Concerning the Riemann and Lebesgue integrals

Mabel J. Foster

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CONCERNING THE RIEMANN AND LEBESGUE INTEGRALS

by

Mabel J. Foster

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>The Riemann Integral</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorems on Sets of Points</td>
<td>28</td>
</tr>
<tr>
<td>Measure of Sets</td>
<td>35</td>
</tr>
<tr>
<td>The Exterior and Interior Measures of a Set</td>
<td>37</td>
</tr>
<tr>
<td>Measurable Functions</td>
<td>41</td>
</tr>
<tr>
<td>The Lebesgue Integral of a Measurable Function</td>
<td>44</td>
</tr>
<tr>
<td>Bibliography</td>
<td>46</td>
</tr>
</tbody>
</table>
THE RIEMANN INTEGRAL

The Riemann Integral is of interest not only from an historical point of view, for it still possesses great importance in Analysis and continues to be the basis upon which practical applications of the Integral Calculus rest. It is defined as follows: Let \((a, b)\) be a closed interval upon which a function \(f(x)\) is defined, single-valued, and bounded. Let \(\pi\) stand for any partition of \((a, b)\) by the points \(a = x_0, x_1, x_2, \ldots, x_n = b\) such that the numbers \(\Delta x_1 = x_1 - a, \Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}\) are each numerically less than or equal to \(\tau\). Let \(\xi_1, \xi_2, \ldots, \xi_n\) be a set of points on the closed intervals \((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\), and let

\[ S_\pi = \sum_{i=1}^{n} f(\xi_i) \Delta x_i = f(\xi_1) \Delta x_1 + f(\xi_2) \Delta x_2 + \cdots + f(\xi_n) \Delta x_n. \]

If the many-valued function of \(\int f(x) \, \mathrm{d}x\) approaches a single limiting value as \(\pi\) approaches zero, then

\[ \int_{a}^{b} f(x) \, \mathrm{d}x = \lim_{\pi \to 0} S_\pi. \]

The class of integrable functions is very large. The investigation of the conditions of integrability is considerably simplified by the introduction of integral oscillation. On any closed interval \((a, b)\), \(f(x)\) has a least upper bound \(B\) and a greatest lower bound \(A\). If \(B - A = \Delta y\) is multiplied by the length of the interval \(\Delta x = |b - a|\), it gives the area of a rectangle, including the graph of \(f(x)\). If the interval is subdivided by a partition \(\pi\), the sum of the products \(|\Delta x| \Delta y\) on the intervals of the partition is called the integral oscillation of \(f(x)\) for the partition \(\pi\), and is denoted by \(O_\pi\). If we call \(\Delta x \Delta y\) the
difference between the upper and lower bounds of \( f(x) \) on the closed intervals \( (x_{k-1}, x_k) \), we have:

\[
O_\pi = |\Delta_1 x| \Delta_y + |\Delta_2 x| \Delta_y + \cdots + |\Delta_n x| \Delta_y = \sum_{k=1}^{n} |\Delta_k x| \Delta_k y. 
\]

**Theorem 1:** If \( S_\pi \) and \( S'_\pi \) are two sums (formed by using different \( f' \)) on the same partition, then

\[
|S_\pi - S'_\pi| \leq O_\pi.
\]

**Proof:** \( S_\pi = \sum_{k=1}^{n} f(S_k') \Delta_k x \) and \( S'_\pi = \sum_{k=1}^{n} f(S_k') \Delta_k x \). Let \( S_\pi - S'_\pi = \sum_{k=1}^{n} \{ f(S_k') - f(S_k') \} \Delta_k x = \sum_{k=1}^{n} | f(S_k') - f(S_k') | \Delta_k x \). But

\[
| f(S_k') - f(S_k') | \leq | f(S_i') - f(S_i') | \Delta_k x.
\]

Hence

\[
| S_\pi - S'_\pi | \leq \sum_{k=1}^{n} | f(S_i') - f(S_i') | \Delta_k x
\]

Then

\[
| S_\pi - S'_\pi | \leq O_\pi.
\]

**Theorem 2:** If \( \pi \) is a repartition of \( \pi \), then

\[
| S_\pi - S'_\pi | \leq O_\pi.
\]

**Proof:** Any interval \( \Delta_k x \) is composed of one or more intervals \( \Delta_k x, \Delta_k x, \) etc., of \( \pi \), and these contribute to \( S_\pi \), the terms

\[
f(S_k') \Delta_k x + f(S_k') \Delta_k x + \cdots.
\]

The corresponding term of \( S_\pi \) is

\[
f(S_k) \Delta_k x = f(S_k) \Delta_k x + f(S_k) \Delta_k x + \cdots
\]

Therefore the difference between the corresponding terms given above is less than or equal to

\[
| f(S_k) - f(S_k') | \Delta_k x.
\]

Then

\[
| S_\pi - S'_\pi | \leq O_\pi.
\]

**Theorem 3:** Every function that is continuous on a closed interval \( (a, b) \) is integrable on \( (a, b) \).

**Proof:** If \( f(x) \) is integrable, then \( \int_{a}^{b} f(x) dx = \lim_{\varepsilon \to 0} S_\varepsilon \) by definition. We know that for any function \( S_\varepsilon \) approaches at least one value (call it \( D \)) as \( \varepsilon \) approaches zero by the theorem, "If \( f(x) \) is any
function defined for any set \( \mathcal{E} \) of which \( a \) is a limit point, then there is at least one value approached by \( f(x) \) as \( x \to a \). Suppose there is another value approached. Call it \( b \), and suppose \( D > G \). Let \( \varepsilon = \frac{D - G}{4} \).

For every \( \varepsilon \) there exists an \( S_p \) such that \( |S_p - D| < \varepsilon \) and such that the corresponding \( \pi_p \) has its largest \( \Delta \kappa \) \( x < \varepsilon \). There also must be an \( S_o \) such that \( |S_o - G| < \varepsilon \) and such that the corresponding \( \pi_o \) has its largest \( \Delta \kappa \) \( x < \varepsilon \). Let \( \pi \) be a partition made up of the points of \( \pi_p \) and \( \pi_o \), and let \( S \) be one of the corresponding sums. Then \( |S - S_o| \leq \sigma \pi \) and \( |S - S_p| \leq \sigma \pi \) because \( \pi \) is a repartition of both \( \pi_o \) and \( \pi_p \). The theorem of uniform continuity states that if a function is continuous on a closed interval \((a, b)\), then for every \( \varepsilon > 0 \) there exists a \( \varepsilon > 0 \) such that for any two values of \( x \), \( x_1 \), and \( x_2 \), on \((a, b)\) with \( |x_1 - x_2| < \varepsilon \), \( |f(x_1) - f(x_2)| < \varepsilon \).

Since \( f(x) \) is continuous, \( \varepsilon \) can be so chosen that if any two values of \( x \) differ by less than \( \varepsilon \), the corresponding values of \( f(x) \) differ by less than \( \frac{\varepsilon}{|b - a|} \). Since the \( \Delta \kappa \) on the partitions \( \pi_p \) and \( \pi_o \) are all less than \( \varepsilon \), the corresponding \( \Delta \kappa \) on the partitions \( \pi_p \) and \( \pi_o \) are all less than \( \frac{\varepsilon}{|b - a|} \). We have then:

\[ O_{\pi_p} = \sum_{x \in K} |\Delta \kappa x| \Delta \kappa y < \sum_{x \in K} |\Delta \kappa x| |b - a| \varepsilon. \]

But \( \sum_{x \in K} \Delta \kappa x = b - a \). Therefore \( O_{\pi_p} \leq \varepsilon \).

Similarly \( O_{\pi_o} \leq \varepsilon \). We have then:

\[
\begin{align*}
|D - S_p| & < \varepsilon \\
|S_o - G| & < \varepsilon \\
|S_p - S| & < \varepsilon \\
|S - S_o| & < \varepsilon.
\end{align*}
\]
Therefore \( |D - G| < 4 \varepsilon \). But this contradicts the statement that 
\[ \varepsilon = \frac{D - G}{4} \]. Hence the supposition that \( f(x) \) is not integrable does not hold.

**Theorem 4:** Every non-oscillating bounded function is integrable.

**Proof:** The proof is identical with that of the preceding theorem through the statements \( |S - S_c| \leq O \pi_c \) and \( |S - S_d| \leq O \pi_d \). Next we let \( U \) and \( L \) be the upper and lower bounds of \( f(x) \). \( \delta \) can be chosen to equal \( \frac{\varepsilon}{U - L} \). Then
\[ O_{\pi_c} = \sum \Delta_k y |\Delta_k x| < \sum \Delta_k y \cdot \delta \]
because \( \Delta_k x < \delta \). Since \( f(x) \) is non-oscillating \( \sum \Delta_k y = U - L \).
Therefore \( O_{\pi_c} < (U - L) \cdot \delta = \varepsilon \). Similarly \( O_{\pi_d} < \varepsilon \).
So again we have: \( |D - S_d| < \varepsilon \), \( |S_c - G| < \varepsilon \), \( |S_D - S| < \varepsilon \), \( |S - S_c| < \varepsilon \).
Therefore \( |D - G| < 4 \varepsilon \). This contradicts the statement that \( \varepsilon = \frac{D - G}{4} \). Our conclusion is that the non-oscillating bounded function \( f(x) \) is integrable.

The lemmas which follow are needed for the proofs of theorems which give conditions of integrability.

**Lemma I:** If \( \pi_i \) is a repartition of \( \pi \), then for any function bounded on a closed interval \((a, b)\) \( O_{\pi_i} \leq O_{\pi} \).

**Proof:** Any interval \( \Delta_k x \) of \( \pi \) is composed of one or more intervals \( \Delta_k' x, \Delta_k'' x \), etc. of \( \pi_i \), and these contribute to \( O_{\pi} \), the terms \( |\Delta_k' x| \Delta_k' y + |\Delta_k'' x| \Delta_k'' y + \ldots \). The corresponding term of \( O_{\pi} \) is \( |\Delta_k x| \Delta_k y = |\Delta_k' x| \Delta_k' y + |\Delta_k'' x| \Delta_k'' y + \ldots \). Each of the terms \( \Delta_k' y, \Delta_k'' y \), etc. is less than or equal to \( \Delta_k y \).
Therefore $\left| \Delta_k x \right| \Delta_k y \geq \sum \left| \Delta_k x \right| \Delta_k y + \cdots$ is less than or equal to $\left| \Delta_k x \right| \Delta_k y$. Hence $O_n \leq O_{n'}$.

**Lemma 2:** If $\pi_o$ is any partition of the closed interval $(a,b)$, and $e_o$ any positive number, then for any bounded function there exists a number $c_o$ such that for every partition $\pi$ whose greatest $\Delta_k x$ is less than $c_o$, $0 \geq O_{\pi_o} + e_o \geq O_{\pi}$.

**Proof:** If $\pi_o$ has $m+1$ partition points, $c_o$ can be chosen to equal $\frac{e}{R \cdot n}$ where $R$ is the oscillation of the function on $(a,b)$. There are at most $n-1$ intervals of $\pi$ which contain as interior points the points of $\pi_o$. Let $\Delta_k x$ denote the lengths of these intervals of $\pi$, and $\Delta_k y$ the lengths of the intervals of $\pi$ which contain as interior points no points of $\pi_o$.

Then $O_{\pi} = \frac{e}{R \cdot n} \sum \left| \Delta_k x \right| \Delta_k y + \frac{e}{2} \sum \left| \Delta_k x \right| \Delta_k y$.

If $\pi'$ is a repartition of $\pi$, obtained by introducing the points of $\pi$, then $\frac{e}{2} \sum \left| \Delta_k x \right| \Delta_k y$ is a subset of the terms whose sum is $O_{\pi'}$. Hence, by the preceding Lemma, $\frac{e}{2} \sum \left| \Delta_k x \right| \Delta_k y \leq O_{\pi'} \leq O_{\pi}$.

