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A DIALECTICAL INVARIANT FOR RESEARCH IN MATHEMATICS EDUCATION

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Abstract

Many current problems in research in mathematics education emerge from pairs of contradictory dialectical categories. In effect, these pairs characterize the problems. When an epistemological study is made to determine the object of research in which a problem is immersed, it is possible to find essential pairs of dialectical categories that become more profound and thus provide enough elements for the determination of appropriate didactic actions to solve the problem under research.

2010 Mathematics Subject Classification: 97D20

Keywords and phrases: dialectical categories, object of investigation, research problems in mathematics education.

Introduction

A central issue in the discipline is the answer to the question: what is mathematics education? *"Many answers to these questions have traditionally been, and continue to be, advanced. Standard reasons include the need to produce another generation of scholars to continue developing the discipline of mathematics, the supply of a cadre of scientists and others such as engineers who need strong mathematical competence, as training in logical thinking and problem solving, as exposure to what is as much a part of cultural heritage as literature or music. All of these, and more, are valid, but a deeper analysis is required"*[3].

Carlos Vasco models mathematics education as an octagon in which mathematics (which he classifies as research, scholars and daily life) is located on the inside and on the outside, forming the sides, are eight disciplines: philosophy, logic, computer science, linguistics, neurology, psychology, anthropology, history of mathematics and epistemology[4].

We believe that mathematics education integrates dissimilar disciplines as they are represented in Figure 1 which clearly illustrates the complexity on which we base the considerations that follow.

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[4] Vasco U.,C.E.(1994). La educación matemática: una disciplina en formación, *Matemática Enseñanza Universitaria*, Vol III, Nro 2, 59-76.

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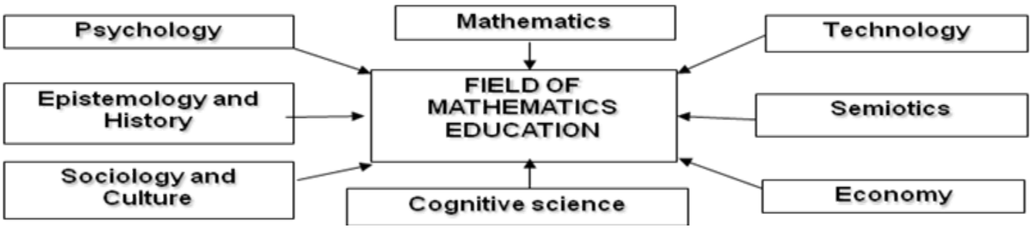

Figure 1

Other important reflections are contained in the following sentences: *"So mathematics education is fashioned to provide appropriate mathematical knowledge, understanding, and skills to diverse student populations"[5]. In this sense we can say that mathematics education is a special type of teaching, engineering in the sense that it is the personalization of the basic mathematical principles to meet the needs of teachers and students[6], some even claim that it is a branch of applied mathematics[7].*

Mathematics education today is a science that at a glance reveals two conflicting features: on the one hand theoretical sublimation and on the other a multiplicity of practical problems that must be resolved by the teacher in the classroom. Of course, this two-faced trait manifests itself in research carried out in different regions of the world. In some cases "theoretical centers" immersed in general issues are at the center of the research being done, and in others there are institutions dealing with the "real" problems that teachers face in the classroom.

A principal point must be made clear: what is a research in mathematics education?

Dialectic theory delimits knowledge formation as an active, complex, ongoing process of organizing and reorganizing conceptual structures rather than an accumulation of fixed truths. Furthermore, in dialectic theory contradiction assumes a central role in the process of change and reorganization that the theory presumes to explain. Whether in the field of cognitive development or in the broader realm of psychology, a dialectical view also assumes that developmental processes are socially and culturally shaped and defined, and that concepts and meanings—whether mathematical or not - evolve in an emergent process of what Vygotsky[8] and Leontiev[9] called a collective activity system. "The latter is understood to operate through the emergence of cognitive conflict within the conceptual system, leading to the ongoing resolution of that conflict in a dialectical manner- which is to say through the

[5] Bass, H. (2005). Mathematics, mathematicians, and mathematics education, *Bulletin Amer. Math. Society*, 42, 417-430 y Ferrini-Mundy, J. and Findell, B. (2001): The mathematics education of prospective teachers of secondary school mathematics: old assumptions, new challenges, *in CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?* Washington DC: Mathematical Association of America.

[6] Wu, H. (2006). How mathematicians can contribute to K-12 mathematics education, *Proceedings of International Congress of Mathematicians*, Madrid 2006, Volume III, European Mathematical Society, Zürich, 2006, 1676-1688 (http://math.berkeley.edu/~wu/ICMtalk.pdf) and "Access and Opportunities to Learn Are Not Accidents: Engineering Mathematical Progress in Your School" by William F. Tate, which is available at http://www.serve.org/downloads/publications/Access AndOpportunities.pdf. The concept of mathematics education as mathematical engineering also sheds some light on Shulman's concept of *pedagogical content knowledge* (see Shulman, L. (1986). Those who understand: Knowledge growth in teaching, *Educational Researcher*, 15, 4-14).

