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Plato on the foundations of Modern Theorem Provers

Ines Hipolito

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Inês Hipolito[1]
Nova University of Lisbon

Abstract**:** Is it possible to achieve such a proof that is independent of both acts and dispositions of the human mind? Plato is one of the great contributors to the foundations of mathematics. He discussed, 2400 years ago, the importance of clear and precise definitions as fundamental entities in mathematics, independent of the human mind. In the seventh book of his masterpiece, *The Republic*, Plato states “arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument” (525c). In the light of this thought, I will discuss the status of mathematical entities in the twentieth first century, an era when it is already possible to demonstrate theorems and construct formal axiomatic derivations of remarkable complexity with artificial intelligent agents — the *modern theorem provers*.

Keywords: Plato; Modern Theorem Provers; Formal Proof.

[1] hipolito.ines@gmail.com

Introduction

According to Platonism, a mathematical proof is the metaphysical view that there are abstract mathematical objects whose existence is independent of human language, thought, and practices. A *full-blooded Platonism* or *platitudinous Platonism* (FBP) asserts that it is possible for human beings to have systematically and non-accidentally true beliefs about a platonic mathematic realm — a mathematical realm satisfying *Existence*, *Abstractness* and *Independence*. Could there be such proof? Could a proof be objective and completely understood, independently of the possibilities of our knowing of truth or falsity?

Plato is one of the great contributors to the foundations of mathematics. He discussed, 2400 years ago, the importance of clear and precise definitions as fundamental entities in mathematics. In the seventh book of his masterpiece, *The Republic*, Plato states "arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number, and rebelling against the introduction of visible or tangible objects into the argument". In the light of this thought, I will discuss the status of mathematical entities in the twentieth first century, an era where it is already possible to demonstrate theorems, construct formal axiomatic derivations of remarkable complexity with artificial intelligent agents — *the modern theorem provers*. A computer-assisted proof is written in a precise artificial language that admits only a fixed repertoire of stylized steps. It is formalized through artificial intelligent agents that mechanically verify, in a formal language, the correctness of the proof previously demonstrated by the human mind.

In contrast, calculi are exactly the kind of feature, which make it appealing for mathematicians. There are two reasons for this: (i.) it can be studied for its own properties and elegance of pure mathematics; and (ii.) can easily be extended to include other fundamental aspects of reasoning. According to Hofstadter, (1979), a proof is something informal, that is, a product of human thought, written in human language for human consumption. All sorts of complex features of thought may be used in proofs, and, thought they may "feel right", one may wonder if they can be logically defended. This is really what formalization is for.

1. Plato's conception of Arithmetic

According to Plato, at the end of the sixth book of *The Republic*, mathematicians' method of thinking is not a matter of intelligence, but rather a matter of *διανοια,* which means *understanding*. This is a definition by Plato that seems to etimologically imply δια (*between*), *νουσ* (*intelligence*) and *δοξα* (*opinion*), as if understanding would be something in between *opinion* and *inteligence*.

In 525a, Plato considers the concept of number, as a non-limited unity trough plurality, since "this characteristic occurs in the case of one; for we see the same thing to be both one and infinite in multitude" (525a). Also, with this conception of plurality as much as unity "thought begins to be aroused within us, and the soul perplexed and wanting to arrive at a decision asks 'What is absolute unity?' This is the way in which the study of the one has a power of drawing and converting the mind to the contemplation of reality." (525a).

As reported by Plato, the reality of calculus is a pure contemplation since in reasoning about numbers there are no visible bodies:

> "Plato — Now, suppose a person were to say to them, Glaucon, 'O my friends, what are these wonderful numbers about which you are reasoning, in which, as you say, there are constituent units, such as you demand, and each unit is equal to every other, invariable, and not divisible into parts,' - what would they answer?
>
> Glaucon —They would answer, as I should think, that they were speaking of those numbers which can only be realized in thought, and there is no other way of handling them." (*Republic*, 526a).