Since $\left| \Delta_k x \right| \leq \frac{e_o}{R \cdot n}$, $\sum \left| \Delta_k x \right| \Delta_k y \leq e_o$

$\left| \Delta_k x \right| \Delta_k y \leq e_o$

Adding $O_{\pi} \leq O_{\pi'} + e_o$.

**Lemma 3:** If $\pi$ is any partition, $O_{\pi}$ is the least upper bound of the expression $S_{\pi'} - S_{\pi''}$, where $S_{\pi'}$ and $S_{\pi''}$ may be any two values of $S_{\pi}$ corresponding to different choices of the $f_{\pi}$. 

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PROOF: We may assume every \( \Delta_k x \) to be positive.

Then \( B S'_\pi - B S''_\pi = B \left\{ S'_\pi - S''_\pi \right\} \).

But \( B S'_\pi = B \left\{ \sum_{\xi=1}^{\xi} + f(\xi_k) \Delta_k x \right\} = \sum_{\xi=1}^{\xi} \left\{ B + f(\xi_k) \right\} \Delta_k x \)
and \( B S''_\pi = B \left\{ \sum_{\xi=1}^{\xi} + f(\xi_k) \Delta_k x \right\} = \sum_{\xi=1}^{\xi} \left\{ B + f(\xi_k) \right\} \Delta_k x \).

Therefore \( B S'_\pi - B S''_\pi = \sum_{\xi=1}^{\xi} \left\{ B + f(\xi_k) - B + f(\xi_k) \right\} \Delta_k x = \sum_{\xi=1}^{\xi} \Delta_k y \cdot \Delta_k x = O_\pi \).

Hence \( B (S'_\pi - S''_\pi) = O_\pi \).

THEOREM 5: A necessary and sufficient condition that a function \( f(x) \), defined, single-valued, and bounded on a closed interval \((a, b)\), shall be integrable on \((a, b)\) is that the greatest lower bound of \( O_\pi \) for this function shall be zero.

PROOF: We shall suppose that \( f(x) \) is integrable. Then \( \int_a^b f(x) \, dx = \int_{a_0}^{a_0} S_f \). We then make use of the theorem which states, "A necessary and sufficient condition that \( f(x) \) shall converge to a finite limit as \( x \to a \) is that for every \( \varepsilon > 0 \) there shall exist a \( V^+(a) \) such that if \( x_1 \) and \( x_2 \) are any two values of \( x \) in \( V^+(a) \), then \( |f(x_1) - f(x_2)| < \varepsilon \). From this we conclude that for every \( \varepsilon \) there exists a \( d_\varepsilon \) such that for every \( d_1 < d_\varepsilon \) and \( d_2 < d_\varepsilon \)

\( |S_{d_1} - S_{d_2}| < \varepsilon \). If \( \pi \) is a partition whose intervals \( \Delta_k x \) are all less than \( d_\varepsilon \), then \( |S'_\pi - S''_\pi| < \varepsilon \) for every \( S'_\pi \) and \( S''_\pi \). This implies that \( O_\pi < \varepsilon \). But if for every \( \varepsilon \) there exists a \( \pi \) such that \( O_\pi < \varepsilon \), then \( B O_\pi = O \). Next we prove that if the lower bound of \( O_\pi \) is zero the function \( f(x) \) is integrable. For any positive quantity \( \varepsilon \) there exists a partition \( \pi_\varepsilon \) such that \( O_{\pi_\varepsilon} < \frac{\varepsilon}{4} \). By
Lemma 2 there exists a $\mathcal{S}_\varepsilon$ such that for every $\pi$ whose intervals are numerically less than $\mathcal{S}_\varepsilon$, $O_\pi \leq O_{\pi'} + \frac{\varepsilon}{2}$. Let $S_{\pi''}$ and $S_{\pi'}$ be any two values of $S_r$ and let $\pi''$ be the partition composed of the points of both $\pi'$ and $\pi''$. Then for any value of $S_{\pi''}$ we have $|S_{\pi'} - S_{\pi''}| \leq O_{\pi'} + \frac{\varepsilon}{2}$, $|S_{\pi'} - S_{\pi''}| \leq O_{\pi''} + \frac{\varepsilon}{2}$.

Therefore $|S_{\pi'} - S_{\pi''}| < \varepsilon$. Hence for every $\varepsilon$ we have a $\mathcal{S}_\varepsilon$ such that for every two values of $S_r$, $\mathcal{S} < \mathcal{S}_\varepsilon$, $|S_{\pi'} - S_{\pi''}| < \varepsilon$.

This implies the existence of $\int_{\mathcal{S}} S_r$ (by the theorem quoted above).

The statement which follows explains the meaning of a term to be used in the next theorem. For every value of $\mathcal{S} > 0$ there is an infinite set of partitions $\pi$, for which the largest $\Delta x$ is less than $\mathcal{S}$, and for each of these there is a value of $O_\pi$. If $O_\mathcal{S}$ stands for any such $O_\pi$, then $O_\mathcal{S}$ is a many-valued function of $\mathcal{S}$.

**Theorem 6:** A necessary and sufficient condition that a function $f(x)$ defined, single-valued, and bounded on a closed interval $(a, b)$ be integrable on $(a, b)$ is that $\int_{\mathcal{S} \to 0} O_\mathcal{S} = 0$.

**Proof:** To prove the condition necessary we suppose $f(x)$ to be integrable. Then $B O_\pi = 0$ (by the preceding theorem). Hence for every $\varepsilon$ there exists a partition $\pi$ such that $O_\pi < \varepsilon$. By Lemma 2 there exists a $\mathcal{S}_\varepsilon$ such that for every $\pi'$ whose greatest $\Delta x$ is less than $\mathcal{S}_\varepsilon$, $O_\pi' < O_\pi + \varepsilon < 2\varepsilon$. Hence $\int_{\mathcal{S} \to 0} O_\mathcal{S} = 0$.

We next prove the condition sufficient. Since $\int_{\mathcal{S} \to 0} O_\mathcal{S} = 0$, and $O_\mathcal{S} > 0$, $B O_\pi = 0$. Hence the function is integrable by the preceding theorem.
**Theorem 7:** A necessary and sufficient condition that a function, defined, single-valued, and bounded on a closed interval \((a, b)\) shall be integrable on \((a, b)\) is that for every pair of positive numbers \(\sigma\) and \(\lambda\) there exists a partition \(\mathcal{P}\) such that the sum of the lengths of those intervals on which the oscillation of the function is greater than \(\sigma\) is less than \(\lambda\).

**Proof:** Suppose that for a given pair of positive numbers \(\sigma\) and \(\lambda\) there exists no such \(\mathcal{P}\) as is required by the theorem. Then \(O_{\mathcal{P}} \geq \sigma \cdot \lambda\) for every \(\mathcal{P}\). But this is contrary to the conclusion of Theorem 5 that \(\mathcal{B} O_{\mathcal{P}} = 0\). The condition, therefore, is necessary.

It remains to prove the condition is sufficient. For a given \(\varepsilon > 0\) we may choose \(\sigma\) and \(\lambda\) so that \(\sigma (b - a) < \frac{\varepsilon}{\lambda} \) and \(\lambda \cdot R < \frac{\varepsilon}{2}\), where \(R\) is the oscillation of the function on \((a, b)\). Let \(\mathcal{P}\) be a partition such that the sum of the lengths of the intervals on which the oscillation of the function is greater than \(\sigma\) is less than \(\lambda\). Then the sum of the terms of \(O_{\mathcal{P}}\) which occur on these intervals is less than \(\lambda \cdot R\), and the sum of the terms of \(O_{\mathcal{P}}\) on the remaining intervals is less than \(\sigma (b - a)\). Therefore \(O_{\mathcal{P}} < \lambda \cdot R + \sigma (b - a) < \varepsilon\). Hence \(\mathcal{B} O_{\mathcal{P}} = 0\). Therefore the integral exists.

**Definition.** The content of a set of points \(\mathcal{L}\) on a closed interval \((a, b)\) is a number \(C_{\mathcal{L}}\) defined as follows: Let \(\mathcal{P}\) be any partition of \((a, b)\), none of the partition points of which are points of \(\mathcal{L}\), and \(D_{\mathcal{P}}\) the sum of the lengths of those intervals of \(\mathcal{P}\) which contain points of \(\mathcal{L}\) as interior points. Then \(\mathcal{B} D_{\mathcal{P}} = C_{\mathcal{L}}\).
**Theorem 8:** A necessary and sufficient condition that a function defined, single-valued, and bounded on a closed interval \((a,b)\) shall be integrable on \((a,b)\) is that for every \(\gamma > 0\) the set of points \(\{x_\gamma\}\) at which the oscillation of \(f(x)\) is greater than or equal to \(\gamma\) shall be of content zero.

**Proof:** To show that the condition is necessary, let us consider the set \(\{x_\gamma\}\) at which the oscillation of \(f(x)\) is \(\geq \gamma\). If an interval contains a point of \(\{x_\gamma\}\) within it, the oscillation of \(f(x)\) in the interval is greater than or equal to \(\gamma\). If a point of \(\{x_\gamma\}\) is the common end-point of two intervals of equal length, the oscillation of \(f(x)\) in at least one of these intervals is greater than or equal to \(\frac{1}{2} \gamma\). Hence the part which these two intervals contribute to \(O_T\) is \(\geq \frac{1}{2} \gamma \cdot \Delta x\).

If we have a partition of \((a,b)\) with equal intervals, the contribution of all those intervals which contain, within them or at an end-point, a point of \(\{x_\gamma\}\), is not less than the product of \(\frac{1}{4} \gamma\) and the sum of the lengths of these intervals. Unless the content of \(\{x_\gamma\}\) is zero, the sum of the lengths of these intervals is greater than some fixed positive number for all the partitions of a symmetrical system. It is therefore necessary for the existence of the integral that the content of \(\{x_\gamma\}\) should be zero.

If at every point of a closed interval \((c,d)\), the oscillation of \(f(x)\) is less than \(\gamma\), then about each point of \((c,d)\) there is an open interval upon which the oscillation is less than \(\gamma\). Therefore there is a partition of \((c,d)\) upon each interval of which the oscillation of \(f(x)\)
is less than $\sigma$, by a theorem which states, "If a closed interval $(a, b)$ is covered by a set of open intervals $[\lambda]$, then $(a, b)$ may be divided into $n$ equal intervals such that each interval is entirely within a $\sigma$.

To prove the condition sufficient we note that if the content of $[\lambda]$ is zero, there exists for every $\lambda$ a partition $\pi$ such that the sum of the lengths of the intervals containing points of $[\lambda]$ is less than $\lambda$. We have just pointed out that the intervals which do not contain points of $[\lambda]$ can be repartitioned into intervals on which the oscillation is less than $\sigma$. Hence the function is integrable by Theorem 7.

The theorems which follow state the properties of definite integrals.

**Theorem 9:** If $a < b < c$, and if a bounded function $f(x)$ is integrable from $a$ to $c$, then it is integrable from $a$ to $b$ and from $b$ to $c$.