[7] Bass, H. (2005). Mathematics, mathematicians, and mathematics education, *Bulletin Amer. Math. Society*, 42, 417-430 y Ferrini-Mundy, J. and Findell, B. (2001). The mathematics education of prospective teachers of secondary school mathematics: old assumptions, new challenges, *in CUPM Discussion Papers about Mathematics and the Mathematical Sciences in 2010: What Should Students Know?* Washington DC: Mathematical Association of America.

[8] Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

[9] Leontiev, A. (1981). *Problems of the development of the mind*. Moscow: Progress.

recognition and articulation of contradictions and inconsistencies, and their mediation in the context of a collective activity"[10].

The teleology of the processes of structural cognitive development, as defined by the major western theorist of cognitive change, Piaget, is understood not only as a continuous movement from "no balance" towards reequilibration, but as progressively directed movement towards "increasing equilibrations," which necessarily require a correspondingly higher organization of cognitive structure. Although it was never affirmed by Piaget, several theorists[11] understand his psychological theory of cognitive development to be fundamentally dialectical. We can identify many parallels between Piaget and the other major theorist of the twentieth century, Vygotsky, at least on the level of the basic, conceptual mechanisms of cognitive development. Vygotsky was in fact an avowed dialectician, who clearly saw cognitive development as "*…a dialectical process, a process in which the transition from one stage to the next occurs not through evolution, but through revolution*"[12].

Ever since Imre Lakatos presented a dialectical view of the development of mathematical knowledge in his *Proofs and Refutations* (1976), the idea of carrying out dialectical processes in the mathematics classroom has often attracted the attention of mathematics educators[13].

On the other hand, the larger scale view of activity provided by this perspective considers learning in terms of fundamental qualitative changes in an activity system as a whole, a process that Engeström calls *expansive learning*. This occurs as a result of deliberate efforts of participants over time to solve inherent conflicts and contradictions that are a part of any activity system. Engeström's theorization does not provide an explicit direction for understanding the place of mathematics within a given activity, nor does it provide details related to the learning process of individuals[14].

Based on his reading of Vygotsky's semiotics, Leontiev's activity theory, and the more recent work of Felix Mikhailov and Evald Ilyenkov, Radford has developed the *Theory of Objectivization* specifically for unpacking nuances and processes of mathematics activity and learning of individuals from a cultural-semiotic activity perspective[15]. In contrast to Engeström, Radford's work focuses on specific aspects of the consciousness, learning and being of individuals as well as on the semiotic and social dimensions of mathematics from an activity perspective. Radford's concept of objectivization is a refinement of Vygotsky's

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[10] Kennedy, N. S. (2006). Conceptual change as dialectical transformation, in Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, pp. 193-200. Prague: PME.

[11] Kitchener, R. (1986). *Piaget's theory of knowledge: Genetic epistemology & scientific reason*. New Haven, MA: Yale University Press.

[12] Vygotsky, [The problem of age], cited in El'konin, D. (1977). Towards the problem of stages in the mental development of the child. In M. Cole (Ed.), *Soviet developmental psychology* (pp. 539-563). New York: Sharpe. p. 542.

[13] For a critical review, see Gila Hanna: The Ongoing Value of Proof, in Luis Puig and A. Gutierrez (Eds) *Proceedings of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, Vol I., 21-34.

[14] Engeström, Y. (2001). Expansive learning at work: toward an activity theoretical reconceptualization, *Journal of Education and Work*, *14*(1), 133-156 and Engeström, Y. (2008).*From teams to knots: Activity-theoretical studies of collaboration and learning at work*, New York: Cambridge University Press.

[15] Radford, L. (2006). Elements of a cultural theory of objectification, *Revista Latinoamericana de Investigación en Matemática Educativa, Special issue on semiotics, culture and mathematical thinking*, pp. 103-129, and Radford, L. (2007). Towards a cultural theory of learning, in Pitta-Pantazi, D., & Philippou, G. (Eds.). *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education (CERME – 5)*, pp. 1782-1797. Larnaca, Cyprus, CD-ROM, ISBN - 978-9963-671-25-0 and Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning, in L. Radford, G Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom and culture* (215-234). Rotterdam: Sense Publishers.

notion of internalization which emphasizes the dialectical relationship between the subject and the cultural object being attended to.

There are other theories that use the dialectical approach from other perspectives. For instance, Dubinsky theorizes that mathematical objects are constructed by reflective abstraction in a dialectic sequence **APOS**, beginning with **A**ctions that are perceived as external, interiorized into internal **P**rocesses, encapsulated as mental **O**bjects developed within a coherent mathematical schema[16]. Drossos, on the other hand, uses the opposite dialectical *assimilation* and *accommodation* when he talks about *adaptation* in the process of cognition[17].

In this paper we show how contradictions, immersed in a large part of the problems related to mathematics education, are a guiding source for didactics modeling that can lead to the solution of the problem researched and to the production of results.

METHODOLOGY AND RESULTS

We will focus our attention on two features of research in education in general and in mathematics education in particular:

1. Difficulties in the emergence of new knowledge.
2. The role played by the dialectical method in interrelating the research subject and the object of research.

How does a researcher discover a particular object of research?

