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Linear programming for the high school student

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T. E. S.
ABSTRACT

In this paper we are concerned with the problem of presenting a unit in linear programming from a motivational standpoint, i.e. assuming the basic notions of sets and inequalities we introduce the student to the ideas of linear programming using a logical and sequential approach in developing a general numerical method for solving linear programming problems. Pedagogically, we feel that this is the way in which students should be introduced to the relatively new subject of linear programming.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Abstract</td>
<td>iii</td>
</tr>
<tr>
<td>Chapter I An Intuitive Approach to the General Linear Programming Problem</td>
<td>1</td>
</tr>
<tr>
<td>Chapter II Possible Algebraic Methods to Solve Linear Programming Problems</td>
<td>30</td>
</tr>
<tr>
<td>Chapter III Developing the Simplex Method</td>
<td>44</td>
</tr>
<tr>
<td>References</td>
<td>100</td>
</tr>
</tbody>
</table>
CHAPTER I
AN INTUITIVE APPROACH TO THE GENERAL LINEAR PROGRAMMING PROBLEM

In today's world, more so than ever before, people in industry, research, business, and other segments of our economy are constantly asking the following question: "How can we conduct our business to insure the greatest amount of financial gain?"

As an example, consider Mr. Jones, a bread manufacturer. Mr. Jones' plants have the capability to produce brands x, y, and z at the daily rates of $x_1$, $y_1$, and $z_1$. The production cost of each of these brands is $x_2$, $y_2$, $z_2$ and the price at which he wishes to sell each of the brands is $x_3$, $y_3$, $z_3$. Assuming that Mr. Jones is a good business man, a question that he might want to answer is just how much bread of each brand should he manufacture to insure the greatest amount of profit if the customer demand for each brand is $x_4$, $y_4$, $z_4$.

In short, we are interested in maximizing or minimizing a particular variable for example, in business we wish to maximize profit and/or minimize cost. In transporting goods, we wish to minimize the number of miles traveled by the trucks and so on. This is the type of problem we will be concerned with. For the sake of having a name for such problems, we will refer to them as allocation problems.
In order to become familiar with mathematical techniques that will prove useful in solving linear programming problems we will first consider some auxiliary ideas and problems. We consider the geometrical interpretation of linear inequalities.

Let us first look at the graph of the solution set for the inequality $y \leq 3x + 4$. (See Figure 1).

Essentially what we have done is to divide the $x - y$ plane into two sets of points; those points that satisfy the inequality and those that do not. If we desire to talk about those points that satisfy the inequality, all we need do is consider those points in the shaded region. This geometrical interpretation will prove to be very helpful later.

Of course, it is quite possible to talk about the set of points satisfying more than one inequality.

For example, we might be interested in the set of points $\{(x, y)\}$ satisfying

\[
\begin{align*}
x &\leq y \\
y &\leq x + 4 \\
and \quad y &\leq -\frac{1}{3}x + 5.
\end{align*}
\]

If we graph these inequalities, we obtain the graph of the solution set satisfying all three conditions simultaneously. In so doing, we have obtained a characterization of a particular set of points. (See Figure 2).

It becomes apparent that we can require that the
Figure 1.
solution set satisfy n of these inequalities for n = 1, 2, ...
k. We can readily see that the nature of the solution set will depend on the particular n inequalities defining it. We could probably ask whether these relations (inequalities) have to be linear. The answer is no. For reasons that we will soon discuss we shall limit ourselves only to the discussion of linear relations.

To aid our discussion let us introduce some terminology. We will refer to the inequalities defining the solution set as constraints. Once we have obtained a solution set defined by a set of constraints, it is then possible to define on this set a particular function, \( F(x, y) \). We will refer to this particular function as the objective function. Later, we will also refer to the solution set as the convex set. At present, we will not use the term convex set because it will be our desire that a convex set have special mathematical properties.

Let us consider a few examples to see what is meant by defining an objective function over the solution set.

Example 1:

Suppose that we have the constraints,
\[
\begin{align*}
y & \geq 0 \\
x & \geq 0 \\
\text{and} \quad y & \leq -2x + 4.
\end{align*}
\]
These three constraints define a solution set A. (See Figure 3).
Let us now define the objective function

\[ F(x, y) = x + y + 2 \]

on region A, i.e. we are interested in the possible values of \( F(x, y) \) when \( (x, y) \) come from solution set A. For example, we have

\[
\begin{align*}
F(0, 0) &= 2 \\
F(1, 1) &= 4 \\
F(2, 0) &= 4 \\
F(0, 1) &= 3 \\
F(1, 0) &= 3 \\
F(0, 2) &= 4 \\
F(0, 4) &= 6 \\
\text{and } F(1, 2) &= 5.
\end{align*}
\]

Assuming that we are interested only on integral values of \( x \) and \( y \). We can make at least two observations.

1. The value of \( F(x, y) \) depends on the choice of \( (x, y) \) in A.
2. We obtained a maximum value of 6 at \( (x, y) = (0, 4) \) and a minimum value of 2 at \( (x, y) = (0, 0) \).

Example 2:

Assume that the constraints are as in Example 1 but that the objective function is now

\[ F(x, y) = y - x + 1. \]
We have

\[
\begin{align*}
F(0, 0) &= 1 \\
F(1, 1) &= 1 \\
F(2, 0) &= -1 \\
F(0, 1) &= 2 \\
F(1, 0) &= 0 \\
F(0, 2) &= 3 \\
F(0, 4) &= 5 \\
F(1, 2) &= 2 \\
\text{and } F(1/2, 2) &= 2 1/2.
\end{align*}
\]

Again we observe that

1. The value of \( F \) depends on our choice of \( x \) and \( y \).

2. The value of \( F(x, y) \) at a particular point \((x_0, y_0)\) depends on the function, \( F(x, y) \).

3. The maximum value of \( F(x, y) \) is 5 attained when \((x, y) = (0, 4)\) and the minimum value of \( F(x, y) \) is -1 attained when \((x, y) = (2, 0)\).\(^1\)

Let us now look at one last example defined by the constraints,

\[
\begin{align*}
y &\geq 2 \\
y &\geq 2x \\
y &\leq 2x + 8 \\
\text{and } x &\geq 0,
\end{align*}
\]

\(^1\)Relative to the values \((x, y)\) used.
and objective function,

\[ F(x, y) = 3x - y + 2. \]

(See Figure 4).

\[ F(x, y) = 3x - y + 2 \] implies

\[ F(0, 2) = 0 \]
\[ F(1, 2) = 3 \]
\[ F(0, 8) = -6 \]
\[ F(1, 4) = 1 \]
\[ F(1, 5) = 0 \]

and \( F(1/2, 6) = -2 \frac{1}{2}. \)

We see that the maximum value of \( F(x, y) \) is 4 and occurs at \((x, y) = (2, 4)\). The minimum value of \( F(x, y) \) is \(-6\) and occurs at \((x, y) = (0, 8)\).²

Let us study the above examples to see if we can infer something about the location of the \((x, y)\) that maximize or minimize the function.

If you were able to observe that the maximum and minimum values of \( F(x, y) \) were obtained using \((x, y)\) on the vertices of the solution sets, your observations were correct.¹

It will be our objective to prove this fact in a subsequent section.

We should now be able to make a few observations.

²Again this is relative.

¹We will consider one special case when the maximum and/or minimum is given by points other than those on the vertices.
1. The regions (solution sets) talked about were always bounded regions.

2. The boundaries of these regions were defined by straight lines.

3. The particular objective function, \( F(x, y) \), that we attempted to maximize and/or minimize was linear.

In the next section we will discuss the implications of each one of the above observations and give reasons as to why the maximum and minimum values of \( F(x, y) \) are given by those points at the vertices of the solution set.

Let us now consider the more general problem defined by the constraints,

\[ x \geq 0 \]

and \( y \geq 0 \)

and the objective function,

\[ F(x, y) = x + y + 2. \]

(See Figure 5).

In this case the region (solution set) is not a bounded set. Let us see what effect this has on our ability to maximize and minimize \( F(x, y) \).

\[ F(x, y) = x + y + 2 \text{ implies} \]

\[ F(0, 0) = 2 \]

\[ F(1, 0) = 3 \]

\[ F(2, 0) = 4 \]

\[ F(3, 0) = 5 \]
\[
\begin{align*}
F(0, 1) &= 3 & F(100, 100) &= 202 \\
F(0, 2) &= 4 & F(0, 1000) &= 1002 \\
F(0, 3) &= 5 & F(1000, 0) &= 1002 \\
F(10, 10) &= 22 & F(1000, 1000) &= 2002 \\
F(0, 100) &= 102 \\
\end{align*}
\]

We should be able to infer from this that it appears as if we have a definite minimum in this region. In the above case the minimum \(F(x, y) = 2\) occurs when \((x, y) = (0, 0)\). However, it appears that \(F(x, y)\) has no definite maximum. For if we say that the maximum occurs when \((x, y) = (1000, 1000)\); we can find, for example, \((x, y) = (2000, 2000)\) such that \(F(2000, 2000) = 4002 > F(1000, 1000) = 2002\). Thus, it is impossible to obtain a maximum in this region.

Let us now consider \(F(x, y) = -x + 2y + 2\) where the solution set is defined by the constraints, \(x \geq 0\) and \(y \geq 0\).

\[
\begin{align*}
F(0, 0) &= 2 & F(0, 4) &= 10 \\
F(1, 0) &= 1 & F(4, 4) &= 6 \\
F(3, 0) &= -1 & F(10, 10) &= 22 \\
F(4, 0) &= -2 & F(100, 0) &= -98 \\
F(0, 1) &= 4 & F(0, 100) &= 202 \\
F(0, 2) &= 6 & F(100, 100) &= 102 \\
F(0, 3) &= 8 & F(0, 1000) &= 2002 \\
F(2, 0) &= 0 & F(1000, 0) &= -998 \\
\end{align*}
\]

A glance at the values obtained and at the general
pattern being followed by the results insures us that \( F(x, y) \) has no specific maximum nor minimum in the region.

Let us briefly consider one last example where the solution set is defined by the constraints,

\[
\begin{align*}
y & \leq 2 \\
\text{and } x & \leq 2,
\end{align*}
\]

and \( F(x, y) = x + y \).

(See Figure 6).

\[
\begin{align*}
F(2, 2) &= 4 & F(0, 1) &= 1 \\
F(1, 1) &= 2 & F(-8, 2) &= -6 \\
F(2, 0) &= 2 & F(2, -8) &= -6 \\
F(0, 2) &= 2 & F(-10, -10) &= -20 \\
F(1, 0) &= 1
\end{align*}
\]

Immediately we observe that in this case we have a specific maximum but no specific minimum.

Intuitively then, we can say that if a region is not bounded; we might have a maximum or a minimum but not both. It is also possible that we have neither a specific maximum nor a specific minimum.

Let us now give a graphic reason as to why what we have observed is true. Let us reconsider example 1 where we have solution set \( A \) defined by \( y \geq 0, x \geq 0, y \leq -2x + 4 \) and where we are considering \( F(x, y) = x + y + 2 \).

In example 1 we determined values \((x, y)\) in region \( A \) such that \( F(x, y) \) was a maximum in one case and a minimum in the other. We can rephrase the problem as follows. Let
\( F(x, y) = k = x + y + 2 \) and determine the point \((x_0, y_0)\) in A that will give us the largest value of \(k\) and then determine the point \((x, y)\) in A that will yield the smallest value of \(k\). \( k = x + y + 2 \), implies that \( y = -x + (k + 2) \). Therefore, for any particular value of \(k\) we can graph this equation as a straight line with slope = \(-1\) and \(y\)-intercept equal to \(k - 2\). What we have then is a family of lines (depending on \(k\)) with the same slope but varying \(y\)-intercepts. (See Figure 7).

Quite clearly, of the lines represented in Figure 7, the line with the largest \(y\)-intercept and thus the largest \(k - 2\) value is line \(l_9\). Line \(l_9\) can not be used since it does not satisfy the condition that it contain values \((x, y)\) in A. The same is true of line \(l_8\). Looking at the graph, we observe that the line that will satisfy the condition that it contain points in A and still have the largest \(k - 2\) value is line \(l_7\). We observe that the \(k - 2\) value is 4 and the point \((0, 4)\) satisfies the equation for this line. Since \(k - 2 = 4\), \(k = 6\); and the point in A that yields this maximum is the point \((0, 4)\). This is in agreement with our conclusion in example 1. Using similar reasoning we observe that line \(l_2\) yields the smallest \(k - 2\) value while at the same time satisfying the condition that it contain points in A. We have from the graph that \(k - 2 = 0\) or \(k = 2\) and that the point in A that yields this minimum value of \(k\) is the point \((0, 0)\). This is also in agreement with the result in example 1.

Let us reconsider example 2 where region A is defined
by the constraints,
\[ y \geq 0 \]
\[ x \geq 0 \]
and \[ y \leq -2 + 4 \]
but where the objective function now is \( F(x, y) = y - x + 1 \).
(See Figure 8).

Using the same procedure as above, we wish to maximize and minimize \( F(x, y) = k = y - x + 1 \) where \((x, y)\) are in \( A \).
Clearly \( y = x + k - 1 \) so we have a family of lines with slope of 1 and \( y \)-intercept of \( k - 1 \). Line \( L_2 \) maximizes \( k - 1 \) and it satisfies the condition that it contain at least one point in \( A \). So \((x, y) = (0, 4)\) maximizes \( F(x, y) \); also, \( k - 1 = 4 \) implies that \( k = 5 \). This result is in agreement with our previous answer. Now line \( L_7 \) satisfies the minimum conditions. Thus, \((2, 0)\) minimizes \( F(x, y) \) and \( k - 1 = -2 \) implies that the minimum value for \( k \) is \(-1\).

Reconsider the problem defined by the constraints \( x \geq 0 \) and \( y \geq 0 \) and the objective function, \( F(x, y) = x + y + 2 \). Recall that in this problem we arrived at the conclusion that we would obtain a specific minimum but no specific maximum. Let us try and verify this conclusion geometrically.
(See Figure 9).

We wish to maximize and minimize \( k, k = x + y + 2 \) or \( y = -x + k - 2 \). These equations define a family of lines with slope of \(-1\) and intercept of \( k - 2 \). Thus line \( L_2 \) defines a minimum that is obtained when \((x, y) = (0, 0)\) and the
minimum value of $k$ is 2. Looking at the graph (Figure 9) we observe that a specific maximum will not be attained. This observation is in agreement with our original observations.

Likewise, we can give a graphic verification of all the conclusions we arrived at in the other examples.

In summary we say

1. If the solution set defined is bounded, it has points $(x_0, y_0)$ and $(x_1, y_1)$ that will maximize and minimize the objective function.

2. If the solution set is not bounded, we can not obtain the maximum and the minimum of an objective function over that region.

3. In trying to find a geometric verification of our conclusions, we have developed a graphic method by which we can in most cases determine two points in the solution set such that $F(x, y)$ attains a minimum at one and a maximum at the other.

Suppose we now consider the problem of maximizing a linear function, $F(x, y)$, over a solution set which is no longer defined by linear constraints.

Example 4:

Let us define a solution set by the following constraints,

$$y \geq (x - 2)^2$$

and

$$y \leq 4,$$
minimum value of \( k \) is 2. Looking at the graph (Figure 9) we observe that a specific maximum will not be attained. This observation is in agreement with our original observations.

Likewise, we can give a graphic verification of all the conclusions we arrived at in the other examples.

In summary we say

1. If the solution set defined is bounded, it has points \((x_0, y_0)\) and \((x_1, y_1)\) that will maximize and minimize the objective function.

2. If the solution set is not bounded, we can not obtain the maximum and the minimum of an objective function over that region.

3. In trying to find a geometric verification of our conclusions, we have developed a graphic method by which we can in most cases determine two points in the solution set such that \( F(x, y) \) attains a minimum at one and a maximum at the other.

Suppose we now consider the problem of maximizing a linear function, \( F(x, y) \), over a solution set which is no longer defined by linear constraints.

Example 4:

Let us define a solution set by the following constraints,

\[
y \geq (x - 2)^2 \\
\text{and } y \leq 4,
\]
and suppose $F(x, y) = y + \frac{1}{2}x$.

(See Figure 10).