This means that arithmetic compels the soul to reach the pure truth trough intelligence. Furthermore, Plato considered the idea of *good* to be the ultimate objective of philosophy: "in the world of knowledge the idea of *good* appears last of all, and is seen only with an effort; and, when seen, is also inferred to be the universal author of all things beautiful and right" (526d). In his perspective, to accomplish the ideal of *good*, it is necessary that one study arithmetic and geometry, since they have two important characteristics. First, they invite thought and lead the mind to reflect and, accordingly, they allow the mind to grasp truth. Second, the advanced parts of mathematics and geometry have the power to draw the soul from becoming to beings: the true use of arithmetic. Therefore, the easiest way for the soul to go from becoming is to pursue the study of arithmetic until one is able to see the natures of numbers with the mind only. Moreover, "arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number and repelling against the introduction of visible or tangible objects into the argument" (525c).

In agreement with Plato's considerations, mathematics has a philosophical importance since mathematics is a tool that helps and exercises the mind to think. This training process will lead to a better understanding and to the accomplishment of the idea of *good*, which is the crucial purpose in Philosophy. Indeed, the main philosophical importance of mathematics is the rewarding it may have in one's understanding of the reality. This is possible because mathematical objects are Forms: they can be completely separated from perceptible objects and they exhibit the same traits that philosophical objects. This means that mathematical objects are not grasped by the senses but by the intellect. In *The Republic*, on the one hand, mathematical objects are intelligible and can be known and, on the other, mathematical axioms are accepted as true without further proof or investigation. This view stands that unlike physical objects, mathematical objects do not exist in space and time; rather there are abstract mathematical objects whose existence is independent of the human mind: just as electrons and planets exist independently of us, so do numbers and sets. And just as statements about electrons and planets are made true or false by the objects with which they are concerned, so are statements about numbers and sets. Correspondingly, one may say that mathematical Platonism may be defined as a conjunction of three theses: *Existence*, *Abstractness*, and *Independence*.

3. Existence, Abstractness and Independence

According to Dummett, 1978, Platonism, as a philosophy of mathematics, is founded on a simile: the comparison between the apprehension of mathematical truth to the perceptions of physical objects, and thus of mathematical reality to the physical universe. Gödel (1995), asserts that Platonism is the view that mathematics describes a non-sensual reality, which exists independently both of acts and [of] the dispositions of the human mind and is the only perceived, and probably perceived very incompletely, by the human mind.

Maddy (1990), stresses that realism of Platonism is the view that mathematics is the scientific study of objectively existing mathematical entities just as physics is the study of physical entities. The statements of mathematics are true or false depending on the properties of those entities, independent of our ability, or lack thereof, to determine which. Parsons (1983), sustains that Platonism means here not just accepting abstract entities or universals but epistemological or metaphysical realism with respect to them. Thus, a platonistic interpretation of a theory of mathematical objects takes the truth or falsity of statements of the theory, in particular statements of existence, to be objectively determined independently of the possibilities of our knowing this truth or falsity.

There are several theses within the mathematical Platonism such as (1) Frege's arithmetic-object argument; (2) Quine's; and (3) a response that is commonly referred as full-blooded or platitudinous Platonism[2] (FBP).

The fundamental idea behind FBP is that it is possible for human beings to have systematically and non-accidentally true beliefs about platonic mathematical realm — a mathematical realm satisfying the *Existence*, *Abstractness* and *Independence* — without that realm in any way influencing us or us influencing it. These three thesis are made possible in virtue of (i.) Schematic Reference: the reference relation between mathematical theories and the mathematical realm is purely schematic, or at least close to purely schematic and (ii.) Plenitude: the mathematical realm is very large, it contains entities that are related to one another in all of the possible ways that entities can be related to one another.

