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Abstract. We discuss a certain very common flaw in the definition of linear independence, which is one of the most important concepts taught in any college or university course of Linear Algebra. This note may be useful to lecturers and students which teach and study Linear Algebra of any level and like the mathematically rigorous approach.

Keywords: linear independence; basis of a vector space;

In many textbooks the definition of linear independence is given in the following form:

Definition 1. A set of vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$ is called linearly independent if

$$k_1\vec{a}_1 + k_2\vec{a}_2 + \cdots + k_n\vec{a}_n = \vec{0} \qquad \Rightarrow \qquad k_1 = k_2 \cdots = k_n = 0 .$$

However, this definition is rather problematic. For example, the well known fact that a square matrix is invertible if and only if its rows are linearly independent, is not true if we use this definition. Indeed, the set $\{\vec{a}, \vec{a}\}$, where $\vec{a} \neq \vec{0}$, is linearly independent according to this definition, since $\{\vec{a}, \vec{a}\} = \{\vec{a}\}$ and $k \cdot \vec{a} = \vec{0}$ implies $k = 0$ since $\vec{a} \neq \vec{0}$. It follows from the definition of the equality of sets that all the elements of a set are distinct. For example, $\{3, 2, 2\} = \{2, 3\}$.

The notions "system of vectors" and "set of vectors" are frequently used as synonyms. However, a system is usually a tuple. Using the word "set" instead of the word "system" (or "tuple") may significantly change the meaning of a definition or of a statement.

Definition 2. A system (i.e., a tuple) of vectors $(\vec{a}_1, \vec{a}_2, ..., \vec{a}_n)$ is called linearly independent if

$$k_1\vec{a}_1 + k_2\vec{a}_2 + \cdots + k_n\vec{a}_n = \vec{0} \qquad \Rightarrow \qquad k_1 = k_2 \cdots = k_n = 0 .$$

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These two definitions, Definition 1 and Definition 2, are not equivalent. Indeed, consider the system $(\vec{a}, \vec{a})$, where $\vec{a} \neq \vec{0}$. Obviously, this system is linearly dependent since $1 \cdot \vec{a} + (-1) \cdot \vec{a} = 0$. However, if we use Definition 1, the set $\{\vec{a}, \vec{a}\}$, where $\vec{a} \neq \vec{0}$, is linearly independent as we noticed above. So, Definition 2 is more suitable for using in theorems, properties etc.

Notice also, that some books begin definitions of linear independence and basis in the following form: the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ are called linearly independent (are called a basis) if... This is also not rigorous since it is not clear what they mean when they say "vectors": a set or a tuple of vectors.

Now let us discuss briefly the definition of basis. In many textbooks appears the following definition: a set of vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\} \subset V$ is called a basis of the vector space V if the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ span V, i.e., $Span\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\} = V$, and the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ are linearly independent. This definition is incorrect. To make it correct one must replace "a set of vectors $\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$" by "a tuple of vectors $(\vec{a}_1, \vec{a}_2, ..., \vec{a}_n)$". The notion of basis of a finite-dimensional vector space must be defined for a tuple (but not a set) of vectors in view of two reasons. First, by definition the vectors of a basis must be linearly independent, and as we explained above, the notion of linear independence must be addressed to a tuple of vectors, not to a set of vectors. The second reason is more important than the first: the order of the vectors in the basis is crucial for relating the unique list of coordinates to any vector. One of the most important properties of basis is the following: the vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$ form a basis of the vector space V if and only if each $\vec{v} \in V$ can be uniquely represented as a linear combination of $\vec{a}_1, \vec{a}_2, ..., \vec{a}_n$, i.e., for each $\vec{v} \in V$ there exists the unique list of scalars $k_1, k_2, ..., k_n$ such that $\vec{v} = k_1\vec{a}_1 + k_2\vec{a}_2 + \cdots + k_n\vec{a}_n$. The column $[\vec{v}]_A = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$ is called the list of coordinates of $\vec{v}$ with respect to the basis $A = (\vec{a}_1, \vec{a}_2, ..., \vec{a}_n)$. Thus, for example, the list of coordinates of $\vec{a}_1$ with respect to the basis $B = (\vec{a}_1, \vec{a}_2, \vec{a}_3)$ is $[\vec{a}_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, however, if we change the order of vectors in the basis B, for example, considering the basis $C = (\vec{a}_2, \vec{a}_3, \vec{a}_1)$, we get the different list of coordinates for $\vec{a}_1$: $[\vec{a}_1]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Of course, it is impossible to identify a vector with its list of coordinates if basis is defined as a set of vectors but not as a tuple of vectors: the sets $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $\{\vec{a}_2, \vec{a}_3, \vec{a}_1\}$ equal, so what is the list of coordinates of $\vec{a}_1$? $\begin{bmatrix}1\\0\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1\end{bmatrix}$? Recall that the mapping $\vec{v} \mapsto [\vec{v}]_A$ is very important since it is an isomorphism between the n-dimensional vector space V (where we fix a certain basis A) over the field F and the vector space $F^n = \left\{ \begin{bmatrix}k_1\\ \vdots \\ k_n\end{bmatrix} \middle| \, k_1,\dots,k_n \in F \right\}$ with naturally defined addition and multiplication by scalars.