**Proof:** Suppose $f(x)$ is not integrable from $a$ to $b$. Then there must be a set of values of $\int_a^b S_r [\lambda]$ such that $\int_a^b S_r = A$, and another set $\int_a^b S_r''$ such that $\int_a^b S_r'' = B$, which is not equal to $A$. Whether $\int_a^b f(x) \, dx$ exists or not, there must be a set of values of $\int_a^b S_r [\lambda]$ such that $\int_a^b S_r = C$. For every $\int_a^b S_r'$ and $\int_a^b S_r''$ there is a $\int_a^b S_r$ such that $\int_a^b S_r' = \int_a^b S_r'' + \int_a^b S_r$. Therefore $A + C$ is a value approached by $\int_a^b S_r$. Similarly we can prove that $B + C$ is a value approached by $\int_a^b S_r$. But we know that $\int_a^b S_r$ has only one value approached because it is given that $f(x)$ is integrable from $a$ to $c$. Hence $\int_a^b f(x) \, dx$ must exist. Similarly $\int_b^c f(x) \, dx$ exists.
**Theorem 10:** If \( a < b < c \) and if a bounded function \( f(x) \) is integrable from \( a \) to \( b \) and from \( b \) to \( c \), then \( f(x) \) is integrable from \( a \) to \( c \) and
\[
\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.
\]

**Proof:** Since \( \int_a^b f(x) \, dx \) and \( \int_b^c f(x) \, dx \) exist, we know by a theorem (A necessary and sufficient condition that \( f(x) \) shall converge to a unique limit \( b \) as \( x \) approaches \( a \) is that for every \( V(b) \) there shall exist a \( V\) such that for every \( x \) in \( V\), \( f(x) \) is in \( V\).)

If the upper bound of \( f(x) \) on \((a, c)\) is \( b \) and its lower bound \( L \), let
\[
\varepsilon = \frac{\varepsilon}{3(b-L)}, \quad \text{and let} \quad \varepsilon \text{ be smaller than the smallest of} \quad \varepsilon', \varepsilon'', \varepsilon'''.
\]
Consider any value of \( \varepsilon'\). If the point \( b \) is one of the partition points, then \( \varepsilon' \) is the sum of one value of \( \varepsilon b S_f \) and one value of \( \varepsilon' \). If \( b \) is not a partition point, let \( \varepsilon b x \) be the length of the interval of \( \varepsilon' \) that contains \( b \). Then for a properly chosen \( \varepsilon b S_f \) and \( \varepsilon' \),
\[
|\varepsilon b S_f + \varepsilon' S_f - \varepsilon f | < \varepsilon b x (b-L) < \frac{\varepsilon}{3} \quad (3)
\]
By combining (1), (2), and (3) we obtain the result that for every \( \varepsilon \) there exists a \( \varepsilon \) such that for every \( S_{\varepsilon'} \),
\[
|\varepsilon S_{\varepsilon'} - \int_{\varepsilon}^{\varepsilon} f(x) \, dx + \int_{\varepsilon}^{\varepsilon} f(x) \, dx| < \varepsilon.
\]
Therefore
\[
\int_{\varepsilon}^{\varepsilon} f(x) \, dx = \int_{\varepsilon}^{\varepsilon} f(x) \, dx + \int_{\varepsilon}^{\varepsilon} f(x) \, dx.
\]

**Theorem 11:** If \( \int_{\varepsilon}^{\varepsilon} f(x) \, dx \) exists, then \( \int_{\varepsilon}^{\varepsilon} f(x) \, dx \) exists and
\[
\int_{\varepsilon}^{\varepsilon} f(x) \, dx = -\int_{\varepsilon}^{\varepsilon} f(x) \, dx.
\]

**Proof:** For every \( \varepsilon \) used in defining \( \int_{\varepsilon}^{\varepsilon} f(x) \, dx \) there
corresponds a sum equal to \(-S\) which is used in defining \(\int_a^b f(x) \, dx\).

Similarly to every \(S\) used in defining \(\int_a^b f(x) \, dx\) there corresponds a sum equal to \(-S\) used in defining \(\int_a^b f(x) \, dx\).

Hence the function \(S_f\) used in the definition of \(\int_a^b f(x) \, dx\) is the negative of the function \(S_f\) used in the definition of \(\int_a^b f(x) \, dx\).

By a theorem which states \(\int_a^b (-f(x)) \, dx = -\int_a^b f(x) \, dx\).

**Theorem 12:** If any two of the following integrals exist, so does the third, and \(\int_a^b f(x) \, dx + \int_a^b \cdot f(x) \, dx = \int_a^b f(x) \, dx\).

**Proof:** This is evident from Theorems 9, 10, and 11.

**Theorem 13:** If \(\int_a^b f(x) \, dx\) exists, then \(\int_{a+n}^b f(x) \, dx\) exists and the two are equal.

**Proof:** This follows at once from the definitions of the integrals as the limits of sums.

**Theorem 14:** If \(f(x)\) is integrable from \(a\) to \(b\), then \(|f(x)|\) is integrable from \(a\) to \(b\) and \(\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx\).

**Proof:** The oscillation of \(|f(x)|\) in any interval cannot exceed that of \(f(x)\) in the same interval. Hence since \(O_n\) of \(f(x)\) has the limit zero, \(O_n\) of \(|f(x)|\) also has the limit zero. That is, \(B \subseteq O_n\) of \(f(x)\) also has the limit zero. That is, \(B \subseteq O_n\) of \(|f(x)|\) also has the limit zero. Hence for every \(S_{\tau} f(x)\) there is a smaller or equal \(S_{\tau} f(x)\), the \(S_{\tau}\) being the same. Therefore \(\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx\), or \(\int_a^b f(x) \, dx\) by the theorem which states, "If \(f(x) \geq f(x)\) in the neighborhood of \(x = \alpha\), then \(\int_{x+\alpha}^b f(x) \geq \int_{x+\alpha}^b f(x)\) if both limits exist."
THEOREM 15: If \( C \) is any constant and if \( f(x) \) is integrable on a closed interval \((a,b)\), then \( Cf(x) \) is integrable on \((a,b)\) and
\[
\int_a^b C f(x) \, dx = C \int_a^b f(x) \, dx.
\]

PROOF: \( S_f = \frac{\Delta x}{x^2} \int f(S_x) \, dx \) is an \( S_f \)-set of the set which defines \( \int_a^b f(x) \, dx \), and \( S_f' = \frac{\Delta x}{x^2} \int C f(S_x) \, dx \) is the corresponding \( S_f \)-set of the set which defines \( \int_a^b C f(x) \, dx \). Hence
\[
\int_a^b C f(x) \, dx = C \int_a^b f(x) \, dx
\]
by the theorem which states that
\[
\int_a^b C f(x) = C \int_a^b f(x).
\]

THEOREM 16: If \( f_1(x) \) and \( f_2(x) \) are any two functions each integrable on a closed interval \((a,b)\), then \( f(x) = f_1(x) \pm f_2(x) \) is integrable on \((a,b)\) and
\[
\int_a^b f(x) \, dx = \int_a^b f_1(x) \, dx \pm \int_a^b f_2(x) \, dx.
\]

PROOF: The proof follows directly from the theorem which states, "If \( \int_{x+x} f_1(x) = b \), and \( \int_{x-x} f_2(x) = b \), and \( b \) being finite, then \( \int_{x}^{x} \{ f_1(x) \pm f_2(x) \} = b, \pm b \)."

THEOREM 17: If \( f_1(x) \) and \( f_2(x) \) are each integrable on a closed interval \((a,b)\) and such that for every value of \( x \) on \((a,b)\) \( f_1(x) \leq f_2(x) \), then
\[
\int_a^b f_1(x) \, dx \leq \int_a^b f_2(x) \, dx.
\]

PROOF: Since \( f_1(x) \leq f_2(x) \) for every value of \( x \) on \((a,b)\), then \( S_i \) is always greater than or equal to \( S_{i,2} \). Therefore
\[
\int_{x}^{x} S_i \geq \int_{x}^{x} S_{i,2} \text{ or } \int_a^b f_1(x) \, dx \geq \int_a^b f_2(x) \, dx
\]
by the theorem, "If \( f_1(x) \leq f_2(x) \) in the neighborhood of \( x = \), then
\[
\int_{x}^{x} f_1(x) \geq \int_{x}^{x} f_2(x)
\]
if both these limits exist."
**THEOREM 18.** (Maximum-Minimum Theorem)

If (1) the product \( f_1(x) \cdot f_2(x) \) and the factor \( f_1(x) \) are integrable on a closed interval \([a,b]\),

(2) \( f_1(x) \) is always positive or always negative on \((a,b)\),

(3) \( M \) and \( m \) are the least upper and the greatest lower bounds respectively of \( f_2(x) \) on \((a,b)\),

then \( m \cdot \int_a^b f_1(x) \, dx \leq \int_a^b f_1(x) \cdot f_2(x) \, dx \leq M \cdot \int_a^b f_1(x) \, dx \),

or \( m \cdot \int_a^b f_1(x) \, dx \leq \int_a^b f_1(x) \cdot f_2(x) \, dx \leq M \cdot \int_a^b f_1(x) \, dx \).

**PROOF:** By Theorem 15,

\[ M \cdot \int_a^b f_1(x) \, dx = \int_a^b M \cdot f_1(x) \, dx \]

and \( m \cdot \int_a^b f_1(x) \, dx = \int_a^b m \cdot f_1(x) \, dx \).

In case \( f_1(x) \) is always positive,

\( m \cdot f_1(x) \leq f_1(x) \cdot f_2(x) \leq M \cdot f_1(x) \).

Hence, by Theorem 17,

\[ \int_a^b m \cdot f_1(x) \, dx \leq \int_a^b f_1(x) \cdot f_2(x) \, dx \leq \int_a^b M \cdot f_1(x) \, dx \]

and therefore

\[ m \cdot \int_a^b f_1(x) \, dx \leq \int_a^b f_1(x) \cdot f_2(x) \, dx \leq M \cdot \int_a^b f_1(x) \, dx \].

Similarly we can prove that when \( f_1(x) \) is always negative

\[ m \cdot \int_a^b f_1(x) \, dx \leq \int_a^b f_1(x) \cdot f_2(x) \, dx \leq M \cdot \int_a^b f_1(x) \, dx \].

As a corollary of the above theorem we have

**THEOREM 19.** (Mean-value Theorem) Under the hypothesis of

Theorem 18 there exists a number \( \kappa \), \( m \leq \kappa \leq M \) such that

\[ \int_a^b f_1(x) \cdot f_2(x) \, dx = \kappa \int_a^b f_1(x) \, dx \].
Corollary. In case \( f_2(x) \) is continuous, there is a value \( \xi \) of \( x \) on \( (a, b) \) such that \( \int_a^b f_1(x) \, dx = f_2(\xi) \int_a^b f_2(x) \, dx \).

In case \( f_1(x) = 1 \), \( \int_a^b f_1(x) \, dx = b - a \),

and the theorem becomes the following:

**Theorem 20.** If \( f(x) \) is any integrable function on the closed interval \( (a, b) \), there exists a number \( M \) lying between the upper and lower bounds of \( f(x) \) on \( (a, b) \) such that \( \int_a^b f(x) \, dx = M(b - a) \),

and if \( f(x) \) is continuous, there is a value \( \xi \) of \( x \) on \( (a, b) \) such that \( \int_a^b f(x) \, dx = f(\xi)(b - a) \).

**Theorem 21.** If \( f_1(x) \) and \( f_2(x) \) are integrable on a closed interval \( (a, b) \) and such that every value of \( x \) on \( (a, b) \) \( |f_1(x)| \leq |f_2(x)| \), then \( \int_a^b f_1(x) \, dx \leq \int_a^b f_2(x) \, dx \).

**Proof:** \( \int_a^b \left( |f_2(x)| - |f_1(x)| \right) \, dx \geq 0 \) since in every interval no value of \( |f_2(x)| - |f_1(x)| \) is negative, and thus the sums of which the integral is the limit are all \( \geq 0 \). We know, by Theorem 14, that \( \int_a^b f_1(x) \, dx \leq \int_a^b f_2(x) \, dx \).

But \( \int_a^b f_2(x) \, dx \leq \int_a^b f_2(x) \, dx \).

Therefore \( \int_a^b f_1(x) \, dx \leq \int_a^b f_2(x) \, dx \).

