Daniell integral

Eddward Melvin Wadsworth
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DANIELL INTEGRAL

BY

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Edward Melvin Wadsworth
CHAPTER I

THEORY OF THE DANIELL INTEGRAL

We shall develop the theory of the integral as given originally by Daniell [2]. The structure starts with a positive linear functional defined on a vector lattice of functions, defined subsequently.

DEFINITION 1.1. A vector space or linear space, V, over a field F is a set of elements called vectors such that f and g in V implies that f + g is in V; f in V, a in F implies af in V; and further:

i) V is an Abelian group under addition;

ii) for all f and g in V and for all a and b in F, a(f + g) = af + ag, (a + b)f = af + bf, (ab)f = a(bf), 1f = f.

DEFINITION 1.2. Let X be an arbitrary set. Suppose f and g are two real-valued functions on X. Then define

\[ f \lor g = \max(f, g) = \left[ (f - g) \lor 0 \right] + g \]

and

\[ f \land g = \min(f, g) = (f + g) - (f \lor g), \]

where 0 is the function identically zero on all of X.

DEFINITION 1.3. Suppose S is a set of real-valued functions on a set X. Then S is called a lattice provided that for all f and g in S, f \lor g and f \land g are in V.
Further, a vector space $V$ of real-valued functions on a set $X$ over the real field is a vector lattice provided that $V$ is also a lattice.

**LEMMA 1.4.** Suppose $L$ is a vector space of real-valued functions on $X$. Then $L$ is a vector lattice provided that for each $h$ in $L$, $h \lor 0$ is in $L$.

**PROOF.** Suppose $f$ and $g$ are in $L$. Then $-g$ is in $L$, so $f - g$ is in $L$, thus $(f - g) \lor 0$ is in $L$ and $[(f - g) \lor 0] + g$ is in $L$. Hence $f \lor g$ is in $L$. Similarly, $f + g$ is in $L$ and $f \lor g$ is in $L$, so $f \land g$ is in $L$.

**DEFINITION 1.5.** Suppose $L$ is a vector lattice of real-valued functions on $X$. Then if $f$ is in $L$ define $|f| = (f \lor 0) + (-f \lor 0)$. Further, let $f^+ = f \lor 0$.

**NOTE 1.6.** This implies that $|f|$ is in $L$ since by 1.4, $(f \lor 0)$ is in $L$ and $(-f \lor 0)$ is in $L$. Note also that this implies that $f^+$ is in $L$ for all $f$ in $L$ and that $|f| = f^+ + (-f)^+$.

**DEFINITION 1.7.** Suppose $V$ is a vector space over a field $F$. A linear functional $I$ from $V$ to $F$ is a function defined such that if $f$ and $g$ are in $V$, and $a$ and $b$ are in $F$, then $I(af + bg) = aI(f) + bI(g)$. If $V$ is a vector lattice of real-valued functions and if $f(x) \geq 0$ for all $x$ in $X$ implies $I(f) \geq 0$, then $I$ is said to be positive. Note that this implies that if $I$ is positive and if $x$ in $X$ then $I(f) \leq I(g)$.
THEOREM 1.8. Suppose $L$ is a vector space of real-valued functions on $X$. Then $L$ is a vector lattice iff for every $f$ in $L$, $g$ is in $L$ where $g(x)$ is $f(x)$ if $f(x) \geq 0$ or $-f(x)$ if $f(x) < 0$. Further, $g = |f|$.

PROOF. Note that by 1.2, $(f \vee 0)(x) = f(x)$ if $f(x) \geq 0$ or $0$ if $f(x) \leq 0$.

**only if:** $L$ a vector lattice implies $|f|$ is in $L$, where $|f|$ is as given in 1.5. Suppose $f$ is in $L$ and suppose $p$ is in $X$. If $f(p) \geq 0$ then $-f(p) \leq 0$ and $(f \vee 0)(p) = f(p)$ and $(-f \vee 0)(p)$ is 0. Hence

$$|f|(p) = f(p) = g(p).$$

If $f(p) < 0$ then $-f(p) > 0$ so $(f \vee 0)(p) = 0$ and $(-f \vee 0)(p) = -f(p)$. Hence,

$$|f|(p) = -f(p) = g(p).$$

But by 1.6, $|f|$ is in $L$, and thus $g$ is in $L$, and $g = |f|$.

