

1-2017

## Examining the interaction of mathematical abilities and mathematical memory: A study of problem-solving activity of high-achieving Swedish upper secondary students

Attila Szabo

Paul Andrews

Let us know how access to this document benefits you.

Follow this and additional works at: <https://scholarworks.umt.edu/tme>

 Part of the [Mathematics Commons](#)

---

### Recommended Citation

Szabo, Attila and Andrews, Paul (2017) "Examining the interaction of mathematical abilities and mathematical memory: A study of problem-solving activity of high-achieving Swedish upper secondary students," *The Mathematics Enthusiast*: Vol. 14 : No. 1 , Article 10. Available at: <https://scholarworks.umt.edu/tme/vol14/iss1/10>

This Article is brought to you for free and open access by ScholarWorks at University of Montana. It has been accepted for inclusion in The Mathematics Enthusiast by an authorized editor of ScholarWorks at University of Montana. For more information, please contact [scholarworks@mail.lib.umt.edu](mailto:scholarworks@mail.lib.umt.edu).

**Examining the interaction of mathematical abilities and mathematical memory: A study of problem-solving activity of high-achieving Swedish upper secondary students**

Attila Szabo<sup>1</sup>, Paul Andrews

Stockholm University

*In this paper we investigate the abilities that six high-achieving Swedish upper secondary students demonstrate when solving challenging, non-routine mathematical problems. Data, which were derived from clinical interviews, were analysed against an adaptation of the framework developed by the Soviet psychologist Vadim Krutetskii (1976). Analyses showed that when solving problems students pass through three phases, here called orientation, processing and checking, during which students exhibited particular forms of ability. In particular, the mathematical memory was principally observed in the orientation phase, playing a crucial role in the ways in which students' selected their problem-solving methods; where these methods failed to lead to the desired outcome students were unable to modify them. Furthermore, the ability to generalise, a key component of Krutetskii's framework, was absent throughout students' attempts. These findings indicate a lack of flexibility likely to be a consequence of their experiences as learners of mathematics.*

Key words: mathematical ability; non-routine problem solving; Krutetskii; mathematical memory; abstraction; generalization; high achieving students; Swedish upper secondary

## **INTRODUCTION**

Typically based on arguments rooted in equity and social justice, much research has focused on the education of low-achieving mathematics students (e.g., Swanson & Jerman, 2006). However, with respect to framing this paper, relatively little attention has been focused on high-achieving students (Leikin, 2014). In this respect, a few studies have addressed the mathematical abilities invoked during problem-solving (Brandl, 2011; Vilkomir & O'Donoghue, 2009) but even fewer have considered the connection between students' memory functions and their mathematical performance (Leikin, Paz-Baruch, & Leikin, 2013; Raghubar, Barnes, & Hecht, 2010). Indeed, no study since Krutetskii (1976) has examined the role of the *mathematical memory* in the context of able students' problem-solving activities.

## **BACKGROUND**

### **Krutetskii's (1976) mathematical abilities**

Understanding students' mathematical abilities has engaged researchers for more than a century. For example, more than one hundred and twenty years ago, Calkins (1894) concluded, with respect to Harvard students, that mathematicians have concrete rather than verbal memories, that there are no differences in ease in memorisation between mathematicians and other students and that, when doing mathematics, there is no significant difference between men and women (Calkins, 1894). Between that time and Krutetskii's (1976) study, psychometric approaches presented mathematical abilities as not only innate

---

<sup>1</sup> attila.szabo@stockholm.se

but fixed, viewpoints that are now not only rejected but which have created a research context in which words like ability have been construed negatively (Adey, Csapó, Demetriou, Hautamäki, & Shayer, 2007), an issue we aim to redress.

As indicated above, a major contribution to our understanding of mathematical abilities emerged from the Soviet psychologist Krutetskii's (1976) longitudinal study of around 200 pupils of differing mathematical achievement levels. His analyses of their problem-solving activities led him to construe mathematical ability as a complex phenomenon comprising four components:

- 1) The ability to *obtain mathematical information* (i.e. formalized perception of mathematical material),
- 2) The ability to *process mathematical information* (i.e. logical thought, rapid and broad generalization of mathematical objects, relations and operations, the ability to curtail the process of mathematical reasoning, flexibility in mental processes, striving for clarity and simplicity of solutions),
- 3) Retaining *mathematical information* (i.e. *mathematical memory*, which is a generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs and methods of problem-solving) and
- 4) A *general synthetic component*, described as a “mathematical cast of mind”.  
(Krutetskii, 1976, pp. 350–351)

For Krutetskii, all students have the propensity to develop these abilities through an engagement with appropriate mathematical activities, although “the specific content of the structure of abilities largely depends on teaching methods, since it is formed during instruction” (Krutetskii, 1976, p. 351). This latter point is particularly important in respect of the distinction between high-achieving and gifted students. For example, while Krutetskii construes mathematical giftedness as an “aggregate of mathematical abilities that opens up for successful performance in mathematical activity” (ibid, p. 77), he does not distinguish between high-achieving and gifted students, because, as is now widely accepted, many of the qualities of those who might once have been described as mathematically gifted can be taught (Leikin, 2014; Øystein, 2011; Usiskin, 2000). Finally, considering that Krutetskii's data were obtained in the 1960s it is natural to ask, five decades on, whether his framework retains its currency. In this respect, we note that researchers have worked with, and continued to develop, his framework throughout this period, as manifested in the work, presented chronologically to emphasise its temporal spread, of Wagner and Zimmermann (1986), Schoenfeld (1992), Garofalo (1993), Sheffield (2003), Sriraman (2003), Heinze (2005), Vilkomir and O'Donoghue (2009), Deal and Wismer (2010), Leikin (2010), Juter and Sriraman (2011).

### **Krutetskii and memory in mathematics**

It is generally accepted that while memory plays a key role in all aspects of mathematical activity (Leikin et al., 2013; Raghobar et al., 2010), the “crucial question is not whether memory plays a role in understanding mathematics but what it is that is remembered and how it is remembered by those who understand it” (Byers & Erlwanger, 1985, p. 261). In this respect, Krutetskii (1976) argues that mathematical memory, which has a higher function than the recollection of multiplication tables or algorithms, concerns mathematical relationships, schemes of arguments, proofs and methods of problem-solving (ibid, p. 300). Additionally, mathematical memory, like all other mathematical abilities, develops within mathematical activities and, with respect to the able student, is selective and

“retains not all of the mathematical information that enters it, but primarily that which is ‘refined’ of concrete data and which represents generalized and curtailed structures. This is the most convenient and economical method of retaining mathematical information” (Krutetskii, 1976, p. 300).