In order to facilitate the wording of the next theorem we will consider a set of points \( X_1, X_2, \ldots, X_{n-1}, X_n = b \) on the closed interval \( (a, b) \) such that \( X_1 - a = x_2 - x_1 = x_3 - x_2 = \cdots = x_{n-1} - x_{n-2} = b - x_n \).

Then \( M_n = \frac{1}{n} \sum_{k=1}^{n} f(\xi_k) \) and we define the mean value of \( f(x) \), \( M_n \), if this limit exists. But \( \frac{b}{n} \sum_{k=1}^{n} f(\xi_k) = \frac{b - a}{n} = \Delta x \). If the definite integral exists, we may
write \( \int_a^b f(x) \, dx = \lim_{n \to \infty} S_n \)

where 
\[
S_n = \frac{b - a}{n} \sum_{k=1}^{n} f(x_k) \Delta x = \frac{b - a}{n} \int_a^b f(x) \, dx = \frac{b - a}{n} f(x_k) = (b - a) M_n
\]

Therefore \( \int_a^b S_n = (b - a) \lim_{n \to \infty} M_n \).

**Theorem 22:** In case the integral of \( f(x) \) exists on the closed interval \((a, b)\),
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{b^2 - a^2}{n} \sum_{k=1}^{n} f(x_k) \Delta x
\]

**Proof:** \( \int_a^b S_n = (b - a) \lim_{n \to \infty} M_n \)
or \( \int_a^b f(x) \, dx = (b - a) \lim_{n \to \infty} M_n \).
Therefore \( \int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{b^2 - a^2}{n} \sum_{k=1}^{n} f(x_k) \Delta x \).

This theorem suggests a method of approximating the value of a definite integral by multiplying the average of a finite number of ordinates by \( b - a \).

**Theorem 23:** If \( f(x) \) is integrable on a closed interval \((a, b)\), and if \( x \) is any point of \((a, b)\), then \( \int_a^b f(x) \, dx \) is a continuous function of \( x \).

**Proof:** We know by Theorem 9 that \( \int_a^b f(x) \, dx \) exists. Therefore we need only to show that \( \int_a^b f(x) \, dx \) exists.

By preceding theorems \( \int_a^b f(x) \, dx \) exists.

where \( \overline{B} \) stands for the least upper bound of \( f(x) \) on the closed interval \((x, x)\) and \( \overline{B} \) for the least upper bound of \( f(x) \) on \((a, b)\).

\( \overline{B} (x' - x) \) approaches zero as \( x' \to x \) because \( \overline{B} \) is a constant. Therefore the expression \( \int_a^b f(x) \, dx \) approaches zero as \( x' \to x \).

or \( \int_a^b f(x) \, dx \) approaches zero as \( x' \to x \).

By the theorem, "If \( f_1(x) \) and \( f_2(x) \) are both positive in the neighborhood of \( x = a \) and if \( f_1(x) = f_2(x) \), then if \( \int_a^b f_1(x) \, dx = 0 \), it follows
that \( \lim_{x \to a} f_x(x) = 0 \).

**Theorem 24:** If \( f(x) \) is continuous on a closed interval \([a, b]\), then \( \int_a^b f(x) \, dx \) (\( a < x < b \)) possesses a derivative with respect to \( x \) such that \( \frac{d}{dx} \int_a^x f(x) \, dx = f(x) \).

**Proof:** Since \( f(x) \) is continuous on \([a, b]\) it is integrable on \([a, b]\). Then by Theorem 23 since \( f(x) \) is integrable on \([a, b]\), \( \int_a^x f(x) \, dx \) is continuous. To form the derivative we consider the expression

\[
\int_a^x f(x) \, dx - \int_a^{x'} f(x) \, dx = \int_{x'}^x f(x) \, dx
\]

as \( x' \) approaches \( x \). By the mean-value theorem \( \int_a^x f(x) \, dx = f(x') (x - x') \), where \( f(x') \) is a value of \( x \) between \( x \) and \( x' \) and is a function of \( x' \). From this, \( \frac{\int_a^x f(x) \, dx}{x' - x} = f(x') \). But as \( x' \) approaches \( x \), \( x' \) also approaches \( x \) and \( f(x') \) approaches \( f(x) \). Therefore

\[
\int_{x'}^x f(x) \, dx = \int_{x'}^x \frac{f(x)}{x' - x} \, dx = \int_{x'}^x f(x) \, dx.
\]

**Theorem 25:** If \( f(x) \) is any continuous function on the closed interval \([a, b]\), and \( F(x) \) any function on this interval such that

\[
\frac{d}{dx} F(x) = f(x),
\]

then \( F(x) \) differs from \( \int_a^x f(x) \, dx \) at most by an additive constant.

**Proof:** Let \( F(x) = \int_a^x f(x) \, dx + \phi(x) \).

Since \( F(x) \) and \( \int_a^x f(x) \, dx \) are both differentiable,

\[
\frac{d}{dx} F(x) = \frac{d}{dx} \left( \int_a^x f(x) \, dx + \phi(x) \right) = \frac{d}{dx} \int_a^x f(x) \, dx + \phi'(x) = \frac{d}{dx} \int_a^x f(x) \, dx + f(x).
\]

By Theorem 24 \( \frac{d}{dx} \int_a^x f(x) \, dx = f(x) \). Hence \( \frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(x) \, dx + \phi'(x) \).

Therefore \( \frac{d}{dx} \phi(x) = 0 \), whence \( \phi(x) \) is a constant by the theorem, "If \( f(x) \) exists and is equal to zero for every value of \( x \) on
the closed interval \((a, b)\), then \(f(x)\) is a constant on that interval".

**THEOREM 26**: If \(f(x)\) is a continuous function on a closed interval \((a, b)\), and \(P(x)\) is such that \(\frac{d}{dx} P(x) = f(x)\), then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

**PROOF**: By Theorem 25,
\[
\int_a^b f(x) \, dx = F(x) + C.
\]
But
\[
\int_a^b f(x) \, dx = C
\]
Therefore
\[
F(b) + C = C
\]
from which
\[
F(b) = C
\]
whence
\[
\int_a^b f(x) \, dx = F(b + c) = F(b) - f(c)
\]

**THEOREM 27**: If \(f_1(x)\) and \(f_2(x)\) are both integrable on a closed interval \((a, b)\), then \(f_1(x) \cdot f_2(x)\) is integrable on \((a, b)\); and, provided there is a constant \(m > 0\) such that \(|f_2(x)| > m > 0\) for \(x\) on \((a, b)\), then \(f_1(x) / f_2(x)\) is integrable on \((a, b)\).

**PROOF**: Let \(f(x) = f_1(x) \cdot f_2(x)\). By a partition \(\pi\) divide the interval \((a, b)\) into intervals \(\Delta x\). Let \(M'_x\) and \(m'_x\), \(M''_x\) and \(m''_x\), \(M'_x\) and \(m'_x\) be the least upper bounds and the greatest lower bounds of \(f_1(x)\) \(f_2(x)\), and \(f(x)\), respectively, in each interval \(\Delta x\). Then
\[
M'_x \geq M'_x \cdot M''_x, \quad m'_x \geq m'_x \cdot m''_x,\quad \text{and} \quad M'_x - m'_x \leq M'_x M''_x - m'_x m''_x.
\]
The last inequality may be written
\[
M'_x - m'_x \leq M''_x (M'_x - m'_x) + m''_x (M'_x - m'_x).
\]
Let \(M'\) and \(M''\) represent the least upper bounds of \(f_1(x)\) and \(f_2(x)\) in the interval \((a, b)\). Then
\[
M'_x - m'_x \leq M''_x (M'_x - m'_x) + M'(M'_x - m'_x).
\]
Then
\[
\sum \left( (M'_x - m'_x) \Delta x \right) \leq M''_x \sum \Delta x + M' \sum (M'_x - m'_x) \Delta x
\]
The two limits of the second member of the inequality are zero, for \(f_1(x)\)
and \( f_a(x) \) are integrable. The first member is then also zero. Hence \( f(x) \) is integrable.

To prove that \( \frac{f_1(y)}{f_2(x)} \) is integrable we need only to prove that \( \frac{1}{f_1(x)} \) is integrable. Consider a partition \( \pi \) such that the oscillation of \( \frac{1}{f_1(x)} \) in each interval \( \Delta x \) will be \( \frac{1}{M_2} - \frac{1}{m_2} = \frac{m_2 - m_2}{m_2 m_2} < \frac{1}{(m_2)^2} (M_2 - m_2) \).

Consequently, \( L \leq \frac{1}{(m_2)^2} (M_2 - m_2) \Delta x \leq \frac{1}{(m_2)^2} \) \( L \leq (M_2 - m_2) \Delta x = 0 \). Hence \( \frac{1}{f_1(x)} \) is integrable. Then \( f_1(x) \cdot \frac{1}{f_2(x)} \) or \( \frac{f_1(x)}{f_2(x)} \) is integrable.

**Theorem 28:** If \( f(x) \) is an integrable function on a closed interval \( (a, b) \), and if \( \phi(y) \) is a continuous function on a closed interval \( [\beta, \delta] \), where \( \beta \) and \( \delta \) are the lower and upper bounds respectively of \( f(x) \) on \( (a, b) \), then \( \phi\{f(x)\} \) is an integrable function of \( x \) on the interval \( (a, b) \).

**Proof:** By the theorem on uniform continuity there exists for every \( \varepsilon > 0 \) a \( \delta_2 \) such that for \( 1 \leq y_1 - y_2 \leq \delta_2 \), 
\[
|\phi(y_1) - \phi(y_2)| < \varepsilon.
\]
Since \( f(x) \) is integrable on \( (a, b) \) it follows, by Theorem 7, that for every positive number \( \lambda \) there is a partition \( \pi \) such that the sum of the intervals on which the oscillation of \( f(x) \) is greater than \( \delta_2 \) is less than \( \lambda \). But from the statement 
\[
|\phi(y_1) - \phi(y_2)| < \varepsilon
\]
this means that the sum of the intervals on which the oscillation of \( \phi\{f(x)\} \) is greater than \( \varepsilon \) is less than \( \lambda \). This proves that \( \phi\{f(x)\} \) is integrable.

**Definition.** A set of points is said to be enumerable if it can be set into a one-to-one correspondence with the positive integral numbers.
THEOREM 29. The points of discontinuity of an integrable function form at most a set consisting of an enumerable set of sets, each of content zero.

PROOF: Let \( \sigma_1, \sigma_2, \sigma_3, \ldots \) be any set of numbers such that \( \sigma_n > \sigma_{n+1} \) and \( \lim_{n \to \infty} \sigma_n = 0 \). By Theorem 8 the set of points \( \bigcup \{ \sigma_n^+ \} \) at which the oscillation of \( f(x) \) is greater than or equal to \( \sigma_n \), is of content zero. Since the set of sets \( \{ \{ \sigma_n^+ \} \} \) includes all the points of discontinuity of \( f(x) \), this proves the theorem.

Definition. If every point of a closed interval \((a, b)\) is a limit point of a set \( [x] \), then \( [x] \) is everywhere dense on \((a, b)\).

THEOREM 30. If a function \( f(x) \) is integrable on a closed interval \((a, b)\), then it is continuous at a set of points which is everywhere dense on \((a, b)\).

PROOF: If the theorem fails to hold, there is a closed interval \((c, d)\) on \((a, b)\) on which the function is discontinuous at every point. By Theorem 29 an integrable function is discontinuous at most on an enumerably infinite set of sets, each of content zero, and such sets of sets fail to contain every point of any interval, by a theorem to that effect.