We are interested in maximizing and minimizing $F(x, y) = k = y + \frac{1}{2}x$. As before, we want to determine the values for $(x, y)$ that yield these maximum and minimum values for $k$. $y + \frac{1}{2}x = k$ defines a family of lines with slope of $-\frac{1}{2}$ so we can represent this family of lines graphically. (See Figure 10).

We observe that the maximum value for $k$ occurs when $(x, y) = (4, 4)$ and the minimum occurs when $(x, y) = (2, 0)$. Thus the maximum value for $k$ is 6 and the minimum value is 1.

It appears that it is quite possible to define a solution set of any size, shape, or form. This comment should not be taken lightly; for even if we know what the solution set looks like, in many cases we will be unable to write the constraints that define the solution set. It is also possible that given the constraints we might not be able to graph the solution set that the constraints describe. If this does happen, the graphic method for obtaining a solution becomes useless.

Let us consider graphically other possible solution sets that might occur. Assume that $F(x, y)$ is linear. (See Figure 11).

We now investigate what happens when we choose $F(x, y)$ to be nonlinear. An example of an $F(x, y)$ that is nonlinear is $F(x, y) = x^2 - 2x + y^2 - 2y$. 
A. Has a maximum and minimum

B. Has only a specific minimum

C. Has no specific maximum or minimum

D. Has specific maximum and minimum, but three values of \( F \) yield maximum value while two values of \( F \) yield minimum value for \( F(x,y) \)

Figure 11
Let the solution set \( H \) be defined by
\[
\begin{align*}
y &> 0 \\
x &> 0 \\
y &\leq 2x + 4 \\
\text{and } y &\leq -3x + 8
\end{align*}
\]
where the objective function is \( F(x, y) = x^2 - 2x + y^2 - 2y \).

\( F(x, y) = x^2 - 2x + y^2 - 2y \) implies
\[
\begin{align*}
F(0, 4) &= 8 \\
F(0, 3) &= 3 \\
F(0, 2) &= 0 \\
F(0, 1) &= -1 \\
F(0, 0) &= 0 \\
F(1, 0) &= -1 \\
F(1, 1) &= -2 \\
F(1, 2) &= -1 \\
F(1, 3) &= 2 \\
F(1, 4) &= 7 \\
F(1, 5) &= 14 \\
F(2, 0) &= 0 \\
F(2, 1) &= -1 \\
\text{and } F(2, 2) &= 0.
\end{align*}
\]

Let us determine point \( A \) and point \( B \). \( A \) lies on \( y = 0 \) and \( y = -3x + 8 \) so \( x = \frac{8}{3} \) and \( y = 0 \). \( F(\frac{8}{3}, 0) = \frac{64}{9} - \frac{48}{9} = \frac{16}{9} \).

\( B \) lies on \( y = 2x + 4 \) and \( y = -3x + 8 \) so \( x = \frac{4}{5} \) and \( y = \frac{28}{5} \). \( F(\frac{4}{5}, \frac{28}{5}) = \frac{16}{25} - \frac{8}{5} + \frac{784}{25} - \frac{56}{5} = 19 \frac{1}{5} \).

Although we have not considered all real numbers \((x, y)\) in \( H \), the solution set, we see that relative to the samples chosen, \( F(1, 1) \) is a minimum and \( F(\frac{4}{5}, \frac{28}{5}) \) is a maximum. We ought to note that in the other examples, where the objective function defined on the solution set was linear and the solution set was bounded, that maximum and minimum points lie on the vertices of the solution set. Looking at the above problem we can readily observe that the minimum no
longer lies on one of the vertices.

Considering this problem in terms of the family of equations, \( F(x, y) = k = x^2 - 2x + y^2 - 2y \), for which we hope to determine maximum and minimum values of \( k \), we have

\[
  k + 2 = x^2 - 2x + 1 + y^2 - 2y + 1 \quad \text{or} \quad * \quad k + 2 = (x - 1)^2 + (y - 1)^2.
\]

Equation * is the equation of a circle with center \((1, 1)\) and radius equal to \(\sqrt{k + 2} \). We then have a family of concentric circles with centers at \((1, 1)\) and varying radii. It is easily seen that the value that will make \( k + 2 \) a minimum is \((x, y) = (1, 1)\). Therefore, we have \( k + 2 = (1 - 1)^2 + (1 - 1)^2 = 0 \) so that \( k = -2 \). This minimum value for \( k \) is in agreement with our previous result. The maximum value of \( k + 2 \) will obviously lie on the largest concentric circle which contains points in \( H \). Thus the maximum value for \( k + 2 \) is given by \((x, y) = (4/5, 28/5)\) and the maximum value for \( k \), \( F(x, y) = k \), is \( 19 1/5 \).

Immediately, we can see the advantage of limiting ourselves to linear \( F(x, y) \). One comment worth making is that in the above case we limited ourselves to a nonlinear \( F(x, y) \) that was easily recognizable. Had we chosen an \( F(x, y) \) such a \( F(x, y) = ax^3 - by^5 + cx - dy + e \), where \( a, b, c, d, \) and \( e \) are real numbers, we can readily see the complications that such an \( F(x, y) \) would lead to.

We should note that the equations and inequalities used in these problems may be thought of as being produced by
a model, whether the model be the production of ice cream or the cost of shipping freight. We should make it clear that we must be able to solve the equations produced by our model. Thus, in most cases, we must compromise, i.e. we might want to use more complicated equations because we feel they more truly represent the real model and hence their solution would more nearly represent the desired answer. However, if we cannot solve the equations, they do us no good even if they were more accurate. Thus we limit ourselves to linear $F(x, y)$ since many real-world problems are amenable to this form and the machinery for solving these problems is readily available.

We agreed to postpone the introduction of the term convex set because we want it to have special mathematical properties that have been violated in looking at some of the solution sets defined by nonlinear constraints. Although in working with linear constraints we could define a convex set as that region in space determined by the constraints, we should reinforce this definition by pointing out that a convex set will always include the entire line segment joining any pair of its points. For illustrations of sets that are convex and those that are not look at figure 13.

Up to this point we have limited ourselves to discussing convex sets defined on the $x - y$ plane. We can easily extend this to consideration of $F(x, y, z)$ defined in 3-dimensional space. As an example consider the convex set
2-DIMENSION

CONVEX

NOT CONVEX

CONVEX

NOT CONVEX

Figure 13

3-DIMENSION

CONVEX

NOT CONVEX

CONVEX

NOT CONVEX

Figure 14
defined by

\[ 0 \leq y \leq 4 \]
\[ 0 \leq x \leq 4 \]
\[ 0 \leq z \leq 4 \]

and where the objective function is \( F(x, y, z) = x + y + z \).

The minimum and maximum values of \( F(x, y, z) \) will occur at the vertices of convex set \( S \). \( F(x, y, z) = x + y + z \) is minimized when \( (x, y, z) = (0, 0, 0) \). The maximum will occur when \( (x, y, z) = (4, 4, 4) \). It can be observed that these values are vertices of the convex set \( S \). In the 3-dimensional case what we have is a family of planes defined by \( F(x, y, z) \) moving in the \( x - y - z \) space and intersecting the convex set defined by the constraints.

Suppose that we have a number of warehouses where we store materials, and we also have a number of stores where we sell our materials. We might wish to ship the material from the warehouses to the stores at minimum cost. If the cost of moving one unit of material is the same no matter how much is shipped, then this is an allocation problem. This particular example would come under the specific heading of transportation problems, a special type of allocation problem.

For the time being we will limit ourselves to working with allocation problems that are graphical solvable at least in part on the \( x - y \) plane. Later we will develop an algebraic method to solve them. Let us now observe how we can solve
simple allocation problems using the methods that have been developed up to now.

Let us consider a few problems.

Problem 1:

Suppose that we wish to make two models of a pistol, a regular model and a deluxe model. Let us say that we can't make more than 2 pistols per hour and that the deluxe model gives us 2 units of profit and the regular model 1 unit of profit. How many of each model should we make per hour to insure the largest profit?

It can easily be seen that we should make 2 of the deluxe model and no regular models. Let us now observe how we can solve this problem using the techniques we have discussed up to now.

Procedure: Let \( y \) = number of deluxe model.
Let \( x \) = number of regular model.

Since we can make no more than two pistols an hour, we have \( x + y \leq 2 \). Quite clearly \( x \geq 0 \) and \( y \geq 0 \) since in this problem it does not make sense to talk about making a negative number of pistols. We have three constraints,

\[
x \geq 0, \quad y \geq 0, \quad \text{and} \quad x + y \leq 2,
\]

which define a convex set \( G \). (See Figure 15).

The profit that we make on each deluxe model is \( 2y \) and the profit on each regular model is \( 1 \cdot x \) or \( x \). The total profit is equal to the sum of the two. \( P = 2y + x \) is what we desire to maximize. Now \( P = 2y + x \) defines a family of lines (see Figure 15). The graph indicates that line \( \ell_2 \) which has a point in \( G \) satisfying it, namely \( (x, y) = (0, 2) \), gives us the largest value of \( P \), \( P = 4 \). In other words, we will make the largest profit if we make no standard model pistols and 2 deluxe model pistols. This is in agreement with what the result should be.
Problem 2:?

A manufacturer makes two models A and B, of a product. Each model must be processed by two machines. To complete one unit of model A machine I must work 1 hour and machine II must work 2 1/2 hours. To complete one unit of model B, machine I and II must work 4 hours and 2 hours respectively. Machine I may not operate more than 8 hours per day and machine II, not more than 12 hours per day. If the profit on model A is $3 per unit and model B $4 per unit, how many units of each model should the manufacturer produce per day to maximize his profit?

Procedure: Let \( x \) = number of units of model A produced per day.
Let \( y \) = number of units of model B produced per day.

We have 1.) \( x \geq 0 \) and \( y \geq 0 \).

The amount of time required in making \( x \) units of model A on machine I is \( 1 \cdot x = x \).

The amount of time required in making \( y \) units of model B on machine I is \( 4y \).

Since machine I can be used no more than 8 hours, 2.) \( x + 4y \leq 8 \).

The amount of time utilized in making \( x \) units of model A on machine II is \( 5/2x \).

The amount of time utilized in making \( y \) units of model B on machine II is \( 2y \).

Since machine II can be used no more than 12 hours, 3.) \( 5/2x + 2y \leq 12 \).

The profit made on model A is the number of units of model A that are made times the profit per unit which is \( 3x \).
The profit made on model B is the number of units of model B that are made times the profit per unit which is $4y$.

The total profit then is given by $3x + 4y$ and this is what has to be maximized. Thus we have

$4.) \ F(x, y) = 3x + 4y$.

Using constraints 1), 2), and 3) we have graphically the convex set T. (See Figure 16).

Now $F(x, y) = 3x + 4y$ defines a family of lines with slope equal to $-\frac{3}{4}$. From the graph we can see that the maximum value of $F(x, y)$ for $(x, y)$ in T occurs when $(x, y) = (4, 1)$. Thus to get the largest profit the manufacturer should make $x = 4$ units of model A and $y = 1$ unit of model B per day. Thus the maximum profit is $F(4, 1) = 3 \cdot 4 + 4 \cdot 1 = $16.

At this time it is appropriate to reflect back and see what has been done. We started by introducing the notion of a solution set, i.e. a set in which all possible solutions lie. This set was defined by using constraints. Having established the solution set we then defined the objective function, $F(x, y)$. The main idea was to determine the set of points $(x, y)$ in the solution set that would yield the maximum and/or the minimum value for $F(x, y)$. We then discussed the various situations that could occur; for example, the solution set could be bounded or it could be unbounded. We determined the implications this would have as far as finding the $(x, y)$ values that maximize and/or minimize the particular function, $F(x, y)$. We intuitively showed, using families of lines, that the maximum and minimum, if they exist, will be given by points that lie on the vertices of the solution set (provided that the solution set is one
which is bounded by straight lines).

We then observed what would happen if we defined a solution set using nonlinear constraints. In most cases we observed that maximums and/or minimum could be found but that the procedure to obtain these values was greatly complicated.

Next, we considered nonlinear $F(x, y)$. In this case we ran into the problem that the values of the solution set that maximized and/or minimized $F(x, y)$ were no longer located at the vertices of the solution set.

We introduced the notion of a convex set, a solution set with the particular property that any line joining any two points of the set will lie completely within the set.

We decided to discuss only problems in which the $F(x, y)$ was linear and in which the convex set was defined by linear constraints. We have thus far considered two problems in which we revealed how we can apply a general procedure of maximizing and/or minimizing a function over a convex set to the solution of simple allocation problems.

There is one statement that we have accepted up to now that we have only intuitively proved; that is, given a linear $F(x, y)$ and a convex set the maximum and/or minimum of $F(x, y)$, with $(x, y)$ in the convex set, will be given by the points that lie on the vertices of the convex set. Let us now present a proof of this fact so that we can justify the use of it on future problems.
Proof:

The proof of the above is illustrated in figure 17. We shall suppose that at the corner $p$ the function takes on its largest corner value, $A$, and at the corner $q$ it takes on its smallest corner value, $B$. Let $v$ be any point of the polygon. Draw a straight line between $p$ and $v$ and continue it until it cuts the polygon again at a point $v$ lying on an edge of the polygon, say the edge between the points $s$ and $t$ (the line may even cut the edge at one of the corner points, the analysis remains unchanged). By hypothesis, the value of the function at any corner point must lie between $A$ and $B$. Let us now show that the value of the function at $u$ must lie between its value at $s$ and $t$. Using elementary concepts from Analytic Geometry we can show that any point on the line segment $s$ and $t$ can be represented as $ws + (1 - w)t$, where $0 < w < 1$. If the value of the function at the points $s$ and $t$ are $V$ and $Z$ (assume that $Z < V$), then at any point between $s$ and $t$, the value will be $wV + (1 - w)Z$ since the function is linear. This value is $wV + Z - Zw = Z + (V - Z)w$, is at least $Z$ (when $w = 0$) and at most $V$ (when $w = 1$).

Using the above result the value of the function at $u$ must lie between its values at $s$ and $t$, and hence between $B$ and $A$. Again by the same result the value of the function at $v$ must lie between its value at $p$ and at $u$, and hence must also lie between $B$ and $A$. Since $v$ was any point of the polygon the proof is complete. It should be emphasized here that $F(x, y)$ must be linear and the constraints defining the convex set must also be linear.

Briefly summarizing, to find the maximum or minimum of the linear function, $F(x, y)$, over a convex set we

1. find the corner points (vertices of the set; there will be a finite number of them,
2. substitute the coordinates of each into the objective function.

The largest of the values so obtained will be the maximum of the function and the smallest will be the
Figure 17.
minimum of the function.

Recall that we commented that there was a special case in which the maximum and/or minimum would be given by points other than the vertices. Let us illustrate what we mean. Suppose that we are asked to maximize \( F(x, y) = x + y \) over the convex set defined by \( x \geq 0, \ y \geq 0 \) and \( x + y \leq 4 \). Now this problem fits the requirements of the above concept so its maximum should occur at one of the corner points of the convex set. Let us consider a geometric solution to the problem. (See Figure 18). Now \( F(x, y) = x + y \) defines a family of lines with slope of \(-1\). Clearly, line \( l_2 \) satisfies the conditions that it contain points in \( A \) and that it gives us the largest value for \( F(x, y) \). The points that maximize \( F(x, y) \) are given by the intersection of line \( l_2 \) and the convex set. Thus we have the set of points on line segment \( AB \) giving us the maximum value for \( F(x, y) \). Notice that this really does not violate the above concept since points \( A \) and \( B \) which are vertices will maximize \( F(x, y) \). Therefore, we should rephrase the above statement as follows:

Given a linear \( F(x, y) \) and a convex set, the maximum and/or the minimum of \( F(x, y) \), with \( (x, y) \) in the convex set, will be given by points that lie on the vertices of the convex set. This is not to say that there might not be other points on the boundary of the convex set that will also maximize and/or minimize \( F(x, y) \).

Let us now apply the above concept in solving some allocation problems.
Problem 3:7

An appliance dealer has a store in Milwaukee, Madison, Beloit, Beaver Dam, and Fort Atkinson. He has 8 extra refrigerators in Milwaukee and 6 extra in Madison. He would like to move 5 of them to Beloit, 5 to Beaver Dam, and 4 to Fort Atkinson. The transportation costs per refrigerator between cities are given in the table

<table>
<thead>
<tr>
<th></th>
<th>Beloit</th>
<th>Beaver Dam</th>
<th>Fort Atkinson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milwaukee</td>
<td>$16</td>
<td>$10</td>
<td>$15</td>
</tr>
<tr>
<td>Madison</td>
<td>$10</td>
<td>$12</td>
<td>$10</td>
</tr>
</tbody>
</table>

How should the refrigerators be distributed to keep transportation costs at a minimum?