Thus, able students retain the contextual information of a problem only during the problem-solving process and, even several months later, can recall the *general method* they used when solving it (Krutetskii, 1976). Such competence is rarely observed in young children, for whom “the relevant and the irrelevant, the necessary and the unnecessary are retained side by side in their memories” (ibid, p. 339). Thus, it is not surprising that mathematical memory, at least as construed by Krutetskii, is easier to observe in mathematically able or older students than in young or low achieving students.

### Mathematical memory according to cognitive theories and neuroscience

According to cognitive neuroscientists, a major function of the human brain is to relate all new information to previous knowledge and experience (Buckner & Wheeler, 2001; Ingvar, 2009; Shipp, 2007). This process relies on the distinctive processes of different parts of the human memory system (Davis, Hill, & Smith, 2000; Olson et al., 2009; Squire, 2004), which, drawing on various sources (e.g. Moscovitch, 1992; Nyberg & Bäckman, 2009; Squire, 2004), can be summarised as in Figure 1.

The system shown in Figure 1 is largely categorized by the length of time information remains in the different partitions and the most fundamental distinction is between short-term and long-term memory, with the function of the former primarily given to working memory. In the working memory takes place a continuous and cyclical process, that includes mathematical problem-solving, focused on discovering new information or, if needed, updating and reforming existing knowledge.

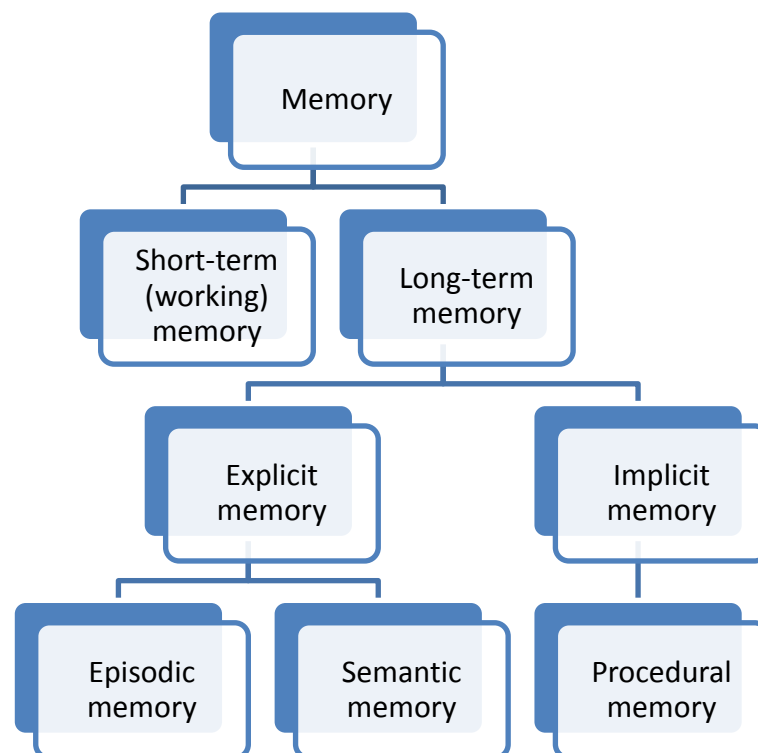


Figure 1: Memory systems – a cognitive model

Long-term memory has two subcategories, depending on the nature of the information stored. The implicit memory stores information about procedures, algorithms and patterns of movement that can be activated when certain events occur (Squire, 2004). It manages information on *how* something is done rather than *why* and is an automated and unconscious response to particular stimuli. Conversely, explicit memory stores information about experiences and facts which can be consciously recalled and explained (e.g. Davis et al., 2000; Gärdenfors, 2010).

Thus, with respect to the learning of mathematics and mathematical problem solving, implicit memory manages the procedures and algorithms that are, by means of *automaticity*, transferred to working memory during the problem-solving process. The development of automaticity, which is an effective but inflexible process that does not interfere with other functions of the working memory, is one of the main goals of the cognitive system (Ingvar, 2009). The overtly conscious role of explicit memory is associated with the ability to create and use mental schemas for problem-solving. Thus, the solving of mathematical problems takes place in working memory and supported by the automatic recall of routine knowledge held in implicit memory and the conscious recall of problem-solving strategies held in explicit memory.

In the above context, it is important to note that Krutetskii's (1976) definition of mathematical memory excludes the recollection of numbers, algorithms and table skills, which, from a perspective of cognitive neuroscience, are automated processes. Consequently, it is not unreasonable to construe mathematical memory – the generalized mathematical methods and relationships altogether with mental schemes for problem-solving – as part of explicit memory. This conjecture is supported by evidence that mathematically able pupils are capable of explaining their use of generalized methods during problem-solving (Davis et al., 2000; Krutetskii, 1976).

Finally, recent studies have identified an interaction between working memory, general giftedness and high achievement in mathematics (Leikin et al., 2013). In particular, a review of developmental and cognitive perspectives on the relationship between working memory and mathematics suggests that some memory processes facilitate learning and that particular aspects of working memory performances are specific to early mathematical learning (Raghubar et al., 2010).

### **Problem solving in mathematics**

Finally, as our interest is in the relationship between students' mathematical abilities and mathematical memory, as defined by Krutetskii (1976), as they solve mathematical problems it is important that both mathematical problem and problem solving are appropriately defined.

With respect to mathematical problems, it is generally accepted that a mathematical problem offers an objective with no immediately obvious means of achievement (Pólya, 1957; Nunokawa, 2005), implying that problem complexity is a function of the problem solver's knowledge, experience and dispositions and not the task itself (Schoenfeld, 1985; Carlson & Bloom, 2005). Thus, as Krutetskii (1976) acknowledges, a familiar task, by dint of that familiarity, is not a problem. The presentation of a problem can vary, although typically it takes one of two forms; it can be located solely within mathematics itself or situated in some representation of the real world (Blum & Niss, 1991; Haylock & Cockburn, 2008). The latter problems are typically set in text and represent a common context within both school mathematics and international assessments like PISA. However, despite being construed similarly throughout the mathematics education literature, they have been prone to variation

in their curricular form and function. For example, they have been used with one-step problems to encourage students' recognition of linguistic structures in the development of solution strategies for solving analogous problems (Bassok, 2001; Philippou & Christou, 1999). More authentically, they have been used to introduce new mathematical content by presenting students with problems for which they have no strategies beyond trial and improvement (Andrews & Sayers, 2012). Finally, in accordance with curricular aims internationally, they have been used to encourage the development of students' problem-solving competence (Andrews & Xenofontos, 2015; Brehmer, Ryve, & Van Steenbrugge, 2015; Palm, 2008). This latter form of word problem, the form we deploy in this paper, is not straightforward as it requires not only the linguistic competence to decode the text but also the ability to extract relevant data, select and implement appropriate operations before interpreting the outcomes against the original context (Nesher, Hershkovitz, & Novotna, 2003; Jitendra, Griffin, Deatline-Buchman, & Sczesniak, 2007; Vilenius-Tuohimaa, Aunola, & Nurmi, 2008). However, such problems have not always had the desired effect, with students often suspending any sense-making in their desire to answer even nonsensical problems (Greer, Verschaffel, Van Dooren, & Mukhopadhyay, 2009).