THEOREM 31. If \( \int_a^b f(x) \, dx = 0 \) for every \( X \) of a closed interval \((a, b)\), then \( f(x) = 0 \) on a set of points everywhere dense on \((a, b)\), and for every \( \sigma > 0 \) the points where \( |f(x)| > \sigma \) form a set of content zero.
PROOF: At every point \( x \) where \( f(x) \) is continuous,
\[
\int_a^x f(x) \, dx = F(x)
\]
by a general statement of Theorem 24.
Since \( \int_a^x f(x) \, dx \) is a constant, \( \frac{d}{dx} \int_a^x f(x) \, dx = 0 \).
Hence \( f(x) = 0 \). According to Theorem 30 \( f(x) \) is continuous at a
set of points which is everywhere dense on \((a, b)\). Hence the zero points
of \( f(x) \) are everywhere dense. At a point of discontinuity the os-
cillation of \( f(x) \) is \( /f(x)\). Hence the points where \( |f(x)| > \alpha \)
form a set of content zero.

**Theorem 32:** If \( \int_a^x f(x) \, dx = \int_a^x \phi(x) \, dx \) for every \( x \)
of the closed interval \((a, b)\), then \( f(x) = \phi(x) \) on a set of points every-
where dense on \((a, b)\), and for every \( \alpha > 0 \) the points where \( |f(x) - \phi(x)| > \alpha \)
forms a set of content zero.

**Proof:** It is given that \( \int_a^x f(x) \, dx = \int_a^x \phi(x) \, dx \)
for every \( x \) on \((a, b)\). Then \( \int_a^x f(x) \, dx - \int_a^x \phi(x) \, dx = 0 \)
or \( \int_a^x [f(x) - \phi(x)] \, dx = 0 \). Then by Theorem 31
\( f(x) - \phi(x) = 0 \), \( f(x) = \phi(x) \) on a set of points every-
where dense on \((a, b)\), and for every \( \alpha > 0 \) the points where
\( |f(x) - \phi(x)| > \alpha \) forms a set of content zero.

The theorems to be listed next are concerned with the com-
putation of definite integrals. When the integral is known to exist, the
limit can be calculated on any properly chosen subset of the \( S_{1^n} \) by
the theorem, "If \( [x'] \) is a subset of \( [x] \), \( \alpha \) being a limit point of \( [x'] \)
and if \( \frac{f(x)}{x \to a} \) exists, then \( \frac{f(x')}{x' \to a} \) exists and
\( \frac{f(x)}{x \to a} = \frac{f(x')}{x' \to a} \)." Consequently if \( S_{1^n}, S_{1^2}, \ldots \) is
any sequence of sums such that \( \lim_{n \to \infty} S_n = 0 \), then \( \lim_{n \to \infty} S_n = \int_a^b f(x) \, dx \).

We find a case of this kind occurring when \( S_n \) is taken as an end-point of the interval \((x_{n-1}, x_n)\) and all the \( \Delta x \) are equal. Then we have

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k \Delta x) \Delta x,
\]

where \( \Delta x = \frac{b-a}{n} \).

**Theorem 33:** If \( f(x) \) is a constant, \( C \), then \( \int_a^b C \, dx = C \, (b-a) \).

**Proof:** The function \( f(x) = C \) is integrable by Theorem 4. Hence

\[
\int_a^b C \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} C \, \frac{b-a}{n} = \lim_{n \to \infty} n \cdot C \, \frac{b-a}{n} = C \, (b-a).
\]

**Theorem 34:** \( \int_a^b e^x \, dx = e^b - e^a \).

**Proof:** The function \( e^x \) is integrable by Theorem 4. Let

\[
S_{\Delta x} = e^{a \Delta x} + e^{a+\Delta x} \Delta x + e^{a+2\Delta x} \Delta x + \cdots + e^{a+(n-1)\Delta x} \Delta x + e^{a+n\Delta x} \Delta x.
\]

Then

\[
\lim_{\Delta x \to 0} S_{\Delta x} = \lim_{\Delta x \to 0} \left( e^{a - e^a} \right) \frac{\Delta x}{e^{a\Delta x}}.
\]

Hence

\[
\int_a^b e^x \, dx = e^b - e^a.
\]

For the proof of the next theorem we shall put the partition points in a geometrical progression. Let \( \left( \frac{a}{b} \right)^\frac{1}{n} = q \), \( \frac{b}{a} = q^n \)

\( \Delta x = a \, g^{-1}, \Delta x = a \, g^{-2}, \Delta x = a \, g^{-3}, \ldots, \Delta x = a \, g^{-n-1} \),

\( S_1 = a, S_2 = a \, q, \ldots, S_n = a \, q^{-1} \),

and obtain the formula:

\[
\int_a^b f(x) \, dx = \lim_{g^{-1} \to 0} \frac{a \, (x - 1) \left[ f(a) + g \, f(aq) + \cdots + g^{-n-1} f(g^{n-1} a) \right]}{g^{-1}}.
\]
THEOREM 35: In all cases where \( m \) is a whole number \( \neq -1 \),
and if \( a > 0 \), \( b > 0 \) for every value of \( m \neq -1 \),
\[
\int_a^b x^{-m} \, dx = \frac{b^{1-m} - a^{1-m}}{m+1}
\]

PROOF:
\[
\int_a^b x^{-m} \, dx = \frac{1}{g'} \int_a^b (\frac{a}{g})^{m+1} \frac{\frac{a}{g} - (\frac{b}{g})^{m+1}}{m+1} \, \frac{\frac{a}{g}}{g'} \, \frac{d\frac{a}{g}}{g'}
\]

\[
= \frac{1}{g'} \int_a^b (\frac{a}{g})^{m+1} \frac{1}{m+1} \, \frac{\frac{a}{g}}{g'} \, \frac{d\frac{a}{g}}{g'}
\]

\[
= \frac{1}{g'} \int_a^b (\frac{a}{g})^{m+1} \frac{1}{m+1} \, \frac{\frac{a}{g}}{g'} \, \frac{d\frac{a}{g}}{g'}
\]

\[
= \frac{1}{g'} \left[ \frac{a^{m+1}}{m+1} - \frac{b^{m+1}}{m+1} \right]
\]

\[
= \frac{b^{1-m} - a^{1-m}}{m+1}
\]

THEOREM 36: \( \int_a^b x^{-1} \, dx = \log b - \log a \) \( (0 < a < b) \).

PROOF: \( \log \) may be written \( x^{-1} \). By the first equation in
Theorem 35, since \( g^{-1} = g^0 = 1 \),
\[
\int_a^b x^{-m} \, dx = \frac{b^{1-m} - a^{1-m}}{m+1}
\]

But \( \frac{a}{b} = g^{-1} \).
Therefore
\[
\frac{b}{a} \log g = \log \frac{a}{b}
\]
from which
\[
m = \frac{\log (\frac{a}{b})}{\log g}
\]

\[
\frac{b}{a} \log g = \log (\frac{a}{b})
\]

Hence
\[
\int_a^b x^{-m} \, dx = \log (\frac{a}{b}) = \log b - \log a.
\]

THEOREM 37: If on a closed interval \((a, b)\) two functions \( f(x) \)
and \( F(x) \) have the property that for every two values of \( x, x_1 \), and \( x_2 \),
where \( a < x_1 < x_2 < b \), \( f(x_1) \cdot (x_2 - x_1) \leq F(x_1) - F(x) \leq f(x_2) \cdot (x_2 - x_1) \)
or if \( f(x_1) \cdot (x_2 - x_1) \leq F(x_2) - F(x_1) \leq f(x_2) \cdot (x_2 - x_1) \), then
(1) if \( f(x) \) is continuous, \( \frac{dF(x)}{dx} = f(x) \); and

(2) whether \( f(x) \) is continuous or not, \( \int_a^b f(x) \, dx \) exists and
is equal to \( F(b) - F(a) \).

**PROOF:** From \( f(x)(x-x) \leq F(x) - F(a) \leq f(x_0) (x-x_0) \)

we obtain \( f(x_0) \leq \frac{F(x) - F(a)}{x_0 - x} \). Since \( f(x) \)

is continuous at \( x = x_0 \), \( \lim_{x \to x_0^+} f(x) = f(x_0) \). Therefore

\[
\int_{a}^{x} \frac{F(x) - F(a)}{x_0 - x} = f(x_0) \]

which proves (I).

We know by Theorem 4 that \( f(x) \) is integrable. On any partition

\( \pi \) whose dividing points are \( x_1, x_2, \ldots, x_m \), we have

\[
f(a) (x_1 - x) \leq f(x_1) (x_1 - x) \leq f(x) (x_1 - x) \leq f(x_2) (x_2 - x) \]

\[
\vdots
\]

\[
f(x_m) (x_m - x) \leq f(b) (b - x) \leq f((x_1, \ldots, x_m) (x_m - x_m) \leq f(b) (b - x_m).

Adding we obtain

\[
f(a) (x_1 - x) + f(x_1) (x_2 - x_1) + \cdots + f(x_m) (x_m - x_m) \leq f(b) (b - x_m).
\]

But

\[
f(a) (x_1 - x) + \cdots + f(x_m) (x_m - x_m) = \int_{a}^{x} f(x) \, dx.
\]

Since this holds for every \( \pi \) we have \( \int_{a}^{x} f(x) \, dx = F(b) - F(a) \).

**THEOREM 38:** (Integration by parts.)

If \( f_i(x) \) and \( f_2(x) \) exist and are continuous on the closed interval \( (a, b) \),

then

\[
\int_{a}^{b} f_i(x) \cdot f_2(x) \, dx = \left[ f_i(x) \cdot f_2(x) \right]_{a}^{b} - \int_{a}^{b} f_i(x) \cdot f_2'(x) \, dx.
\]

**PROOF:** \( \int_{a}^{b} [f_i(x) \cdot f_2(x)] \, dx = \int_{a}^{b} f_i(x) \cdot f_2(x) + \int_{a}^{b} f_i(x) \cdot f_2'(x) \) \( f_2(x) \).

Therefore \( \int_{a}^{b} [f_i(x) \cdot f_2(x)] \, dx = \int_{a}^{b} f_i(x) \cdot f_2(x) \, dx + \int_{a}^{b} f_i(x) \cdot f_2'(x) \, dx \).

(\( f_i(x) \) and \( f_2(x) \) are continuous because of the existence and continuity
of \( f(x) \) and \( f'(x) \), and therefore the integral exists.)

But
\[
\int_a^b \{ f(x) \cdot f'(x) \} \, dx = f(a) \cdot f'(a) - f(b) \cdot f'(b)
\]
by the theorem which states, "If \( f(x) \) is a continuous function on a closed interval \( (a, b) \) and \( F(x) \) is such that \( \frac{d}{dx} F(x) = f(x) \), then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]
Then
\[
\int_a^b f(x) \cdot f'(x) \, dx = f(a) \cdot f'(a) - f(b) \cdot f'(b) \]
or
\[
\int_a^b f(x) \cdot f'(x) \, dx = \left[ f(x) \cdot f'(x) \right]_a^b - \int_a^b f'(x) \cdot f(x) \, dx.
\]
Hence
\[
\int_a^b f(x) \cdot f'(x) \, dx = \left[ f(x) \cdot f'(x) \right]_a^b - \int_a^b f'(x) \cdot f(x) \, dx.
\]

**Theorem 39.** (Integration by substitution.)

If \( y = \phi(x) \) has a continuous derivative at every point of the closed interval \( (a, b) \) and \( f(y) \) is continuous for all values taken by \( y = \phi(x) \) as \( x \) varies from \( a \) to \( b \), then
\[
\int_a^b f(y) \, dy = \int_{\phi(a)}^{\phi(b)} f(u) \, du, \tag*{where \( A = \phi(a), \ B = \phi(b). \)}
\]

**Proof:** By Theorem 25
\[
\int_a^b f(y) \, dy = \int_{\phi(a)}^{\phi(b)} \left\{ \int_{\phi(u)}^{\phi(v)} f(y) \, dy \right\} \, du + C
\]
But
\[
\int_a^b \left\{ \int_{\phi(u)}^{\phi(v)} f(y) \, dy \right\} \, du + C = \int_{\phi(a)}^{\phi(b)} f(u) \, du + C
\]
by the theorem, "If \( f(x) \) exists and is finite for \( x = x_0 \), and \( f(x) \) is continuous at \( x = x_0 \), and if \( f'(y) \) exists and is finite for \( y = f(x) \), then
\[
\frac{d}{dx} \left\{ f(x) \right\} = f'(x) = \frac{df}{dx}.
\]
Hence
\[
\int_{\phi(a)}^{\phi(b)} f(u) \, du = \int_{\phi(a)}^{\phi(b)} f(y) \, dy = \int_{\phi(a)}^{\phi(b)} f(y) \, du + C
\]
(by substitution)

If \( x = a \), \( C = 0 \). Then let \( x = b \). Then
\[
\int_{\phi(a)}^{\phi(b)} f(u) \, du = \int_{\phi(a)}^{\phi(b)} f(y) \, dy
\]

**Theorem 40:**
\[
\int_a^b f(x) \, dx = \int_{\phi(a)}^{\phi(b)} f(y) \, dy
\]
where \( x = \phi(y) \), \( a = \phi(A) \), \( b = \phi(B) \); provided that both integrals exist, and that \( \phi(y) \) is non-oscillating and has a finite derivative.
whenever the least upper bound of $\Delta_k x$ for each $n$ approaches zero as $n$ approaches $+\infty$. Let $\Delta y = \frac{y_k - y_{k-1}}{n}$, $y_k = A + \kappa \Delta y$.