Moreover, *Existence*, *Abstractness* and *Independence* seem to be validated by two other thesis (i.) mathematical theories embed collections of constraints on what the ontological structure of a given "part" of the mathematical (ii.) The existence of *any* such appropriate part of the mathematical realm is sufficient to make the said theory true of that part of that realm. In agreement with Roger Penrose,

> Platonic existence, as I see it, refers to the existence of an objective external standard that is not dependent upon our individual opinions nor upon our particular culture. Such 'existence' could also refer to things other than mathematics, such as to morality or aesthetics, but I am here concerned just with mathematical objectivity, which seems to be a much clearer issue...
>
> Plato himself would have insisted that there are two other fundamental absolute ideals, namely that of the *Beautiful* and that of the *Good*. I am not at all adverse to admitting the existence of such ideals, and to allowing the Platonic world to be extended so as to contain absolutes of this nature (Penrose, 2007).

[2] This third response has been most fully articulated by Mark Balaguer (1998) and Stewart Shapiro (1997).

The concept *Existence* in mathematical objects is rather controversial. If this view is true, it will dig up the physicalist idea that the reality is exhausted by the physical and will also put great pressure on many naturalistic theories of knowledge[3], since there is little doubt that the human mind possesses mathematical knowledge. Burgess has defended anti-nominalism. *Anti-nominalism* is, simply, the rejection of nominalism. As such, anti-nominalists endorse ontological commitment to mathematical entities, but refuse to engage in speculation about the metaphysical nature of mathematical entities that goes beyond what can be supported by common sense and science (Burgess 1983, and Burgess and Rosen 1997, 2005). Anti-nominalism, is the conjunction of *Existence* and *Abstractedness*, and its consequences are not as strong as of Platonism. Some views such as intuitionistic, are anti-nominalistic about being a Platonism, for the existence of mathematical objects are true however, these objects depend on or are constituted by mathematicians.

On the *Truth-value realism* perspective[4], Mathematical objects must exist since they are true of all. Mathematical objects have the existence in virtue of its unique and objective truth-value independent of whether we can know it and whether it follows logically from our current mathematical theories. Mathematical objects have a unique and objective truth-value. Expressly, "a sentence proper is a proper name, and its *Bedeutung*[5], if it has one, is a truth-value: the True or the False" (Beaney 1997, 297). This is clearly a metaphysical view, but not an ontological view, as Platonism, since truth-values are not committed to the flow from an ontology that demands *Existence*.

For another hand, there is the *working realism* methodological view that entails mathematics should be practiced as if Platonism was true (Bernays, 1935, Shapiro, 1997), considering that working realism is first and foremost a view within mathematics itself about the correct methodology of the discipline, and Platonism, rather, is a philosophical view. Nevertheless, as reported by Hoystein (2013), this two are very related theories since, assuming Platonism is true, then (1) language of mathematics is a classical first-order language; (2) provided it is legitimate to reason classically about any independently existing part of reality, classical rather than intuitionistic logic; (3) since Platonism ensures that mathematics is discovered rather than invented, there would be no need for mathematicians to restrict themselves to constructive methods and axioms, which establishes non-constructive methods; (4) impredicative definitions are legitimate whenever the objects being defined exist independently of our definitions, and this is how we assure the impredicative definitions; (5) if mathematics is about some independently existing reality, then every mathematical problem has a unique and determinate answer, which provides at least some motivation for Hilbertian optimism. As we can see, the truth of mathematical Platonism has important consequences within mathematics itself.

On this matter, Frege developed an argument — the *Fregean argument* — which is based on two premises: (1) The singular terms of the language of mathematics purport to refer to mathematical objects, and its first-order quantifiers purport to range over such

[3] In the nominalist perspective there exist no abstract or universal objects: this position denies the existence of abstract objects – objects that do not exist in space and time.
[4] The notion of a *truth value* has been explicitly introduced into logic and philosophy by Gottlob Frege—for the first time in 1891, and most notably in his seminal paper (1892).
[5] *meaning*

objects, this is, *S* to be true, must succeed in referring or quantifying, a classical semantics perspective; (2) most sentences accepted as mathematical theorems are true (regardless of their syntactic and semantic structure). By *Classical Semantics* and *Truth*, Frege considers that some simple numerical identities are objectively true because such identities allow the application of natural numbers in representing and reasoning about reality, especially the non-mathematical parts of reality. Other versions of Fregean argument are sometimes stated as the notion of ontological commitment, such as the Quine's Criterion[6].