Problem solving "is an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine" (Cai & Lester, 2005, p. 221). Various frameworks have been proposed for analysing and describing the components of such activity, the best known of these, focused on supporting the teaching of problem-solving skills, is Pólya's (1957) *understand* the problem, *devise a plan*, *implement* the plan, and *reflect*, a process that is still found to encourage the development of problem-solving competence (Hensberry & Jacobbe, 2012). Subsequent problem-solving frameworks have typically been refinements of Pólya's work; such as Mason, Burton and Stacey's (1982) three-phase, Kapa's (2001) six-phase or Nunokawa's (2005) three-phase models. Others, such as the four component – orientation, organisation, execution and verification – model proposed by Garofalo and Lester (1985), have been explicitly meta-cognitive in their descriptions of the problem-solving process, as has Singer and Voica's (2013) four phase process in which one *decodes* the problem, *represents* the problem in appropriate mathematical forms, *processes* what one already knows to test the model currently under scrutiny and, finally, *implements* a solution strategy. Such frameworks not only show much similarity but are also sufficiently general to accommodate the distinction between mathematical and application problems (Haylock & Cockburn, 2008).

## THE STUDY

As indicated above, the aim of this study was to investigate the interaction of students' mathematical abilities and mathematical memory, as defined by Krutetskii (1976), as they solved mathematical problems. Such aims are not straightforwardly realised because problem-solving tasks are expected to uncover the mathematical competences necessary for solving them rather than the recall of previously solved problems. In other words, tasks need to minimise the impact of analogical reasoning invoked by what Nogueira de Lima and Tall (2008) have described as 'met-befores'. Moreover, good problems should be challenging, attainable and allow for different solution strategies (Hiebert et al., 2003; Roberts & Tayeh, 2007).

### The tasks

To accommodate different learner preferences with respect to problem types (Juter & Sriraman, 2011; Krutetskii, 1976), two mathematical problems, one geometric and one algebraic, were devised. Both problems, shown below, underwent substantial a-priori testing

with a group, different from the experimental group discussed below, of high-achieving students, which confirmed their suitability for the study.

**Problem 1:** In a semicircle, as shown in Figure 2, are drawn two additional semicircles. Is the perimeter of the large semicircle longer, shorter or equal to the sum of the perimeters of the two smaller semicircles? Justify your answer.

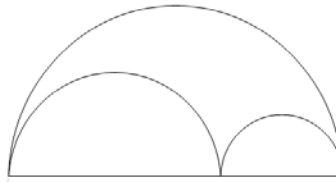


Figure 2: The diagram associated with Problem 1

**Problem 2:** Mary and Peter want to buy a CD. At the store they realise that Mary has 24 SEK less and Peter has 2 SEK less than the price of the CD. Even when they put their money together they would not be able to afford the CD. What is the cost of the CD and how much money has each person?

### **Participants**

The participants were students, aged 16–17 years, in the first year of post-compulsory school in Sweden. They were following an advanced mathematics programme, admission to which was based on their performance on a mathematical entrance examination, which situated them among the top 5% of students nationally. They were considered to be high-achieving in that all had participated in an acceleration programme while in compulsory school and completed the first course of post-compulsory school mathematics with the highest possible grades.

Prior to the investigation reported below and in order to familiarise students with the study, the first author spent approximately 30 hours, over a period of four months, as a participant observer in their mathematics classroom. During this time he interacted with students as they solved problems. Through this students came to see him as a mathematical friend rather than a teacher and, importantly, came to trust him as an observer of how they solved their mathematical problems. At the end of this process, and in consultation with the teacher, six students, three boys and three girls, were invited to participate in the study reported here.

### **Clinical interviews**

Students were invited to participate in individual clinical interviews, which have three primary purposes (Ginsburg, 1981). The first is the discovery of cognitive processes, the second the specification of those processes, which “is usually a complex inferential process” (Ginsburg, 1981, p. 6) and the third is the “evaluation of levels of competence”. Moreover, as in the study we report below, “the distinctions among these aims may be blurred, and more than one aim may be involved” (Ginsburg, 1981, p. 5). Such interviews typically begin with interviewees being invited to solve mathematical tasks in a think-aloud manner, being prompted to describe the steps taken and the reasons behind them.

In this study, students were interviewed in a private room close to but separate from their ordinary classroom. First Problem 1 was posed and then, after a short break, Problem 2. During this time students were encouraged not only to write down as much as they could but also ‘think aloud’ whenever possible. When appropriate, the interviewer posed supplementary questions to facilitate the process and students were encouraged to take as much time as they felt was necessary. To minimise the participants’ influence on each other,

all interviews were undertaken during a single day. No participant needed longer than 14 minutes to solve either problem. Finally, at the end of the problem solving element of the interview students were asked to reflect on their two problem solving attempts. This was undertaken to allow, as far as is practicable, additional opportunities for students to articulate their mathematical thinking (Øystein, 2011). All students' writings and utterances were captured by an electronic pen that makes digital records of both writing and speech.

### Data analysis

Prior to data collection, the two tasks used in this study were piloted on a class of students different from that to which our six participants belonged. Analyses of their responses indicated the following. Firstly, as might be expected, Krutetskii's *general synthetic component* or 'mathematical cast of mind' was unobservable in students' problem-solving behaviours and excluded from the analysis. Of the remaining three broad abilities, the abilities to *obtain mathematical information* and *retain mathematical information* were defined as originally by Krutetskii. However, our view was that the ability to *process mathematical information* was better construed as two broad abilities. That is, we separated those elements focused on meta-cognitive or regulative abilities from those focused on the processes of mathematical generalisation, not least because Krutetskii himself saw the latter a key component of mathematical understanding, arguing that understanding depends on a "rapid and broad generalization of mathematical objects, relations and operations" (Krutetskii, 1976, p. 341). These four broad abilities can be seen below.

