how these two ideas are similar and dissimilar, with particular emphasis on how they are dissimilar. For this to happen, though, exemplary motivating examples need to be devised that allow students to develop their concept images of area and signed area. This may be an area where technology could be very useful, as students can explore a large number of examples, draw conclusions based on the examples, and then test out their ideas.

Limitations of the Study

This study was qualitative in nature, so there was a small number of participants involved in the study. While small numbers allowed for a rich description of each participant's understanding of the definite integral, other potentially rich participant statements may have been missed. Also, because of the small number of participants involved, the results are not generalizable to all undergraduate calculus students. However, the results could be an indicator of trends to look for in all undergraduate calculus students.

Interview methods may influence the way participants react. Examples would be time constraints on interviews and rapport between the participant and interviewer. This could make it difficult to discern what participants were actually thinking while they worked on a task or responded to a question.

The interviews took place early in the fall semester, so the participants had not studied mathematics for at least three months. With this much time between their initial exposure to the definite integral and the interviews, some of the participants' understanding may have suffered attrition. On the other hand, students were taking Calculus II at the time of the study, so they may have had in-class reminders that influenced their recollections of material related to the definite integral.

Directions for Future Research

As a result of this study several issues and questions are raised that provide direction for future research. As alluded to in the presentation of the results, any one of the themes that emerged from the study would make an interesting hypothesis to investigate further. Also, several questions arising from this study are worthy of future research; specifically:

- What experiences are necessary for students to come to understand the definite integral as a limit of a summing process?
- How do students come to see the distinction between the area of a region bounded by the graph of a function and the definite integral?
- Why are students reluctant to use graphical techniques to compute definite integrals presented algebraically?
- What is the relationship between students' understandings of function itself and their conceptualizations of the definite integral as a function?
- How do students' understandings of the Riemann sum change over time?
- When does the total change viewpoint of the definite integral become integrated into students' concept image of the definite integral?

- What role can technology play in helping students better understand the definite integral?
- Why do some students with below average understanding have a sense of the harder concepts, while the above average students do not show this sense? Have the above average students just settled into a mindset that works for classroom purposes, while the below average students are still struggling to understand the definite integral well enough and so are looking at more of the viewpoints?

In addition, studies similar to this one need to be conducted with groupings of students from different calculus experiences. Examples of such experiences could be different educational settings, programs using and not using reform methods, or programs emphasizing varying viewpoints when introducing the concept of the definite integral. A longitudinal study of a small number of students also needs to be undertaken to map out growth in student understanding of the definite integral.

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APPENDIX A

VITA

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Complete responsibility for assigned classes.
Implementing alternative teaching methods into classes.
Reforming the Math for Elementary Teachers course.

1995-1999 — Teaching Assistant, The University of Montana, Missoula, MT
Complete responsibility for assigned classes.
Implemented the use of technology and alternative teaching methods into classes.
Team taught Euclidean and Non-Euclidean Geometry course.

1992-1995 — Instructor, Lincoln University, Jefferson City, MO
Complete responsibility for assigned classes.
Implemented the use of technology into classes.

1990-1992 — Teaching Assistant, The University of Iowa, Iowa City, IA
Complete responsibility for teaching Precalculus and Calculus I.
Assisted in the Mathematics Tutorial Lab.

1989-1990 — Student Assistant to the Mathematics Tutorial Lab Coordinator,
The University of Iowa, Iowa City, IA

1988-1989 — Teaching Assistant, The University of Iowa, Iowa City, IA
Led discussion sections of Basic Algebra II.
Assisted in the Mathematics Tutorial Lab.

1985-1988 — Teaching Assistant, Luther College, Decorah, IA
Tutored and graded homework for Basic Algebra, Linear Algebra, Probability, Statistics, and Ordinary Differential Equations.

COURSES TAUGHT

Intermediate Algebra
Applied Algebra
College Algebra
Trigonometry
Precalculus
Applied Calculus
Calculus I, II, and III
Mathematics for Elementary Teachers
Introduction to Abstract Mathematics
TI-85 Graphics Calculator
Euclidean and Non-Euclidean Geometry (as part of a team)

PROFESSIONAL INTERESTS

Conducting research in how undergraduate students learn mathematics.
Teaching undergraduate mathematics courses.
Developing courses using technology in an appropriate manner.
Incorporating alternative learning methods into mathematics courses.