The formulation of a research problem in mathematics education is frequently related to difficulties in the teaching-learning process of a mathematical topic at some level of education. It proceeds consciously or unconsciously through a process of abstraction of contradictory dialectical categories[18]. We consider that a contradiction exists only if it has a witness. That means that a contradiction does not exist by itself, but only with reference to a cognitive system[19]. According to Piaget (1974, p. 161), the awareness of a contradiction is only possible at the level at which the subject becomes able to overcome it[20]. We consider that most of the problems in the teaching of mathematics are characterized by a contradiction between dialectical pairs of students' knowledge and their level of achievement.

In the theory of situations, the term 'dialectic' refers to the method used by a cognitive system (teacher, student) to manage the contradictions between its expectations concerning the output from the system it attempts to control (the student-milieu system, the teacher-milieu, respectively) and the feedback. Feedback is communication of information. The process of dialectic turns this information into knowledge: out of the contradiction, something positive is attained that explains the contradiction and generates ways of avoiding it in the future.

[16] Dubinsky. E. & MacDonald, M. A. (2001). APOS: A Constructivist Theory of Learning in Undergraduate Mathematics Education Research, in D. Holton *et al.* (Eds.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study*, Dordrecht: Kluwer, 273-280 http://www.math.kent.edu/~edd/ICMIPaper.pdf

[17] Drossos, C. A.(1987). Cognition, Mathematics and Synthetic Reasoning, *General Seminar of Mathematics*, Department of Mathematics, University of Patras, Greece vol. 13, 107-151.

[18] Godino, J. D.; C. Batanero and V Font (2007).The onto-semiotic approach to research in mathematics education, *ZDM Mathematics Education*, 39:127–135.

[19] Cf. Grize, J.B.; Piéraut-le Bonniec (1983). La Contradiction, Paris: PUF and Balacheff, N.(1991).Treatment of refutations: aspects of the complexity of a constructive its approach to mathematics learning, E. von Glasersfeld (ed.) (1991). Radical Constructivism in Mathematics Education, 89-110, Kluwer Academic Publishers.

[20] Piaget, J. (1974*). Les Relations entre Affirmations et Négations*, Recherches sur la contradiction, Vol. 2, Paris: PUF, p. 161.

Resolution of contradictions in each case (either in the *situation of action* or in the *situation of formulation*) brings some positive new knowledge about the situations: a better way of expressing one's ideas or an improved strategy[21].

Usually this situation emerges in the classroom, almost always far removed from the possibility of solving it by means of a scientific process. Ignoring the difficulties in learning, it can be characterized and didactical solutions can be sought focusing on the identification of the object of research in the process of epistemologization as described as follows.

Primary contradiction → Research problem → Object of research

This triple indicates the path starting from a primary contradiction (which is evident), and may be referred to as an external contradiction. When this path for seeking scientific knowledge in mathematics education is assumed, then it is possible to find, perhaps through a series of steps, better refinements towards possible resolutions to the problem being researched, i.e., a succession of contradictions $\{C_n\}$, from which we have:

External contradiction = $C_0 \rightarrow C_1 \rightarrow \ldots \rightarrow C_n$ = Fundamental contradiction

From the problematic situation that has been detected, we build the research design, and with its help a better understanding of the research problem, closely related to the object of study and the proposed objective. As refinements are achieved, we will find a better approach to an **object to study** or valid **object of research** and, of course, toward a **field of a** relevant **action**.

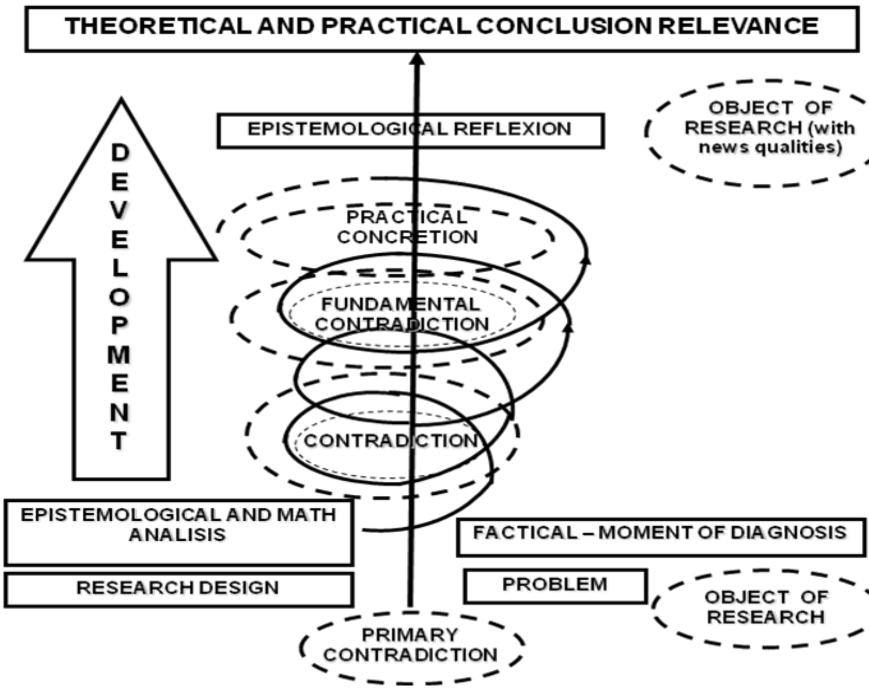

Figure 2

Throughout this article the system of contradictory pairs will appear repeatedly because of their invariant nature, even after conclusion of the investigation, that is, the new object of research that is defined above must contain, in a natural way, a pair of new dialectical contradictions, qualitatively higher.