Before plunging into the problem it might be appropriate to take an intuitive guess at what the answer should be.

It may first appear that we need to use more than two variables in working out this problem; but we can let $x$ represent the number of refrigerators to be shipped from Milwaukee to Beloit, then $5 - x$ will represent the number of refrigerators to be shipped from Madison to Beloit. If we let $y$ represent the number of refrigerators to be shipped from Milwaukee to Beaver Dam, $5 - y$ will be the number of refrigerators to be shipped from Madison to Beaver Dam. Since the total extra refrigerators in Milwaukee is 8 and if $x$ and $y$ are shipped to Beloit and Beaver Dam respectively, the remainder or $8 - x - y$ will be shipped to Fort Atkinson.
Using similar reasoning the number of refrigerators shipped from Madison to Fort Atkinson is $6 - (5 - x) - (5 - y)$ or $x + y - 4$.

Let us make a table for the number of refrigerators shipped from Milwaukee and Madison to the respective three locations:

<table>
<thead>
<tr>
<th></th>
<th>Beloit</th>
<th>Beaver Dam</th>
<th>Fort Atkinson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milwaukee</td>
<td>$x$</td>
<td>$y$</td>
<td>$8 - x - y$</td>
</tr>
<tr>
<td>Madison</td>
<td>$5 - x$</td>
<td>$5 - y$</td>
<td>$x + y - 4$</td>
</tr>
</tbody>
</table>

We can easily see using the above tables that the total cost of shipping these refrigerators is

$$16x + 10y + 15(8 - x - y) + 10(5 - x) + 12(5 - y) + 10(x + y - 4)$$

or

$$x - 7y + 190.$$ 

Thus we wish to minimize $F(x, y) = x - 7y + 190$.

Noting that $x$ and $y$ be integers, the only constraints are:

$$x \geq 0, y \geq 0, 8 - x - y \geq 0, 5 - x \geq 0, 5 - y \geq 0\text{ and } x + y - 4 \geq 0.$$ 

The convex set determined by these constraints is shown in figure 19.

We find the coordinates of the corner points in the usual algebraic way and check them to determine which will minimize the cost, $x - 7y + 190 = F(x, y)$. 
\[ F(4, 0) = 194 \quad F(3, 5) = 158 \]
\[ F(5, 0) = 195 \quad F(0, 5) = 155 \]
\[ F(5, 3) = 174 \quad F(0, 4) = 162 \]

Thus \((x, y) = (0, 5)\) gives \(F(x, y)\) a value of 155 and this is the minimum. If the refrigerators are shipped according to the table below the dealer will have minimum transportation cost of $155.

<table>
<thead>
<tr>
<th></th>
<th>Beloit</th>
<th>Beaver Dam</th>
<th>Fort Atkinson</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milwaukee</td>
<td>0</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Madison</td>
<td>5</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Problem 4.7

According to a well-known nursery rhyme Jack Spratt could eat no fat, his wife could eat no lean. Suppose Jack needed at least 4 lbs. of lean meat per week and his wife needed at least 3 lbs. of fat per week. Their diet consists of beef and pork. Each pound of pork is 0.6 fat and 0.4 lean. Each pound of beef is 0.2 fat and 0.8 lean.

The Spratts have a very small refrigerator and therefore can not buy more than 9 pounds of meat per week. If pork costs $0.75 per pound and beef costs $1.00 per pound, find out how many pounds of beef and how many pounds of pork the Spratts should buy per week to minimize the cost.

Procedure:

Let \(x = \) number of lbs. of beef bought per week.
Let \(y = \) number of lbs. of pork bought per week.
Now some of the relationships that we find are the following.

The total number of pounds of meat bought is \( x + y \). This has to be no greater than 9 lbs. So we have 1) \( x + y \leq 9 \).

Since Jack can eat only lean, the total amount of lean meat he will get in a week is \( 0.8x + 0.4y \). This has to be greater than or equal to 4 lbs., his weekly need. We have 2) \( 0.8x + 0.4y \geq 4 \).

Since Jack's wife can eat only fat, the total amount of fat meat she will get in a week is \( 0.2x + 0.6y \). This has to be greater than or equal to 3 lbs. We have 3) \( 0.2x + 0.6y \geq 3 \).

The total cost of the meat bought during one week is \( 1x + 0.75y \). This is \( F(x, y) \), \( F(x, y) = x + 0.75y \), which we wish to minimize.

The constraints are \( x \geq 0, y \geq 0, x + y \leq 9, \) \( 0.8x + 0.4y \geq 4 \) and \( 0.2x + 0.6y \geq 3 \) and they define convex set \( C \). (See Figure 20).

We solve for the corner points using ordinary algebra and check the corner points to determine which one will minimize the cost, \( x + 0.75y = F(x, y) \)

- \( F(1, 8) = 1 + 0.75(8) = 7 \)
- \( F(3, 4) = 1(3) + 0.75(4) = 6 \)
- \( F(6, 3) = 1(6) + 0.75(3) = 7 1/2 \)

We see that \((3, 4)\) gives \( F(x, y) \) a value of 6 which is the minimum. Hence Jack should buy 3 lbs. of beef and 4 lbs. of pork to insure that he will be paying a minimum price of $6 and yet be fulfilling his families diet requirements.

Exercises to consider:

Ex. 1 Two men, Joe and George, are working in a small
factory. Since they both work as part-time help, they may each be employed for any number of hours. Joe's wages are $2 per hour. George receives $13 per hour. Joe produces 10 nuts and 4 bolts per hour. George can produce 5 nuts and 7 bolts per hour. A rush order for 50 nuts and 50 bolts comes in. Because of the expense of running the machines, no more than 11 man-hours can be spent on the manufacture of these nuts and bolts. How many hours should Joe work, and how many hours should George work to keep the payroll at a minimum?

Answer: Joe should work 2 hours, George, 6 hours.

Ex. 2 Aspirin and Buffirin are made on 2 machines I and II. To make 1 lb. of aspirin, machine I must be used 1 hour and machine II must be used 2 1/4 hours. To make 1 lb. of Buffirin, machine I must be used 3 hours and machine II must be used 3/4 hours. No machine may operate more than 12 hours per day. A profit of $2 per lb. is made on aspirin and $3 per lb. on Buffirin. How many pounds of each should be made per day to maximize the profit? What will the profit be?

Answer: 4 1/2 pounds of aspirin and 2 1/2 pounds of Buffirin; profit $16.50.

Ex. 3 A grocer is buying soap powder from the wholesaler. He is considering stocking two brands, Sudso and Brighto. He makes a profit of 10¢ a box on Sudso and 20¢ a box on Brighto. Customer buying statistics show that at least 3 times as much Sudso is sold as Brighto. The grocer has, at most, 900 square inches of shelf space for soap. It takes 20 square inches to store a box of Sudso and 30 square inches to store a box of Brighto. How many boxes of each kind of soap should the grocer stock to maximize his profit? What will the profit be?

Answer: 30 boxes of Sudso and 10 boxes of Brighto; profit $5.

Ex. 4 A manufacturer produces two different models of steam irons, the Deluxe and the Superior. Each model must be processed by three machines, I, II, III. To complete one unit of each model the three machines must work the number of hours indicated in the following table
No machine may operate more than 12 hours per day. The profit on the Deluxe model is $3 per unit and the profit on each unit of the Superior model is $5. How many of each unit should be manufactured per day to maximize the profit? What will the profit be?

Answer: 4 Deluxe model and 8 Superior model; profit $52.
CHAPTER II
POSSIBLE ALGEBRAIC METHODS TO SOLVE
LINEAR PROGRAMMING PROBLEMS

As we have pointed out not all allocation problems lend themselves to a simple graphic solution. As the number of variables increases to three in the defining constraints and objective function, we have to obtain a solution in 3-dimensional space, a feat that is not impossible but highly impractical. If we wish to solve problems with four or more variables, the graphic method is rendered useless.

It is for the above reason that we will attempt to derive an algebraic method to solve these problems. It will be one of our primary goals to develop a method that can easily be converted into a feasible computer method. We will attempt to arrive at a method using geometric representations.

Consider the convex set \( H \) defined by
\[
\begin{align*}
y &\geq 0 \\
y + 4/3x &\leq 8 \\
y + 5/7x &\leq 5 \\
y + 4/9x &\leq 4 \\
x &\geq 0
\end{align*}
\]
and the objective function \( F(x, y) = Y + 2x \). (See Figure 21). Remembering that the maximum and minimum value of \( F(x, y) \) are given by those points on the corners of the convex set, it becomes apparent that one way of attacking
Figure 21
this problem is to determine the intersection \((x, y)\) of all pairs of lines, where by lines we mean the edges of the half-planes determine by the constraints. In this particular case, the equations of the lines would be: line \(l_1(y = 0)\), line \(l_2(y + 4/3x = 8)\), line \(l_3(y + 5/7x = 5)\), line \(l_4(y + 4/9x = 4)\) and line \(l_5(x = 0)\). Once having determined the points of intersection we would decide which of these points lie in the convex set. It would then suffice to establish which one of the points maximizes or minimizes \(F(x, y)\).

To illustrate, let us consider the problem at hand.

\[
\begin{align*}
l_1 \cap l_2 &= \{(6, 0)\} \\
l_1 \cap l_3 &= \{(7, 0)\} \\
l_1 \cap l_4 &= \{(9, 0)\} \\
l_1 \cap l_5 &= \{(0, 0)\} \\
l_2 \cap l_3 &= \{(63/13, 20/13)\} \\
l_2 \cap l_4 &= \{(9/2, 2)\} \\
l_2 \cap l_5 &= \{(0, 8)\} \\
l_3 \cap l_4 &= \{(63/17, 40/17)\} \\
l_3 \cap l_5 &= \{(0, 5)\} \\
l_4 \cap l_5 &= \{(0, 4)\}
\end{align*}
\]

The next thing to do is to exclude all those points, \((x, y)\), that violate at least one of the constraints, i.e. those points that do not lie in the convex set.

\((9, 0)\) is excluded since it violates \(y + 4/3x \leq 8\).

\((7, 0)\) is excluded since it violates \(y + 4/3x \leq 8\).
(0, 5) is excluded since it violates $y + \frac{4}{9}x \leq 4$.
(0, 8) is excluded since it violates $y + \frac{4}{9}x \leq 4$.
(9/2, 2) is excluded since it violates $y + \frac{5}{7}x \leq 5$.
We therefore have to consider maximizing $F(x, y)$
over the set \{(0, 0), (6, 0), (0, 4), (63/17, 40/17),
(63/13, 20/13)\}. $F(x, y) = y + 2x$ implies
\[
F(0, 0) = 0 \quad F(63/17, 40/17) = 166/17 = 9.77
\]
\[
F(0, 4) = 4 \quad \text{and} \quad F(63/13, 20/13) = 146/20 = 7.30
\]
\[
F(6, 0) = 12.
\]
We observe that the maximum occurs at (6, 0) and
that the maximum is 12. This would be the value obtained
had we gone through this problem using the graphic method.
(See Figure 21).

In the case involving 3 variables in the constraints
and the objective function we would have to consider the
intersection of planes. Thus each time we want to determine
a point of intersection we would have to work with three
equations (these equations would be obtained from the con­
straints, as we did in the above example). Similarly, if
we had four variables involved we would have to work with
four equations at one time, and so on.

Immediately we can observe a few difficulties.

1. As the number of constraints increase we would
have to consider more and more combinations of
equations to determine the corner points of the
convex set. The number of combinations that we
would have to consider is given by $c = \frac{n!}{p!(n-p)!}$
where $n$ is the number of constraints given and $p$
is the number of variables involved.

2. In trying to solve for points of intersection by pairing up equations there are cases in which we would run into equations that are inconsistent and thus for which we are unable to determine a point of intersection. (See Figure 22). Of course, as you can probably observe, all we have to do in the case when this occurs is to disregard the equations that are inconsistent when it comes to pairing them up to find their point of intersection. As an example, look at lines $l_1$ and $l_2$ in Figure 22. The difficulty here lies in being able to determine inconsistency when working on a computer.

As you can probably observe the amount of mathematical computation is tremendous and the problem of being able to determine inconsistencies adds certain disadvantages to the method just discussed. It is at this time appropriate to mention that it is desirable to develop a method that does not involve large amounts of computation, since long and tedious computations increase computer operation time which is costly and increase the chances of obtaining errors of the type encountered in computer work.

The above method will work but let us attempt to find a method that is more economical and that can be more readily computerized.

Let us once more consider the figure defined by the constraints,

\[ y \geq 0 \]
\[ y + \frac{4}{3}x \leq 8 \]
\[ y + \frac{5}{7}x \leq 5 \]
\[
y + \frac{4}{9}x \leq 4
\]
\[
x \geq 0
\]
(See Figure 23).

For now let us only concern ourselves with the problem of determining the maximum of the objective function.

Let us consider the problem defined by the above constraints and the objective function, \( F(x, y) = y + 2x \).

We should be able to observe that the ideal algebraic method would enable us to start on the boundary of the convex set \( H \) and go from one corner point to another. (See Figure 23). In so doing, we could evaluate the objective function at each of these points and thus determine which point maximizes the function. The question is, can we arrive at a method that will enable us to do this?

Refer to Figure 23. Suppose that we have been able to determine that point 0 lies on the boundary of the convex set \( H \) and is a corner point for this set. We really do not have to start at a corner point but for convenience let us assume that we have. We would like a way of getting from 0 to A then to B and so on. One way of finding point A is to find the intersection of line \( l_1 \) and line \( l_2 \), i.e. O and A lie on line \( l_1 \) and A lies on line \( l_2 \). So, we can move from point 0 to point A. To go from point A to Point B, we find the intersection of line \( l_2 \) and line \( l_3 \), since A and B lie on line \( l_2 \) and B lies on line \( l_2 \). To go from B to C we find the intersection of line \( l_3 \) and line \( l_4 \). To go from point C
to point D we obtain the intersection of lines \( l_4 \) and \( l_5 \).

Many questions arise. Why did we choose the intersection of lines \( l_1 \) and \( l_2 \)? Why not choose the intersection of lines \( l_1 \) and \( l_4 \)? Why did we fail to consider the intersection of lines \( l_2 \) and \( l_4 \)? Let us in the next few paragraphs attempt to answer these questions as well as others that might arise as we go along.

Suppose that we are at point O. Now going from point O in the direction of A constitutes increasing the variable \( x \). Since \( H \) is defined by inequalities (constraints) the increase in \( x \) can be no larger than the smallest restriction imposed on \( x \) while moving along line \( l_1 \), i.e. in the positive \( x \)-direction. Now being at point A we need not worry about anymore increase in \( x \) since \( x \) has attained its maximum relative to the constraints. We now consider movement in the positive \( y \)-direction along line \( l_2 \). Being at point A our movement in that direction is restricted by line \( l_3 \). Thus to determine the next point (point B) we determine the intersection of lines \( l_3 \) and \( l_2 \). As you can easily observed from the drawing the \( x \) value has decreased in going from A to B. This would normally be something to wonder about but in this case since we are merely concerned with being able to move from one corner point to another we need not worry about the decrease. \( Y \) has not yet attained a maximum so let us consider again movement in the positive \( y \)-direction along \( l_3 \) starting now at point B.
Using the same reasoning as before, movement from B in the positive y-direction is restricted by line \( l_4 \); thus to determine the next point (point C) we determine the intersection of lines \( l_3 \) and \( l_4 \). The maximum in the positive y-direction still has not been attained. Now movement in the positive y-direction along \( l_4 \) appears to be restricted by line \( l_2 \), but we can not use line \( l_2 \) again because if we did we would violate two principles: 1) \( y \) would not be increasing and 2) most important, the convex set would no longer be as defined. Therefore, the only other possible restriction on \( y \) is that lines \( l_4 \) and \( l_5 \) intersect (point D). We have thus completed the cycle of starting at one point and marching through the entire set of corner points. One fact to keep in mind is that this method is still quite dependent on the graph we are working with.

Let us apply this method to the problem we considered a little earlier.