**if:** $g$ in $L$ implies $L$ a vector lattice. Suppose $f$ is in $L$. Then $g$ is in $L$ and $f \vee 0 = \frac{1}{2}(f + g)$ is in $L$ and by 1.4, $L$ is a vector lattice.

DEFINITION 1.9. Define $\mathbb{R}^*$, the extended real number system, to be the real numbers with $\infty$ and $-\infty$ adjoined with the following conventions: if $a$ is in $\mathbb{R}^*$ but neither $\infty$ nor $-\infty$ then,

(i). $a + \infty = \infty + a = \infty$. $a + (-\infty) = -\infty + a = -\infty$.

(ii). $a(-\infty) = -\infty$ if $a > 0$. $a(\infty) = \infty$ if $a > 0$.

(iii). $-\infty < a < \infty$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
(iii). \( \infty + \infty = \infty. \) \(-\infty + (-\infty) = -\infty. \) \( \infty - (-\infty) = \infty. \) \(-\infty - (\infty) = -\infty. \)

(iv). an infinite sum with one or more terms \( \infty \)
and no terms of \(-\infty \) is equal to \( \infty \).

We may notice here that in \( \mathbb{R}^* \), every increasing sequence of real numbers has a limit, where we define \( \lim a_n = \infty \) if the sequence is not bounded.

**DEFINITION 1.10.** Suppose that \( I \) is a positive linear functional defined on a vector lattice \( L \) of real-valued functions on \( X \). Then \( I \) is called a Daniell functional or a Daniell integral if whenever \( \{ f_n \} \) is an increasing sequence of functions in \( L \) and \( f \) is in \( L \) and is such that

\[
f(x) \leq \lim f_n(x) \text{ for all } x \text{ in } X, \text{ then } I(f) \leq \lim I(f_n).
\]

**LEMMA 1.11.** Suppose \( \{ u_n \} \) is a sequence of non-negative functions in \( L \). Suppose that \( I \) is a Daniell integral and further that \( g \) in \( L \) is such that \( g(x) \leq \sum_n u_n(x) \) for all \( x \) in \( X \). Then \( I(g) \leq \sum_n I(u_n) \).

**PROOF.** Define \( f_n(x) = \sum_{i=1}^n u_i(x) \) for all \( x \) in \( X \). Then \( f_n \) is in \( L \) for all \( n \) and \( \lim f_n(x) = \sum_n u_n(x) \) for all \( x \) in \( X \). Now, \( g(x) \leq \sum_n u_n(x) \) which is \( \lim f_n(x) \) for all \( x \) in \( X \). Thus \( I(g) \leq \lim I(f_n) \). But

\[
I(f_n) = I(u_1) + \ldots + I(u_n) = \sum_{i=1}^n I(u_i).
\]
Hence, \[ \lim I(f_n) = \sum_n I(u_n) \]
so \[ I(g) \leq \sum_n I(u_n). \]

**DEFINITION 1.12.** Let \( L_u \) be the class of all those extended real-valued functions on \( X \) each of which is the limit of a monotone increasing sequence of functions in the vector lattice \( L \).

Now we have enlarged our vector lattice \( L \) on which a Daniell Integral \( I \) has been defined to \( L_u \) which we show is only a lattice. We wish to show that \( I \) can still be defined.

**LEMMA 1.13.** Suppose \( f \) and \( g \) are in \( L_u \) and \( a \) and \( b \) are non-negative real numbers. Then \( af + bg \) is in \( L_u \), \( f \vee g \) is in \( L_u \), \( f \wedge g \) is in \( L_u \), and hence, \( L^u \) is a lattice containing \( L \).

**PROOF.** Suppose \( \{f_n\} \) and \( \{g_m\} \) are monotone increasing sequences of functions in \( L \) such that \( f(x) = \lim f_n(x) \) for all \( x \) in \( X \) and such that \( g(x) = \lim g_m(x) \) for all \( x \) in \( X \). Then \( af(x) = a \lim f_n(x) = \lim af_n(x) \) for all \( x \) in \( X \) so \( af \) is in \( L_u \). Also, \( bg \) is in \( L_u \). In addition, 
\[ af(x) + bg(x) = \lim af_n(x) + \lim bg_m(x) = \lim (af_n(x) + bg_n(x)). \]
Thus, \( af + bg \) is in \( L_u \).