(O) The ability to *obtain and formalize mathematical information*

e.g. the student is thinking of, draws or notes the structure of the problem or determines the relationship between the entities and the variables from the context of the problem

(P) The ability to *process mathematical information*

e.g. the student performs well-known methods for problem-solving by using logical, systematic and sequential thinking

(G) The ability to *generalize mathematical objects, relations and operations*

e.g. the student transforms numerical approaches into operations with conventional symbols and thereby obtains a general solution of the given problem

(M) The ability to *retain mathematical information, i.e. mathematical memory*

e.g. the student remembers a method for calculating the area of a geometric shape or recalls that a given task can be written as an equation or an inequality.

As mentioned, the digital recording of the problem-solving activities resulted in an exact linear reproduction of the participants written solutions, drawings and verbal utterances. This was especially useful when performing qualitative content analysis of the empirical material, inspired by Graneheim and Lundman (2004) and van Leeuwen (2005). Each episode that lasted at least one second was scrutinised for evidence of those abilities that could be observed in the written solutions, drawings and verbal utterances. In this manner, some episodes were multiply-coded.

### RESULTS

The clinical interviews revealed that students had not seen the problems previously and that the tasks, fulfilling one of the major requirements for the study, were non-routine and challenging. Taking a broad sweep of the data for each student's problem-solving attempts, it



became apparent that each fell into three main phases, which occurred chronologically, that structure what follows.

### *The orientation phase*

The first, the *orientation* phase, occurred at the start of the problem-solving activity and shows students exploiting abilities related to obtaining and formalising mathematical information as well as mathematical memory. By way of example, Table 1 shows the data for Heather's orientation phase on Problem 1.

<b>Time</b>	<b>Digital recording</b>	<b>Transcription of observations and interviews Heather</b>	<b>Observed abilities</b>
0:00	<i>sits quietly, doesn't draw nor write anything</i>	<i>from the interview</i> I: And what were you, what were you thinking about in the beginning? Heather: Hum (...) how to calculate the area of these kinds of figures. I: The perimeter. Heather: Yes, the perimeter, I mean perimeter. Yes, if you could see any connection type. Yes, I needed one (...) common variable or just variable.	O, M
0:25		<i>from the observation</i> Heather: Ah, this is the large one (...) and this length. Thus, this length, so it is (...) its perimeter.	O
0:48		<i>from the observation</i> Heather: The perimeter is around the whole...	O, M
0:54		<i>from the observation</i> Heather: Hum (...) okay.	O, M
1:22		<i>from the observation</i> Heather: How is one supposed to calculate the perimeter?	M
1:30	$d \cdot \pi$		O, M

Table 1: Heather's orientation phase for Problem 1

As can be seen, Heather began her solution attempt by saying and apparently doing nothing for 25 seconds. However, during her clinical interview she recalled that during this period she was thinking about how to calculate the perimeter of the semicircles. Thus, while the observational evidence appeared to yield little, her interview showed, in her thinking about the perimeter of circles and the procedure for calculating them, evidence of her obtaining (O) mathematical information and exploiting her mathematical memory (M). In the following episodes, which started at 0:25 and ended after 1:30, observational data showed her thinking aloud, albeit in a hesitant and initially unsure manner, about the mathematical relationships connecting the perimeter of a circle and its diameter. Finally, she wrote down the formula for the perimeter of a circle. As with the first episode, throughout these episodes she exhibited

both the ability to identify information (O) and exploit mathematical memory (M). Like Heather, all students began their solution processes by reading the problem and thinking quietly for a while before writing or drawing something. Thus, it is not unreasonable to assume that this *orientation* phase may be a characteristic of able students; that is, they do not expect to start to solve a problem before they have obtained and formalised its embedded mathematical information and recalled what they believe are the appropriate methods.

### *The processing phase*

The *orientation* phase was directly followed by a phase in which students *processed* the mathematical information formalised earlier. That is, they begin to undertake mathematical operations, think logically and apply mathematical reasoning. In this respect, the data in Table 2, showing Larry's *processing* phase, exemplify this well with respect to Problem 2:

Time	Digital recording	Transcription of observations and interviews Larry	Observed abilities
1:22	Mary = $x - 24$ Peter = $x - 2$	Larry: The CD has a price and we can call that price $x$ , we write the price of the CD (...) then Mary has (...) Oh, $x$ minus 24. And then Peter has $x$ minus 2.	O
1:42	$(x - 24) + (x - 2) =$ $2x - 26 = ? < x$	Larry: And $x$ minus 24, oh, plus $x$ minus 2, oh, it will not be $x$ , but if, it might be. Hmm (...) it will be $2x$ minus 26 (...) Larry: And it will be (...) something less than $x$ . I just put a question mark here (...)	P
2:35  3:50	Mary < 2  25 – 25.99 24 < $x$ < 26	Larry: But, well, it feels as if Mary has 24 less and Peter has 2 less than the CD's price and even if they put their money together (...) This means of course that Mary cannot have more than 2 SEK. Since she could not even add up with Peter's (...) Hum (...) 2 SEK, then Mary has less than 2 SEK. Larry: Eh...Thus... the CD cost from 24 to maybe 25.99. It may cost between 24 and... It may not cost 26... It must cost ... 26 is greater than $x$ , and $x$ is greater than 24.	P

Table 2: Larry's processing phase at Problem 2

As can be seen from Table 2, Larry engaged in considerable thinking out loud while processing information. In the first episode he formalised (O) the structure of the problem, by using  $x$  to represent the unknown, which in this case was the cost of the CD. In so doing his notation, Mary =  $x - 24$ , was mathematically poor but his comments indicated a clear understanding as to his mathematically correct intentions. This was followed (at 1:42) by his constructing an inequality by adding and then simplifying the two expressions. In this case, his oral comments showed a clear understanding of both his goal and his mathematical representation of the problem, which he was able to manipulate and simplify (P). Later, at

2:35, despite what seemed like uncertainty with respect to the formal solving of the inequality, he argued in a logical, systematic and sequential way to obtain an accurate solution (P).

### *The checking phase*

The third phase was situated at the very end of the problem-solving activities, when students were observed to *check* their methods and solutions. In this respect, Table 3 shows how Linda checked her solution to the second problem.