$\phi(y_k) - \phi(y_{k-1}) = \Delta_k x$. Hence $\Delta_k x = \phi'(y_k) \Delta y$,

where $\eta_k$ lies between $y_k$ and $y_{k-1}$. This is true by the theorem which states: "If $f(x)$ is continuous on the closed interval $(x_1, x_2)$, and if the derivative exists at every point between $x$ and $x$, then there is a value of $x$, $x = \xi$ between $x_1$ and $x_2$ such that $f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$." If $\int_{x_1}^x = \phi'(y_k)$, it will lie between $\phi(y_k)$ and $\phi(y_{k-1})$; moreover the $\Delta_k x$'s are all of the same sign or zero; and since the hypothesis makes $\phi(y)$ uniformly continuous, their least upper bound approaches zero as $n$ approaches $+\infty$. Therefore

$$\int_a^b f(x) \, dx = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^n f(S_k) \Delta_k x$$

$$= \lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^n \left\{ \phi'(\eta_k) \right\} \phi'(y_k) \Delta y = \int_a^b \phi'(y) \, dy$$

provided the latter integral exists.

Hence $\int_a^b f(x) \, dx = \int_a^b \phi'(y) \, dy$.

**Corollary.** The preceding theorem remains true if $\phi(y)$ has a finite number of oscillations.

**Proof:** Suppose the maximum and minimum values of $\phi(y)$ are $d_1, d_2, d_3, \ldots, d_m$, corresponding to the values of $y, A_1, A_2, \ldots, A_m$.

Then

$$\int_a^b f(x) \, dx = \int_a^b f(x) \, dx + \int_b^a f(x) \, dx + \ldots + \int_{A_m}^{A_{m-1}} f(x) \, dx$$

$$= \int_a^b \phi'(y) \, dy + \int_b^a \phi'(y) \, dy + \ldots + \int_{A_m}^{A_{m-1}} \phi'(y) \, dy$$

$$= \int_a^b \phi'(y) \, dy.$$
As was previously stated, the class of integrable functions is very large. Even such functions as the following are integrable:

1. \( f(x) = 0 \) if \( x \) is irrational, \( f(x) = \frac{1}{x^2} \) if \( x = \frac{m}{n} \).

2. \( f(x) = \frac{(x)}{x^2} + \frac{(2x)}{2^2} + \ldots + \frac{(nx)}{n^2} \)

where \((x)\) denotes the positive or negative excess of \( x \) over the nearest integer, and \((x) = 0\) when \( x \) is half-way between two integers.

\( f(x) \) is integrable in the interval \((0,1)\).

3. \( f(x) \) defined for the interval \((0,1)\) as follows: For \( \frac{1}{2} < x \leq 1 \), let \( f(x) = 1 \); for \( \frac{1}{2^2} < x \leq \frac{1}{2} \), let \( f(x) = \frac{1}{x^2} \); and generally, for \( \frac{1}{2^n} < x \leq \frac{1}{2^{n+1}} \), let \( f(x) = \frac{1}{x^2} \); and \( f(0) = 0 \).

The following functions are not integrable \((R)\):

1. \( f(x) = \frac{1}{x} \) on the interval \((0,1)\).

2. \( f(x) = 0 \) if \( x \) is irrational, \( f(x) = 1 \) if \( x \) is rational.

3. \( f(x) = 0 \) if \( x \) is rational, \( f(x) = 1 \) if \( x \) is irrational.

The Riemann definition of the integral of a bounded function, defined for a closed interval \((a,b)\), may be extended to the case of a bounded function \( f(x^{(\omega)},x^{(\nu)}) \) defined in a rectangular cell \((a^{(\omega)},a^{(\nu)};b^{(\omega)},b^{(\nu)})\), or more generally to the case of a bounded function \( f(x^{(\omega)},x^{(\nu)},\ldots,x^{(p)}) \), of \( p \) variables, defined in a \( p \)-dimensional cell.
THEOREMS ON SETS OF POINTS

In the present century Riemann's Integral has, for the purposes of theoretical investigations, been largely superseded by the more general formulation of Lebesgue. The theory of Lebesgue integration has as its foundation the conception of the measure of a set of points. In order to understand the theory of measure it is necessary to define certain terms and to state theorems concerning sets of points and intervals.

A set of points is said to be bounded if every point $x$ in it is such that $|x| < A$, where $A$ is some fixed positive number.

A point $P$ is said to be a limit point of a set $G$ if there are points of $G$, other than $P$, in every neighborhood of $P$.

A set of points which includes all its limit points is called a closed set. It will in general be assumed that a closed set is bounded.

A set of points is said to be an isolated set when no point of the set is a limit point of the set.

If every point of a set $G$ is a limit point of the set, $G$ is said to be dense in itself.

A set that is both closed and dense in itself is called perfect.

A point $P$ of a set $G$ is said to be an interior point of $G$ if a neighborhood of $P$ can be determined all the points of which are points of $G$. However, if the set $G$ is bounded, and contained in a fundamental interval, a point of $G$ on the boundary of the interval may be regarded as an interior point of $G$ relatively to the interval if a neighborhood of
the point exists such that every point of that neighborhood which is in
the fundamental interval is a point of G.

An open set is one in which every point is an interior point. A
set is open relatively to a fundamental interval when every point is an
interior point relatively to the interval.

If all the points of a set $H$ are points of a set $G$, $H$ is said to
be a part, or component, of $G$. Those points of $G$ not belonging to $H$ form a
set $G-H$. The set $G-H$ is called the complement of $H$ with respect to $G$
and is denoted by $C_0(H)$. If the set $G$ consists of all the points of the
fundamental interval which contains $H$, then the set $G-H$ is called the
complement of $H$, and is denoted by $C(H)$.

The theorems on sets of points used here are given without proof
since these are largely routine and may be found in any of the treatises
on sets of points.

**Theorem 1.** The complement $C(G)$ of a closed set $G$ with respect
to a closed interval in which $G$ is contained is an open set relatively
to the interval. Conversely, the complement of an open set contained in a
closed interval is a closed set.

A point $P$ of a set $H$ is said to be an interior point of $H$ re-
latively to a perfect set $G$ if it is a point of $G$ and is such that a
neighborhood of $P$ exists for which all the points of $G$ in that neigh-
borhood are also points of $H$.

A set $H$, all the points of which are interior points of $H$ re-
latively to a perfect set $G$, is said to be open relatively to $G$. 
THEOREM 2. The complement $C(H)$ of a closed set with respect to a perfect set $G$ which contains $H$ is open relatively to $G$. Conversely, the complement with respect to the perfect set $G$ of a set $H$ contained in it, and open with respect to it, is a closed set.

If $G, G, \ldots, G_n$ denotes a number of sets of points, the set which contains every point that belongs to one or more of the given sets will be denoted by $M(G, G, \ldots, G_n)$. That set which contains all those points which belong to every one of the given sets will be denoted by $D(G, G, \ldots, G_n)$.

THEOREM 3. If $G$ and $G$ are closed sets, both the sets $M(G, G)$ and $D(G, G)$ are closed.

The theorem can be extended to the case of any finite number of closed sets. If the number of closed sets is indefinitely great, so that they form a sequence $G, G, \ldots, G, \ldots$ of such sets, the set $M(G, G, \ldots, G, \ldots)$ is not necessarily closed. The set $D(G, G, \ldots, G, \ldots)$, however, if it exists, is closed.

THEOREM 4. If $O, O, O, \ldots$ is a finite number, or a sequence, of open sets, then $M(O, O, O, \ldots)$ is also an open set.

THEOREM 5. If $O, O, \ldots, O_n$ is a finite number of open sets, the set $D(O, O, \ldots, O_n)$, if it exists, is also an open set.

The preceding theorem does not hold for the case of an infinite number of open sets.

THEOREM 6. If $G, G, \ldots, G, \ldots$ is a sequence of closed sets, each of which contains the next, there exists a closed set $G_\omega$, each point
of which belongs to $G_\omega$, for every value of $\omega$.

The corresponding statement does not necessarily hold for a sequence \{\(G_\omega\)\} of sets, each of which contains the next, when the sets are not closed.

The properties of a set of intervals are closely connected with the properties of sets of points.

**Theorem 7.** Every set of non-overlapping intervals in a bounded, or unbounded, interval is either finite or forms an enumerable set.

**Theorem 8.** Every isolated set is enumerable.

**Theorem 9.** Every set of intervals contained in a finite interval can be replaced by a set of non-overlapping intervals of which the interior points are the same as those of the given set.

If the given set is taken to be a set of open intervals, the theorem is equivalent to the statement that the set of points, each of which belongs to one or more open intervals of the given set, is itself an open set.

Every point of a finite interval \(a, b\) which is not interior to an interval of a non-overlapping set of open intervals in \(a, b\) is

1. a common end-point of two intervals of the given set; or
2. a point interior to, or at an end of, an interval not belonging to the given set, this interval containing no point which is interior to any interval of the set; or
3. a limit point, on both sides, of end-points of intervals of the given set; or
(4) an end-point of an interval of the given set, and also a limit point, on one side, of end-points of intervals of the given set.

If either $a$ or $b$ is an end-point of an interval, it is said to belong to set (1). The points described in (2) and (3) may be called external points of the given set; and if $a$ or $b$ is a limit point of end-points, it will be considered an external point. The points described in (4) may be called semi-external points.

**Theorem 10.** Those points of the interval $(a, b)$ which are not points of a given set of non-overlapping open intervals form a closed set of points.

**Theorem 11.** Every closed set of points in the interval $(a, b)$ is the complement of a non-overlapping set of open intervals.

The most general linear closed set of points in an interval $(a, b)$ consists of (1) the end-points of a set of non-overlapping intervals, (2) limit points of such end-points, and (3) the points interior to intervals every point of which belongs to the closed set.

The open intervals belonging to the set $C(G)$, complementary, relatively to $(a, b)$, to a given closed set $G$ in $(a, b)$, are said to be contiguous to, or complementary to, $G$.

If the set $G$ is non-dense, no interval exists in $(a, b)$ which consists entirely of points of $G$. (A non-dense set is one such that no interval exists in which the set is everywhere dense.) The set of contiguous intervals is then everywhere dense in $(a, b)$, since no interval can be determined in $(a, b)$ so as to contain no points of the set of
contiguous intervals. Thus: Every linear non-dense closed set in an interval \((a,b)\) consists of the end-points of the intervals of an everywhere dense set of open intervals and of the limit points of such end-points.

