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Rational Number Operations on the Number Line

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Abstract

The Common Core State Standards make clear that teachers should use the number line to represent integer operations in Grades 6, 7 and 8, continuing the development of number line understanding begun in earlier grades. In this paper, we will investigate the basics of the colored chip and number line models for representing integers and their operations. We will then draw connections between the two models. Finally, we will argue that the number line model is an important one that should be taught alongside or in lieu of the chip model in order to deepen students' understanding operations with integers, rational numbers, and real numbers in general.
Keywords: Rational number, operations, Common Core, instruction, number line, vectors

Introduction

Many teachers are familiar with the "colored chip" model for teaching integer concepts and integer operations. Briefly, different colored chips are used to represent positive or negative units that can then be combined in ways that model operations with integers. However, the CCSS-M make it clear that teachers should use the number line to represent integer operations as well; in fact Standard 7.NS.A.1 in Grade 7 states students:

7.NS.A.1: Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

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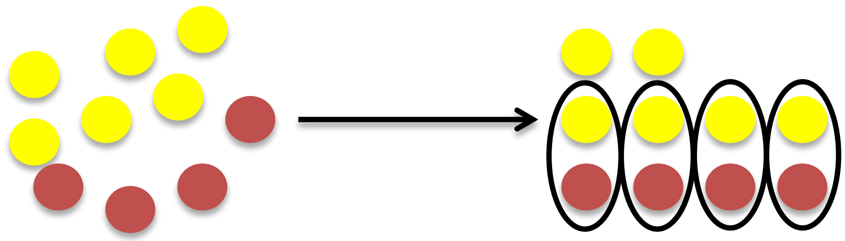
Clearly, addition and subtraction should be represented on the number line. But if teachers are only familiar with using the chip model for integer operations, then how do they make the transition to using the number line? This article will explore the connection between these two representations for addition and subtraction and make a strong case for using both as multiple representations of a challenging mathematical concept.

There are several reasons for representing operations using the number line, not the least of which is that students must learn to perform operations with rational numbers, not just integers. Since the colored chip model can only reasonably illustrate integers and their operations, another model should be used that incorporates all rational numbers and is also mathematically sound.

In this paper, we will investigate the basics of the colored chip and number line models for representing integers and their operations. We will then resolve the challenge of connecting the two models. Finally, we will argue that the number line model is an important one that should be taught alongside or in lieu of the chip model in order to deepen students' understanding of operations with integers, rational numbers, and real numbers in general.

The Colored Chip Model: Basics

Generally, the colored chip model for integers uses one colored chip to represent a positive unit, and another colored chip to represent a negative unit. In this paper, we will use a circular yellow chip to represent a positive unit and a circular red chip to represent a negative unit. Readers familiar with the colored chip model will recall the notion of a "zero pair", that is, that a pair consisting of a positive and negative unit "cancels" out to make zero and therefore does not affect the value of the quantity shown.



The value of the chips shown is "+2" as four yellow chips (+4) and four red chips (−4) combine to make 0.

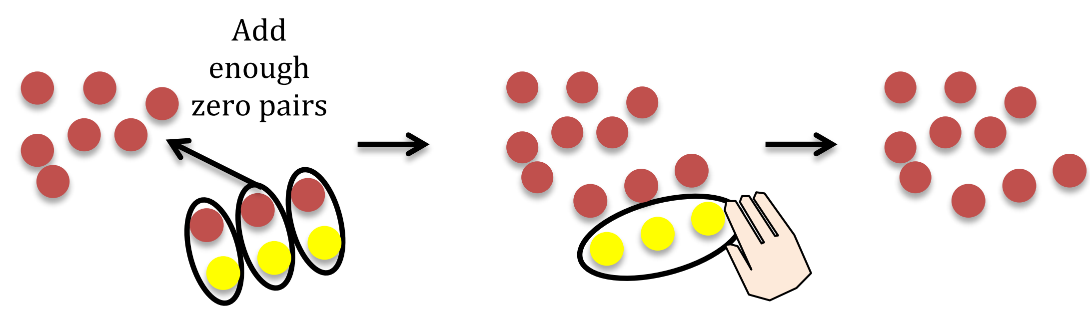
Figure 1: Zero Pairs with Colored Chips

Addition with the Colored Chip Model

The operation of addition with the chip model is rather simple, employing a "grouping together" interpretation of addition. This means that when an addition statement $a + b$ is presented, we interpret the statement as meaning to start with quantity a, introduce quantity b ("add b") and then determine the resulting value after this action, removing zero pairs that are formed. It is important to note here that addition is interpreted as an action, i.e. something "is done" or "happens" to the quantity a.