Maximize \( F(x, y) = y + 2x \)

where \( y \geq 0, y + 4/3x \leq 8, y + 5/7x \leq 5, y + 4/9x \leq 4, \) and \( x \geq 0 \).

Because we will be considering points on the boundary of the convex set these constraints may be written as the following equations \( y = 0, y + 4/3x = 8, y + 5/7x = 5, y + 4/9x = 4 \) and \( x = 0 \).

For convenience, again assume that we start at point 0 on the drawing, i.e. at the point \((x, y) = (0, 0)\).
(See Figure 23). We first consider movement in the positive x direction, i.e. we are increasing the variable x. We are moving along the line y = 0. Now y = 0 implies from \( y + \frac{4}{3}x \leq 8 \), \( y + \frac{5}{7}x \leq 5 \), and \( y + \frac{4}{9}x \leq 4 \) that \( x \leq 6 \), \( x \leq 7 \), and \( x \leq 9 \). Since \( y + \frac{4}{3}x \leq 8 \) gives us the smallest restriction on x. A is determined by finding \( \{ y = 0 \cap y + \frac{4}{3}x = 8 \} \). Thus A = \( (6, 0) \) and \( F(A) = F(6, 0) = 12 \). We now consider moving in the positive y-direction, i.e. we are increasing the variable y. Remember that we are now on the line \( y + \frac{4}{3}x = 8 \). Now \( y + \frac{4}{3}x = 8 \) implies from \( y + \frac{5}{7}x \leq 5 \) and \( y + \frac{4}{9}x \leq 4 \) that \( y \leq \frac{20}{13} \) and \( y \leq 2 \). Since \( y + \frac{5}{7}x \leq 5 \) gives us the smallest restriction on y, B is determined by finding \( \{ y + \frac{4}{3}x = 8 \cap y + \frac{5}{7}x = 5 \} \). So B = \( \left( \frac{63}{13}, \frac{20}{13} \right) \) and \( F(B) = F(\frac{63}{13}, \frac{20}{13}) = 7.30 \).

Considering again movement in the positive y-direction and remembering that we are on the line \( y + \frac{5}{7}x = 5 \), we have from \( y + \frac{4}{9}x \leq 4 \) that \( y \leq \frac{40}{17} \). C is determined by finding \( \{ y + \frac{5}{7}x = 5 \cap y + \frac{4}{9}x = 4 \} \). C = \( \left( \frac{63}{17}, \frac{40}{17} \right) \) and \( F(C) = F(\frac{63}{17}, \frac{40}{17}) = 166/17 \). Again moving in the positive y-direction remembering that we are on the line \( y + \frac{4}{9}x = 4 \), we have only the constraint that \( x \geq 0 \). Thus D is determined by \( \{ x = 0 \cap y + \frac{4}{9}x = 4 \} \). So D = \( (0, 4) \) and \( F(D) = F(0, 4) = 4 \).

The method just described above lacks a lot of refinement and still leaves a great many questions unanswered. It is still very dependent on the graph in arriving at a
solution but it does give us a way of evaluating the objective function at the corner points of the convex set. We would hope that by adding refinements to this method that we could arrive at an algebraic method of solving linear programming problems.

Let us then attempt to add refinements to the method just introduced. The method described enabled us to start at a corner point on the boundary of the convex set and to move from corner point to corner point. After determining the coordinates for the corner points we evaluated the objective function at those points. We were able to determine the maximum value of the objective function after checking all the corner points. It can be seen that the ideal method would enable us to determine when we have reached the point that will maximize the function so that we can stop the procedure as soon as possible. For example, if we look at the problem that we considered, we see that the maximum was actually obtained when we reached point A. Had we had a method like the one just described we would have been able to tell that A was the point that maximizes the function without having to determine all of the points, B, C, and D.

We will thus attempt to develop a method that will allow us to stop as soon as possible after the maximum has been obtained. Let us consider the following example.
Maximize \( F(x, y) = y + 4x \)
where \( y \geq 0, 2x - y \leq 12, 3x + y \leq 23, \frac{1}{3}x + y \leq 7 - x + y \leq 3 \) and \( x \geq 0 \). (See Figure 24).

For reference, the graphic solution has been obtained. Suppose we are able to determine that point 0 lies on the boundary of the convex set, i.e. the point \( (x, y) = (0, 0) \) is a corner point.

Procedure:

\[ F(0, 0) = 0 \]

Let the variable \( x \) increase along the line \( \ell_1(y = 0) \). Now \( y = 0 \) implies from \( 2x - y \leq 12, 3x + y \leq 23, \frac{1}{3}x + y \leq 7 \) and \(-x + y \leq 3\) that \( x \leq 6, x \leq 23/6, x \leq 21, \) and \( x \geq 3 \).

We determine \( A \) by finding \( \ell_1 \cap \ell_2 = y = 0 \cap 2x - y = 12 \). \( A = (6, 0) \). \( F(A) = 24 \).

We now allow the variable \( y \) to increase along the line \( \ell_2(2x - y = 12) \). Now \( 2x - y = 12 \) implies from \( 3x + y \leq 23, \frac{1}{3}x + y \leq 7 \), and \(-x + y \leq 3\) that \( y \leq 2, y \leq 30/7, \) and \( y \leq 18 \).

We determine \( B \) by finding \( \ell_2 \cap \ell_3 = 2x - y = 12 \cap 3x + y = 23 \). \( B = (7, 2) \) and \( F(B) = 30 \).

We again allow the variable \( y \) to increase along the line \( \ell_3(3x + y = 23) \). Now \( 3x + y = 23 \) implies from \( \frac{1}{3}x + y \leq 7 \) and \(-x + y \leq 3 \) that \( y \leq 5 \) and \( y \leq 26 \).

To determine \( C \) we find \( \ell_3 \cap \ell_4 = 3x + y = 23 \cap \frac{1}{3}x + y = 7 \). \( C = (6, 5) \) and \( F(C) = 26 \).

Here we stop. Why? We know that the point that maximizes \( F(x, y) \) is \( (x, y) = (7, 2) \). The reason behind this is easily seen. Once \( F(x, y) \) has reached a relative maximum with respect to the points that come just before and just after it. The value of \( F(x, y) \) will never attain
any value larger than that obtained at the point where it had its relative maximum. Suppose A, B, C, and D are corner points of a convex set in that order. Suppose \( F(A) < F(B) \) and \( F(B) > F(C) \) and \( F(C) < F(D) \) where \( F(D) > F(B) \). Graphically in the two dimensional case we would have: See Figure 25.

From the drawing it follows that it is not possible for the above to happen since if it did we would no longer have a convex set. The analog of this can be found in the three-dimensional case where the role of the lines in figure 25 is played by planes. This is a very special property of convex sets and should be remembered in working with linear programming problems.

Let us now suppose that we wish to maximize the function \( F(x, y) = 4y + x \) over the convex set illustrated in figure 26. Note that the solution is shown. Assuming that we are given that point 0 is a corner point of the convex set it would be ridiculous to go from point 0 to point A, then to points B, C, and D respectively. We can go directly from point 0 to point D and thus establish our solution more readily. What we want to point out is that there is nothing special about starting in the positive \( x \)-direction first. We can just as easily start in the positive \( y \)-direction and in a lot of cases it will be a lot more convenient if we do so, as would be the case in figure 26. A question might be, how do we determine which
variable to increase first? If it is the case that by increasing the variables we can increase the value of the objective function, the answer is to look at the objective function, as in the case when \( F(x, y) = 4y + x \). Here it would be more beneficial to increase \( y \) first since for each unit of change in the variable \( y \) its contribution to the value of \( F(x, y) \) is quadrupled as opposed to the case for which each unit of change in \( x \) increased \( F(x, y) \) only by the one unit. This is not to say that this is the only possible way to arrive at a solution. In general, it will assure us of a faster convergence to a solution.

Let us now observe what potential our method has when we work in 3-dimensional space, i.e. when we have to maximize a function of three variables, \( F(x, y, z) \).

Suppose we are asked to maximize \( F(x, y, z) = 2x + \frac{11}{4}y + 3z \) where the convex set is defined by

\[
\begin{align*}
x & \geq 0, \quad y \geq 0, \quad z \geq 0 \\
2x + y + z & \leq 8 \\
2x + y + 2z & \leq 10 \\
\text{and} \quad \frac{1}{2}x + y + z & \leq 4.
\end{align*}
\]

(See Figure 27). It is our objective here to show that the maximum occur at point \( E = (8/3, 2/3, 2) \), that \( F(E) = 13 \frac{1}{6} \) is a maximum, and that the method we have developed will establish this as our solution.

Let us again assume that we are given point \( (0, 0, 0) \) as the initial solution.
Procedure:

\[ F(0, 0, 0) = 0 \]

Which variable should we increase first and how much may we increase it? Looking at the objective function, \( F(x, y, z) = 2x + 11/4y + 3z \), we decide to increase \( z \) first. Since we are moving along the \( z \)-axis we know that \( x = 0 \) and \( y = 0 \). Now \( x = 0 \) and \( y = 0 \) imply from \( 2x + y + z \leq 8 \), \( 2x + y + 2z \leq 10 \), and \( 1/2x + y + z \leq 4 \) that \( z \leq 8 \), \( z \leq 5 \) and \( z \leq 4 \). To determine \( G \) we find \( \{ x = 0 \cap y = 0 \cap 1/2x + y + z = 4 \} \). \( G = (0, 0, 4) \) and \( F(G) = 12 \).

Once at \( G \) we can increase the \( x \)-variable or the \( y \)-variable. Suppose we decide to increase \( x \). We are then increasing \( x \) along the line defined by \( (y = 0 \cap 1/2x + y + z = 4) \), i.e. the line \( 1/2x + z = 4 \). We thus determine which constraint restricts increases of \( x \) the most. Now \( 1/2x + z = 4 \) and \( y = 0 \) imply from \( 2x + y + 2z \leq 10 \) and \( 2x + y + z \leq 8 \) that \( x \leq 2 \) and \( x \leq 8/3 \). To determine \( P \) we find \( \{ y = 0 \cap 2x + y + 2z = 10 \cap 1/2x + z = 4 \} \). \( F = (2, 0, 3) \) and \( F(F) = 13 \).

At \( F \) as at \( G \) we have two possible lines to follow, the line \( (y = 0 \cap 2x + y + 2z = 10) \) or the line \( (2x + y + 2z = 10 \cap 1/2x + y + z = 4) \). Suppose it is our desire to increase \( x \) along the line \( (y = 0 \cap 2x + y + 2z = 10) \). Now \( y = 0 \) and \( 2x + y + 2z = 10 \) imply from \( 2x + y + z \leq 8 \) and \( z \geq 0 \) that \( x \leq 3 \) and \( x \leq 5 \). To determine \( D \) we find \( \{ y = 0 \cap 2x + y + 2z = 10 \cap 2x + y + z = 8 \} \). \( D = (3, 0, 2) \) and \( F(D) = 12 \). We therefore must stop following this line since \( F(F) > F(D) \). We know then that either \( F \) yields the maximum or we must increase \( x \) along the line \( (1/2x + y + z = 4 \cap 2x + y + 2z = 10) \). The restrictions on increasing \( x \) along this line are given by \( (1/2x + y + z = 4 \cap 2x + y + 2z = 10 \cap 2x + y + z \leq 8) \) and by \( (1/2x + y + z = 4 \cap 2x + y + 2z = 10 \cap z \geq 0) \). From these we get that \( x \leq 8/3 \) and \( x \leq 4 \). To determine \( E \) we find \( \{ 1/2x + y + z = 4 \cap 2x + y + 2z = 10 \} \). \( E = (8/3, 2/3, 2) \) and \( F(E) = 13 1/6 \). Note that \( F(E) > F(F) \).

At \( E \) as before we have two possible lines to follow that leading to \( B \) and that leading to \( D \). Obviously we do not want to go to \( D \) since \( F(D) = 12 < 13 1/6 \). To determine \( B \) we follow the line \( (1/2x + y + z = 4 \cap 2x + y + z = 8) \). Since we have just one restriction on \( x \) that is \( z \geq 0 \), to determine \( B \) we find \( \{ 1/2x + y + z = 4 \cap 2x + y + z = 8 \cap z = 0 \} \). \( B = (8/3, 8/3, 0) \) and \( F(B) = 12 2/3 \). Hence, we can say that point \( E \) is the point that maximizes \( F(x, y, z) \).
It has been our intention to shorten mathematical computation and to arrive at a method that does not depend on a graph to arrive at a solution. We have failed at both of these. Our method does what we want it to do in that it allows us to evaluate the objective function at the corner points and to stop once we have obtained a maximum but it fails to meet the above two desirable characteristics. From here we will attempt to develop another method similar to this one which has these two characteristics in addition to the other characteristics.
CHAPTER III
DEVELOPING THE SIMPLEX METHOD

It thus becomes apparent that we desire to develop a method by which we can

1. Determine the corner points of the convex set with less mathematical computation.
2. March from one corner point to another in some definite and optimum order.
3. Determine when we have reached the corner point that will maximize the objective function.
4. Perform the operation of maximization with the least degree of difficulty.
5. Eliminate having to depend on a graph for our solution.
6. Enable allocation problems to be amenable to computer solution

At this time we should look back at the constraints that define the convex set over which we are maximizing. Suppose we are given the constraint $2x + y \leq 5$ where $x \geq 0$ and $y \geq 0$. (See Figure 28).

The above constraints define convex set $H$. The question is, can we describe convex set $H$ in another way that might be more useful to us? Let us recall that up to now we have defined convex set $H$ by $(x \geq 0 \cap y \geq 0 \cap 2x + y \leq 5)$.

Suppose we choose an arbitrary point $(x_0, y_0)$ in $H$. 

What do we know about $x_0$ and $y_0$? We know essential two things.

1. $x_0 \geq 0$ and $y_0 \geq 0$
2. $1/2x_0 + y_0 \leq 5$.

Now from 1. we see that $x_0$ and $y_0$ are nonnegative. From 2., $1/2x_0 + y_0 \leq 5$, we know that there exists a number $w_0$, $w_0 \geq 0$ such that $1/2x_0 + y_0 + w_0 = 5$. For example $(1, 1)$ is an element of $H$; $1/2 \cdot 1 + 1 \leq 5$ implies there exists $w_1$, $w_1 \geq 0$ (in particular $w_1 = 7/2$) such that $1/2 + 1 + w_1 = 5$. $(8, 1)$ is an element of $H$; $1/2 \cdot 8 + 1 \leq 5$ implies there exists $w_2$, $w_2 \geq 0$ (in particular $w_2 = 0$) such that $4 + 1 + w_2 = 5$. The same statement can be made for any $(x, y)$ in $H$. Let us now consider a point not in $H$. For example, $(6, 5)$ is not an element of $H$. This implies that $1/2 \cdot 6 + 5 > 5$, i.e. there exists $v_1$, $v_1 < 0$ such that $1/2 \cdot 6 + 5 + v_1 = 5$ (in particular $v_1 = -3$ would do the trick for the above example). Therefore, for $(x_0, y_0)$ not an element of $H$ there exists a negative number $v_0$ such that $1/2x_0 + y_0 + v_0 = 5$. It now becomes apparent that we can define $H$ in two equivalent ways.

1. $H = \{ (x, y) \mid x \geq 0, y \geq 0, \text{ and } 1/2x + y \leq 5 \}$
2. $H = \{ (x, y) \mid x \geq 0, y \geq 0 \text{ and such that there exists a } w, w \geq 0, \text{ so that } 1/2x + y + w = 5 \}$, i.e. for $(x_1, y_1) \in H$ there exist $w_1$, $w_1 \geq 0$ such that $1/2x_1 + y_1 + w_1 = 5$.

Quite clearly, $w$ is a variable since its value may
vary for different \((x, y)\) in \(H\). It should be pointed out here that two different points in \(H\) may have the same \(w\) value associated with them but in general this does not have to be the case. For example, \((8, 1)\) is an element of \(H\) and \(\frac{1}{2} \cdot 8 + 1 + w_1 = 5\) implies \(w_1 = 0\); \((0, 5)\) is an element of \(H\) and \(\frac{1}{2} \cdot 0 + 5 + w_2 = 5\) implies \(w_2 = 0\). Now for \((1, 1)\) an element of \(H\), \(\frac{1}{2} \cdot 1 + 1 + w_2 = 5\) implies \(w_3 = 7/2\). We say then that \(w\) is a variable and it depends on \(x\) and \(y\).