[21] More details in Sierpinska, A. Theory of situations as a means to overcome the 'procedures vs understanding' dilemma in mathematics teaching, MATH 645: Theory of Situations/Lecture 2 (in http://annasierpinska.wkrib.com/pdf/TDSLecture%202.pdf consulted on February 3th of 2012).

In Figure 3 we can see the actions of a teacher in the process of didactic transposition, as a simple case of the previous figure. In fact, a path from the concrete to the abstract is shown; both categories are dialectically connected through the activities of human beings whose actions are intrinsically related to the activity setting which represents a multi-faceted, yet organized, whole. Abstraction is a process of making sense of such concrete situations by discovering new meanings in order to establish connections amongst the different elements of the whole[22].

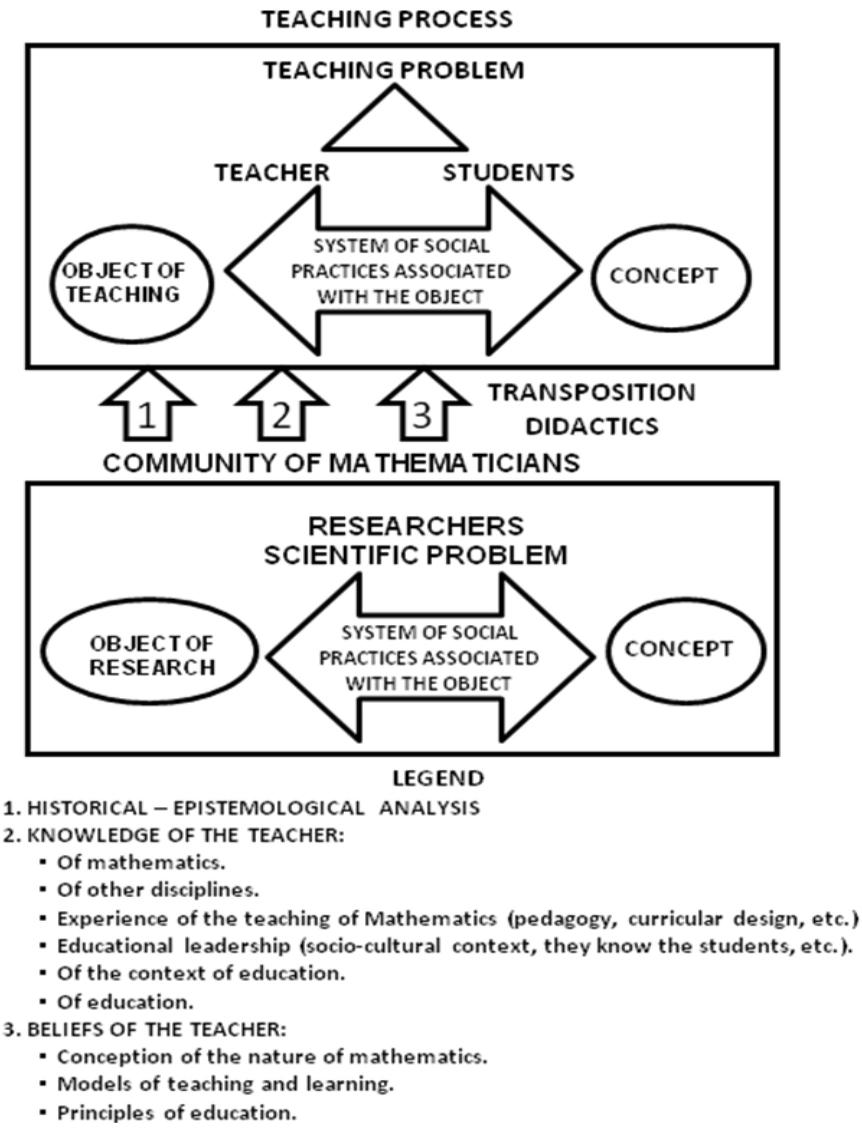

Figure 3

In Figure 4, an example is shown in which we can appreciate a refinement made in a recent research experience in the teaching of geometry. From the external contradiction or **contradiction 0** the process led the researcher to the fundamental contradiction or **contradiction n** in three steps:

[22] Ozmantar, M. F. and J. Monaghan (2007). A Dialectical Approach to the Formation of Mathematical Abstractions, *Math. Education Research J.* Vol. 19, No. 2, 89–112.