APPENDIX B

TASKS

Task 1

Evaluate the following definite integrals. (Please refrain from using your calculator's integration feature for this problem.)

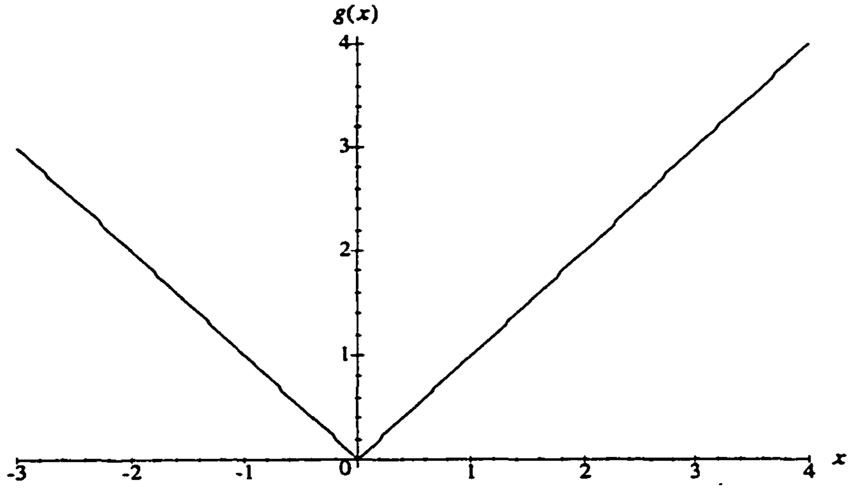
a) $\int_{-1}^{2}\left(3x^2 - x + 1\right)dx$

b) $\int_{0}^{1} e^{t}dt$

c) $\int_{0}^{5}\lfloor x \rfloor dx$ ($\lfloor x \rfloor$ represents the greatest integer function)

Task 2

Let g be the function shown graphically below; evaluate $\int_{-2}^{3} g(x)\,dx$.



Task 3

Evaluate $\int_{-2}^{3}|x|dx$.

Task 4

Let f be the function shown graphically below. Explain how you would compute the area of the region bounded by the function f and the x-axis from $x = 1$ to $x = 7$.

Remark: The function used to generate the graph for this task is given below. Some participants were allowed to see the function after they complete their response based solely on the graph.

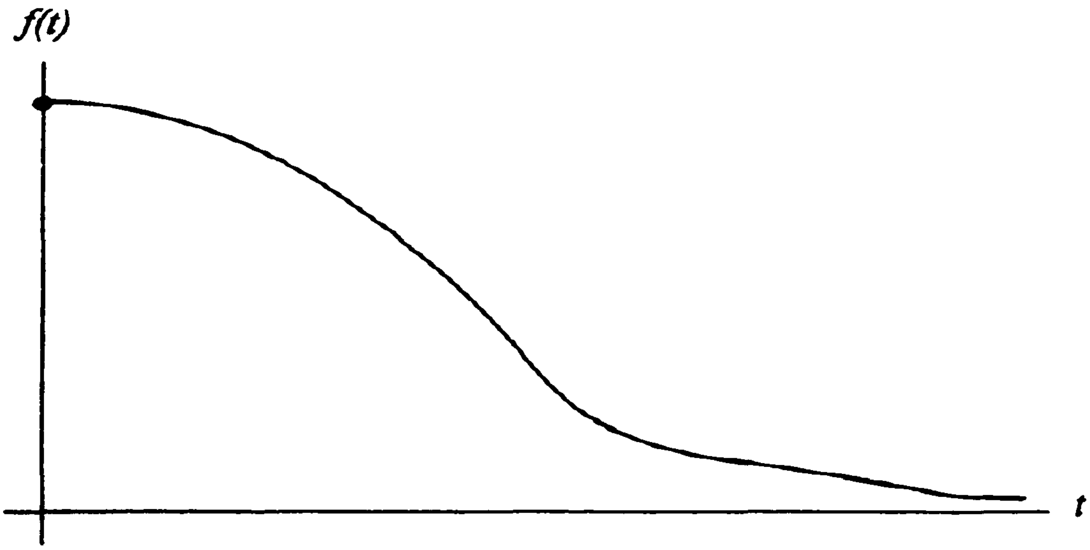
$$f(x) = \begin{cases} \frac{\sqrt{3}}{3}(x-1), & 0 \le x < 4 \\ \sqrt{\frac{9-(x-4)^2}{3}}, & 4 \le x \le 7 \end{cases}$$