The important fact to remember is that if we are given: \(x \geq 0, y \geq 0\), and the constraint \(ax + by \leq c\) then we can describe the convex set defined by the above constraints as

\[
ax + by + w = c \quad \text{where} \quad x \geq 0, y \geq 0 \text{ and } w \geq 0.
\]

The variable \(w\) that was introduced to arrive at \(ax + by + w = c\) is called a slack variable, the reason for the name being obvious, since it takes up the slack between the left-hand and the right-hand sides of the inequality \((ax + by \leq c)\).

Let us now consider an example so that we may observe the usefulness of the slack variable that we have introduced. Suppose we have the following constraints

\[
\frac{1}{3}x + y \leq 4 \\
x + y \leq 7
\]

where \(x \geq 0\) and \(y \geq 0\). These constraints define convex set \(G\). (See Figure 29).

Is there any way of determining the corners of the
convex set $G$ without having to determine specifically what
${x = 0 \cap y = 0}$, ${x = 0 \cap 1/3x + y = 4}$, ${x = 0 \cap x + y = 7}$,
${y = 0 \cap 1/3x + y = 4}$, ${y = 0 \cap x + y = 7}$ and
${x + y = 7 \cap 1/3x + y = 4}$ are and then having to eliminate
the $(x, y)$ that are not in the convex set?

Recall that convex set $G$ may be defined by the equivalent form,

$$G = \{(x, y) \mid x \geq 0, y \geq 0, w_1 \geq 0 \text{ and } w_2 \geq 0 \text{ such that } 1/3x + y + w_1 = 4 \text{ and } x + y + w_2 = 7 \}.$$ 

In other words, $G = S \cap T$ where $S = \{(x, y) \mid x \geq 0, y \geq 0,$
$w_2 \geq 0 \text{ such that } x + y + w_2 = 7 \}$ and $T = \{(x, y) \mid x \geq 0,$
$y \geq 0, w_1 \geq 0 \text{ such that } 1/3x + y + w_1 = 4 \}.$

Let us for a moment consider the points of intersection, $O$, $A$, $B$, $C$, $D$, and E. (See Figure 29). Let us find out just what relationship exists between the points of intersection and the equations we have just defined; namely, $1/3x + y + w_1 = 4$ and $x + y + w_2 = 7$.

$O = (0, 0)$ implies from the equations that $w_1 = 4$
and $w_2 = 7$.

$A = (7, 0)$ implies from the equations that
$w_1 = 5/3$ and $w_2 = 0$.

$B = (12, 0)$ implies from the equations that
$w_1 = 0$ and $w_2 = -5$.

$C = (9/2, 5/2)$ implies from the equations that
$w_1 = 0$ and $w_2 = 0$. 
D = (0, 4) implies from the equations that $w_1 = 0$
and $w_2 = 3$.

E = (0, 7) implies from the equations that $w_1 = -3$
and $w_2 = 0$.

There are a few very important observations that
should be made.

1. Exactly two of the variable are equal to zero
at each point of intersection. This would imply
that given the equations (assuming that the con­
straints defining the equations contain two
variables) all we need do is set two of the
variables equal to zero and solve for the re­
mainng variables. The result would be a point
of intersection.\footnote{We should point out here that it is not necessarily
ture that exactly two are zero, the condition should read,
that at least two are zero.}

2. We notice that points B and E are not corner
points of the convex set. We should also notice
that from B = (12, 0) it is implied that $w_1 = 0$
and $w_2 = -5$. We have thus violated the condition
that $w_2$ be nonnegative. The same is true for
point E. This would imply that once we have
found the points of intersect by using the pro­
cedure described by 1., that we should exclude
those points that violate the condition that
$x \geq 0$, $y \geq 0$, and $w_1 \geq 0$ for $i = 1, 2$.

It thus appears that we have come up with a method
that allows us to determine the corner points of the convex
set with relative ease.

It is now appropriate to give an intuitive reason
as to why the above is true. When we set $x = 0$ and $y = 0$ we
are on line $l_1(y = 0)$ and line $l_2(x = 0)$ and $l_1 \cap l_2 = (0, 0)$.\footnote{We should point out here that it is not necessarily
true that exactly two are zero, the condition should read,
that at least two are zero.}
When $y = 0$ and $w_2 = 0$ we are on line $l_1(y = 0)$ and line $l_4(x + y = 7)$ and $l_1 \cap l_4 = (7, 0)$. When $y = 0$ and $w_1 = 0$ we are on line $l_1(y = 0)$ and line $l_3(1/3x + y = 4)$ and $l_1 \cap l_3 = (12, 0)$. When $w_1 = 0$ and $w_2 = 0$ we are on lines $l_3(1/3x + y = 4)$ and $l_4(x + y = 7)$ and $l_3 \cap l_4 = (9/2, 5/2)$. When $x = 0$ and $w_1 = 0$ we are on line $l_2(x + 0)$ and line $l_3(1/3x + y = 4)$ and $l_2 \cap l_3 = (0, 4)$. When $x = 0$ and $w_2 = 0$ we are on lines $l_2(x + 0)$ and $l_4(x + y = 7)$ and $l_2 \cap l_4 = (0, 7)$. The reason for excluding points B and E was given above, but essentially they are excluded because they do not lie in the convex set, i.e. they violate at least one of the equations by making either $w_1$ or $w_2$ negative.

Consider the example that gave us so much difficulty in the 3-dimensional case. (See Figure 27). The constraints defining the convex set were $x \geq 0, y \geq 0$,

\[
2x + y + z \leq 8,
\]

\[
2x + y + 2z \leq 10,
\]

and \[
1/2x + y + z \leq 4.
\]

The coordinates for the points were $O = (0, 0, 0)$, $A = (0, 4, 0)$, $B = (8/3, 8/3, 0)$, $C = (4, 0, 0)$, $D = (3, 0, 2)$, $E = (8/3, 2/3, 2)$, $F = (2, 0, 3)$ and $G = (0, 0, 4)$. We will attempt to arrive at these values using the above method.

We first introduce slack variables $w_1$, $w_2$, and $w_3$ where $w_1 \geq 0$ for $i = 1, 2, 3$,

\[
2x + y + z + w_1 = 8
\]

\[
2x + y + 2z + w_2 = 10
\]
\[ \frac{1}{2}x + y + z + w_3 = 4. \]

Procedure

\( x = 0, \ y = 0, \) and \( z = 0 \) implies \( w_1 = 8, \ w_2 = 10, \) and \( w_3 = 4. \) \((0, 0, 0)\) is a corner point.

\( x = 0, \ y = 0, \) and \( w_1 = 0 \) implies \( z = 8, \ w_2 = -6, \) and \( w_3 = -4. \) Exclude this as a corner point since \( w_2 \neq 0 \) and \( w_3 \neq 0. \)

\( x = 0, \ y = 0, \) and \( w_2 = 0 \) implies \( z = 5, \ w_1 = 3, \) and \( w_3 = -1. \) Exclude this since \( w_3 \neq 0. \)

\( x = 0, \ y = 0, \) and \( w_3 = 0 \) implies \( z = 4, \ w_1 = 4, \) and \( w_2 = 2. \) \((0, 0, 4)\) is corner point C.

\( y = 0, \ z = 0, \) and \( w_1 = 0 \) implies \( x = 4, \ w_2 = 2, \) and \( w_3 = 2. \) \((4, 0, 0)\) is corner point C.

\( y = 0, \ z = 0, \) and \( w_2 = 0 \) implies \( x = 5, \ w_1 = -2, \) and \( w_3 = 3/2. \) Exclude this since \( w_1 \neq 0. \)

\( y = 0, \ z = 0, \) and \( w_3 = 0 \) implies \( x = 8, \ w_1 = -8, \) and \( w_2 = -6. \) Exclude this since \( w_1 \neq 0 \) and \( w_2 \neq 0. \)

\( x = 0, \ z = 0, \) and \( w_1 = 0 \) implies \( y = 8, \ w_2 = 2, \) and \( w_3 = -4. \) Exclude this since \( w_3 \neq 0. \)

\( x = 0, \ z = 0, \) and \( w_2 = 0 \) implies \( y = 10, \ w_1 = -2, \) and \( w_3 = -4. \) Exclude this since \( w_1 \neq 0 \) and \( w_3 \neq -4. \)

\( x = 0, \ z = 0, \) and \( w_3 = 0 \) implies \( y = 4, \ w_2 = 6, \) and \( w_1 = 4. \) \((0, 4, 0)\) is corner point A.

\( x = 0, \ w_1 = 0, \) and \( w_2 = 0 \) implies \( z = 2, \ y = 6, \) and \( w_3 = 4. \) Exclude this since \( w_3 \neq 0. \)
x = 0, \( w_1 = 0 \), and \( w_3 = 0 \). We get two inconsistent equations so we have no point of intersection.

\[ x = 0, \quad w_2 = 0, \quad \text{and} \quad w_3 = 0 \] implies \( z = 6, \quad y = -2, \) and \( w_1 = 4 \).

Exclude this since \( y \neq 0 \).

\[ y = 0, \quad w_1 = 0, \quad \text{and} \quad w_2 = 0 \] implies \( z = 2, \quad x = 3, \) and \( w_3 = 1/2 \).

\( (3, 0, 2) \) is corner point D.

\[ y = 0, \quad w_1 = 0, \quad \text{and} \quad w_3 = 0 \] implies \( x = 8/3, \quad z = 8/3, \) and \( w_2 = -2/3 \).

Exclude since \( w_2 \neq 0 \).

\[ y = 0, \quad w_2 = 0, \quad \text{and} \quad w_3 = 0 \] implies \( x = 2, \quad z = 3, \) and \( w_1 = 1 \).

\( (2, 0, 3) \) is corner point F.

\[ z = 0, \quad w_1 = 0, \quad \text{and} \quad w_2 = 0 \] implies two inconsistent equations so we have no points of intersection.

\[ z = 0, \quad w_1 = 0, \quad \text{and} \quad w_3 = 0 \] implies \( x = 8/3, \quad y = 8/3, \) and \( w_2 = 2 \).

\( (8/3, 8/3, 0) \) is corner point B.

\[ z = 0, \quad w_2 = 0, \quad \text{and} \quad w_3 = 0 \] implies \( x = 4, \quad y = 2, \) and \( w_1 = -2 \).

Exclude since \( w_1 \neq 0 \).

\[ w_1 = 0, \quad w_2 = 0, \quad \text{and} \quad w_3 = 0 \] implies \( x = 8/3, \quad y = 2/3, \) and \( z = 2 \).

\( (8/3, 2/3, 2) \) is corner point E.

It is now apparent that we could evaluate \( F(x, y, z) \) at points O, A, B, C, D, E, F and G and determine which point maximizes the function.

Although we have performed an extreme amount of mathematical computations above we have at least achieved one of our objectives, that of being able to solve a linear programming problem without graphic aids.

In order to cut down the amount of mathematical
computations we will be adding refinement to the above method which involves the slack variables. Again, we should point out that we are still searching for a method that can be computerized.

Recall that in the above problem involving \( x \geq 0, y \geq 0, z \geq 0, w_1 \geq 0, w_2 \geq 0 \) and \( w_3 \geq 0 \) where
\[
\begin{align*}
2x + y + z + w_1 &= 8 \\
2x + y + 2z + w_2 &= 10 \\
\frac{1}{2}x + y + z + w_3 &= 4
\end{align*}
\]
that we arbitrarily assigned the value zero to three of the six variable and solve for the remaining three. Upon repeating this procedure we obtained 20 possible solutions. These twenty solutions are called basic solutions. After inspection we found out that twelve of the basic solutions contained negative values for the variables and therefore were not feasible since they did not satisfy the nonnegative requirement. The remaining eight solutions are called the basic feasible solutions since they satisfy the conditions of the constraints and the condition that they be nonnegative. In looking at one of the basic solutions obtained the variables that were assigned the value of zero, i.e. those required to be zero, are called the nonbasic variables. Those that are not required to be zero, although they might be zero, will be called the basic variables. For each of the above basic solutions then, we have three nonbasic variables and three basic variables.
Let us attempt to develop a method of solving a linear programming problem using the concept of slack variables and the concept of basic and nonbasic variables. Let us do this using a simple example. Suppose we wish to maximize \[ F(x, y) = 2x + y \]
where \[ x \geq 0, \ y \geq 0, \ 1/3x + y \leq 4 \]
\[ x + y \leq 6 \]
and \[ 1/2x - y \leq 2. \]
(See Figure 30).

Introducing the slack variables we have \( x \geq 0, \ y \geq 0, \ w_1 \geq 0, \ w_2 \geq 0, \ w_3 \geq 0 \), where we wish to maximize
\[ F(x, y) = 2x + y \]
and
\[ 1/3x + y + w_1 = 4 \]
\[ x + y + w_2 = 6 \]
and \[ 1/2x - y + w_3 = 2. \]

Setting \( x + 0 \) and \( y + 0 \) we arrive at the first basic feasible solution. We have
\[ x = 0, \ y = 0, \ w_1 = 4, \ w_2 = 6, \ w_3 = 2, \text{ and } F(0, 0) = 0. \]

Now in this case \( x \) and \( y \) are nonbasic variables the \( w_1 \), \( i = 1, 2, 3 \), are basic variable. We can write the above equations as
\[ w_1 = 4 - 1/3x - y \]
\[ w_2 = 6 - x - y \]
\[ w_3 = 2 - 1/2x + y \]
to indicate that the variables on the right hand side of the equation are nonbasic, those on the left are basic.
Using the ideas developed earlier we decide to increase the variable \( x \) since it contributes more toward maximizing \( F(x, y) \). Recall that we want our method to move from one corner point to another. This means that if we increase \( x \), i.e. if \( x \) becomes nonzero, by the ideas discussed earlier, one of the variables that is now basic will have to become zero, i.e. become nonbasic. In particular, looking at figure 30, we are at point 0 increasing \( x \) should take us to A where \( x \) is nonzero and one of the \( w_i \) is zero, i.e. nonbasic. We must find out then which one of the \( w_i \) \( x \) will replace as a basic variable. This we determine by finding out which one of \( w_1 = 4 - 1/3x - y \), \( w_2 = 6 - x - y \), and \( w_3 = 2 - 1/2x + y \) restricts \( x \)'s increase the most. For example, if \( x \) replaces \( w_1 \) as a basic variable \( x = 12 - 3y - 3w_1 \) since \( y \) and \( w_1 \) will be nonbasic \( x = 12 \). If \( x \) replaces \( w_2 \) as a basic variable \( x = 6 - y - w_2 \), since \( y \) and \( w_2 \) will be nonbasic \( x = 6 \). If \( x \) replaces \( w_3 \) as a basic variable \( x = 4 + 2y - 2w_3 \), since \( y \) and \( w_3 \) will be nonbasic \( x = 4 \). Thus it appears that \( x \) should replace \( w_3 \) as the basic variable. We might argue that increasing \( x \) the most, i.e. letting \( x = 12 \) should make the value of the objective function the largest, but notice what will happen. From \( w_1 = 4 - 1/3x - y \) we would have that \( w_1 = 0 \). From \( w_2 = 6 - x - y \) we would have \( w_2 = -6 \), i.e. we are no longer in the convex set. It should now become apparent why we choose the \( w_1 \) that restricts \( x \) the most as the basic variable \( x \) will replace.
Let us get back to the problem. $x$ replacing $w_3$ as a basic variable implies from $w_3 = 2 - \frac{1}{2}x + y$ that $x = 4 + 2y - 2w_3$. Since we wish the nonbasic variable to appear on the right hand side of the equations we have

\[ w_1 = 4 - \frac{1}{3}x - y = 4 - \frac{1}{3}(4 + 2y - 2w_3) - y \]
\[ w_2 = 6 - x - y = 6 - (4 + 2y - 2w_3) - y \]
\[ x = 4 + 2y - 2w_3 \]

or

\[ w_1 = \frac{8}{3} - \frac{5}{3}y + \frac{2}{3}w_3 \]
\[ w_2 = 2 - 3y + 2w_3 \]
\[ x = 4 + 2y - 2w_3 \]

Since $y = 0$, $w_3 = 0$ we have $w_1 = \frac{8}{3}$, $w_2 = 2$ and $x = 4$. $A = (4, 0)$. Now $F(x, y) = 2x + y = 2(4 + 2y - 2w_3) + y = 8 + 5y - 4w_3$ implies

1. $F(A) = 8$ since $y = 0$ and $w_3 = 0$.