The points of a non-dense closed set \(G\) consist in general of three classes:

1. Those which are common end-points of two contiguous intervals abutting on one another;

2. semi-external points of the set of contiguous intervals; that is, points which are end-points of one interval, and also limit points, on one side, of end-points; and

3. external points; that is, such as are not end-points of any contiguous interval, but are limit points, on both sides, of such end-points.

The point \(a\) or \(b\), if it belongs to \(G\), may be regarded as belonging to (1) or (3), according as it is, or is not, an end-point of a contiguous interval. Those points which belong to (1) are isolated points of \(G\). If no such points exist, every point of \(G\) is a limit point; \(G\) then is perfect. It follows that: Every non-dense perfect linear set \(G\) consists of the end-points of an everywhere dense set of non-overlapping intervals, contiguous to \(G\), no two of which abut on one another, together with the limit points of these end-points.

Let \(S_1, S_2, \ldots, S_n, \ldots\) be a sequence of sets of points such that each set \(S_{n+1}\) is contained in the preceding one \(S_n\). The set \(S_\infty\), or
\( D(E_1, S_1, \ldots, E_n, \ldots) \), consisting of points each of which belongs to all the sets of the sequence, is said, when such a set exists, to be the inner limiting set of the sequence of sets. If any sequence \( G_1, G_2, \ldots, G_n, \ldots \) of sets be given, and we take \( E_1 = G_1, S_1 = D(G_1, G_2), S_2 = D(G_1, G_2, G_3), \ldots \), and in general \( S_m = D(G_1, G_2, \ldots, G_m) \), the inner limiting set \( E_\infty \), when it exists, also defines \( D(G_1, G_2, \ldots, G_m, \ldots) \) and may be regarded as defined by the given sets \( G_1, G_2, \ldots \). A point that belongs to all the sets \( G_1, \ldots, G_m \) belongs to all the sets \( E_1, S_2, \ldots, S_m, \ldots \); and a point that belongs to all the latter sets belongs to all the former. Thus we see that the set of points common to all the sets of any given sequence of sets is an inner limiting set.

When \( E_1, S_1, \ldots, S_m, \ldots \) is a sequence of sets, each one of which \( S_m \) is contained in the next \( S_{m+1} \), the set \( S_\infty \), or \( M(S_1, S_2, \ldots, S_m, \ldots) \), which consists of the set of those points each of which belongs to all the sets, from and after some value of \( n \) dependent on the particular point, is said to be the outer limiting set of the sequence. If we take \( S_1 = G_1, S_2 = M(G_1, G_2), S_3 = M(G_1, G_2, G_3), \ldots \), the outer limiting set \( S_\infty \) defines \( M(G_1, G_2, \ldots, G_m, \ldots) \), and thus the set of all points that belong to at least one of the sets of any given sequence is an outer limiting set.

From Theorem 6 we know that when the sets \( E_1, S_1, \ldots \) are all closed sets, the inner limiting set always exists, and is a closed set.
THE PROBLEM OF ASSIGNING DEFINITE NUMERICAL MEASURES TO SETS OF POINTS

The problem of assigning definite numerical measures to sets of points requires that the measure of a set shall satisfy the following conditions:

(1) The linear measure of the set of points in an interval \((a, b)\) is taken to be \(b - a\) whether the set includes neither, one, or both of the end-points, \(a\) and \(b\), of the interval.

(2) The measure of the sum of a finite number of sets, no two of which have a point in common, is to be the sum of the measures of the sets.

(3) The measure of the sum of an enumerably infinite sequence of sets, no two of which have a point in common, is to be the limiting sum of the measures of the sets, whenever that limiting sum exists.

A linear set of points \(O\), open relatively to an interval \((a, b)\) which contains the set, is the sum of a unique enumerable set of open intervals. According to (3), the measure of \(O\) must be taken to be the limiting sum of the measures of the intervals of this set. This limiting sum is finite and cannot exceed \(b - a\); for the sum of the first \(n\) intervals, arranged in order, is less than \(b - a\) by the sum of the lengths of the finite set of intervals complementary to them.

Every open interval may be regarded as consisting of the points of an enumerable set of closed intervals, no two of which overlap one another, although they may have an end-point in common. Hence any open
set may be regarded as consisting of the points of an enumerable set of
non-overlapping closed intervals. The measure of the open set can then
be regarded as the limiting sum of the measures of such a set of non-
overlapping closed intervals. Since the set of closed intervals which
constitutes a given open set \( O \) is not unique, it must be shown that the
measure of \( O \) is independent of the particular sets of such intervals.
This can be proved by the following:

**Theorem 12.** If an open set \( O \) consists of all the points of an
enumerable set \( [\mathcal{J}_m] \) of non-overlapping closed intervals, and if \( [\mathcal{J}'_m] \)
is any other set of non-overlapping closed intervals such that every
point of \( O \) belongs to at least one of the closed intervals \( [\mathcal{J}'_m] \), the
limiting sum of the measures of \( [\mathcal{J}'_m] \) cannot be less than that of the
measures of \( [\mathcal{J}_m] \).

The unique measure of an open bounded set \( O \) is thus defined as
the limiting sum of the measures of the closed intervals of a set, all
the points of which constitute the set \( O \).

The measure of a bounded closed set \( G \) is then defined to be the
excess of the measure of the fundamental interval, in which \( G \) is con-
tained, over the measure of the open set \( C(G) \), which is the complement of
\( G \) with regard to the fundamental interval. All the points of the closed
set \( G \) are in the finite set of intervals complementary to the intervals
\( \mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n \). Thus the measure of \( G \), denoted by \( m(G) \), is the lower limit
of the sum of the measures of this set, as \( n \) is indefinitely increased.
THEOREM 13. If \( O_1, O_2, \ldots \) is a sequence of open sets all contained in a finite interval, then
\[
m(O_1) + m(O_2) + \cdots + m(O_n) + \cdots = m\left(\bigcup (O_1, O_2, \ldots)\right).
\]

THEOREM 14. If \( O_1, O_2, \ldots O_n, \ldots \) is a sequence of open sets all contained in a finite interval, and such that no two of them have a point in common, then
\[
m(O_1) + m(O_2) + \cdots + m(O_n) + \cdots = m\left(\bigcup (O_1, O_2, \ldots)\right).
\]

The Exterior and Interior Measures of a Set

Let \( S \) be any linear set of points in a given interval \((a, b)\), and let a finite, or infinite, set of non-overlapping open intervals be defined, such that every point of \( S \) is in one of these intervals. Either point \( a \) or point \( b \) is regarded as an interior point of any interval of which it is an end-point. The set of intervals constitutes an open set which contains \( S \). The sum, or limiting sum, of the lengths of the intervals has a definite value, not greater than \( b - a \). The lower boundary of this sum, or limiting sum, for all possible such sets of intervals, is a number which is called the exterior measure of the set \( S \), and this may be denoted by \( m(a)(S) \).

Let \( C(S) \) be the complementary set of \( S \), relatively to the interval \((a, b)\) in which \( S \) is contained. If \( m(a)[C(S)] \) represents the exterior measure of \( C(S) \), the number \( b - a - m(a)[C(S)] \) defines the interior measure of the set \( S \); it may be denoted by \( m(j)(S) \).

The above definitions are equivalent to the following statements: The exterior measure \( m(a)(S) \) of a bounded set \( S \) is the lower
boundary of the measures of open sets which contain \( G \). The interior measure \( m_+ (G) \) of a bounded set \( G \) is the upper boundary of the measures of closed sets contained in \( G \). This can be explained as follows: By definition, the exterior measure \( m_- \{ C(G) \} \) is the lower boundary of open sets which contain \( C(G) \). Then
\[
 b - a = m_- \{ C(G) \}, \text{or } m_+ (G), \text{is the upper boundary of the measures of the closed sets complementary to these open sets. It is evident then that every bounded set of points has definite exterior and interior measures. When the exterior and interior measures of a set \( G \) are equal, the set \( G \) is said to be \textit{measurable}, and the number \( m_+ (G) = m_- (G) \) is defined to be the \textit{measure} of \( G \). If a set \( G \) is measurable, \( C(G) \) also is.

That the above definition satisfies the conditions for measure previously listed, whenever it is applicable, can be shown as follows: If \( \alpha \) is a set of non-overlapping intervals enclosing a set of points \( G \), measurable or not, and \( \beta \) is a similar set for \( C(G) \), then all the points of the fundamental interval are enclosed in the set made up of \( \alpha \) and \( \beta \). It follows that
\[
 m(\alpha) + m(\beta) \geq I, \text{where } I \text{ is the measure of the interval. Since } m_- (G) \text{ and } m_- \{ C(G) \} \text{ are the lower boundaries of } m(\alpha) \text{ and } m(\beta), \text{respectively, } m_- (G) + m_- \{ C(G) \} \geq I, \text{ and therefore}
\]
\[
 m_- (G) \geq m_+ (G).
\]

From the definition of a measurable set we can make the following statement: A set of points \( G \) is measurable if an open set \( O \), containing \( G \), and a closed set \( H \), contained in \( G \), can be so determined that
\[
 m(0) - m(H) \text{ is less than an arbitrarily chosen positive number } \varepsilon. \]
**THEOREM 15.** A necessary and sufficient condition that a set of points $G$ shall be measurable is that its points can be enclosed in a finite, or an infinite, set $\mathcal{C}$ of open intervals and that $C(G)$ can be similarly contained in a set $\mathcal{D}$, such that the sum, or limiting sum, of the measures of the open intervals which contain all those points which are common to $\mathcal{C}$ and $\mathcal{D}$ is arbitrarily small.

**THEOREM 16.** Every bounded open set and every bounded closed set is measurable.

**THEOREM 17.** Every enumerable set of points is measurable, and its measure is zero.

**PROOF:** Let $P, P_1, \ldots, P_n, \ldots$ denote the points of the set. Each point $P_m$ can be enclosed in an open interval with $P_m$ as center and of measure $\frac{\epsilon}{2^m}$. By Theorem 13 the open set of points consisting of all the points belonging to one or more of these intervals has its measure $\leq \sum \frac{\epsilon}{2^m}$, or $\leq \epsilon$. It follows that the exterior measure of the set is zero, and thus that the interior measure is also zero, and that the set is measurable, with measure equal to zero.

It is interesting to note that, in accordance with the preceding theorem, the set of all the rational numbers can be covered by a set of intervals the sum of whose measures is less than any arbitrarily chosen positive number.

**THEOREM 18.** If $G, G_1, G_2, \ldots, G_n, \ldots$ is an enumerable, or a finite, sequence of measurable sets, all contained in a finite interval, the set $M(G, G_1, G_2, \ldots, G_n, \ldots)$ is measurable.
**Theorem 19.** If $G_1, G_2, \ldots, G_\infty$ are measurable sets all contained in a bounded domain, and no two of the sets have a point in common, then the set $(G_1 + G_2 + \cdots)$ is measurable and its measure is the limiting sum of the measures of the sets.

**Theorem 20.** If a measurable set $G_1$ contains another measurable set $G_2$, the set $G_1 - G_2$ is measurable, and its measure is $m(G_1) - m(G_2)$.

**Theorem 21.** If $G_1, G_2, \ldots, G_\infty$ are all measurable sets, the set $D(G_1, G_2, \ldots, G_\infty)$ is also measurable.

**Theorem 22.** If $H$ is the set of points each of which belongs to an infinite number of the measurable sets $G_1, G_2, G_3, \ldots$, the set $H$ is measurable.

**Theorem 23.** If $K$ is the set of points each of which belongs to all the measurable sets $G_1, G_2, G_3, \ldots$, where $\alpha$ has a definite value for each point of $K$, the set $K$ is measurable.