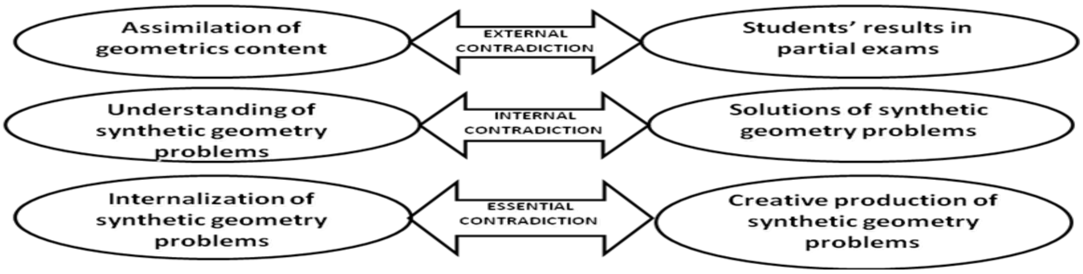

Figure 4

On a more general level, Asano notices several differences between dialectics and didactics of mathematics, on the basis that they are concerned with different kinds of objects, intermediates and forms; however, his observations principally treat differences in methods and it is important to keep in mind a long standing appreciation in this regard. Plato talks about mathematics and dialectics in the simile of the line and subsequent pages of Book VI of *The Republic*[23]. Plato's educational priorities also reflected his distinct pedagogy. Challenging the Sophists (who prized rhetoric, believed in ethical and epistemological relativism, and claimed to teach "excellence"), Plato argued that training in "excellence" was meaningless without content. Plato doubted whether a standard method of teaching existed for all subjects, and he argued that morally neutral education would corrupt most citizens. He preferred the dialectical method over the Sophists' rhetorical pedagogy.

Before continuing, let us look at some of the aspects that determine quality in research in mathematics education. A result of quality in research in mathematics education can be characterized as follows:

1. It shows new relationships and regularities that the researcher reveals in the process of resolving the problem.
2. Along with scientific background results, it makes a difference in the resolution of similar problems.
3. It works as a systemic feature of transformation of the process being modeled.
4. The model supports the theoretical contributions of the thesis or research result as well as the essence of the text being written. Some contributions can be:
    - Problem solving.
    - Mathematical technique
    - Mathematical education theory
    - Mathematical exposition
    - Mathematical pedagogy
    - Mathematical vision and visualization
    - Rigorous mathematics
    - Beautiful mathematics
    - Elegant mathematics
    - Creative mathematics
    - Intuitive mathematics

[23] Cf. Asano, K. (1993). Degrees of Reality in Plato: Part I." Aichi (Philosophy) Vol. 10: 131–118; (1994). Degrees of Reality in Plato: Part II. The Hannan Ronshu (*Journal of Hannan University*) *Humanities & Natural Science*, Vol. 30, No. 2 (Sep.): 17–34; (1996). Two Arguments for Forms in Plato: Conflicting Appearances and One over Many. The Hannan Ronshu (*Journal of Hannan University*) Humanities & Natural Science, Vol. 32, No. 1 (June): 79–96; (1997A). The Simile of the Line in Plato's Republic VI. Sapientia (*The Eichi University Review*) No. 31: 207–34; (1997B). A Study of Plato's Metaphysics in the Republic." Ph.D. diss., University of Texas, Austin and (1998). Mathematics and Dialectic in Plato's Republic, Sapientia (32):117-142.

These results, seen as new knowledge for science, are identified by the educational community of the sciences as a **theoretical contribution** to research which may or may not lead to the obtention of a scientific degree.

Research reports in the area of mathematics education have revealed argumentative inadequacies of researchers when they attempt to prove the scientific novelty of their theoretical contribution. What this paper proposes is a general procedure to promote understanding for the modeling of the theoretical scientific contribution of research in the pedagogical sciences in general and in mathematics education in particular[24].

Conception of theoretical contribution in research in mathematics education.

The results of research should contain contributions on the theoretical and practical levels. Contributions on the theoretical level are embodied in models, definitions, concepts, characterizations, revelation of phenomena of a periodic nature, among others. Practical contributions include methodologies, strategies, techniques, procedures, among others, and they work as tools for the implementation of the theory. The main contribution is the theoretical conception or theoretical contribution that underpins the research.

Both the identification of a problem and its resolution are supported in the dialectical approach to knowledge. The formulation of the scientific problem is the expression of a contradiction between objects and/or phenomena contrasting the current state and the desired state, which we have previously called an external contradiction. This external contradiction is revealed in the initial diagnosis of the object based on empirical data obtained by means of an exploration of an actual situation and a bibliographic review that allows the construction of theoretical foundations for the problem as revealed in the state of the art. As a result, the object of study and field of action (unit of analysis) in an initial stage of the research process may be described.

Ways of promoting the dialectical method of knowledge for the construction of a theoretical conception which is the basis for the resolution of a scientific problem

Here we will see the methodological value of the dialectical method. The dialectical approach allows the analysis of the most essential aspects of the object, analysis which consists of determining those contradictory elements which are present in it. The opposites are mutually exclusive aspects of the object that at the same time question each other. The mutual relationship between the opposites constitutes a contradiction.

The contradiction plays its role as source of the movement and the development of the object and the phenomenon. Contradictions of the object or phenomenon with other objects or phenomena are considered external contradictions. Internal contradictions are formed between opposing aspects of the object itself or given phenomenon in themselves. When is not possible to refine an internal contradiction, then it plays the decisive role in the development of the object or phenomenon being in this case the fundamental contradiction.