On the other hand, *Abstractness* asserts that a mathematical object is said to be abstract just in case it is non-spatiotemporal, and therefore causally inefficacious. In Plato's thought, this distinction embodies the distinction between Forms, and Sensibles. In the seventeenth century, Locke's idea of an abstract object — as one that is formed from concrete ideas — was rejected by Berkeley and then by Hume. However, even for Locke there was no suggestion that the distinction between abstract ideas and concrete corresponded to a distinction among objects. In the twentieth century, Frege insisted that the objectivity and aprioricity of the truths in mathematics entail that numbers are neither material beings nor ideas in the mind. In The *Foundations of Arithmetic* (1974), Frege concludes that numbers are neither external "concrete" things nor mental entities of any sort. Later in *The Thought* (1952) he asserts that thoughts belong to a "third realm". Similar claims have been made by Bolzano, and later by Brentano, and his pupil Husserl.

In most recent attempts to distinguish concrete objects from abstract, Putnam (1975) makes the case for abstract objects on scientific grounds. Bealer (1993) and Tennant (1997) present *a priori* argument for the necessary existence of abstract entities. The dispute over the existence of *abstracta* is reviewed in Burges and Rosen (1997). A general theory of abstract objects is developed in Zalta (1983, 1999).

Independence is less evident than the other two claims. What does it mean for an object to be independent? Does it mean it is self-representative? An object seems to be independent when what it represents does not depend on any intelligent agent, or an agent's thought, reason, cognition or representation.

So far one may say that we have understood how mathematical objects assure their *Existence* and their *Abstractedness*. However, are mathematical entities sufficiently true to claim independence? Is it possible to entail mathematical objects as an objective concept? If *Independence* means that (1) mathematical objects are mind-independent; that (2) reality includes objects not subject to intentionality; that (3) to be referred requires a definition of truth; and that (4) there is only one correct description of the reality, then the trivial forms of Platonism are likely to satisfy the claim, and thus qualify Platonism with the property of *Independence.*

On this account, there are at least some objects, in reality, that could be perceived by the human mind via definition. On this assumption, an object is said to be independent when it is self-representative. Furthermore, we may also think of some other examples of self-representation. The Epimenides' paradox "All Cretans are liers" therefore, "I am a liar", *ergo* "This statement is false". This paradox relies on some form of self-reference. Any language

[6] A first-order sentence (or collection of such sentences) is ontologically committed to such objects as must be assumed to be in the range of the variables for the sentence (or collection of sentences) to be true.

capable of expressing some basic syntax can generate self-referential sentences. A language containing a truth predicate and this basic syntax will thus have a sentence L such that L implies $\neg Tr(\ulcorner L \urcorner)$ and vice versa.

Also Gödel's incompleteness theorems use mathematical reasoning in exploring mathematical reasoning itself. What the theorem states and how it is proved are, as a matter of fact, two different things. The theorem asserts that all consistent axiomatic formulation of number theory includes *undecidable* propositions. In the same way that Epimenides' paradox is a self-referential, also Gödel's axiom is a self-referential mathematics statement[7]. In Gödel–numbering, numbers are made to stand for symbols and sequences of symbols. Transporting the Epimenides' Paradox into number-theoretical formalism, the Epimenides' Paradox does not say

This statement of number-theory is false.

but that

This statement of number-theory does not have any proof.