Task 5

If $G(x) = \int_0^x (\lambda - 1)^2 d\lambda$, then $G(5) = ?$

Task 6

Let f be the function shown graphically below.



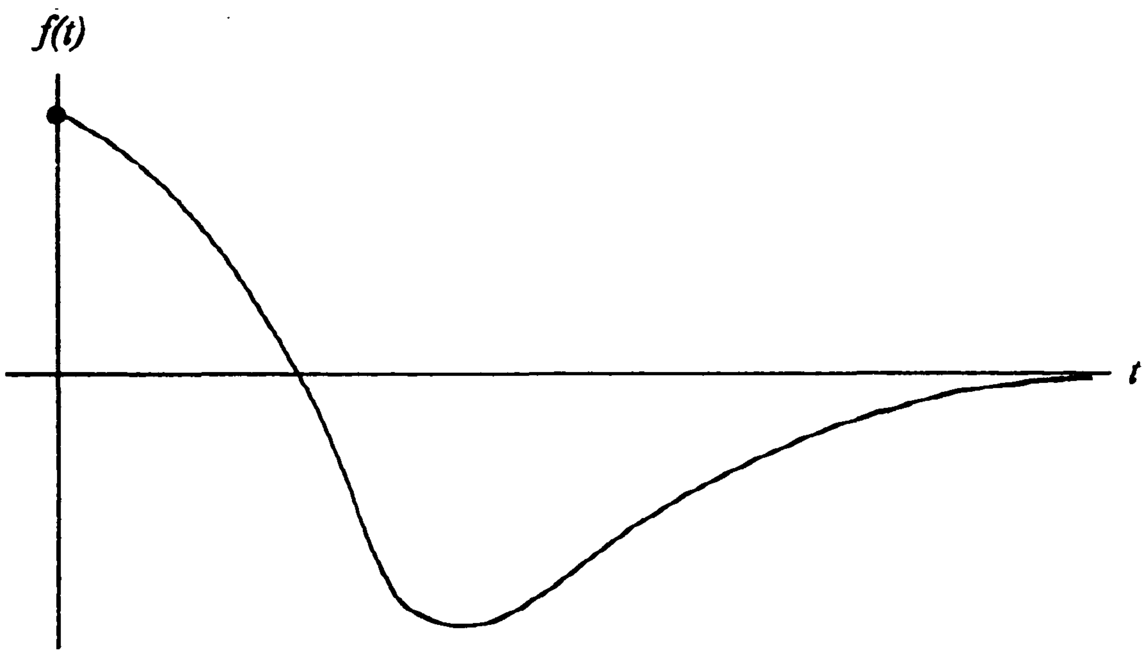
a) For $x \geq 0$, predict what will happen to the area of the region bounded by the graph of f and the t-axis from $t = 0$ to $t = x$ as x moves to the right. Explain why.

b) For $x \geq 0$, predict what will happen to the values of the function G defined by

$G(x) = \int_0^x f(t)\,dt$ as x moves to the right. Explain why.

Task 7

Let f be the function shown graphically below.