2. If we increase $y$, $F(x, y)$ will increase since the sign on $5y$ is positive.

Thus we now want to increase $y$. Increasing $y$ will make either $w_1$, $w_2$, or $x$ nonbasic. Again we check for the restrictions on $y$ they are $y = \frac{8}{5}$, $y = \frac{2}{3}$, and $y = -2$. The last one we disregard since $y \geq 0$. We have $y$ replacing $w_2$ as a basic variable. From $w_2 = 2 - 3y + 2w_3$ we have $y = \frac{2}{3} - \frac{1}{3}w_2 + \frac{2}{3}w_3$ we have then

\[ w_1 = \frac{8}{3} - \frac{5}{3}y + \frac{2}{3}w_3 = \frac{8}{3} - \frac{5}{3}(\frac{2}{3} - \frac{1}{3}w_2 + \frac{2}{3}w_3) + \frac{2}{3}w_3 \]

\[ y = \frac{2}{3} - \frac{1}{3}w_2 + \frac{2}{3}w_3 \]
\[ x = 4 + 2y - 2w_3 = 4 + 2\left(\frac{2}{3} - \frac{1}{3}w_2 + \frac{2}{3}w_3\right) \]

or

\[ w_1 = \frac{14}{9} + \frac{5}{9}w_2 - \frac{4}{9}w_3 \]
\[ y = \frac{2}{3} - \frac{1}{3}w_2 + \frac{2}{3}w_3 \]
\[ x = \frac{16}{3} - \frac{2}{3}w_2 - \frac{2}{3}w_3 \]

Since \( w_2 = 0 \) and \( w_3 = 0 \), we have \( w_1 = \frac{14}{9} \), \( y = \frac{2}{3} \), and \( x = \frac{16}{3} \). \( B = (\frac{16}{3}, \frac{2}{3}) \). \( F(x, y) = 2x + y \)

\[ = 2\left(\frac{16}{3} - \frac{2}{3}w_2 - \frac{2}{3}w_3\right) + \left(\frac{2}{3} - \frac{1}{3}w_2 + \frac{2}{3}w_3\right) \]
\[ = \frac{34}{3} - \frac{5}{3}w_2 - \frac{2}{3}w_3 \]

implies

1. \( F(B) = \frac{34}{3} \) since \( w_2 = 0 \) and \( w_3 = 0 \).

2. Since \( w_2 \) and \( w_3 \) are zero and can only increase increasing either \( w_2 \) or \( w_3 \) we will decrease \( F(x, y) \). Therefore, we can no longer increase the objective function.

The maximum value of \( F(x, y) \) then is given by

\( F(\frac{16}{3}, \frac{2}{3}) = \frac{34}{3} \). Observe, that this method took us from one corner point to another until the maximum was obtained.

Let us reconsider the following problem that gave us so much difficulty when we attempted to obtain a solution.

Maximize \( F(x, y, z) = 2x + \frac{11}{4}y + 3z \)

where \( 2x + y + z \leq 8 \), \( 2x + y + 2z \leq 10 \), \( \frac{1}{2}x + y + z \leq 4 \), \( x \geq 0 \), \( y \geq 0 \), and \( z \geq 0 \).

We convert this to the equivalent problem of maximizing \( F(x, y, z) = 2x + \frac{11}{4}y + 3z \)
where \( x \geq 0, y \geq 0, z \geq 0, w_i \geq 0, i = 1, 2, 3 \) and

\[
2x + y + z + w_1 = 8
\]
\[
2x + y + 2z + w_2 = 10
\]
and \( 1/2x + y + z + w_3 = 4 \).

If there are any questions as to why or how any of the following steps are being done refer to the preceding problem.

Step 1:

Setting \( x = 0, y = 0, z = 0 \), we obtain the first basic feasible solution.

\[
x = 0, y = 0, z = 0, w_1 = 8, w_2 = 10, \text{ and } w_3 = 4.
\]

Here \( F(0, 0, 0) = 0 \).

Step 2:

As in the preceding problem we write the above set of equations as

\[
w_1 = 8 - 2x - y - z
\]
\[
w_2 = 10 - 2x - y - 2z
\]
\[
w_3 = 4 - 1/2x - y - z.
\]

We look at the objective function and decide to increase the variable \( z \).

Step 3:

\( z \) will replace one of the basic variable so we look for restrictions. They are

\[
z = 8
\]
\[
z = 5
\]
and \( z = 4 \).

\( z \) will replace \( w_3 \) as a basic variable. Thus we have
\( w_1 = 8 - 2x - y - z = 8 - 2x - y - (4 - 1/2x - y - w_3) \)
\( w_2 = 10 - 2x - 2z = 10 - 2x - y - 2(4 - 1/2x - y - w_3) \)
\( z = 4 - 1/2x - y - w_3 \)

or
\( w_1 = 4 - 3/2x + w_3 \)
\( w_2 = 2 - x + y + 2w_3 \)
\( z = 4 - 1/2x - y - w_3. \)

**Step 4:**

The values of the basic variables are now \( w_1 = 4 \), \( w_2 = 2 \), \( z = 4 \). We are at the point \((0, 0, 4)\). We now look at the objective function.

\[
F(x, y, z) = 2x + 11/4y + 3z = 2x + 11/4y + 3(4 - 1/2x - y - w_3)
= 12 + 1/2x - 1/4y - 3w_3.
\]

Since \( x, y, \) and \( w_3 \) are nonbasic, i.e. equal to zero, \( F(0, 0, 4) = 12. \)

Looking at \( F(x, y, z) \), we observe that by increasing \( x \) the value of the objective function will increase.

**Step 5:**

\( x \) will replace one of the basic variables so we look for restrictions. They are

\( x = 8/3 \)
\( x = 2 \)

and \( x = 8. \)

\( x \) will replace \( w_2 \) as a basic variable. Thus we have

\( w_1 = 4 - 3/2x + w_3 = 4 - 3/2(2 + y - w_2 + 2w_3) + w_3 \)
\[
x = 2 + y - w_2 + 2w_3 \\
z = 4 - 1/2x - y - w_3 = 4 - 1/2(2 + y - w_2 + 2w_3) - y - w_3
\]
or
\[
w_1 = 1 - 3/2y + 3/2w_2 - 2w_3 \\
x = 2 + y - w_2 + 2w_3 \\
z = 3 - 3/2y + 1/2w_2 - 2w_3.
\]

Step 6:

The values of the basic variables are now \(w_1 = 1,\)
\(x = 2,\) and \(z = 3.\) We are at the point \((2, 0, 3).\) We now consider the objective function.

\[
F(x, y, z) = 2x + 11/4y + 3z = 2(2 + y - w_2 + 2w_3) + 11/4y + 3(3 - 3/2y + 1/2w_2 - 2w_3) = 13 + 1/4y - 1/2w_2 - 2w_3.
\]

Since \(y, w_2,\) and \(w_3\) are nonbasic, \(F(2, 0, 3) = 13.\)

Looking at \(F(x, y, z)\) we observe that by increasing \(y\) the value of the objective function will increase.

Step 7:

\(y\) will replace one of the basic variables. The restrictions on \(y\) are

\[
y = 2/3 \\
y = -2 \text{ (not considered since } y \geq 0) \\
and \quad y = 2.
\]

\(y\) will replace \(w_1\) as a basic variable. Thus we have

\[
y = 2/3 - 2/3w_1 + w_2 - 4/3w_3 \\
x = 2 + y - w_2 + 2w_3 = 2 + (2/3 - 2/3w_1 + w_2 - 4/3w_3) - w_2 + 2w_3
\]
\[
\begin{align*}
z &= 3 - \frac{3}{2}y + \frac{1}{2}w_2 - 2w_3 = 3 \\
&\quad - \frac{3}{2}(2/3 - 2/3w_1 + w_2 - 4/3w_3) + 1/2w_2 - 2w_3 \\
\text{or} \\
y &= 2/3 - 2/3w_1 + w_2 - 4/3w_3 \\
x &= 8/3 - 2/3w_1 + 2/3w_3 \\
z &= 2 + w_1 - w_2.
\end{align*}
\]

Step 8:

The values of the basic variable are now \( y = 2/3 \), \( x = 8/3 \), and \( z = 2 \). We are now at the point \((8/3, 2/3, 2)\). We look at the objective function

\[
\begin{align*}
F(x, y, z) &= 2x + 11/4y + 3z = 2(8/3 - 2/3w_1 = 2/3w_3) \\
&\quad + 11/4(2/3 - 2/3w_1 + w_2 - 4/3w_3) \\
&\quad + 3(2 + w_1 - w_2) = 79/6 - 1/6w_1 - 1/4w_2 \\
&\quad - 7/3w_3.
\end{align*}
\]

Since \( w_1, w_2, \) and \( w_3 \) are nonbasic, \( F(8/3, 2/3, 2) = 79/6 \).

Looking at \( F(x, y, z) \) we observe that we can not increase any of the nonbasic variables without decreasing \( F(x, y, z) \) so we stop. The maximum is therefore \( F(8/3, 2/3, 2) = 79/6 \).

The method used in going through the preceding problems was developed in 1947 by G. B. Dantzig. It is a systematic procedure for solving linear programming problems. This procedure, the simplex method, has become the most effective general method for handling linear programming problems. Much research has gone into improving and refining the simplex method. Since it is an iterative procedure it is readily adaptable for use on the computer.
Its importance has become recognized in many fields such as industry, agriculture, transportation, economics and engineering to name a few.

Thus far we have only discussed the problem of maximizing a function over a convex set. As you are probably aware minimization of functions is just as important in a great majority of the applied problems. Let us now attempt to tie the two procedures together, i.e. let us attempt to find out whether there exists some relationships between the problem of minimizing a function and that of maximizing a function related to the function being minimized.

Suppose we are to minimize the function, \( F(x, y) = -2x + y \)

where
\[
\frac{1}{2}x - y \leq 1 \\
-\frac{1}{2}x + y \leq 3 \\
x + y \leq 4
\]

and \( x \geq 0, \ y \geq 0 \).

To use the simplex method we convert the problem to that of minimizing the function, \( F(x, y) = -2x + y \)

where
\[
\frac{1}{2}x - y + w_1 = 1 \\
-\frac{1}{2}x + y + w_2 = 3 \\
x + y + w_3 = 4
\]

and \( x \geq 0, \ y \geq 0, \ w_1 \geq 0, \ w_2 \geq 0, \ w_3 \geq 0, \ i = 1, 2, 3 \). Look at the graphic solution, figure 31.

To obtain the initial basic feasible solution we set \( x = 0, \ y = 0 \) and get \( w_1 = 1, \ w_2 = 3, \) and \( w_3 = 4 \). As before
we write the above equations as

\[ w_1 = 1 - 1/2x + y \]
\[ w_2 = 3 + 1/2x - y \]
\[ w_3 = 4 - x - y. \]

We are at the point \((0, 0)\) and \(F(0, 0) = 0\).

We look at the objective function, \(F(x, y) = -2x + y\) and decide to increase \(x\) since increasing \(x\) will decrease \(F(x, y)\). The restrictions on \(x\) are

\[ x = 2 \]
\[ x = -6 \text{ (no restriction since } x \geq 0) \]
\[ x = 4. \]

We replace the basic variable \(w_3\) by \(x\). We have from \(w_1 = 1 - 1/2x + y\) that \(x = 2 + 2y - 2w_1\).

We have then

\[ x = 2 + 2y - 2w_1 \]
\[ w_2 = 3 + 1/2(2 + 2y - 2w_1) - y \]
\[ w_3 = 4 - (2 + 2y - 2w_1) - y \]

or

\[ x = 2 + 2y - 2w_1 \]
\[ w_2 = 4 - w_1 \]
\[ w_3 = 2 - 3y + 2w_1. \]

Since \(y\) and \(w_1\) are nonbasic, we have \(x = 2, w_2 = 4, \) and \(w_3 = 2\). \(F(x, y) = -2x + y\) implies that

\[ F(x, y) = -2(2 + 2y - 2w_1) = y = -4 - 3y + 4w_1. \]

Thus \(F(x, y) = -4\) at \((2, 0)\) and by looking at the objective function we observe that it is possible to decrease \(F(x, y)\).
by increasing $y$.

The restrictions for increasing $y$ are

$y = -1$ (no restriction since $y \geq 0$)

(no restriction from $w_2 = 4 - w_1$)

$y = 2/3$.

$y$ will then replace $w_3$ as a basic variable. From

$w_3 = 2 - 3y + 2w_1$ we have $y = 2/3 + 2/3w_1 - 1/3w_3$, we have

then that

$x = 2 + 2(2/3 + 2/3w_1 - 1/3w_3) - 2w_1$

$w_2 = 4 - w_1$

$y = 2/3 + 2/3w_1 - 1/3w_3$

or

$x = 10/3 - 2/3w_1 - 2/3w_3$

$w_2 = 4 - w_1$

$y = 2/3 + 2/3w_1 - 1/3w_3$.

Since $w_1$ and $w_3$ are nonbasic $x = 10/3$, $w_2 = 4$, and

$y = 2/3$. $F(x, y) = -2x + y$ implies $F(x, y) = -2(10/3 - 2/3w_1$

$-2/3w_3) + (2/3 + 2/3w_1 - 1/3w_3) = -6 + 2w_1 + w_3$. Thus

$F(x, y) = -6$ at $(10/3, 2/3)$ and

observing $F(x, y)$ we notice that if $w_1$ or $w_3$ are increased
the value of the objective function will increase. We have
reached a minimum, $F(x, y) = -6$, where $(x, y) = (10/3, 2/3)$.

We now consider the problem of maximizing $-F(x, y)$,
where $F(x, y) = -2x + y$ as above; and the constraints are
also as above. $-F(x, y) = 2x - y$.

Setting $x = 0$ and $y = 0$ we have the initial basic
feasible solution (0, 0) and \( F(0, 0) = 0 \).

\[

d_1 = 1 - \frac{1}{2}x + y \\
d_2 = 3 + \frac{1}{2}x - y \\
d_3 = 4 - x - y.
\]

We look at the objective function, \( -F(x, y) = 2x - y \) and observe that if the variable \( x \) increases the value of \( -F(x, y) \) will increase. The restrictions on \( x \) are

\[
x = 2 \\
x = -6 \text{ (no restriction)} \\
x = 4.
\]

We replace \( w_1 \) by \( x \) as a basic variable. From \( w_1 = 1 - \frac{1}{2}x + y \) we get \( x = 2 + 2y - 2w_1 \), so that

\[
x = 2 + 2y - 2w_1 \\
w_2 = 4 - w_1 \\
w_3 = 2 - 3y + 2w_1.
\]

We are at the point (2, 0). \( -F(x, y) = 2x - y \) implies \( -F(x, y) = 2(2 + 2y - 2w_1) - y = 4 + 3y - 4w_1 \).

\( -F(x, y) = -4 \) at (2, 0) and increasing \( y \) will increase \( -F(x, y) \).

The restrictions on \( y \) are

(no restrictions from \( x = 2 + 2y = 2w_1 \))

(no restrictions from \( w_2 = 4 - w_1 \))

\[
y = \frac{2}{3}.
\]

\( y \) now replaces \( w_3 \) as a basic variable.

We have

\[
y = \frac{2}{3} + \frac{2}{3}w_1 - \frac{1}{3}w_3 \text{ so that} \\
x = \frac{10}{3} - \frac{2}{3}w_1 - \frac{2}{3}w_3 \\
w_2 = 4 - w_1
\]
\[ y = \frac{2}{3} + \frac{2}{3}w_1 - \frac{1}{3}w_3. \]

At this corner point \( x = \frac{10}{3} \) and \( y = \frac{2}{3} \). \(-F(x, y) = 2x - y\) implies \(-F(x, y) = 2\left(\frac{10}{3} - \frac{2}{3}w_1 - \frac{2}{3}w_3\right) - (\frac{2}{3} + \frac{2}{3}w_1 - \frac{1}{3}w_3)\)

\[ = 6 - 2w_1 - w_3. \]

\(-F(x, y) = 6\) at \( (\frac{10}{3}, \frac{2}{3}) \) and looking at the coefficients of \( w_1 \) and \( w_3 \) we observe that \(-F(x, y)\) can not be increased any further. The maximum is \(-F(\frac{10}{3}, \frac{2}{3}) = 6\).