Our use of dialectic follows ancient Greek thought. Unlike the more recent Hegelian use of the term that anticipates a synthesis of opposites, we want to revitalize an earlier sense of dialectic that predates Plato and views dialogue, discourse and dispute themselves as deepening our understanding of the world[25]. Dialectic is a kind of juxtaposition of ideas,

[24] Concepción, R. y Rodríguez, F. (2005). Un procedimiento para elaborar el aporte teórico de la tesis de doctorado en Ciencias Pedagógicas basado en el enfoque sistémico-estructural, Universidad de Holguín, Cuba.

[25] Parmenides' (510 BC). Foundational poem is seen as a starting point for the ongoing development of the idea of dialectic: *"There is need for you to learn all things ... both the unshaken heart of persuasive Truth and the opinions of mortals, in which there is no true reliance ... that the things that appear must genuinely be, being always, indeed, all things"* in Diels, H. & Kranz, W. (1951). *Die*

often literally a debate, rather than a resolution or synthesis. Understandings emerge by means of holding in creative tension ideas that can even seem paradoxical.

We assume, accordingly, that the resolution of the research problem with a dialectical approach should reveal the existence of contradictory pairs of elements present in the object of research (fundamental contradiction) and their resolution through a third procedural element, concurrent and simultaneous with the other two, and such that through its introduction it is possible to accelerate the inherently dynamic nature of a dialectical contradiction. This can be achieved throughout the research process[26].

The challenge imposed by the analysis of the object using a dialectical approach is that of specifically determining the primary contradictory pair and of discovering a third element which is also contradictory to the original couple and thus stimulate the transformation of the problem. This analysis permits the characterization of the object of research and of the field of action through a model or theoretical conception, which becomes an instrument of optimization and forms the basis of the proposal or contribution of the research.

Ways of building the theoretical conception or model which constitutes the essential theoretical contribution of the research

The theoretical model or the modeling of the theoretical conception is a construction that the researcher creates or develops starting from his theoretical knowledge of the object of research and the field of action, it is, as all kinds of models are, the idealization (abstraction-realization) the researcher makes in order to transform the process.

The conception of a theoretical contribution is defined as a personal construction of the researcher, product of the abstraction of the object and process that seeks to transform, in which the latter is reproduced in its totality by means of the relationship between contradictory elements that can accelerate the movement and development of that process in a given social historical context.

The realization of a theoretical contribution in itself constitutes the manner of achieving new concrete knowledge as thought through by the researcher. According to Alvarez de Zayas, the representation or theoretical construction can be presented as a theoretical model and its totalizing conception must be achieved in order to conceive this as a system[27]. That is to say, the construction of the model is favored if it is treated with a systemic approach. So it is advisable, following this approach, to build the model of the theoretical contribution based on the general characteristics of systems.

Components of the system: These are the fundamental elements that characterize the model and which are essential to resolve the problem.[28] They must include concepts and categories that the researcher has discovered in order to abstract the object or phenomenon that is modeled. The components are the contradictory pair and the third co-existing element that causes or resolves the fundamental contradiction as well as other elements such as dimensions or variables that permit the understanding of the object or phenomenon to be modeled. The components of the system should acquire their own personality in the object or process to be idealized and are contextualized to the activity in which this contradiction and its resolution take life.

Fragmente der Vorsokratiker. Berlin: Weidmann, translated into English by Kathleen Freeman in her *Ancilla to the Pre-Socratic Philosophers*. Oxford: Basil Blackwell, 1962, p. 246.

[26] Álvarez de Zayas C.(2000). Metodología de la investigación científica. Cómo se modela la investigación científica (in digital format)

[27] Ibidem

[28] Ibidem

Structures and their functional relationships: These provide the framework for interaction and organization among the components of the system, necessary to assure its functions. The structure guides the procedure or mechanism that sparks the activity or process that is modeled. Relationships explain the dynamics of behavior.

Hierarchy: This is the degree of interaction between subsystems.

The process of elaboration or conception of the theoretical contribution requires a scientific abstraction and represents the essential qualitative jump that a researcher must make in order to make contributions to the science being researched.

Personal experience and the meta cognitive diagnosis of how researchers operate in the elaboration of a theoretical conception in research in the pedagogical sciences, reveal that such research unfolds like a process of successive scientific abstractions which are facilitated by means of a procedure that orients the phases and actions of this process and are necessary to attain the objective.

General procedure for the construction of a theoretical contribution.

The procedure that sets out to elaborate a theoretical conception or contribution is supported by the postulates of the theories of the systemic and dialectic approaches to knowledge construction and scientific modeling. It is important to keep in mind that the application of this procedure in itself does not guarantee arriving at a theoretical contribution if the researcher does not have well defined the establishment of the theory that serves as base for the research process. The procedure is a manner of establishing the components, their relation and structure in the modeling of the theoretical contribution and facilitating the search for arguments that explain it.

The procedure consists of questions, actions and phases. The questions are formulated with a metacognitive intention of orienting the investigator towards the reflective understanding of the actions that are chosen for each phase. The phases integrate the actions in the process of scientific elaboration, in the manner of a generalizing succession of analysis and synthesis.

The elaboration of the theoretical conception should demonstrate the logic of the scientific reasoning followed by the researcher in its construction and passes through four phases or moments.