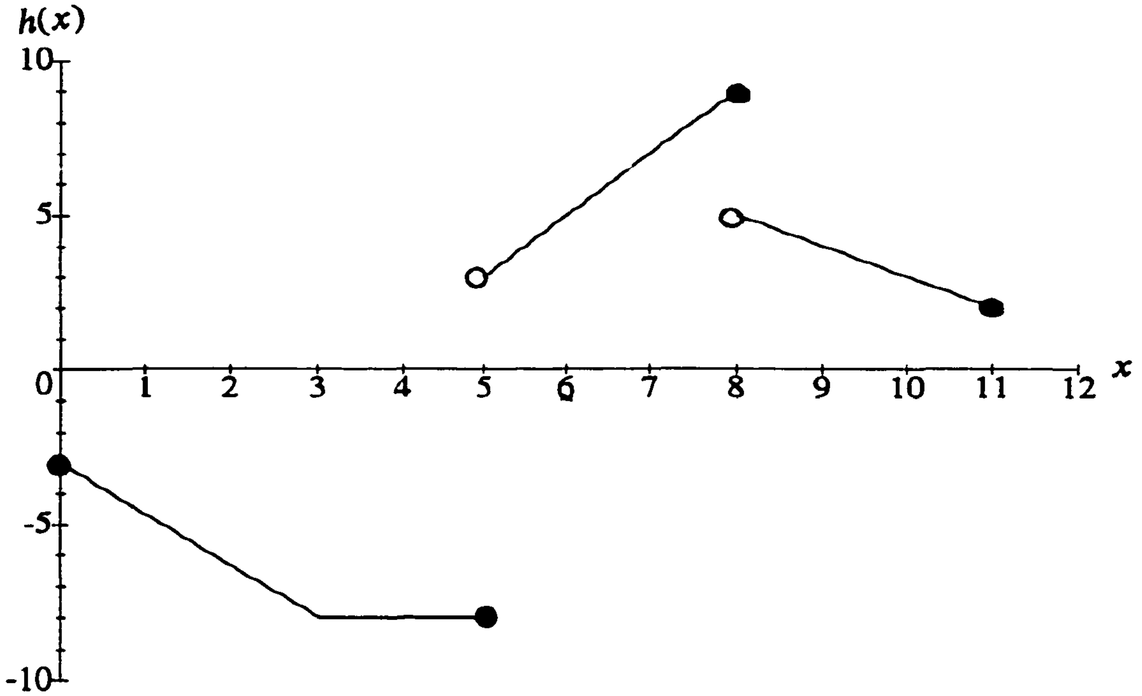
a) For $x \geq 0$, predict what will happen to the area of the region bounded by the graph of f and the t-axis from $t = 0$ to $t = x$ as x moves to the right. Explain why.

b) For $x \geq 0$, predict what will happen to the values of the function G defined by

$G(x) = \int_0^x f(t)\,dt$ as x moves to the right. Explain why.

Task 8

Let h be the function shown graphically below.



List the following from smallest to largest, and justify your result.

a) $\int_0^{11} h(x)dx$ b) $\int_5^{11} h(x)dx$ c) $\int_0^{5} h(x)dx$ d) $\int_5^{5} h(x)dx$ e) $\int_3^{8} h(x)dx$

Task 9

On the piece of graph paper provided, please sketch the graph of a function f on some interval $[a,b]$ of your choice such that $\int_a^b f(x)dx = 2$.

Task 10

On the piece of graph paper provided, please sketch the graph of a function f on some interval $[a,b]$ of your choice such that $\int_a^b f(x)\,dx = 2$ and $\int_a^b |f(x)|\,dx = 4$.

Task 11

Let F be some antiderivative of f. Given $\int_1^2 f(t)\,dt = 1.25$, $\int_0^2 f(t)\,dt = 0.75$, and $F(1) = 0.3$, determine the values for $\int_0^1 f(t)\,dt$, $F(0)$, and $F(2)$.