Time	Digital recording	Transcription of observations and interviews Linda	Observed abilities
4:38		Linda: We know that it has to be possible to take off 24 SEK, because Mary has money.	P
4:52	1 25 23	Linda: She has in fact 1 SEK. The price of the CD is 25 SEK. Peter has 23 SEK.	P
5:24		Linda: It is all right, 25 is less than 26.	P

Table 3: Linda's checking of Problem 2

As seen in Table 3 there is more evidence of Linda thinking aloud than of her writing. At 4:38 she talks of the need, to account for Mary's conditions, to be able to subtract 24 SEK from the cost of the CD. At 4:52 she then concludes, having already shown that  $x < 26$ , that Mary must have just one SEK and Peter, therefore, must have 23 because the cost of the CD must be at least 25 SEK. Throughout, she is articulating her mathematical reasoning (P), concluding that "it is all right, 25 is less than 26".

These three phases were observed in every problem-solving activity, leading us to conclude that:

- Problem-solving activity typically begins with an *orientation phase* which engages both the ability to *obtain* and *formalize mathematical information* and *mathematical memory*. These abilities are intimately interrelated and, with the methods used in the present study, it is difficult to differentiate them.
- Following the *orientation phase* is a *processing phase* during which students typically draw on the ability to *process mathematical information*. During this phase can often be observed *logical thought*, *flexibility* in mental processes, *striving for clarity* and *simplicity* of solutions, and the ability to *curtail mathematical reasoning*.
- Every activity ends with a *checking phase* of *processing mathematical information*. At this stage, students check the appropriateness of their problem-solving methods and solutions by a re-actualisation of the problem context.

### **What happens if the selected method does not lead to the desired outcome?**

In ten of the twelve cases, evidence indicated that when their first chosen method failed to lead directly to a solution, and despite evidence of personal stress, students typically returned to the *orientation phase*. That is, they began a second round of formalizing the mathematical information and using their mathematical memory, a phase that would then be followed by a new *processing phase*. Some participants went through this phase-shifting process three times. This was most evident with students' solutions to the second problem, when four participants who used similar methods made the same mistake at the inequality  $2x - 26 < x$ , as exemplified in the extract from Earl's data presented in Table 4.

Time	Digital recording	Transcription of observations and interviews Earl	Observed abilities
4:47	$x - 2 + x - 24 < x$ $2x - 26 < x$	Earl: Which means that $x$ has to be less than 26.	P
5:55	$x - 13 < x$	<i>notes quietly</i>	P
6:08	<i>crosses out</i> <del><math>x - 13 &lt; x</math></del>	<i>notes quietly</i>	P

Table 4: Earl's attempt to solve  $2x < 26 < x$ 

As seen in Table 4, during the episode beginning at 4:47, Earl correctly sums the two expressions to obtain the correct inequality. The recording of his thinking out loud also shows that he has interpreted the inequality correctly to assert that  $x$ , the cost of the CD, "has to be less than 26". However, he then tried to divide the inequality by two and made a calculative error. After a few seconds he realised something was wrong and, although he said nothing, crossed out  $x - 13 < x$ .

Importantly, perhaps reflecting an apparent insecurity, Earl returned to the *orientation* phase, as shown in Table 5. For the second time he extracted and formulated relevant information (O) and commented that he is trying to get "everything together in a more proper way", confirming his confidence in his goal. He then processed this information (P) to construct exactly the same inequality he had earlier. Interestingly, his conclusion with respect the inequality came just 80 seconds after he had crossed out his previous attempt.

Time	Digital recording	Transcription of observations and interviews Earl	Observed abilities
7:28	$x - 24 = \text{Maria}$ $x - 2 = \text{Peter}$ $x = \text{CD price}$	Earl: I write, so I have everything together in a more proper way.	O
7:57	$2x - 26 < x$	<i>notes quietly</i>	P

Table 5: Earl's second orientation phase

At this point, having realised that his new inequality was the same as his first, Earl became insecure once again and returned for a third *orientation* phase, as shown in Table 6. This time, with the same objective, he formulated the problem differently (O). In this case, drawing on the assumption that Maria had  $y$  SEK, he correctly concluded that Peter would have  $y + 22$ . However, as can be seen at 10:10, when he began trying to process this formulation he made an error with the inequality sign. This led, although his processing of the incorrect inequality was mathematically correct, to the incorrect solution for  $y$  at the end of this episode. At this point he crossed out a second solution. Finally, having realised his mistake, he reversed the inequality but failed to understand its significance – that  $y$ , Maria's sum, must be one SEK. At this point Earl became visibly anxious and stopped. After some seconds and additional contemplation, he understood the significance of  $y < 2$  and completed the problem.

Time	Digital recording	Transcription of observations and interviews Earl	Observed abilities
9:35	$y + 22 = \text{Peter}$ $y = \text{Maria}$ $y + 24 = x$	<i>notes quietly</i>	O
10:10	$y + 22 + y > y + 24$ $2y + 22 > y + 24$ $y > 2$	<i>notes quietly</i>	P
10:38	<i>crosses out</i> <del><math>y &gt; 2</math></del>	<i>notes quietly</i>	P
10:44	$y < 2$	Earl: I come back to the same inequalities...	P

Table 6: Earl's third orientation phase

As indicated earlier, four participants, drawing on similar approaches, construed  $2x - 26 < x$  as being equivalent to  $x - 13 < x$ . In this context it is important to underline that every participant was familiar with solving inequalities. Thus, it is natural to wonder why high-achieving students should consistently make such simple mistakes. In this respect the clinical interviews revealed that students became stressed their formalisation yielded an inequality rather than the expected equation. For example, Earl commented that "I was a little surprised when it was (...) an inequality to be solved". Similar sentiments were expressed by, for example, Linda, who said that "I get so... When I start with equations ... then I really want to solve it with equations" and Heather, who observed that "I just didn't know how to formulate it, right here... It is clear that it had been somewhat easier if there was a usual equation, I don't know". In this way the interviews indicated that the participants became insecure and disturbed mainly because they were not expecting to deal with inequalities after starting to solve the problem with equations. Thus, Sebastian's comment that "this kind of tasks usually requires an equation" resonated closely with his peers' perspectives. Interestingly, the analyses also revealed that even in the light of evidence that their strategies were working less well than expected, students persisted with the same orientation strategies.

### **The absence of the ability to generalize mathematical objects, relations and operations**

As discussed earlier, Krutetskii (1976) highlighted a particular form of mathematical memory focused on generalised mathematical relationships. This we operationalised as the ability (G) to *generalize mathematical objects, relations and operations*

e.g. the student transforms numerical approaches into operations with conventional symbols and thereby obtains a general solution of the given problem.

Throughout the data of this particular ability was conspicuous in its absence, although there were three occasions, involving Erin, Sebastian and Larry, where students' numerical approaches had the potential to result in such generalizations. In these circumstances, on completion of their numerical solutions, students were invited to generalize their results. An example of this can be seen in Table 7, which shows an excerpt from Erin's solution of Problem 1.