We should observe that

1. The maximum of \(-F(x, y) = -(\text{the minimum of } F(x, y))\).

2. The maximum of \(-F(x, y)\) occurs at the same point as the minimum of \( F(x, y) \).

Therefore, we need only concern ourselves with the problem of being able to maximize (minimize) functions.

Before going any further we should obtain some machinery (mathematical techniques) that will enable us to perform the computations involved in the simplex method with greater ease. We will introduce the following technique because it is very adaptable to the simplex method and to use on the computer.

Suppose we want to solve the following system of equations.

\[ 2x + 3y = 17 \]
\[ 5x - 4y = 8 \]

Although there are numerous methods that may be used
to arrive at a solution let us illustrate the following method.

1 \quad 2x + 3y = 7 \quad \text{and} \\
2 \quad 5x - 4y = 8.

Step 1:

Multiply equation 1 by 5/2 and subtract it from equation 2. We get

3 \quad 2x + 3y = 7 \quad \text{and} \\
4 \quad -23/2 y = -69/2.

Step 2:

Multiply equation 4 by 6/23 and add it to equation 3. We get

5 \quad 2x = 8 \quad \text{and} \\
6 \quad -23/2 y = -69/2.

Step 3:

Divide equation 5 by 2 and equation 6 by -23/2. We get

\begin{align*}
  x &= 4 \quad \text{and} \\
  y &= 3.
\end{align*}

This is the solution to the original system of equations.

Consider the following system of equations.

\begin{align*}
  1 \quad 3x - 6y + 7z &= 3 \\
  2 \quad 9x - 5z &= 3 \\
  3 \quad 5x - 8y + 6z &= -4
\end{align*}

Step 1:

Multiply equation 1 by 3 and subtract it from equation 2. Multiply equation 1 by 5/3 and subtract it from 3. We have
\[
\begin{align*}
4 & \quad 3x - 6y + 7z = 3 \\
5 & \quad 18y - 26z = -6 \\
6 & \quad 2y - 17/3z = -9
\end{align*}
\]

**Step 2:**

Multiply equation 6 by 3 and add it to equation 4. Multiply equation 6 by 9 and subtract it from equation 5. We get

\[
\begin{align*}
7 & \quad 3x - 10z = -24 \\
8 & \quad 25z = 75 \\
9 & \quad 2y - 17/3z = -9
\end{align*}
\]

**Step 3:**

Interchange equations 8 and 9.

\[
\begin{align*}
10 & \quad 3x - 10z = -24 \\
11 & \quad 2y - 17/3z = -9 \\
12 & \quad 25z = 75
\end{align*}
\]

**Step 4:**

Multiply equation 12 by 10/25 and add it to 10. Multiply equation 12 by 17/75 and add it to 11. We have

\[
\begin{align*}
13 & \quad 3x = 6 \\
14 & \quad 2y = 8 \\
15 & \quad 25z = 75
\end{align*}
\]

**Step 5:**

Divide equation 13 by 3, equation 14 by 2 and equation 15 by 25. We get

\[
\begin{align*}
x & = 2 \\
y & = 4 \\
z & = 3
\end{align*}
\]
The above method then consists of isolating one variable to each equation by performing successive multiplications, additions, and subtractions.

We observe that the variable do not serve any real purpose in the above method; they just served as identification. This suggests that we may write the above system of equation as

\[
\begin{bmatrix}
3 & -6 & 7 & 3 \\
9 & 0 & -5 & 3 \\
5 & -8 & 6 & -4
\end{bmatrix}
\]

where we call this array a matrix. By performing row operations on this matrix, we can arrive at a solution to the original system of equations. Let us illustrate this.

We commence with

\[
\begin{bmatrix}
3 & -6 & 7 & 3 \\
9 & 0 & -5 & 3 \\
5 & -8 & 6 & -4
\end{bmatrix}
\]

Multiply row 1 by 3 and subtract it from row 2. Multiply row 1 by 5/3 and subtract it from row 3.

\[
\begin{bmatrix}
3 & -6 & 7 & 3 \\
18 & -26 & -6 \\
2 & -17/13 & -9
\end{bmatrix}
\]

Multiply row 3 by 3 and add it to row 1. Multiply row 3 by 9 and subtract it from row 2.

\[
\begin{bmatrix}
3 & 0 & -10 & -24 \\
0 & 0 & 25 & 75 \\
0 & 2 & -17/3 & -9
\end{bmatrix}
\]

Interchange rows 2 and 3.

\[
\begin{bmatrix}
3 & 0 & -10 & -24 \\
0 & 2 & -17/3 & -9 \\
0 & 0 & 25 & 75
\end{bmatrix}
\]

Multiply row 3 by 10/25 and add it to row 1. Multiply row 3 by 17/75 and add it to row 2.

\[
\begin{bmatrix}
3 & 0 & 0 & 6 \\
0 & 2 & 0 & 8 \\
0 & 0 & 25 & 75
\end{bmatrix}
\]

Divide row 1 by 3, row 2 by 2 and row 3 by 25.
The above method that we have illustrated is called the Gauss-Jordan elimination method. It can readily be applied to the problem of solving systems of equations and has the added property that it can readily be computerized.

Sometimes it is desirable to use a process known as normalization to avoid the numbers in the matrix from increasing. Adding normalization to the Gauss-Jordan method is just a refinement of the method. We will illustrate this process using an example.

Suppose we want to solve the system of equations,

\[
\begin{align*}
3x - 5y + 4z &= 17 \\
6x + y - 3z &= 1 \\
2x - y + z &= 7.
\end{align*}
\]

We have in matrix form

\[
\begin{bmatrix}
1 & 3 & -5 & 4 & 17 \\
6 & 1 & -3 & 1 \\
2 & -1 & 1 & 7
\end{bmatrix}
\]

To normalize the element of the second row and third column we divide row 2 by -3. We get

\[
\begin{bmatrix}
3 & -5 & 4 & 17 \\
-2 & -1/3 & 1 & -1/3 \\
2 & -1 & 1 & 7
\end{bmatrix}
\]

Using the Gauss-Jordan method we multiply the second row by 4 and subtract it from row 1.

\[
\begin{bmatrix}
11 & -11/3 & 0 & 55/3 \\
-2 & -1/3 & 1 & -1/3 \\
4 & -2/3 & 0 & 22/3
\end{bmatrix}
\]

Subtract row 2 from row 3 and we get

\[
\begin{bmatrix}
11 & -11/3 & 0 & 55/3 \\
-2 & -1/3 & 1 & -1/3 \\
4 & -2/3 & 0 & 22/3
\end{bmatrix}
\]

To normalize the element of the third row and the second column we divide row 3 by -2/3. We have

\[
\begin{bmatrix}
11 & -11/3 & 0 & 55/3 \\
-2 & -1/3 & 1 & -1/3 \\
-6 & 0 & 0 & -11
\end{bmatrix}
\]

Multiply row 3 by 11/3 and add it to row 1. Multiply row 3 by 1/3 and add it to row 2.
To normalize the element of the first row, first column we divide row 1 by -11.

Multiply row 1 by 4 and add it to row 2. Multiply row 1 by 6 and add it to row 3.

From this matrix we get that $x = 2$, $z = 4$, and $y = 1$.

As pointed out already, the above method is easily adaptable to the computer and it is primarily for this reason that we have introduced it. A secondary reason is that it allows us to work with system of equations in tableau form.

We will now apply the Gauss-Jordan elimination method to the problem of solving linear programming problems.

Suppose we want to maximize $F(x, y, z) = 5x + 4y + 6z$ where $x > 0$, $y > 0$, $z > 0$ and

\[
\begin{align*}
    x + y + z & \leq 100 \\
    3x + 2y + 4z & \leq 210 \\
    3x + 2y & \leq 150.
\end{align*}
\]

This problem can be transformed to the equivalent problem of maximizing $c$,

\[
c = 5x + 4y + 6z \text{ where } x \geq 0, y \geq 0, z \geq 0, w_1 \geq 0, i = 1, 2, 3 \text{ and }
\]

\[
\begin{align*}
    x + y + z + w_1 & = 100 \\
    3x + 2y + 4z + w_2 & = 210 \\
    3x + 2y + w_3 & = 150.
\end{align*}
\]
For convenience we rephrase the problem to that of determining \( x \geq 0, \ y \geq 0, \ z \geq 0, \ w_i \geq 0, \ i = 1, 2, 3 \) so that
\[-5x - 4y - 6z + ow_1 + ow_2 + ow_3 + c = 0 \]
and \( c \) has its maximum value.

We will solve this problem using the Gauss–Jordan elimination method, as well as the method we have been using up to now to solve linear programming problems. This we do so that we can observe that what we are doing parallels what we would do using the first method.

We first write the four equations in the matrix form known as a simplex tableau.

\[
\begin{array}{ccccccccc}
  x & y & z & w_1 & w_2 & w_3 & c \\
\hline
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 100 \\
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 210 \\
  1 & 2 & 4 & 0 & 1 & 0 & 0 & 150 \\
  3 & 2 & 0 & 0 & 0 & 1 & 0 & 150 \\
\end{array}
\]

We obtain the initial solution by letting \( x = 0, \ y = 0, \) and \( z = 0. \) We have \( w_1 = 100, \)
\( w_2 = 210, \) and \( w_3 = 150. \) The first row corresponds to the objective function. The new basic variable will be the variable with the smallest negative coefficient. Circle that column. Note that increasing \( z \) are

To start the computations we set \( x, \ y, \) and \( z \) equal to zero.

We have \( x = 0, \ y = 0, \) and \( z = 0 \)

1. \( w_1 = 100 - x - y - z - 100 \)
2. \( w_2 = 210 - 3x - 2y - 4z \)
3. \( w_3 = 150 - 3x - 2y = 150 \)
4. \( c = 0 + 5x + 4y + 6z = 0. \)

Looking at equation 4 we observe that if we increase \( z \) the value of \( c \) will increase the most.

The restrictions on increasing \( z \) are

\( z = 100 \)
z will force c to increase. To determine which one of the basic variables \( (w_1, w_2, w_3) \) z will replace. Divide each positive element on the z column excluding the first element into the respective element in the constant row. Circle the row giving the smallest quotient. The reason that we consider only the positive values in the z column is that non-positive values of z impose no restriction on z, i.e. z can become infinitely large. z will replace \( w_2 \) as a basic variable. We normalize the pivot, the element at the intersection of the circled row and the circled column, by dividing row 3 by the pivot element.

\[
\begin{array}{ccccccc}
-5 & -4 & -6 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 \times 100 \\
B. 3/4 & 1/2 & 1 & 0 & 1/4 & 0 & 0 \times 105/2 \\
3 & 2 & 0 & 0 & 0 & 1 & 0 \times 150 \\
\end{array}
\]

Using the Gauss-Jordan elimination method to remove the coefficients of z from all the rows except the

\[
z = 210/4 = 52 1/2
\]

(no restrictions from equation 3). z will then replace \( w_2 \) as a basic variable. From \( w_2 = 210 - 3x - 2y - 4z \) we have that \( z = 105/2 - 3/4x - 1/2y - 1/4w_2 \).

Look at table B. Note the third row. The third row tells us that

\[
3/4x + 1/2y + z + w_2 = 105/2,
\]

i.e. \( z = 105/2 - 3/4x - 1/2y - 1/4w_2 \).

Normalizing the third row has given us an expression equivalent to

\[
z = 105/2 - 3/4x - 1/2y - 1/4w_2.
\]
Using the Gauss-Jordan elimination method to remove the coefficients of \( z \) from all the rows except the third row we have

\[
\begin{bmatrix}
-1/2 & -1 & 0 & 0 & 3/2 & 0 & 1 & 3/5 \\
1/4 & 1/2 & 0 & 1 & -1/2 & 0 & 0 & 95/2 \\
3/4 & 1/2 & 1 & 0 & 1/4 & 0 & 0 & 105/2 \\
3 & 2 & 0 & 0 & 0 & 1 & 0 & 150
\end{bmatrix}
\]

The equations we now have are

1. \(-1/x - y + 3/2w_2 + c = 315\)
2. \(1/4x + 1/2y + w_1 - 1/4w_2 = 95/2\)
3. \(3/4x + 1/2y + z + 1/4w_2 = 105/2\)
4. \(3x + 2y + w_3 = 150\).

Note that these equations are equivalent to equations 9-12 on the opposite side of the page.

We look at the first row and choose the variable with the smallest negative coefficient as the variable to become the new basic variable.

Circle that column. \( y \) is selected as the variable to become the new basic variable. We determine the restrictions on \( y \) by dividing the positive coefficients in the \( y \) column into the respective

We thus have

5. \( w_1 = 100 - x - y \\
   - (105/2 - 3/4x \\
   - 1/2y - 1/4w_2)\)

6. \( z = 105/2 - 3/4x \\
   - 1/2y - 1/4w_2 \)

7. \( w_3 = 150 - 3x - 2y \)

8. \( c = 0 + 5x + 4y \\
   + 6(105/2 - 3/4x \\
   - 1/2y - 1/4w_2), i. \)

9. \( w_1 = 95/2 - 1/4w \\
   - 1/2y + 1/4w_2 \)

10. \( z = 105/2 - 3/4x \\
    = 1/2y - 1/4w_2 \)

11. \( w_3 = 150 - 3x - 2y \)

12. \( c = 315 + 1/2x \\
    + y - 3/2w_2 = 315 \)

Looking at equation 12 we observe that increasing \( y \) will increase \( c \). The restrictions on \( y \) increasing are \( y = 95 \) \( y = 105 \) \( y = 75 \).

\( y \) will thus replace \( w_3 \) as a basic variable. \( w_3 = 150 - 3x - 2y \) implies that \( y = 75 - 3/2x - 1/2w_3 \).
elements in the constant column. We select the smallest quotient and circle the row on which it was obtained. y will replace \( w_3 \) as basic variable. We normalize the pivot element by dividing row 4 by the pivot element.

\[
\begin{bmatrix}
-1/2 & -1 & 0 & 0 & 3/2 & 0 & 1 & 315 \\
1/4 & 1/2 & 0 & 1 & -1/4 & 0 & 0 & 95/2 \\
3/4 & 1/2 & 1 & 0 & 1/4 & 0 & 0 & 105/2 \\
3/2 & 1 & 0 & 0 & 0 & 1/2 & 0 & 75
\end{bmatrix}
\]

Using the Gauss-Jordan elimination method we eliminate the coefficients of the y variable from every row except the fourth row. We have

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 3/2 & 1/2 & 1 & 390 \\
-1/2 & 0 & 0 & 1 & -1/4 & -1/4 & 0 & 10 \\
0 & 0 & 1 & 0 & 1/4 & -1/4 & 0 & 15 \\
3/2 & 1 & 0 & 0 & 0 & 1/2 & 0 & 75
\end{bmatrix}
\]

Note that the equations represented in the above matrix are equivalent to equations 17-20.

Looking at the first row of the matrix we notice that there are no negative values entered. The conclusion then is that we have

Since \( y = 75 - 3/2x - 1/2w_3 \) we have

13. \( w_1 = \frac{95}{2} - \frac{1}{4}x - \frac{1}{2}(75 - \frac{3}{2}x - \frac{1}{2}w_3) \) + \( \frac{1}{4}w_2 \)

14. \( z = \frac{105}{2} - \frac{3}{4}x - \frac{1}{2}(75 - \frac{3}{2}x - \frac{1}{2}w_3) - \frac{1}{4}w_2 \)

15. \( y = 75 - \frac{3}{2}x - \frac{1}{4}w_2 \)

16. \( c = 315 + \frac{1}{2}x + (75 - \frac{3}{2}x - \frac{1}{2}w_3) - \frac{1}{4}w_2 \), \( i.e. \)

17. \( w_1 = 10 + \frac{1}{2}x + \frac{1}{4}w_2 + \frac{1}{4}w_3 = 10 \)

18. \( z = 15 - \frac{1}{4}w_2 + \frac{1}{4}w_3 = 15 \)

19. \( y = 75 - \frac{3}{2}x - \frac{1}{2}w_3 = 75 \)

20. \( c = 390 - x - \frac{3}{2}w_2 - \frac{1}{2}w_3 = 390 \).