Task 12

A farmer's harvest rate (in bushels per minute) is measured at ten-minute intervals over the course of an hour. This data is given below. Explain how you could estimate the total number of bushels harvested over the 60-minute time period. Demonstrate your plan as you explain it.

time (minutes)	0	10	20	30	40	50	60
harvest rate (bu/min)	0	8	10	14	12	12	6

Task 13

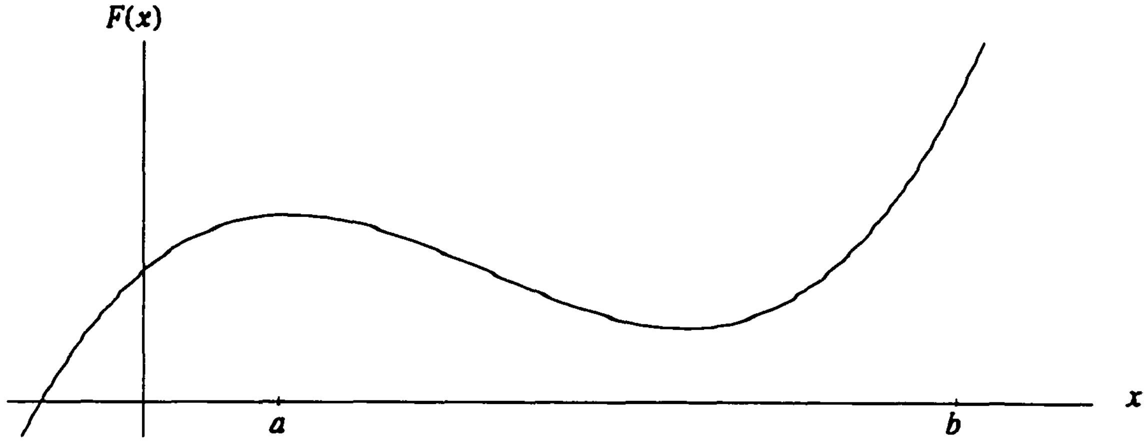
Suppose that, in a one-dimensional setting, mass is given by density times length. Consider an ultra thin rod of length 100 centimeters, and suppose the density of this rod, measured in grams per centimeter, is $\rho(x)$ at x centimeters from the beginning of the rod. How could you determine the mass of this rod?

Task 14

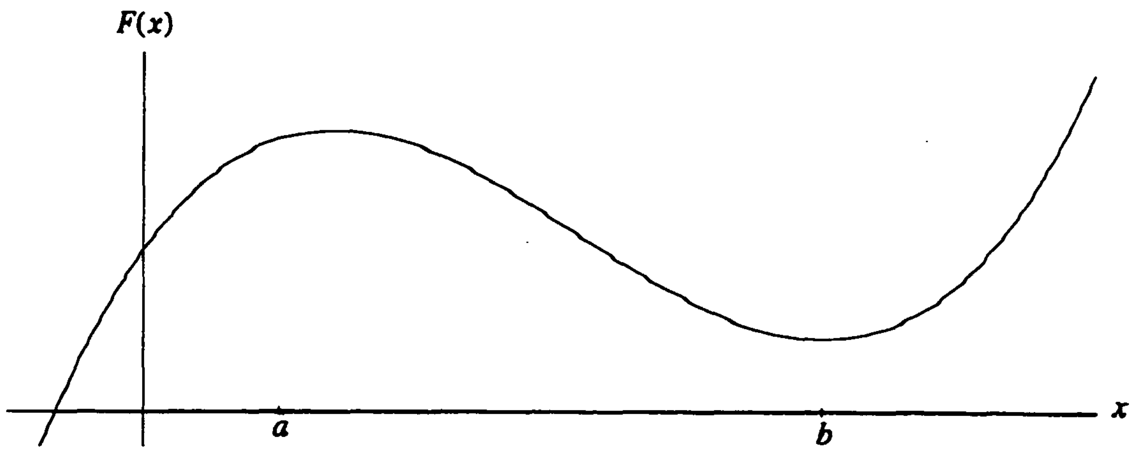
What does $\int_a^b f(x)\,dx$ represent?

Task 15

Let F be some antiderivative of f, and suppose the graph of F is as shown below. Indicate on the diagram what $\int_a^b f(x)\,dx$ represents.



Remark: Only one participant was asked to look at the second graph given below.