**Theorem 24.** If $G_\infty$ is the inner limiting set of a sequence $\{G_n\}$ of measurable sets $G_1, G_2, \ldots$, each of which contains the next, then $m(G_\infty) = \lim_{n \to \infty} m(G_n)$.

**Theorem 25.** If $G_\infty$ is the outer limiting set of a sequence $\{G_n\}$ of measurable sets $G_1, G_2, \ldots$, each of which is contained in the next, and if $G_\infty$ is a bounded set, $m(G_\infty) = \lim_{n \to \infty} m(G_n)$.
MEASURABLE FUNCTIONS

A function \( f(x) \), defined in any interval \((a, b)\), is said to be measurable, provided that, for every value of \( A \), the set of points \( x \), of \((a, b)\), at which \( f(x) > A \), is a measurable set of points.

**Theorem 26.** If \( f(x) \) is a measurable function, defined at each point of a given domain, the set of points for which \( A < f(x) < B \); \( A \leq f(x) < B ; A \leq f(x) \leq B ; f(x) < A ; f(x) \leq A \) are all measurable, whatever real numbers \( A \) and \( B \) may denote, provided \( A < B \).

**Proof:** Let \( A \) have the values \(-N, -N_1, -N_2, \ldots, -N_m, \ldots\), successively, of a sequence such that \( N_m \) increases indefinitely as \( m \to \infty \). The set \( E_m \), for which \( f(x) > -N_m \), is measurable, by hypothesis, for every value of \( m \). The domain for which \( f(x) \) is defined is the outer limiting set of the sequence \( \{ E_m \} \) of measurable sets, and is therefore measurable by Theorem 25. The set of points for which \( f(x) \leq A \), being complementary, relatively to the domain of the function, to the measurable set for which \( f(x) > A \), is measurable. If \( \{ A_m \} \) is a monotone increasing sequence of numbers converging to \( A \), all the sets for which \( f(x) \leq A_m \) are measurable, and their outer limiting set, for which \( f(x) < A \), is consequently measurable. The sets for which \( f(x) < A \) and \( f(x) \leq A \) being measurable, it follows that the sets for which \( f(x) = A \) is measurable. The set \( f(x) = A \) is identical with the set \( f(x) < B \). The sets for which \( f(x) < B \) and \( f(x) < A \) being measurable, their difference, the set for which \( A \leq f(x) < B \), is also measurable. The set for which \( A < f(x) < B \) is then
measurable as is the set for which \( A \leq f(x) \leq B \).

**Theorem 27.** A function \( f(x) \) is measurable if the set of points \( x \) is measurable, for which \( \alpha < f(x) < \beta \), for every pair \( \alpha, \beta \), of real numbers which belong to a given set, everywhere dense in the indefinite interval \( (-\infty, \infty) \). The given set may be taken to be enumerable.

**Proof:** Let \( A \) and \( B \) be any real numbers such that \( A < B \). The number \( A \) can be expressed as the upper limit of a sequence \( \{ \alpha_n \} \) of increasing numbers, all of which belong to the everywhere dense set; and the number \( B \) can be taken to be the lower limit of a sequence \( \{ \beta_n \} \) of diminishing numbers. The set \( \varepsilon \) for which \( \alpha_n < f(x) < \beta_n \) is measurable for each value of \( n \) by the preceding theorem. The inner limiting set \( \{ \varepsilon_n \} \) of the sequence is the set for which \( A \leq f(x) \leq B \), and this set is consequently measurable. Since this is the case whatever values \( A \) and \( B \) may have, \( f(x) \) is measurable.

**Theorem 28.** If \( \Phi_1, \Phi_2, \ldots, \Phi_n \) is a finite set of functions that are measurable in a measurable domain, and if \( F(\Phi_1, \Phi_2, \ldots, \Phi_n) \) is a function that is continuous relatively to \( (\Phi_1, \Phi_2, \ldots, \Phi_n) \) for all values of \( \Phi_1, \Phi_2, \ldots, \Phi_n \), then \( F(\Phi_1, \Phi_2, \ldots, \Phi_n) \) is measurable in the given domain.

**Proof:** (for bounded functions) Suppose the values of \( \Phi_1, \Phi_2, \ldots, \Phi_n \) are in the interval \( (-N, N) \). Let a partition \( (c_0, c_1, \ldots, c_m) \) be fitted on to the interval \( (-N, N) \), where \( c_0 = -N \), \( c_m = N \), and suppose the breadth \( c_n - c_{n-1} \) of each interval is less than \( \eta \). Let the function \( \nu_n \) be defined, corresponding to each function \( \Phi_n (n = 1, 2, \ldots, n) \),
by the conditions \( \forall_a = \phi_a \leq \alpha_a \) at every point at which \( \alpha_a \leq \phi_a < c_a \),
for \( a = 1, 2, \ldots, m - 1 \), and \( \forall_a = c_m \) where \( \phi_a = c_m \). Then
\[ 0 \leq \phi_a - \forall_a < \zeta, \]
and the function \( \forall_a \) taking only the values in the
finite set \( c_a, c, \ldots, c_m \), is measurable in the given domain. Since
\[
F(\phi, \phi_2, \ldots, \phi_m) \text{ is continuous in the closed domain, } (-N, -N, \ldots; N, N, -)
\]
| \( F(\phi_1, \phi_2, \ldots, \phi_m) - F(\forall_1, \forall_2, \ldots, \forall_m) \) | < \( \varepsilon \), provided \( \zeta \) is
taken sufficiently small; \( \varepsilon \) being arbitrarily chosen. The function
\( F(\forall_1, \forall_2, \ldots, \forall_m) \) has only a finite set of values, and is measurable.
If \( U \) and \( L \) are its upper and lower boundaries, respectively, we have
\[ L - \varepsilon < F(\phi_1, \phi_2, \ldots, \phi_m) < U + \varepsilon \]
in the whole domain. Let \( A \) and \( B \) be any two numbers in the interval \( (L, U) \); then the set of points for
which \( A < F(\forall_1, \forall_2, \ldots, \forall_m) < B \) is measurable. Let \( \varepsilon \) have successively
the values in a sequence \( \{\varepsilon_\alpha\} \) which converges to zero; then
there exists a corresponding sequence \( \{\zeta_\alpha\} \) of values of \( \zeta \), which
converges to zero. The set of points \( E_\alpha \), for which
\[ A < F(\forall_1, \forall_2, \ldots, \forall_m) < B, \]
is measurable for each value of \( \zeta_\alpha \) in \( \{\zeta_\alpha\} \).
Each point of the set for which \( A < F(\phi_1, \phi_2, \ldots, \phi_m) < B \) belongs to
all the measurable sets \( E_\alpha \), from and after some particular value of \( x \),
and therefore the set is measurable by Theorem 23. Hence
\( F(\phi_1, \phi_2, \ldots, \phi_m) \) is measurable in the given domain.

From the preceding we may state:

**Theorem 23.** The sum, or the product, of any finite number of
measurable functions, defined in a measurable domain, is a measurable
function.
The Lebesgue Integral of a Measurable Function

Let $f(x) = 1$ at all points of a measurable set $E$. The measure $m(E)$ is said to define the integral $\int_E f(x) \, dx$, of the function $f(x)$, over the set $E$. If $c$ is any number and $f(x) = c$ at all points of $E$, then $\int_E f(x) \, dx$ is defined to be the number $cm(E)$. Next let $E_1, E_2, \ldots, E_m$ be a finite number of measurable sets, no two of which have a point in common. Let $f(x) = c_\infty$ at the points of $E_\infty$, for $\infty = 1, 2, \ldots, m$; where $c_1, c_2, \ldots, c_m$ are assigned numbers. Then if $E$ denotes the set $E_1 + E_2 + \ldots + E_m$, the integral $\int_E f(x) \, dx$ is defined as the sum $\sum_{\infty} c_\infty m(E_\infty)$. Next let $f(x)$ be a measurable function, defined for the points of a measurable set $E$, and bounded in that set. Let $U$ and $L$ denote the upper and lower boundaries, respectively, of $f(x)$ in $E$. Let the interval $(L, U)$ be divided into $\infty$ parts $(a_0, a_1), (a_1, a_2), \ldots, (a_{\infty-1}, a_\infty)$; where $a_0 = L, a_\infty = U$, and such that the greatest of these parts $a_\infty - a_{\infty-1} < \gamma (\infty = 1, 2, \ldots, \infty)$. Let $E_\infty$ be that part of $E$ for all points of which $a_{\infty-1} \leq f(x) < a_\infty$ where $\infty = 1, 2, \ldots, \infty-1$. Let $E_\infty$ be that part of which $a_{\infty-1} \leq f(x) \leq a_\infty$. Let $\phi_\gamma(x)$ be the function which has the value $a_{\infty-1}$ at all points of $E_\infty$, for $\infty = 1, 2, \ldots, m$; and let $v_\gamma(x)$ be the function which has the value $a_\infty$ at all points of $E_\infty$, for $\infty = 1, 2, \ldots, \infty$. Then, according to the definition given above,

$$\int_{(E)} \phi_\gamma(x) \, dx = \sum_{\infty} a_{\infty-1} m(E_\infty)$$

and

$$\int_{(E)} v_\gamma(x) \, dx = \sum_{\infty} a_\infty m(E_\infty).$$
Therefore $0 \leq \int_{(E)} \psi_{\eta m}(x) \, dx - \int_{(E)} \phi_{\eta m}(x) \, dx < \eta_m m(E)$.

If we assign to $\eta$, successively, the values in a sequence $\{\eta_m\}$ of diminishing numbers, for which $\eta_m \to 0$ as $m \to \infty$, it is seen that $\int_{(E)} \phi_{\eta m}(x) \, dx$ does not diminish, and that $\int_{(E)} \psi_{\eta m}(x) \, dx$ does not increase as $m \to \infty$. These two sets consequently converge to a common limit. Hence $\lim_{m \to \infty} \int_{(E)} \phi_{\eta m}(x) \, dx = \lim_{m \to \infty} \int_{(E)} \psi_{\eta m}(x) \, dx$ is defined to be the value of the Lebesgue Integral $\int_{(E)} f(x) \, dx$, of $f(x)$, taken over the measurable set $E$. That the number so defined is independent of the particular mode in which the interval $(L, U)$ is subdivided may be proved as follows: Let $\overline{\phi}_{\eta m}(x)$, $\overline{\psi}_{\eta m}(x)$ be the functions which correspond, in a second mode of subdivision, to $\phi_{\eta m}(x)$, $\psi_{\eta m}(x)$ defined above. There is no loss of generality in taking the sequence $\{\eta_m\}$ to be the same in the two cases. Suppose the two subdivisions of $(L, U)$, corresponding to $\eta_m$, to be superimposed, and let $\overline{\phi}_{\eta m}(x)$ be the function corresponding to $\phi_{\eta m}(x)$ and $\overline{\psi}_{\eta m}(x)$.

Then $0 \leq \int_{(E)} \overline{\phi}_{\eta m}(x) \, dx - \int_{(E)} \phi_{\eta m}(x) \, dx < \eta_m m(E)$.

and $0 \leq \int_{(E)} \overline{\psi}_{\eta m}(x) \, dx - \int_{(E)} \phi_{\eta m}(x) \, dx < \eta_m m(E)$.

Hence $| \int_{(E)} \phi_{\eta m}(x) \, dx - \int_{(E)} \overline{\phi}_{\eta m}(x) \, dx | < 2 \eta_m m(E)$.

As $m \to \infty$, $2 \eta_m \cdot m(E) \to 0$; and therefore

$\lim_{m \to \infty} \int_{(E)} \phi_{\eta m}(x) \, dx = \lim_{m \to \infty} \int_{(E)} \overline{\phi}_{\eta m}(x) \, dx$. 

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