Equation 20 allows us to say that no further increase in c is possible. We have that the maximum
reached a maximum. From the table of $F(x, y)$ is
we have

$$F(0, 75, 15) = 390.$$  

c = 390  
z = 15  
and $y = 75$.

The maximum is $c = 390$ and it is given by the point $(0, 75, 15)$.

It will soon become apparent just how much of a time and space saver the simplex tableau form for solving linear equations is.

We consider a few exercises to illustrate the advantages afforded by the method just introduced.

Exercise 1:

Maximize $3x - y$

where

$$x - y \leq 1$$
$$-x + y \leq 1$$
$$x \leq 2$$
$$2x + 2y \leq 7$$

and $x \geq 0$, $y \geq 0$.

We may write this down as the problem of maximizing $c$, where
\[-3x + y + ow_1 + ow_2 + ow_3 + ow_4 + c = 0.\]

The convex set over which we are maximizing \(c\) is defined by

\[
\begin{align*}
    x - y + w_1 &= 1, \\
    -x + y + w_2 &= 1, \\
    x + w_3 &= 2, \\
    2x + 2y + w_4 &= 7,
\end{align*}
\]

and \(x \geq 0, y \geq 0, w_i \geq 0, \ i = 1, 2, 3, 4.\)

Step 1: Select the most negative element in the first row. Circle the column where this element was found.

\[
\begin{pmatrix}
-3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\
2 & 2 & 0 & 0 & 0 & 1 & 0 & 7
\end{pmatrix}
\]

Step 2: Determine the quotients of the positive elements of the column circled with the corresponding elements of the constant column. Circle the row giving you the smallest quotient.

Step 3: Normalize by dividing the circled row by the pivot element. Eliminate all the coefficients from the circled column except for the element now occupying the pivot position using the Gauss-Jordan elimination method.

\[
\begin{pmatrix}
0 & -2 & 3 & 0 & 0 & 0 & 1 & 3 \\
1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\
0 & 1 & -1 & 0 & 1 & 0 & 0 & 1 \\
0 & 4 & -2 & 0 & 0 & 1 & 0 & 5
\end{pmatrix}
\]

Step 4: Determine the column in which the most negative element of the first row lies. Circle that column. Determine the quotients of the positive elements in that column with the elements in the constant column. Circle the row giving the smallest quotient. Normalize using the pivot element and eliminate using the Gauss-Jordan method.
Step 5: Since the first row contains no negative elements, it is not possible to increase the value of \( c \) any further. The value of \( c \) is given by the first element of the last row. The value for \( x \) is given by the second element of the last row. The value of \( y \) is given by the fourth element of the last row. The maximum then is \( c = 5 \) and it is obtained when \( x = 2 \) and \( y = 1 \), i.e. at the point \((2, 1)\).

Exercise 2: The reasons for the steps will be analogous to those in the above two problems.

Maximize: \( c = 2x_1 + x_2 \)

where
\[
\begin{align*}
    x_1 + x_2 &\leq 3 \\
    x_1 + 5x_2 &\leq 10 \\
    2x_1 &\leq 3 \\
    10x_1 + 5x_2 &\leq 40
\end{align*}
\]

and \( x_i \geq 0 \) for \( i = 1, 2 \).

We rewrite the problems as

Maximize \( c \), where
\[
\begin{align*}
-2x_1 - x_2 + w_1 + ow_1 + ow_2 + ow_3 + ow_4 + c &= 0 \\
    x_1 + x_2 + w_1 &= 3 \\
    x_1 + 5x_2 + w_2 &= 10 \\
    2x_1 + w_3 &= 3 \\
    16x_1 + 5x_2 + w_4 &= 40
\end{align*}
\]

where \( x_i \geq 0 \ i = 1, 2, w_j \geq 0 \ j = 1, 2, 3, 4 \).
We stop since the first row contains only nonnegative elements. The maximum for $c$, $c = 9/2$, is obtained when $x = 3/2$ and $y = 3/2$.

Exercises to consider.

Exercise 1. Maximize $F(x, y) = \frac{1}{2}x + y$

where

$1/3x - y \leq 0$

$-4x + y \leq 0$

$3/2x + y \leq 11/2$

and $x \geq 0$, $y \geq 0$.

Answer: $F(1, 4) = 4 \frac{1}{2}$
Exercise 2: Maximize $F(x, y) = 2x + y$
where
$-2x + y \leq 0$
$2/5x - y \leq 0$
$y \leq 5/2$
and $x \geq 0, y \geq 0$.
Answer: $F(25/4, 5/2) = 15$

Exercise 3: Minimize $F(x, y) = -1/2x - y$
where
$3x - y \leq 0$
$4x + y \leq 21$
$-x + 3y \leq 11$
$-4x + y \leq 0$
and $x \geq 0, y \geq 0$.
Answer: $F(4, 5) = -7$

In the problems that we have considered we used the simplex method to solve maximization problems involving constraints containing only inequalities of the "less than or equal to" type. The initial basic feasible solution that started off the iterations could easily be found by setting $x = 0$ and $y = 0$, i.e. making $x, y$ nonbasic variables. In general, however, not all linear programming problems are this way.

We will not discuss problems that are not solvable by the technique developed up to this point. It will suffice for us to point out that such problems do exist and that adaptation can be made to the simplex method so that most of these problems are solvable. There are other problems that are encountered in using the simple method, such problems as degeneracy where we find ourselves going in a never-ending loop in which a sequence of basic feasible solutions corresponding to the same value of the objective function is repeated over and over without ever reaching the final solution. This situation will occur at times when there is a tie in determining the smallest quotient obtained by dividing the positive element in the column of variable to be increased into the corresponding elements in the constant column. There are techniques that are used to get around some of these degeneracy problems.
As we have pointed out the simplex method does lend itself to the computer. What follows is a general program to solve linear programming problems using the simplex method and also included are some problems that were evaluated using the program.

SIMPLEX METHOD METHOD FOR SOLVING LINEAR PROGRAMMING PROBLEMS FORTRAN II MUST BE USED

THE LIMIT OF THE PROGRAM IS AS FOLLOWS
MAX. NUMBER OF VARIABLES INCLUDING SLACK VARIABLES = 12
MAX. NUMBER OF EQUATIONS INCLUDING THE OBJECTIVE FUNCTION = 20
THE INITIAL SIMPLEX TABLEAU CORRESPONDING TO THE GIVEN EQUATIONS MUST BE PUT INTO THE FORM

\[
\begin{array}{cccc}
A(1,1) & A(1,2) & \ldots & A(1,J_J) \\
A(2,1) & A(2,2) & \ldots & A(2,J_J) \\
\vdots & \vdots & \ddots & \vdots \\
A(J_I,1) & A(J_I,2) & \ldots & A(J_I,J_J)
\end{array}
\]

WHERE THE ELEMENTS ARE DEFINED FROM THE EQUATIONS

DIMENSION A(12,20), W(12), L(12)

READ IN
II=TOTAL NUMBER OF THE GIVEN EQUATIONS INCLUDING THE OBJECTIVE FUNCTION.
JJ=TOTAL NUMBER OF ROWS OF THE ABOVE SIMPLEX TABLEAU

108 READ 1, II, JJ
1 FORMAT (18I4)

READ THE ELEMENTS OF THE MATRIX ROW BY ROW.

DO 9 I=1, II
9 READ 4, (A(I,J), J=1, JJ)
4 FORMAT (7F10.4)
PUNCH 111
111 FORMAT (18HTHE INITIAL MATRIX)
DO 99 I=1, II
PUNCH 150, I
99 PUNCH 4, (A(I,J), J=1, JJ)

READ IN THE SUBSCRIPT FOR THE SLACK VARIABLE ON ROW (I) WHERE I IS NOT EQUAL TO 1 AND III.

READ 1, (L(I), I=2, II)

NEXT STATEMENT FOR INITIALIZATION

KKK=0

NEXT STATEMENTS FOR SEARCHING FOR THE COLUMN AT WHICH THE MOST NEGATIVE ENTRY APPEARS IN THE OBJECTIVE FUNCTION.

40 K=1
44 J=0
   W(K)=0.
   L(K)=0
42 J=J+1
   IF (J-JJ) 41, 45, 45
41 IF (A(K,J)) 43, 42, 42
43 IF (W(K)-A(K,J)) 42, 42, 47
47 W(K)=J
   GO TO 42

TEST FOR L(K). IF L(K) IS EQUAL TO ZERO, THAT IS, ALL THE ENTRIES EXCEPT THE EXTREME RIGHT ONE EITHER IN THE FIRST ROW ARE POSITIVE, GO TO ST. 62 FOR FURTHER EXAMINATION.

45 IF (L(K)) 46, 62, 46

FIND THE PIVOT COLUMN

46 KJ=L(K)

TEST EVERY ENTRY IN THE PIVOT COLUMN TO SEE IF IT IS POSITIVE OR NOT.

DO 120 I=2, II
   IF (A(I,KJ)) 120, 120, 121
120 CONTINUE

IF ALL THE ENTRIES IN THE PIVOT COLUMN ARE ZERO OR NEGATIVE NUMBERS, 'UNBOUNDED' IS GOING TO BE TYPED.

PUNCH 130
130 FORMAT (9HUNBOUNDED)
GO TO 70
THE FOLLOWING STATEMENTS ARE FOR COMPUTING THE RATIO DEFINED WHICH IS USED TO DETERMINE THE LOCATION OF THE PIVOT

121 I=1
   JK=0
50 I=I+1
   IF (I=II) 52, 52, 56
52 IF (A(I,KJ)) 50, 50, 51
51 X=A(I,JJ)/A(I,KJ)
   IF (JK) 55, 53, 55
55 XMN=X
   JK=I
   GO TO 50

THE NEXT STATEMENT INDICATES THE PIVOT ELEMENT BEFORE NORMALIZATION

56 X=A(JK,KJ)
   L(JK)=KJ

NEXT STATEMENTS FOR CALCULATING THE NEW ROWS ABOVE THE PIVOT ROW

DO 57 I=1, II
57 W(I)=A(I,KJ)
   IJ=JK-1
   DO 59 I=1, IJ
   DO 59 J=1, JJ
      IF (A(JK,J)) 58, 59, 58
58 IF (W(I)) 580, 59, 580
580 A(I,J)=A(I,J)-W(I)*(A(JK,J)/X)
   CONTINUE

NEXT STATEMENTS FOR CALCULATING THE NEW ROWS BELOW THE PIVOT ROW

IJ=JK+1
   DO 61 I=IJ, II
   DO 61 J=1, JJ
      IF (A(JK,J)) 60, 61, 60
60 IF (W(I)) 600, 61, 600
600 A(I,J)=A(I,J)-W(I)*(A(JK,J)/X)
   CONTINUE

NEXT STATEMENTS FOR NORMALIZATION

DO 205 J=1, JJ
205 A(JK,J)=A(JK,J)/X
   KKK=KKK+1
PUNCH 101
101 FORMAT (///46HITERATION OBJ, FUNCTION NEW BASIC VAR.)
PUNCH 105, KKK, A(K,JJ), L(JX)
PUNCH 117
105 FORMAT (IX, 14, 6X, F15.2, 10X, I4)
117 FORMAT (///10HTHE MATRIX)
DO 901 I=1, II
PUNCH 150, I
901 PUNCH 4, (A(I,J), J=1,JJ)
GO TO 44

NEXT STATEMENT FOR TESTING TO SEE IF IT IS THE FIRST
ROW ON WHICH ALL THE ENTRIES ARE POSITIVE EXCEPT THE
EXTREME RIGHT ONE. IF IT IS SO, THAT MEANS, NO FURTHER
IMPROVEMENT ON THE SOLUTION CAN BE MADE? GO TO ST. 70
AND THE ANSWER WILL BE TYPED OUT

62 IF (K-1) 70, 70, 70

TYPE OUT SOLUTION

70 PUNCH 8, A(1,JJ)
8 FORMAT (///13HOBJ. FUNCTION, F20.8/)
PUNCH 7
7 FORMAT (23HVARIABLE VALUE)
DO 71 I=2, II
71 PUNCH 5, L(I), A(I,JJ)
5 FORMAT (I4, F20.8)
GO TO 108
150 FORMAT (///35X, 4HR0W, 12/)
END

SOLVED PROBLEMS

THE INITIAL MATRIX

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ROW 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-10.0000</td>
<td>-11.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td><strong>ROW 2</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0000</td>
<td>4.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>9.0000</td>
</tr>
<tr>
<td><strong>ROW 3</strong></td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
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<td>2.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>8.0000</td>
</tr>
<tr>
<td><strong>ROW 4</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0000</td>
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THE MATRIX

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OBJ. FUNCTION 26,500,000

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THE INITIAL MATRIX

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ITERATION   OBJ. FUNCTION   NEW BASIC VAR.
1             12.00          3

THE MATRIX

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ITERATION   OBJ. FUNCTION   NEW BASIC VAR.
2             13.00          1

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ROW 4
0.0000 1.5000 1.0000 0.0000 -.5000 2.0000 3.0000
OBJ. FUNCTION 13.16
NEW BASIC VAR. 2

THE MATRIX

ROW 1
0.0000 0.0000 0.0000 .1666 .2500 2.3333 13.1666
ROW 2
0.0000 1.0000 0.0000 .6666 -1.0000 1.3333 .6666
ROW 3
1.0000 0.0000 0.0000 .6666 0.0000 -.6666 2.6666
ROW 4
0.0000 0.0000 1.0000 -.9999 1.0000 0.0000 2.0000
OBJ. FUNCTION 13.16666600

VARIABLE VALUE
2 .66666666
1 2.66666660
3 2.00000010

THE INITIAL MATRIX

ROW 1
-100.0000 -200.0000 -50.0000 0.0000 0.0000 0.0000 0.0000
ROW 2
5.0000 5.0000 10.0000 1.0000 0.0000 0.0000 1000.0000
ROW 3
10.0000 8.0000 5.0000 0.0000 1.0000 0.0000 200.0000
ITERATION 1

OBJ. FUNCTION

NEW BASIC VAR.

5000.00

2

THE MATRIX

ROW 1

150.000 0.000 75.0000 0.0000 25.0000 0.0000 5000.0000

ROW 2

-1.2500 0.0000 6.8750 1.0000 -0.6250 0.0000 877.0000

ROW 3

1.2500 1.0000 .6250 0.0000 .1250 0.0000 25.0000

ROW 4

3.7500 0.0000 -3.1250 0.0000 -.6250 1.0000 375.0000

OBJ. FUNCTION 5000.00000000

VARIABLE  VALUE

4  875.00000000
2  25.00000000
6  375.00000000

THE INITIAL MATRIX

ROW 1

-0.5000 -1.0000 0.0000 0.0000 0.0000 0.0000 0.0000

ROW 2

0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000

ROW 3

0.4000 -1.0000 1.0000 0.0000 0.0000 0.0000 0.0000

ROW 4

3.0000 -1.0000 0.0000 1.0000 0.0000 0.0000 0.0000

ROW 4

1.0000 1.0000 0.0000 0.0000 1.0000 0.0000 0.0000

0.0000 11.0000

0.0000 11.0000
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**The Matrix**

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**Notes:**
- The matrix is presented in a tabular format, with rows and columns indicating the values for each iteration.
- The objective function and new basic variable are specified for each iteration.
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<th>NEW BASIC VAR.</th>
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ITERATION 4

OBJ. FUNCTION 8.99

NEW BASIC VAR. 7

THE MATRIX

ROW 1
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0.0000 8.9999

ROW 2
0.0000 0.0000 1.0000 0.0000 0.0666 0.9333 0.0000
0.0000 5.3999

ROW 3
0.0000 0.0000 0.0000 1.0000 -1.6666 2.6666 0.0000
0.0000 8.0000

ROW 4
0.0000 0.0000 0.0000 1.0000 -2.0000 1.0000
0.0000 3.0000

ROW 5
0.0000 0.0000 0.0000 0.0000 2.3333 -3.3333 0.0000
1.0000 9.0000

ROW 6
0.0000 0.0000 0.0000 1.3333 -2.3333

ROW 7
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0.0000 6.0000
### OBJ. FUNCTION

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### THE INITIAL MATRIX

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### ITERATION

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### THE MATRIX

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**OBJ. FUNCTION**

315.00

**THE INITIAL MATRIX**

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### Notes

- **Objective Function**: 245.41
- **New Basic Variable**: 4
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REFERENCES

Texts


Journals