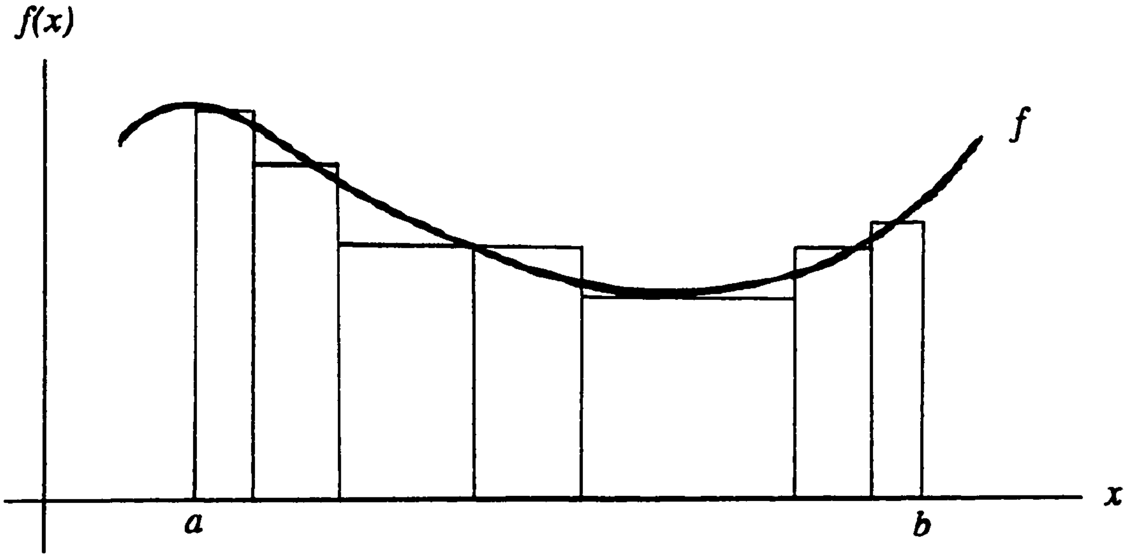


Task 16

Suppose a fellow student does not know any definitions for the definite integral. How would you go about explaining to this student the definition, as you know it, of the definite integral?

Task 17

Let f be the function shown graphically below.



How might this picture be related to the area of the region bounded by the graph of f and the x-axis from $x = a$ to $x = b$?

How might this picture be related to a definition of the definite integral?

Task 18

What role do limits play in the development of the definite integral?

Task 19

Let $f(x) = e^x$. Explain what relationship, if any, exists between the definite integral of f

and the sequence $S_n = \dfrac{e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}} + e^{\frac{n}{n}}}{n}$, where n is a natural number.

Remark: The equivalent form of the sequence given below was only used as an alternative.

$$S_n = \frac{1}{n}e^{\frac{1}{n}} + \frac{1}{n}e^{\frac{2}{n}} + \cdots + \frac{1}{n}e^{\frac{n-1}{n}} + \frac{1}{n}e^{\frac{n}{n}}$$

Task 20

Given the function f defined by $f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \end{cases}$, sketch the graph of the function

$F(x) = \int_0^x f(t)\,dt,\ 0 \le x \le 2.$

APPENDIX C

STUDENT CONSENT FORM

Student Consent Form

In recent years many research studies have been undertaken to learn about how undergraduate calculus students develop an understanding of calculus. Much work has been done concerning the topics of functions, limits, continuity, and derivatives; however, relatively little work has been done on undergraduate calculus students' understanding of the definite integral. This study aims to investigate this topic as part of a research project for the mathematics education doctoral program for Todd Oberg.

I hereby give my consent to be interviewed and to have each interview tape-recorded and my work on any problems videotaped for Todd Oberg's research project. I will be interviewed two or three times during the month of September, 1998, each interview 60 to 90 minutes in length. These interviews will take place outside of class time on my own time. I give my written consent for the taped and written materials to be used in Todd Oberg's research analysis and in any presentations which rely on such data. I understand that these materials are confidential, and will be coded to provide anonymity. My Calculus II instructor will NOT have access to any of the materials. The tapes and written material will be kept in the possession of Todd Oberg. Questions or concerns about this research project may be directed to either Todd Oberg (Math Dept., 243-4485, at Corbin Hall Room 353, or obergt@selway.umt.edu) or Jim Hirstein -- project advisor (Math Dept., 243-2661, at Math Building Room B7B, or hirstein@selway.umt.edu).