Phases of the construction of a theoretical conception

First phase: Determination of the process or activity object of transformation. This initial phase is crucial for research, because it is when a researcher identifies the process that he or she intends to model, something that requires a profound theoretical preparation on the object and the field of research. The meaning of this process or activity for the research to be carried out is conceptualized.[29].

Second phase: Determination of elements that characterize the process or activity object of transformation and are essential to resolve the problem (components of the model).

Taking into account that the theoretical conception will be modeled with a systemic approach, it is necessary to determine all the elements that make up the model without omitting:

1. The process to be modeled
2. The dimensions of the process that is to be modeled
3. The contradictory pair (fundamental contradiction) present in the object

[29] Observations on the use of internet and the electronic library are interesting in Barry, C. A. (1997). Information skills for an electronic world: training doctoral research students, en *Journal of Information Science*, 23 (3), 225-238.

4. The third element of procedural nature or means for expediting the resolution of the contradiction
5. The context in which the fundamental contradiction occurs and is resolved

Third phase: Organization of the structure of the theoretical conception (structure of the model)

When the elements of the model have been determined, functional relationships and the hierarchy of the system must be analyzed. For the modeling, the components may denote categories or very short phrases linked in such a way that they show the dynamic and the feedback of the system.

Fourth phase: Explanation of the process or activity that is to be modeled (dynamic)

This phase presents the arguments that the new theoretical concepts supply in support of the transformation of practice, although the model must "speak for itself". As all systems they should generate a higher systemic quality that does not belong of any element in particular; the interrelationships between components must be such that if they affect one of them, they will affect the whole system and, in consequence, will not develop new and higher characteristics and qualities. For example, see a didactical model in Figure 5.

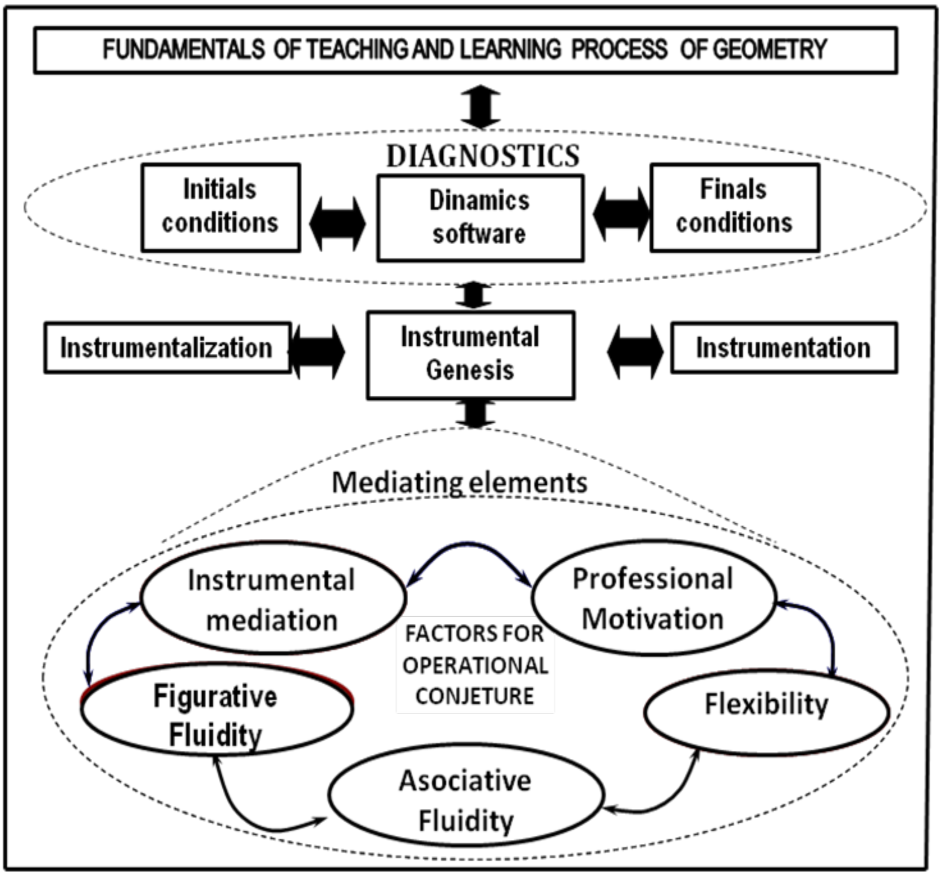

Figure 5: A didactical model[30]

Indications for the elaboration of a heuristic procedure of introduction to practice.

[30] Wilson, C. (2012). Operational conjecture. A didactical model for teaching and learning of geometry in engineering's careers. Thesis in option to the Scientifics grade of Mathematics Education. University of Holguin. (the first author was advisor himself)

In the structure of the procedure, first the questions are formulated; they lead to actions and finally the phases or stages emerge.

The questions are the most dynamic part of the procedure, they give way to a situation in which each researcher performs actions depending on his/her possibilities and needs without becoming an algorithm composed of linear sequences, but rather a process of construction of the scientific text and context that allows the researcher to discover some things and perhaps to return to previous phases to reconsider, or even to find out that phases may overlap.