Subtraction with the Colored Chip Model

Representing the operation of subtraction with the colored chip model takes more ingenuity. Typically, the subtraction sentence $a - b$ is represented with a "take away" interpretation of subtraction: we begin with the minuend, a units, and attempt to take away the subtrahend, b units. It is important to note here that subtraction is again interpreted as an action: something is *being done* to the minuend. However, when there is not enough to take away, as in the case of $-7 - 3$ (i.e., you cannot take away 3 positives from 7 negatives), zero pairs come into play. The idea is to start with the minuend, represented here as 7 negatives, and then to introduce as many zero pairs as needed so that 3 positives can be removed. The result is then tallied up, and the difference of 10 negatives, or -10, is deduced. This is illustrated in Figure 2.



When there are not enough chips to take away, a student first adds in zero pairs, enough so that she can take away the subtrahend. The student then removes the indicated number of chips and determines the result. Here, we subtract $-7-3$ by adding in at least three zero pairs, so that three positives $(+3)$ can be removed.

Figure 2: Subtracting when there is not enough to take away.

Two more cases of subtraction are illustrated in Figure 3, in which including zero pairs is used when there are not enough to take away.

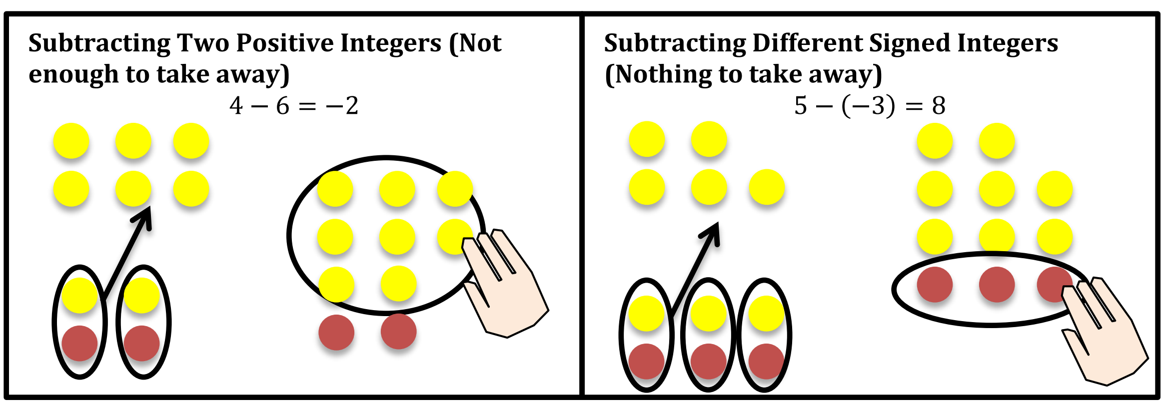


Figure 3: Subtracting Integers with the Colored Chip Model and Zero Pairs

By comparing number sentences and the results, teachers can use the colored chip models for addition and subtraction in tandem to illustrate the fact that $a-b=a+(-b)$, that is, that subtraction is equivalent to adding the opposite. Notice that the actions involved in obtaining the result in each case are different. The interested reader should explore this relationship.

The Number Line Model: Basics

The number line model for representing numbers posits that every real number is represented by a single point on a continuous line that extends indefinitely in each direction, with positive numbers located to the right of 0, and negative numbers located to the left of 0. The formality of the argument is beyond this paper, but the number line can be thought of as a model that justifies the existence of real numbers (e.g. irrational numbers, which cannot be easily represented by other means, can be considered to be locations on the number line).

With the number line model and integers, and furthermore with rational numbers, numbers have a bit of a dual nature: they are represented with points on the number line, but also with vectors. A *vector* can be thought of as simply a directed line segment, usually represented by an arrow. On the number line, the length of the arrow is the absolute value of the number it represents, while the direction in which the arrow points represents the sign of the number. Important with vectors, and for many challenging to understand at first, is that the only things that characterize a vector—and hence the number it represents—are its direction and length. Thus, the same vector (number) can be represented in different locations, depending on the starting point (the *tail* of the vector) and the ending point (the *head* of the vector; see Figure 4. Note that the zero-vector might be represented as just a dot over the point 0.