Even though the researcher anticipates no risks to students participating in this study, The University of Montana requires that all consent forms include the following statement.

> In the event that you are injured as a result of this research you should individually seek appropriate medical treatment. If the injury is caused by the negligence of the University or any of its employees, you may be entitled to reimbursement or compensation pursuant to the Comprehensive State Insurance Plan established by the Department of Administration under the authority of M.C.A., Title 2, Chapter 9. In the event of a claim for such injury, further information may be obtained from the University's Claims Representative or University Legal Counsel.
>
> *(reviewed by University Legal Counsel, July 6, 1993)*

I understand that I am guaranteed anonymity in this project and that I will be assigned a code name in this research. I understand that the taped material will contain no identifiable features identifying me as a participant in this research project. I understand that participation is strictly voluntary and that I may withdraw from the project at any time, without penalty.

(signature)____ (date)____________

(please print name here)________________________________

APPENDIX D

BACKGROUND SURVEY

SURVEY

Please complete the following survey to the best of your ability. The requested information is to be used for selecting possible participants in Todd Oberg's Ph.D. dissertation study on integration in calculus. If you are selected to be a possible participant in the study, Todd will contact you by September 15. At that time, a more specific discussion about the study and your role in it will take place.

When completing the survey please print or write legibly. Thank you for taking the time to complete this survey.

Name:

Phone number (where you can be reached over the next two weeks):

Home town and state:

Age:

Year in school: *Freshman* *Sophomore* *Junior* *Senior* *Other*

Major(s):

What math course did you take prior to taking Calculus I?

When did you take Calculus I? (If you repeated Calculus I, list only information pertaining to the last time you took the course.)

Where did you take Calculus I?

What book did you use for Calculus I?

Was the chapter on applications of the definite integral covered in that Calculus I course?

Yes *No* *Do Not Know*

What grade did you receive in Calculus I?

A *B* *C* *D* *F* *Do Not Know*

Have you repeated Calculus I? *Yes* *No* If yes, briefly state why.

Have you previously attempted Calculus II? *Yes* *No* If yes, briefly explain.

APPENDIX E

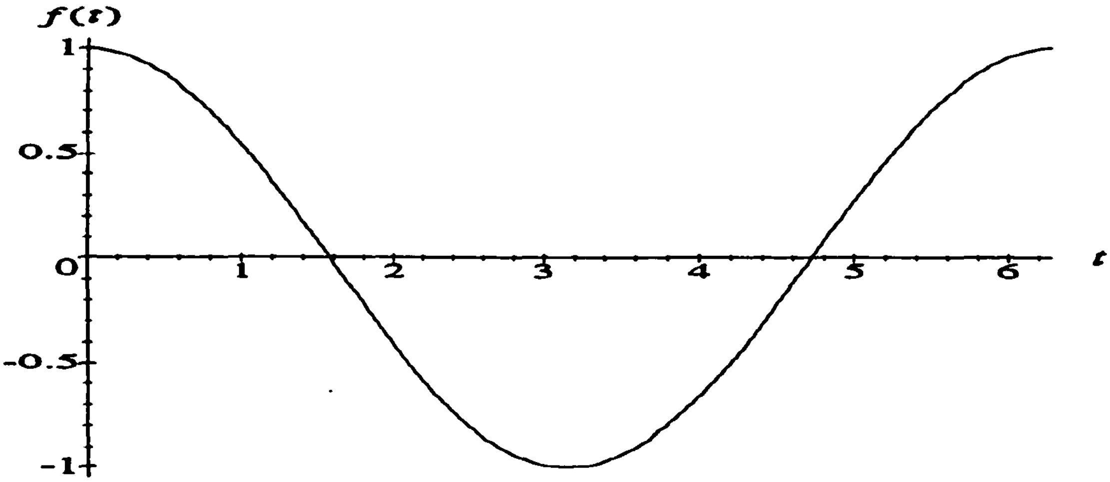
PARTICIPANT SELECTION TASKS

Name:

For this part of the survey please, complete each of the following problems to the best of your ability. If you are not sure about how to proceed with a particular problem, please think about the problem for a few moments, then write down what you believe needs to be done. Finally, show all of your work within the allotted space.