Since \( f_n \vee g_n \) is in \( L \) for all \( n \) we know that 
\[ \lim (f_n \vee g_n) \]  is in \( L_u \) if \( f_n \vee g_n \) is monotone increasing.
To see this suppose \( p \) is in \( X \). Then

\[
  f_n(p) \leq f_{n+1}(p) \leq (f_{n+1} \lor g_{n+1})(p)
\]

and

\[
  g_n(p) \leq g_{n+1}(p) \leq (f_{n+1} \lor g_{n+1})(p)
\]

so that

\[
  (f_n \lor g_n)(p) \leq (f_{n+1} \lor g_{n+1})(p).
\]

Thus \( f_n \lor g_n \) is monotone increasing. Then if

\[
  \lim f_n(p) = \infty \text{ or } \lim g_n(p) = \infty, \lim (f_n \lor g_n)(p) = \infty
\]

and \( (f \lor g)(p) = \infty \) so in this case

\[
  (f \lor g)(p) = \lim (f_n \lor g_n)(p).
\]

Now suppose that \( \lim f_n(p) \neq \infty \) and \( \lim g_n(p) \neq \infty \);
then \( \lim (f_n \lor g_n)(p) \neq \infty \) and \( (f \lor g)(p) \neq \infty \). Now define

\[
  h_n = f_n \lor g_n \text{ for all positive } n.
\]

Then \( h_n \) is in \( L \) for all \( n \) and \( \lim h_n(p) = \lim (f_n \lor g_n)(p) \). Suppose

\[
  \lim h_n(p) > \max(\lim f_n(p), \lim g_n(p)) = \max(f(p), g(p)).
\]

Then, since \( h_n \) is monotone, there exists a positive integer

\( N \) such that \( n \geq N \) implies \( h_n(p) > \max(f(p), g(p)) \). But

\[
  \{f_n\}, \{g_n\} \text{ monotone increasing implies } \lim g_n(p) \geq g_n(p)
\]

and \( \lim f_n(p) \geq f_n(p) \) for all \( n \). Thus

\[
  h_n(p) > \max(f_i(p), g_i(p)) \text{ for all } i \text{ and for all } n \geq N.
\]

Thus we have a contradiction.

Therefore, \( \lim h_n(p) \leq \max(f(p), g(p)) \). But \( h_n(p) \geq f_n(p) \)
for all \( n \) implies \( \lim h_n(p) \geq f(p) \), and \( h_n(p) \geq g_n(p) \)
for all \( n \) implies \( \lim h_n(p) \geq g(p) \) which implies

\[
  \lim h_n(p) \geq \max(f(p), g(p)), \text{ so } \lim h_n(p) = \max(f(p), g(p)).
\]

Hence, \( f \lor g \) is in \( L_u \). Similarly, \( f \land g \) is in \( L_u \). Clearly

\( L \) is contained in \( L_u \), and hence \( L_u \) is a lattice containing \( L \).
LEMMA 1.14. Suppose \( \{ f_n \} \) and \( \{ g_m \} \) are monotone increasing sequences such that \( f_n \) and \( g_m \) are in \( L \) for all \( n \) and \( m \). Suppose also that \( \lim f_n \leq \lim g_m \). Then 
\[
\lim I(f_n) \leq \lim I(g_m).
\]
Further if \( f \) is in \( L_u \), \( f_n \uparrow f \), \( g_m \uparrow f \), then 
\[
\lim I(f_n) = \lim I(g_m).
\]

PROOF. Choose \( n \) arbitrary but fixed. Then 
\[
f_n \leq \lim f_n \leq \lim g_n \quad \text{so} \quad I(f_n) \leq \lim I(g_m) \quad \text{for each} \quad n.
\]
Hence 
\[
\lim I(f_n) \leq \lim I(g_m).
\]
Suppose \( f \) is in \( L_u \), \( f_n \uparrow f \), \( g_m \uparrow f \). Then 
\[
\lim f_n = f \leq \lim g_m
\]
so 
\[
\lim I(f_n) \leq \lim I(g_m).
\]
But 
\[
\lim g_m = f \leq \lim f_n
\]
so 
\[
\lim I(g_m) \leq \lim I(f_n).
\]
Thus 
\[
\lim I(f_n) = \lim I(g_m).
\]

DEFINITION 1.15. Suppose \( f \) is in \( L_u \). Then there exists an increasing sequence \( \{ f_n \} \) such that \( f_n \) is in \( L \) for all \( n \) and \( f = \lim f_n \). Then by 1.14 we may define 
\[
I(f) = \lim I(f_n).
\]
We shall now show that \( I \) is additive on \( L_u \).