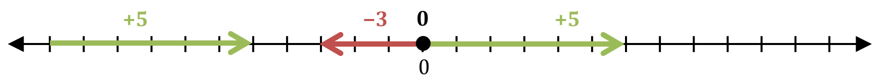
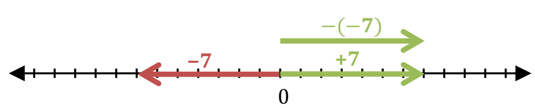


Figure 3: A vector's length and direction determine the number it represents. Thus, the same number can be represented in different ways. This is crucial to understanding operations on the number line.

Figure 4: Basic Representation of Integers with Vectors on a Number Line

Important in using the number line model is an understanding of *opposites*. The Common Core Standards introduce the notion of opposites in Grade 6 (6.NS.B.6), noting that introducing a negative sign in front of a quantity results in a new quantity located on the other side of zero at the same distance of the original quantity. Thus, we accept as a property of working with vectors on the number line that introducing a negative sign (taking the opposite) effectively changes the direction of the original arrow. See Figure 5 for an illustration.



The opposite of -7 must be $+7$ since it must be located at the same distance from zero as -7 but on the other side. Thus, $-(-7) = 7$.

Figure 5: Opposites lie on opposite sides of the number line but at equal distances.

While number line models have been used in the past for integer operations, many of them rely on using motion for representing the operation in question, for instance, walking forwards and backwards to represent addition and subtraction. However, we quickly encounter problems with representing situations like subtracting a negative number; for this, an arbitrary rule akin to "walking backwards in a negative direction means walking forwards" is often introduced. While in any model representing operations can be troublesome and often involves introducing rules, we wish to do so only when completely necessary and in a way that is consistent with students' previous and later mathematical learning.

Addition with the Number Line Model

We represent addition with vectors by placing vectors "head to tail" and determining the resulting vector. This applies to any of the cases illustrated earlier with the colored chip model, and is illustrated in Figure 6. Something immediately apparent is that positive and negative units combine to "cancel out" their directions, similar to what happens with the colored chip model. Readers should model for themselves what happens when a number and its opposite are added together to make 0.

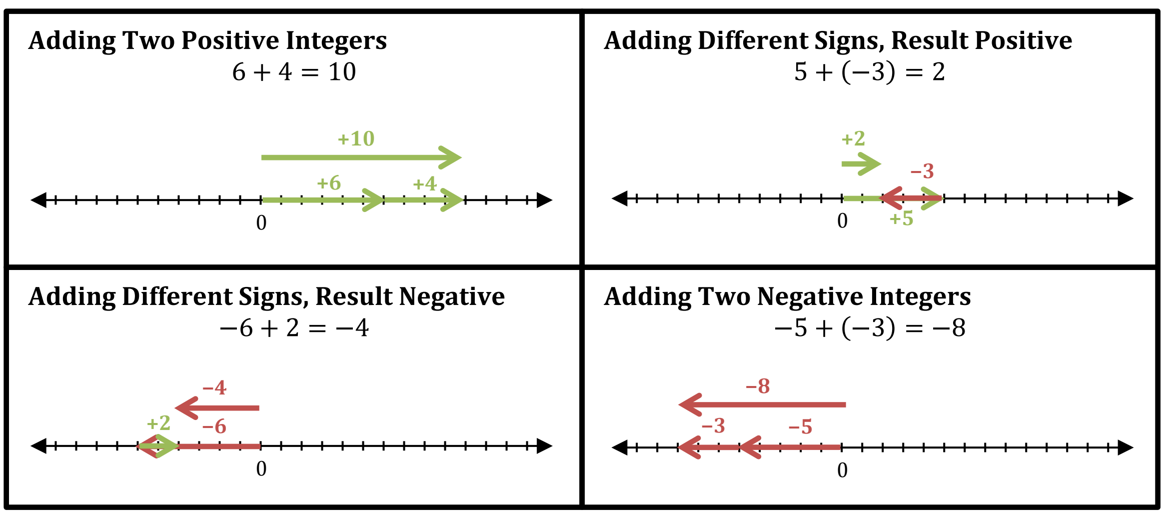


Figure 6: Addition with the Number Line (Vector) Model

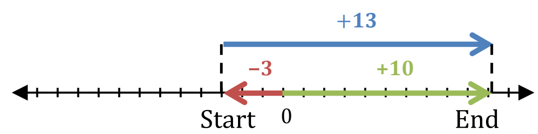
Subtraction with the Number Line Model

If subtraction on the number line is interpreted as take away, it is not clear what is being taken away at all, and we run into the same earlier issue in not always having enough to take away. Furthermore, while it is perhaps possible to represent a zero pair on the number line and take away the corresponding subtrahend, this representation is cumbersome and inconsistent with the way vector subtraction is represented in higher mathematics. The key then, is to use a different interpretation when visualizing subtraction on the number line.