1. If $G(x)=\int_0^x 2\sin\theta\, d\theta$, then $G\left(\frac{3\pi}{2}\right)=?$ (Please refrain from using your calculator's integration feature for this problem.)

2. Let f be the function shown graphically below.



Describe what will happen to the definite integral $\int_0^b f(t)\,dt$ as the value for the upper limit of integration b starts at 0, and then is allowed to increase to 2π.

3. What does $\int_a^b f(x)\,dx$ represent?

4. On the piece of graph paper provided, please sketch the graph of a function f on some interval $[a,b]$ of your choice such that $\int_a^b f(x)\,dx = 2$.

APPENDIX F

INTERVIEW SCHEDULE FOR PARTICIPANTS

Interview schedule for the first interview (week of 9/21/1998):

Tuesday			
Joan	7 - 8:30 p.m.	mathematics education classroom	
Wednesday			
Stan	3:30 - 5 p.m.	conference room	
Lynn	7 - 8:30 p.m.	mathematics education classroom	
Friday			
Rob	3:30 - 5 p.m.	conference room	
Tina	5:30 - 7 p.m.	conference room	

Interview schedule for the second interview (week of 9/28/1998):

Monday			
Lynn	7 - 8:30 p.m.	mathematics education classroom	
Tuesday			
Joan	1 - 2:30 p.m.	conference room	
Wednesday			
Rob	3:30 - 5 p.m.	conference room	
Thursday			
Stan	4:30 - 6 p.m.	conference room	
Friday			
Tina	5:30 - 7 p.m.	conference room	

Interview schedule for the third interview (week of 10/5/1998):

Monday			
Joan	1 - 2:30 p.m.	conference room	
Lynn	7 - 8:30 p.m.	mathematics education classroom	
Tuesday			
Tina	11:10 - 12:10 p.m.	conference room	(Only time she had available.)
Wednesday			
Rob	3:30 - 5 p.m.	conference room	
Thursday			
Stan	4:30 - 6 p.m.	conference room	

Note: Most of the first and second interviews lasted about 1 hour and 20 minutes with the third interviews lasting about 1 hour.

APPENDIX G

ANALYSIS MATRIX

Task	Joan	Lynn	Rob	Stan	Tina
1a)	c	c	c	c	c
1b)	c	c	c	c	c
1c)	c, a	c, a	c	c, a	c, a, s
2	a	a	[b]	s, a	a
3	c	c	[b]	c	c
2 & 3[a]	a	c, a, o	c, a	a	u
4	a, s, c, o	a, s, c, o	c, o, a, s	a, c	a, s, c, o
5	c, f	c, f	c, f	c, a, f	c, f
6a)	a, o	a	a, o	a	a
6b)	f, a	f, a	f, a	f, a	f, a
7a)	a	a, c	a	a	a
7b)	f, a	f, a, c	f, a	f, a	f, a
8	a, o	a, o	o, a	a, o	a, c, u[c]
9	a	a	a	a	a
10	a	a	a	a	a
11	a, o, u[c]	o, a, c, t	a, o, c	a, o, f, c	a, o, t
12	a, c, s	a, s, c, o	a, s, c	s, a	s, c
13	a, c	c	c, s, a	u	–
14	a, c	a, c	a, o	a	a, t
15	t, c	a, c	a	a, t	a
16	a, c	c	c, a	a, f	c, a
17	a, s	a, s, o	a, s	c, a, s	a, s
18	c, s, a, o	o	c, u[c]	u	u
19	c, s	–	s	–	–
20	a, f, o	a, f, c	a, f	c, f, a	–

Note. c = computation, a = area, s = summation or accumulation, t = total change, f = function, o = abstract object, u = unsuccessful with no identifiable viewpoint, – = task that was not presented

[a]For when Tasks 2 and 3 were considered simultaneously.

[b]Rob combined Tasks 2 and 3 immediately.

[c]This applied to only a portion of the task.