We could agree that proofs are demonstrations within fixed systems of propositions. However, this statement of number theory does not have any proof in the system. Gödel's sentence is *unprovable* within the system, for there are true statements of number theory, which its methods of proof are too weak to demonstrate in the system; therefore, the system is incomplete. In fact, what Gödel showed was that provability is a weaker notion than truth no matter what axiomatic system is involved, and therefore, no fixed system could represent the complexity of the whole numbers[8]. Actually, if a system, such as the one defined in *Principia Mathematica*, is (i.) consistent, this is, *contradiction free*; and (ii.) complete, this is, every true statement of number theory could be derived within the framework drawn up in the P.M., how would it be possible to justify the methods of reason on the bases of that same methods of reasoning? If such a proof could be found using only methods inside a system then, the system itself would be inconsistent.

As we discussed on the first part of this essay, Plato considered the knowledge of arithmetic a *conditio* for philosophical knowledge since both require universal truths, accessible to the human mind by acquaintance to the incorporeal intelligible realm, — in opposition of *doxai*, which belong to a sensible sphere, — in resolving the problem of reality, knowledge and human existence.

Plato, in his considerations on arithmetic and geometry assesses epistemological issues, such as what and how the mind know things; and metaphysics, on their ontological status as things. As we discussed, a mathematical proof is a mathematical object that qualifies to the properties of *Existence*, *Abstractedness* and *Independence*. A proof is, a consequence of human reasoning, written in human language for human consumption. What is still left to

[7] For more detail on this issue, see Hofstadter (1979).
[8] Cf. Nagel and Newman, (2001).

discuss is whether a computer assisted proof — a proof that results from an artificial intelligent agent — may or may not qualify to those properties.

3. Modern Theorem Provers

A computer assisted proof is a proof in which every logical inference has been checked all the way back to the fundamentals axioms of mathematics (Hales, 2008). It is written in a precise artificial language that admits only a fixed repertoire of stylized steps (Harrison, 2008).

Proof assistants or *computer theorem provers* are the artificial intelligent agents that mechanically verify, in a formal language, the correctness of the proof previously demonstrated by the human mind. In fact, with this artificial system, the user is allowed to set up a mathematical theory, define properties and realize logical reasoning (Geuvers, 2009).

Nowadays, there is a large number of computer provers, that can check or construct computer assisted proofs, such as the HOL Light for classical and higher order logic, based on a formulation of *type theory*; and Coq. The theorem provers allow the expression of mathematical assertions; mechanically checks proofs; helps to find computer assisted proofs; and extracts a certified program from the constructive proof of its formal specification. Other theorem provers are Mizar, PVS, Otter/Ivy, Isabelle/Isar, Affa/Agda, ACL2, PhoX, IMPS, Metamath, Theorema, Lego, Nuprl, Ωmega, B method and Minlog.

According to Wiedijk (2006), theorem provers:

- are designed for the formalization of mathematics, or, if not designed specifically for that, have been seriously used for this purpose in the past;
- are special at something. These are the systems that in at least one dimension are better than all the other systems in the collection. They are the leaders in the field.

In recent years, several theorems have been formally verified by an artificial intelligent agent. There are many to choose from, however, some significant ones are:

(1) The *Four Colour Theorem*, states that, given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. The theorem was proven in the late 19th century (Heawood 1890); however, proving that four colors suffice turned out to be pointedly harder. A number of false proofs and false counterexamples have appeared since the first statement of the four color theorem in 1852. Nevertheless, it was proven, using a computer, in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved using a computer**.** In 2005, Benjamin Werner and Georges Gonthier formalized a proof of the theorem inside the Coq proof assistant.

(2) The *Jordan Curve Theorem*, states that every simple closed curve in the plane separates the complement into two connected nonempty sets: an interior region and an exterior. The Jordan curvetheorem is named after the mathematician Camille Jordan, who found its first proof[9]. In 1905, O. Veblen declared that this theorem is "justly regarded as a most important step in the direction of a perfectly rigorous mathematics". According to Courant and Robbins, "The proof given by Jordan was neither short nor simple, and the surprise was even greater when it turned out that Jordan's proof was invalid and that considerable effort was necessary to fill the gaps in his reasoning". According to Hales (2007), Jordan's proof is essentially correct... Jordan's proof does not present the details in a satisfactory way. But the idea is right, and with some polishing the proof would be impeccable. In 1978, Dostal and Tindell in their essay "The Jorden Curve Theorem Revisited", wrote "however, notwithstanding substantial simplifications achieved in the elementary proof of JCM by these and other authors, the theorem has remained and will probably always remain, difficult to establish by purely elementary means". This proof was formalized, in 2005, using HOL Light.