Time	Digital recording	Transcription of observations and interviews Erin	Observed abilities
4:52	$2.5 \cdot 3 = 7.5$ $7.5/2 = 3.75$	<i>notes quietly</i>	P
5:37	$3.75 + 6.75 = 10.5$	Erin: I get 10.5... so they have the same size, I believe... I: It seems that they have the same size. Erin: Yes.	P
5:48		I: Do you think that this applies for all circles of this kind? Erin: Hum... maybe, I don't know. I haven't thought about that, if this applies for all circles...	
...		<i>she is reasoning about general methods of problem-solving</i>	
6:30		I: If we would change the proportions? If they were... If this short segment went there... Or if we move the point where the semi-circles are tangent to each other... Erin: Uh, yes, but I would, not because ... I do not really know how to say it. But I think that this is always true. I do not know how to prove it in some way if I have to use a general method.	

Table 7: Erin's solution of Problem 1

The data of Table 7 show that despite various invitations, Erin did not go on to generalize the results from her numerical solution. When asked, at 5:48, if she believed her solution would apply to all circles, she replied that she did not know. At 6:30, when she was offered additional hints with the aim of prompting a generalized solution, she confessed that she was unfamiliar with using general methods. In other words, she was unable to express the numerical relations between the diameters and semicircles in a generalized, abstract way. In short, the ability to generalize mathematical objects, relations and operations (G) could not be observed at the participants in the present study.

## DISCUSSION

One of the main objectives of this study was to investigate the interaction of Krutetskii's (1976) four key mathematical abilities during problem-solving. To achieve this, we employed an adaptation of those abilities and, following six high-achieving upper secondary students' attempts to solve two unfamiliar problems, identified three phases of problem-solving activity. These were the *orientation phase*, where the ability to obtain and formalize mathematical information interacted with mathematical memory, the *processing phase*, where the formalized mathematical data were processed according to mathematical rules and principles, and a *checking phase*, in which further processing was undertaken. The evidence also showed that where students experienced failure, typically during their processing phases,

they invoked further phases of orientation. In these three phases can be seen clear resonance with earlier problem solving frameworks, such as, for example, Mason et al.'s (1982) entry, attack and review or Pólya's (1957) four principles of problem solving. In this latter case, the first two of Pólya's principles, understand the problem and devise a plan, were reflected in the *orientation* phase, while his carrying out the plan and reflecting and looking back matched closely the processing and checking phases. However, despite these clear resonances, which validate the analyses, the main contribution of this study lies in its highlighting how different forms of mathematical ability were invoked at different stages of the problem-solving process. A key aspect of this concerns what Krutetskii (1976) called the mathematical memory and the ability to generalise mathematical relations and operations.

With respect to the mathematical memory, the data yielded little explicit evidence of its being invoked independently of other forms of ability, with only 5% of episodes exhibiting this ability on its own. More typically, as shown above, it was observed in conjunction with other abilities, usually the ability to obtain and formalize the mathematical information, during the *orientation* phase. In such instances it was observed in 12% of episodes. However, despite its relatively low rate of occurrence, the role of the mathematical memory seemed to play two pivotal roles in the ways these six students' approached challenging, non-routine mathematical problems. Firstly, perhaps unsurprisingly, students selected their methods during the *orientation* phase. Secondly, having committed themselves to a particular method or approach during this first phase, they seemed to find it difficult to abandon or even modify their methods. Moreover, for some students, selecting ineffective methods led to stress, time delay and unexpected errors during problem-solving. That being said, while an inappropriate choice of method tended to create problems, the counter was also true. For example, with respect to the first problem, Linda, who was able to relate the problem to a generalized method involving well-defined and well-rehearsed procedures (the result of earlier generalizations), solved the problem in a most effective manner.

These findings with respect to method selection allude to some important consequences. Firstly, students' methods – chosen during the *orientation* phase at the start of the process – seem to be retained in working memory throughout the problem-solving process (Ingvar, 2009; Nyberg et al., 2003). Secondly, the abilities manifested by these six students seemed closer to the nonflexible and conformist strategies of the high-achiever (Brandl, 2011) than the flexible and out-of-the-box-thinking of the mathematically gifted (Leikin, 2014). That is, throughout their problem-solving activity, students' reasoning seemed more imitative than creative (Lithner, 2008), not least because their typical approaches to the first problem were numerical with little evidence of any generalised strategies, while their perspectives on the second were clouded by a failure to adapt an expected equation to the reality of an inequality.

However, the relatively fast selection of problem-solving methods and the lack of flexibility when using them can also be explained by some basic functions of the human brain (e.g. Buckner & Wheeler, 2001; Ingvar, 2009; Nyberg et al., 2003; Shipp, 2007). For example, one of the brain's functions is to assimilate new information in relation to previous experience; this means that the number of possible interpretations of a given problem decreases and the information-processing accelerates (Ingvar, 2009; Shipp, 2007). Thus, it is not unreasonable to assume that at the *orientation* phase, when searching for relevant information, participants were influenced by their previous experiences and acted in ways with which they were familiar – in the particular case of the second problem by drawing on their experiences that word-problems always yield equations. Another basic function of the human brain is to automate knowledge. While such automated processes are effective they tend to be inflexible, in that an automated process tends not to influence or be influenced by other working memory processes (Ingvar, 2009; Shipp, 2007). Therefore, it might be reasonable to assume

that when their formalization of the second problem led to inequalities rather than the expected equations, students were unable to adapt their automated equation-solving procedures.

In addition to the infrequent occurrence of the mathematical memory, the data yielded no evidence of the *ability to generalize* mathematical relations and operations. That is, not one participant, among those three who had the opportunity, was able to generalize their numerically-derived solutions. This is somehow surprising as the participants were extremely high-achievers and belonged to a group of students who, according to Krutetskii (1976), should be able to generalize numerical approaches. However, even though a proper investigation of the ability to generalize mathematical relationships or problem-solving methods was not the main focus of the present study, these findings might merit further discussion. Thus, it should be mentioned, that Krutetskii identified four levels of the ability to generalize, from the highest level, at which gifted students generalize “mathematical material correctly and immediately, “on the spot”” (Krutetskii, 1976, pp. 255) to the lowest, at which less able students cannot “generalize mathematical material according to essential features even with help from experimenter” (Krutetskii, 1976, p. 254). Therefore, it is not unreasonable to assume that the ability of generalization of the three mentioned students was below the highest level during our study.