Finally a pre experiment or a case study must be carried out in which the procedure is introduced into practice, and then interpret or verify the results according to the research paradigm used.

The procedure described should allow the elaboration of an argument that validates the proposed didactical model and, most importantly, the research methodology used to extract the theoretical and practical contributions that raise the object of research to a higher qualitative level. Finally all the results should constitute an argument for the **scientific novelty** that resolved a **gap** and therefore contributed to the **epistemological development** of the research. (See Figure 6 referent to our geometric example.)

Stages of the generation of conjectures	Actions for the generation of conjectures	Guiding questions for the generation of conjectures	Resources or heuristic means used
Construction of the basic elements	**Begin with the construction of basic elements**	**1. What elements should make up the figure or geometric locus?**	**HP: Related to the elements of the figure** **HR: Based on representation of the figure used to analyze it**
	Determine what figures or elements can be generated.	**2. How can I represent the desired figure or geometric locus?**	
	Identify the necessity of each basic element.	**3. Are the basic elements used necessary and sufficient?**	
Movements (transformations) of the elements to initiate a search	**Identify the relations between the basic elements constructed and how they influence one another**	**1. How are the components of the system related to one another?**	**HS: Consists of identifying the basic elements that have been constructed and of observing their behavior using software** **HS: The movement of the mathematical objects is part of the strategy** **HS: Execution of a plan for obtaining a possible solution**
	Move each basic component in coherent manner.	**2. What possible movements can be made?**	
	Identify the pertinence of each basic element.	**3. Is each basic element constructed pertinent?**	
Generation of variants of loci or figures	**Identify the actions that can be performed with each basic element**	**1. Which movements generate geometric figures and loci?**	**HP: Compare the movements of the figure using software in the construction of geometric loci.**
	Look for and determine all the variations that can be generated	**2. Have all the variants been identified?**	

Conjecture what geometric results can be obtained	**Determine the possible figures or geometric loci**	**1. What part of the figure or geometric locus is sustained by the initial basic elements? 2. Can new conjectures be obtained, with other basic elements?**	**HR: Understands the problem and explains or supports it based on the initial basic elements. HP: Relates the problem to other problems.**
Check with paper and pencil	**Manually determine the figure or locus**	**1. What level of coincidence does the verification give rise to? 2. What differentiates the static form obtained from the conventional procedure with the dynamic form employed?**	**HS: Analyze the concept solution using pencil and paper and using dynamic software**

Figure 6. HP (Heuristics procedure), HS (Heuristics strategy) and HR (Heuristics rule)

Conclusions

This is an ascent to the concrete -a process of making meaning by establishing connections amongst elements of the whole- and this is precisely what dialectics is. As Douady affirms, the necessity of organizing a critical work on these cultural answers and also as the acknowledgement of their necessary contribution to the *milieus* with which the students interact, the dialectics between *media* and *milieus* play the essential role.

For instance, new properties of the triangle were found when it was regarded, not in itself, but in connection with the circle. Each triangle can be divided in two right triangles; each one of which can be considered as belonging to some circle. Here the sides and angles appear in totally different interrelations, which were revealed to the eyes of the researcher only by this new relationship. This is a dialectic technique, the technique of theoretical thought. The connection between the triangle and the circle can only be seen as an idea that presupposes the possibility of mentally transforming a triangle into a component of the circle, i.e., reduction of one to the other (of the particular to the general). Only with a transformation, a mental reduction of one figure to another could new properties be detected in the triangle which then laid the foundations of what was a new theory. These properties cannot be revealed by "considering" the triangle in itself and the establishment of the connections defined (reduction of the different one) requires thinking through the concepts.

Theoretical systems, in particular mathematical theories, are always changing, and this includes the scientific theories concerning mathematics education. The words of Karl Popper have a special meaning. *Scientific theories are perpetually changing. This is not due to mere chance but might well be expected, according to our characterization of empirical science"*[31]. Remark: The scientific theories in mathematics education, given the dynamic nature of mathematics education in a context mediated by the modern world, shows this constant change mediated by the laws of dialectics.

The theoretical and practical contributions of an investigation constitute two levels of the concrete that are thought about in the scientific activity organized by the researcher. The theoretical conception or theoretical contribution is built on incorporated scientific

[31] Popper, K. (2009). *The Logic of Scientific Discovery*, Rutledge, New York, p. 50 (of the first English edition on 1959)

knowledge which supports the outcome of the research; it is an essential conclusion that contributes to the science.

The progress that has been made in recent decades is nothing less than phenomenal. In little more than one quarter of a century there have been great epistemological changes, accompanied by a flowering of the tools, techniques and theoretical perspectives that supported them. Cognitive science and socio-cultural research in mathematical education have matured and are becoming more robust; fields that at first seemed to be related almost as thesis and antithesis have, over the last decade or so, generated a synthesis that seems even more promising in terms of its ability to help explain questions concerning (mathematical) thinking, teaching and learning. The same can be said for the artificial distinction between quantitative and qualitative methods that becomes less important when formulating central research questions[32].

In conclusion, we suggest that by solving the research problem we deepen the fundamental understanding of learning, which also helps us in the resolution of many practical issues of teaching. If we start paying serious attention to previous issues, the problems of theory and philosophy will be easier to address and resolve.

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