(3) The *Odd Order Theorem* precisely says that groups that have an odd number of elements are solvable. Finite groups can be factored somewhat like integers, though for a more complicated multiplication. The basic group factors, called simple groups, also come in many more shapes than the basic integer factors, the prime numbers. Solvable groups, however, can be factored down to primes, like integers. They're called this because they correspond to solvable polynomial equations. Feit and Thompson classically demonstrated this proof in 1963. Recently, it was formalized, in 2012, using *Coq* System.

Discussion

The history of mathematical reasoning began with the attempt to mechanize the thought processes. Aristotle codified syllogisms and Euclid codified geometry. Frege and Peano worked on combining formal reasoning with the study of sets and numbers. David Hilbert worked on stricter formalizations of geometry than Euclid's. All of these efforts were directed towards clarifying what one means by "proof".

In 1940, giant electronic brain catalysed the convergence of three previously disparate areas: theory of axiomatic reasoning, the study of mechanical computation and Psychology of intelligence. Since then, there has been a restless progress of in computer science and artificial agency. What does this has to do with Plato's theory of arithmetic explained previously in this essay? Following his that perspective, a proof is a statement about the reality that satisfies the condition of *Existence*, *Abstractness* and *Independence.* Mathematical objects exist objectively; this means they (1) are not subject to spatiotemporal causality and are intentionality independent; and (2) that to be referred they demand a definition of truth, which is the one and only correct description of reality.

Modern theorem provers seem to satisfy all of the above conditions in the accomplishment of a computer assisted proof. The great enterprise of Artificial Intelligence is to find out what sort of rules could possibly capture intelligent reasoning. If we assume a

[9] For decades, mathematicians generally thought that first rigorous proof was carried out by Oswald Veblen. However, this notion has been challenged by Thomas C. Hales and others.

platonistic perspective, the machine seems to be much able to perform an arithmetic proof. The following table aims to be self-explanatory on that matter:

Formal System	Classical Mathematical
Interactive Theorem Proving	The Human Mind
Symbolic language	Natural language
Computer assisted proof	Classical proof
Inanimate	Animate
Inflexible	Flexible
Mind Independent	Mind depended
Strings of symbols	Statements
Produced by typographical rules	Theorems are proven
Not subject to Spatiotemporal causality	Subject on spatiotemporal causality
Objectivity	Subjectivity
No intuition	Intuition
So trivial, beyond reproach	“Feels right” — not always provable
Astronomical size	Complexity

According to what we asserted about this issue, we may accurately acknowledge that the inferential result or consequence of a computer assisted proof, satisfies Plato’s conditions to intelligibles, since a computer assisted proof’s satisfies the preconditions:

(i.) $\exists x Mx$ (Existence).

(ii.) Non-spatiotemporal and (therefore) causally inefficacious (Abstractedness).

(iii.) Independent of intelligent agents and their language, thought, and practices (Independence).

Conclusions

For the reasons stated, we may recognize that the conditions for an agent to perform a calculus of reason are pretty much achieved. In fact, a programed inanimate, intelligent agent with symbolic language is able to demonstrate a proof; a computer assisted proof witch uses strings of symbols produced by typographical rules. From a symbolic language, an inanimate, inflexible, and mind-independent agent in the end accomplishes a proof that is nor subject to spatio-temporal causality. It is an objective, non-intuitive proof that is so trivial, it is beyond reproach. This computer assisted proof is, accordingly, a rebellion “against the introduction of visible or tangibles objects into the argument”.

The result of such computer assisted proof is an existent, abstract and independent self-representative proof. A mathematical object whose existence is independent of the human mind: the objective mathematical entity, Plato very much aspired for.

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