Furthermore, earlier studies have shown that with appropriate interventions the ability to generalise develops over time (Sriraman, 2003, 2004a, 2004b, 2004c). In particular, when they are offered, over a continuous period of several months and expected to solve individually and reflectively, a series of tasks focused on the same generality, talented students begin to engage in *convergent thinking* (Guilford, 1967). During this time they become able to discern “invariant principles or properties, as well as to formulate generalizations from seemingly different situations by focusing on structural properties during abstraction” (Tan & Sriraman, 2017, p. 118). In this manner, generalization can be regarded as an outcome of convergent thinking, which assembles structural similarities in ways that eliminates superficial similarities (Tan & Sriraman, 2017). Consequently, it is not unreasonable to assume that the failure to generalize of the students reported here may be a consequence more of the procedures of the present study – only one problem and relatively little time for reflection – than their mathematical abilities. Moreover, acknowledging that the individual structure of mathematical abilities is dependent on and formed by the instruction students receive (Krutetskii, 1976), it is unlikely that our study’s participants would have experienced teaching from which convergent thinking and therefore the ability to generalize could have been developed (Sriraman, 2003, 2004a, 2004b, 2004c; Tan & Sriraman, 2017).

Finally, the findings of this study confirm Krutetskii's (1976) observation that during the initial or *orientation* phase it is difficult to distinguish the ability to obtain and formalize mathematical information from the mathematical memory. According to the cognitive model, problem-solving involves parallel processes in the working memory and information-units from the working memory disappear after approximately 30 seconds – to counter that, one has to repeat and re-actualize the information in the working memory (e.g. Buckner & Wheeler, 2001; Nyberg et al., 2003; Olson et al., 2009). Consequently, it is not unreasonable to assume that mathematical problem-solving starts with parallel processes in the working memory and that the (in the present study identified) *orientation* phase is associated with the above mentioned abilities. However, since the information-units are retrieved at an extremely high speed to the working memory – often by automated processes – the methods used in this study were not sufficient to distinguish the information-units connected to respective abilities.



We end by concluding that the relationship between problem-solving competence and Krutetskii's (1976) abilities, including the mathematical memory, remains complex and in need of further study. That said, the findings of this study highlight two important phenomena. Firstly, that the mathematical memory has a key function during the *orientation* phase of any problem solving. Secondly, unless students are inducted into flexible ways of managing prior experiences of mathematics or engaged in problem-solving activities which enable convergent thinking and thereby facilitate the discernment of generalizations (Tan & Sriraman, 2017), some students currently identified as high-achieving will fail to acquire the higher order abilities associated with mathematical giftedness (Leikin, 2014; Øystein, 2011; Usiskin, 2000).

## REFERENCES

- Adey, P., Csapó, B., Demetriou, A., Hautamäki, J., & Shayer, M. (2007). Can we be intelligent about intelligence? Why education needs the concept of plastic general ability. *Educational Research Review*, 2(2), 75–97.
- Andrews, P., & Sayers, J. (2012). Teaching linear equations: Case studies from Finland, Flanders and Hungary. *The Journal of Mathematical Behavior*, 31(4), 476–488.
- Andrews, P., & Xenofontos, C. (2015). Analysing the relationship between the problem-solving-related beliefs, competence and teaching of three Cypriot primary teachers. *Journal of Mathematics Teacher Education*, 18(4), 299–325.
- Bassok, M. (2001). Semantic alignments in mathematical word problems. In D. Gentner, K. J. Holyoak, & B. N. Kokinov (Eds.), *The analogical mind: Perspectives from cognitive science* (pp. 199–253). Cambridge, MA: The MIT Press.
- Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – state, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, 22(1), 37–68.
- Brandl, M. (2011). High attaining versus (highly) gifted pupils in mathematics: a theoretical concept and an empirical survey. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 1044–1055). University of Rzeszow.
- Brehmer, D., Ryve, A., & Van Steenbrugge, H. (2015). *Scandinavian Journal of Educational Research*, 59(1), 1–17.
- Buckner, R. L., & Wheeler, M. E. (2001). The cognitive neuroscience of remembering. *Nature Neuroscience Reviews*, 2(9), 624–634.
- Byers, V., & Erlwanger, S. (1985). Content and form in mathematics. *Educational Studies in Mathematics*, 16(3), 259–281.
- Cai, J., & Lester, F. K. Jr. (2005). Solution representations and pedagogical representations in Chinese and U.S. classrooms. *The Journal of Mathematical Behavior*, 24(3), 221–237.
- Calkins, M. W. (1894). A study of the mathematical consciousness. *Educational Review*, VIII, 269–286.
- Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent problem-solving framework. *Educational Studies in Mathematics*, 58(1), 45–75.
- Davis, G., Hill, D., & Smith, N. (2000). A memory-based model for aspects of mathematics teaching. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the*

- International Group for the Psychology of Mathematics Education*, 2 (pp. 225–232). Hiroshima: Hiroshima University.
- Deal, L. J., & Wismer, M. G. (2010). NCTM principles and standards for mathematically talented students. *Gifted Child Today*, 33(3), 55–65.
- Garofalo, J. (1993). Mathematical problems preferences of meaning-oriented and number-oriented problem solvers. *Journal for the Education of the Gifted*, 17(1), 26–40.
- Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education*, 16(3), 163–176.
- Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: aims, rationales, techniques. *For the Learning of Mathematics*, 1(3), 4–11.
- Greer, B., Verschaffel, L., Van Dooren, W., & Mukhopadhyay, S. (2009). Introduction: Making Sense of Word Problems: Past, Present, and Future. In L. Verschaffel, B. Greer, W. Van Dooren, & S. Mukhopadhyay (Eds.), *Words and Worlds: Modelling Verbal Descriptions of Situations* (pp. xi-xxviii). Rotterdam: Sense.
- Graneheim, U. H., & Lundman, B. (2004). Qualitative content analysis in nursing research: concepts, procedures and measures to achieve trustworthiness. *Nurse Education Today*, 24(2), 105–112.
- Guilford, J. P. (1967). *The nature of human intelligence*. New York: McGraw-Hill.
- Gärdenfors, P. (2010). *Lusten att förstå: om lärande på människans villkor* [The desire to understand: about learning based on the human condition]. Stockholm: Natur & Kultur.
- Haylock, D., & Cockburn, A. (2008). *Understanding mathematics for young children*. London: Sage.
- Heinze, A. (2005). Differences in problem solving strategies of mathematically gifted and nongifted elementary students. *International Education Journal*, 6(2), 175–183.
- Hensberry, K. K. R., & Jacobbe, T. (2012). The effects of Polya's heuristic and diary writing on children's problem solving. *Mathematics Education Research Journal*, 24(1), 59–85.
- Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., ... Stigler, J. (2003). *Teaching Mathematics in Seven Countries: Results from the TIMSS 1999 Video Study*. U.S. Department of Education. Washington, DC: National Center for Education Statistics.
- Ingvar, M. (2009). Hjärnbarkens funktion [The function of the cerebral cortex]. In L. Olson, A. Josephson, M. Ingvar, L. Brodin, B. Ehinger, G. Hesslow, ... S. Aquilonius (Eds.), *Hjärnan* [The brain] (pp. 29–46). Stockholm: Karolinska Institutet University Press.
- Jitendra, A., Griffin, C., Deatline-Buchman, A., & Sczesniak, E. (2007). Mathematical word problem solving in third-grade classrooms. *The Journal of Educational Research*, 100(5), 283–302.
- Juter, K., & Sriraman, B. (2011). Does high achieving in mathematics = gifted and/or creative in mathematics? In B. Sriraman & K. H. Lee (Eds.), *The elements of creativity and giftedness in mathematics* (pp. 45–66). Rotterdam: Sense Publishers.
- Kapa, E. (2001). A metacognitive support during the process of problem solving in a computerized environment. *Educational Studies in Mathematics*, 47(3), 317–336.

- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago, IL: The University of Chicago Press.
- Leikin, R. (2010). Teaching the mathematically gifted. *Gifted Education International*, 27(2), 161–175.
- Leikin, R. (2014). Giftedness and high ability in mathematics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 247–251). Springer Netherlands.
- Leikin, M., Paz-Baruch, N., & Leikin, R., (2013). Memory abilities in generally gifted and excelling-in-mathematics adolescents. *Intelligence*, 41(5), 566–578.
- Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67(3), 255–276.
- Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Wokingham: Addison-Wesley.
- Moscovitch, M. (1992). Memory and working-with-memory: a component process model based on modules and central systems. *Journal of Cognitive Neuroscience*, 3(4), 257–267.
- Nesher, P., Hershkovitz, S., & Novotna, J. (2003). Situation model, text base and what else? Factors affecting problem solving. *Educational Studies in Mathematics*, 52(2), 151–176.
- Nogueira de Lima, R., & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, 67(1), 3–18.
- Nunokawa, K. 2005. Mathematical problem solving and learning mathematics: What we expect students to obtain. *The Journal of Mathematical Behavior*, 24(3-4), 325–340.
- Nyberg, L. & Bäckman, L. (2009). Det episodiska minnet [The episodic memory]. In L. Olson, A. Josephson, M. Ingvar, L. Brodin, B. Ehinger, G. Hesslow, ... S. Aquilonius (Eds.), *Hjärnan* [The brain] (pp. 105–114). Stockholm: Karolinska Institutet University Press.
- Nyberg, L., Marklund, P., Persson, J., Cabeza, R., Forkstam, C., Petersson, K. M., & Ingvar, M. (2003). Common prefrontal activations during working memory, episodic memory, and semantic memory. *Neuropsychologia*, 41(3), 371–377.
- Olson, L., Josephson, A., Ingvar, M., Brodin, L., Ehinger, B., Hesslow, G., ... Aquilonius, S. (2009). *Hjärnan* [The brain]. Stockholm: Karolinska Institutet University Press.
- Øystein, H. P. (2011). What characterizes high achieving students' mathematical reasoning? In B. Sriraman & K. H. Lee (Eds.), *The elements of creativity and giftedness in mathematics* (pp. 193–216). Rotterdam: Sense.
- Palm, T. (2008). Impact of authenticity on sense making in word problem solving. *Educational Studies in Mathematics*, 67(1), 37–58.
- Philippou, G. N., & Christou, C. (1999). Teachers' conceptions of mathematics and students' achievement: A cross-cultural study based on results from TIMSS. *Studies in Educational Evaluation*, 25(4), 379–398.
- Pólya, G. (1957). *How to solve it*. New Jersey: Princeton University Press.
- Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences*, 20, 110–122.

- Roberts, S., & Tayeh, C. (2007). It's the thought that counts: Reflecting on problem solving. *Mathematics Teaching in the Middle School*, 12(5), 232–237.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 165–197). New York, NY: Macmillan.
- Sheffield, L. (2003). *Extending the challenge in mathematics: Developing mathematical promise in K-8 students*. Thousand Oaks, CA: Corwin Press.
- Shipp, S. (2007). Structure and function of the cerebral cortex. *Current Biology*, 17(12), 443–449.
- Singer, F., & Voica, C. (2013). Problem-solving conceptual framework and its implications in designing problem-posing tasks. *Educational Studies in Mathematics*, 83(1), 9–26.
- Squire, L. R. (2004). Memory systems of the brain: A brief history and current perspective. *Neurobiology of Learning and Memory*, 82(3), 171–177.
- Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations: The problem-solving experiences of four gifted students. *The Journal of Secondary Gifted Education*, XIV(3), 151–165.
- Sriraman, B. (2004a). Discovering a mathematical principle: The case of Matt. *Mathematics in School*, 33(2), 25–31.
- Sriraman, B. (2004b). Discovering Steiner Triple Systems through problem solving. *The Mathematics Teacher*, 97(5), 320–326.
- Sriraman, B. (2004c). Reflective abstraction, unframes and the formulation of generalizations. *The Journal of Mathematical Behavior*, 23(2), 205–222.
- Swanson, H. L., & Jerman, O. (2006). Math disabilities: A selective meta-analysis of the literature. *Review of Educational Research*, 76(2), 249–274.
- Tan, A.G. & Sriraman, B. (2017). Convergence in Creativity Development for Mathematical Capacity. In R. Leikin & B. Sriraman (Eds.), *Creativity and Giftedness – Interdisciplinary perspectives from mathematics and beyond* (pp. 117–133). Switzerland: Springer International Publishing.
- Usiskin, Z. (2000). The development into the mathematically talented. *Journal of Secondary Gifted Education*, 11(3), 152–162.
- Wagner, H., & Zimmermann, B. (1986). Identification and fostering of mathematically gifted students: Rationale of a pilot study. *Educational Studies in Mathematics*, 17(3), 243–260.
- van Leeuwen, T. (2005). *Introducing social semiotics*. London: Routledge.
- Vilenius-Tuohimaa, P. M., Aunola, K., & Nurmi, J. (2008). The association between mathematical word problems and reading comprehension. *Educational Psychology*, 28(4), 409–426.
- Vilkomir, T., & O'Donoghue, J. (2009). Using components of mathematical ability for initial development and identification of mathematically promising students. *International Journal of Mathematical Education in Science and Technology*, 40(2), 183–199.