One of the prominent features of the CCSS-M is the progressive development of ideas across grade levels. Subtraction with rational numbers on the number line takes advantage of this by interpreting subtraction not as take away but as a question of "comparison", referred to here as a *compare* problem.[2] Specifically, we consider a subtraction problem $a - b$ as an equivalent "unknown addend" problem, that is, "$a - b = \square$" means "$b + \square = a$." In other words, $a - b = \square$ is equivalent to asking, "What do I add to b to get to a on the number line?" We then use our understanding of addition of vectors to answer the question.

For example, when subtracting in a problem like $10 - (-3)$, we rewrite the problem as $-3 + \square = 10$. This becomes the problem of which vector one would add to -3 in order to end up at the vector that points to $+10$. Notice the reliance on the connection between addition and subtraction for this strategy, as well as the understanding of vector addition. We start by representing -3 with a vector as the starting place and then representing 10 with a vector as the ending place, and then think about which vector would bring us from -3 to $+10$. This is illustrated in Figure 7.

[2] Specifically, in the CCSS-M/CGI framework, this is the *compare, difference unknown* interpretation of subtraction.



We represent the problem $10 - (-3)$ as asking, "What do I add to -3 to get up to $+10$?" We see that a vector of length 13 pointing in the positive direction will suffice. This vector represents $+13$.

Figure 7: Representing Subtraction with "Adding On" on the Number Line

Figure 8 illustrates two more examples of using the comparison interpretation of subtraction and the missing addend strategy for finding the difference.

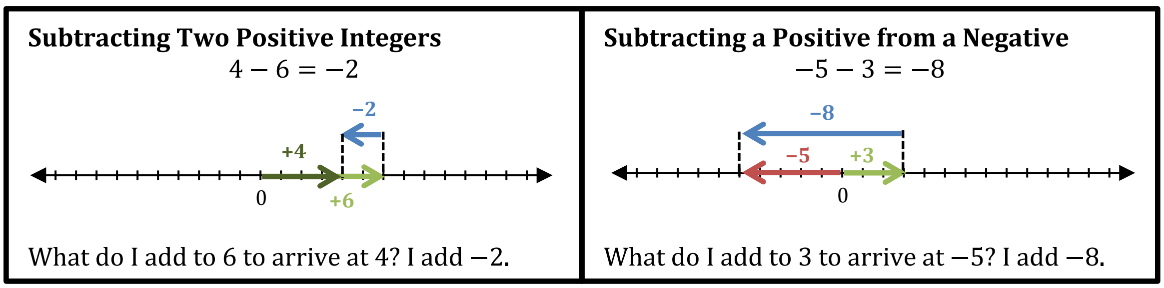


Figure 8: Subtracting Integers on the Number Line with the Compare Interpretation of Subtraction.

The number line model can be used to show that the result of subtracting a number and adding its opposite are the same, much like with the chip model. Note that the two operations $a - b$ and $a + (-b)$ look very different on the number line; the reader is encouraged to explore these two situations.

Connection Between the Two Models

The connection between addition with colored chips and addition with vectors is fairly clear. For example, we see that in the case of adding positive and negative units, "canceling" to create zero pairs corresponds to moving in an opposite direction on the number line, effectively backtracking in either the positive or negative direction. Figure 9 shows the same example of adding positive and negative numbers using each model.

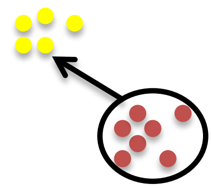
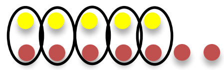
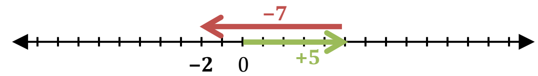
Colored Chip Model: $5 + (-7)$		**Number Line Model:** $5 + (-7)$
	 We see we can make 5 zero pairs by "canceling", with 2 negatives leftover for a sum of -2.	 In the number line model, we see that 5 of the negative units contained in -7 take you down to 0, and then there are 2 more negative units, which leaves us at -2.

Figure 9: Connecting Addition Using the Different Models.

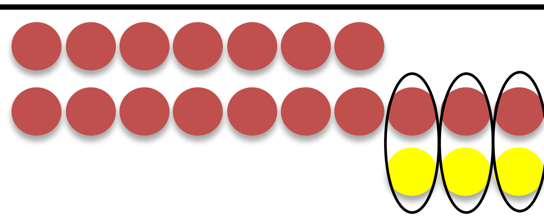
When it comes to subtraction, we still must address the issue that the colored chip model and number line model have been presented using different interpretations of subtraction, one with take away and the other with compare. The connection between these models lies in using the compare interpretation of subtraction with the colored chip model as well.

Solving a problem of the form $a - b$ with colored chips and the compare interpretation of subtraction involves asking the same question, "What must I add to b to get the value a?" Astute readers will already see the connection to the number line interpretation, but we illustrate with the example $-7 - 10$. We rewrite this as the missing addend problem $-10 + \square = -7$, and try to figure out what the value of $\square$ is. We illustrate the solution to the problem in Figure 10.

Target: -7

Start: -10

Add On: $+3$



We see that we would add 3 positive units to eliminate 3 of the negative units in -10 to get to -7. That is, we find $-7 - 10 = \square$ by solving $-10 + \square = -7$.

Figure 10: Seeing Colored Chip Subtraction as Missing Addend, as opposed to Take Away.

Notice that this strategy still involves the concept of zero pairs, but it avoids the issue of "not having enough to take away," and needing to introduce zero pairs. Figure 11 illustrates the two different models with the problem $10 - (-5)$:

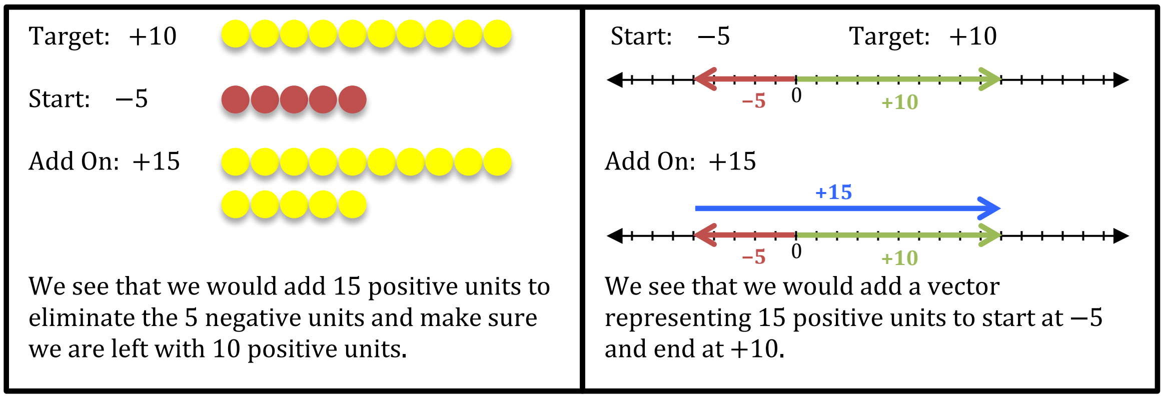


Figure 11: Adding On with Chips and on Number Line.

The picture clarifies the connection between representing the missing addend strategy for solving the subtraction problem using each model.

The compare strategy with chips yields an explicit connection to the number line model for integer operations. Clearly, the missing addend strategy of finding how many chips must be added to the subtrahend to yield the minuend is related to asking which vector must be added to one vector on the number line (the subtrahend) to obtain another vector (the minuend). The two models can be summarized as follows:

Colored Chip Model Compare Subtraction

To solve $a - b = \square$:

1. Reframe the problem as $b + \square = a$.
2. Set out a chips(positive or negative as necessary). This is the "Target."
3. Set out b chips. This is the "Start."
4. Determine what to add to b to yield a. This is the

Number Line Model Compare Subtraction

To solve $a - b = \square$:

1. Reframe the problem as $b + \square = a$.
2. Mark a as a vector starting at 0 and ending at a. This is the "Target."
3. Mark b as a vector starting at 0 and ending at b. This is the "Start."

"Add On."

5. Understand that the "Add On" represents the difference between the two numbers.

4. Determine what vector needs to be added to the head of vector b to end up at the head of vector a. This is the "Add On."
5. Understand that the "Add On" represents the difference between the two numbers.

Advantages of the Vector/Number Line Model

There are several potential advantages to using the number line model and the corresponding interpretations of the basic number operations presented here.

1. The Number Line Model can be used to model operations with Rational Numbers

One of the major drawbacks with using the integer chip model is that it is only feasible to use it to represent integers. When students move to problems like $-\frac{1}{2}-\frac{3}{4}$, they simply need to accept on faith that the rules developed for integers extend to other numbers as well. However, the number line model will apply to operations with all positive and negative numbers since they can all be represented using vectors.

2. Relationship Between Operations

Throughout K-6, the CCSS-M are continually focusing on the algebraic relationships between operations. Addition and subtraction are introduced simultaneously as "inverse operations," as are multiplication and division. The missing addend strategy for solving a compare subtraction problem on the number line or with colored chips makes explicit use of this relationship and can further enhance students' understanding of this relationship.

3. Distance as Absolute Value of the Difference

The CCSS-M explicitly lay the foundation and set the expectation that students understand the quantity $|a-b|$ as representing the distance between the numbers a and b

on the number line. To arrive at this idea, one can use the number line interpretation of subtraction to explicitly illustrate that each of $a - b$ and $b - a$ represents the difference between the numbers a and b, but with different directions. For example, while $8 - (-4) = 12$, the distance between 8 and -4, it is also the case that $-4 - 8 = -12$, which effectively gives the distance between -4 and -8 but in the opposite direction.

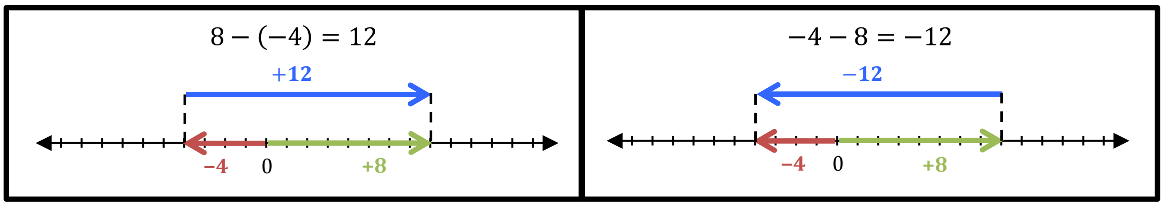


Figure 12: Subtracting both $8 - (-4) = 12$ and $-4 - 8 = -12$ shows that each gives the distance, only with a direction attached. Thus the absolute value removes the directionality and yields the distance.

4. Multiplication as scaling

Those experienced with the chip model will be familiar with the difficulty in representing problems like "$(-2) \times (-3) = \square$" or "$(-8) \div (-2) = \square$" using colored chips. We argue here that these can be represented with relative ease with the number line model if one appeals to the interpretation of multiplication as *scaling*, which is introduced in Grade 5 (5.NF.B.5.a)

We can apply the interpretation of multiplication as scaling to solve a problem like "$(-2) \times (-3) = \square$" by simply interpreting it as, "What is the vector that is *the opposite* of the vector that is 2 *times as long* as the vector -3?" Clearly, this is the opposite of the vector -6, i.e. the answer is positive 6. Here we interpret the negative sign of the factor -2 as meaning "the opposite of", as discussed earlier.

For division, yet again we appeal to the relationship between operations: a problem like "$(-8) \div (-4) = \square$" can be reframed as "$\square \times (-4) = (-8)$." That is, we ask, "By what factor do I scale the vector (-4), and in what direction, to obtain the vector (-8)?"

5. Consistency with Later Vector Operations

One final argument for using the number line model for rational number operations is the fact that the interpretations of the four operations with vectors on the number line align with the corresponding operations for vectors in 2- and 3-dimensions that students encounter in higher mathematics courses. Geometrically, such vectors are added by placing vectors head to tail and subtracted by considering a missing addend. This will correspond to adding and subtracting the components of the vectors, but the geometric interpretation is similar to the number line model interpretations. Furthermore, the number line model for multiplication described above can be connected to the geometric interpretation of scalar multiplication.

Conclusion

The number line model for rational numbers and their operations plays an extremely important role in the Common Core in Grades 6 and 7 and onward. While many teachers will be familiar with the chip model, or a number line model that interprets operations with motion, many will benefit from the primer on using vectors given here. Teachers are encouraged to continue exploring the relationship between the two models, and to use their professional teaching acumen to discover ways to incorporate the number line into instruction on rational number operations.

Resources:

Common Core Standards Writing Team. (2013, July 4). *Progressions for the Common Core State Standards in Mathematics (draft). Grades 6–8, The Number System; High School, Number.* Tucson, AZ: Institute for Mathematics and Education, University of Arizona.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.