LEMMA 1.16. Suppose that \( f \) is in \( L_u \), \( g \) is in \( L_u \), \( a \geq 0 \), \( b \geq 0 \). Then 
\[
I(af + bg) = aI(f) + bI(g).
\]
Further, if \( f \leq g \) then \( I(f) \leq I(g) \).
PROOF. (1). To show that \( f \leq g \) implies \( I(f) \leq I(g) \).

We know that \( f \) and \( g \) in \( L_u \) implies there exist sequences \( \{f_n\} \) and \( \{g_m\} \) of functions in \( L \) such that \( f_n \uparrow f \) and \( g_m \uparrow g \). \( f \leq g \) implies that \( \lim f_n \leq \lim g_m \). Thus

\[
I(f) = \lim I(f_n) \leq \lim I(g_n) = I(g).
\]

(2). \( I(af + bg) = I(a \lim f_n + b \lim g_m) \)

\[
= I(\lim af_n + \lim bg_m) = I(\lim (af_n + bg_n)) = \lim I(af_n + bg_n)
\]

\[
= \lim (I(af_n) + I(bg_n)) = \lim aI(f_n) + \lim bI(g_n)
\]

\[
= a \lim I(f_n) + b \lim I(g_n) = aI(f) + bI(g).
\]

LEMMA 1.17. Suppose \( f \) is a non-negative extended real-valued function on \( X \). Then \( f \) is in \( L_u \) iff there exists a sequence \( \{f_n\} \) of non-negative functions in \( L \) with \( f = \sum_{n=1}^{\infty} f_n \). Further, \( I(f) = \sum_{n} I(f_n) \).

PROOF. \textbf{If:} This follows from the definition.

\textbf{Only if:} Suppose that \( f \) is in \( L_u \). Then there exists a sequence \( \{g_n\} \) of functions in \( L \) such that \( g_n \uparrow f \).

We may assume that each of the \( g_n \)'s are non-negative since if not then we can choose \( g_n \rightleftharpoons 0 \) instead of \( g_n \). Let \( f_1 = g_1 \), and \( f_n = g_n - g_{n-1} \) for \( n \geq 2 \). Then

\[
\sum_{n=1}^{k} f_n = \sum_{n=2}^{k} (g_n - g_{n-1}) + g_1 = g_k.
\]

Thus

\[
\sum_{n} f_n = \lim_{k} \sum_{n=1}^{k} f_n = \lim_{k} g_k = f.
\]

Therefore,

\[
I(f) = \lim I(g_n) = \lim I(\sum_{i=1}^{n} f_1) = \lim \sum_{i=1}^{n} I(f_1) = \sum_{i=1}^{\infty} I(f_1).
\]
LEMMA 1.18. Suppose $f_n$ is in $L_u$ for all $n$, each $f_n$ is non-negative and that $f = \sum_n f_n$. Then $f$ is in $L_u$ and

$$I(f) = \sum_n I(f_n).$$

PROOF. For each integer $n$, there exists a sequence \( \{g_{n,m}\} \) of non-negative functions in $L$ with $f_n = \sum_m g_{n,m}$. Then

$$f = \sum_n f_n = \sum_n \sum_m g_{n,m} = \sum_{n,m} g_{n,m}.$$  

But this implies that $f$ is the sum of a series of non-negative functions of $L$. Hence, by 1.17, $f$ is in $L_u$. Further,

$$I(f) = \sum_{n,m} I(g_{n,m}) = \sum_n \sum_m I(g_{n,m}) = \sum_n I(\sum_m g_{n,m}) = \sum_n I(f_n).$$

$I$ is defined on all functions in $L_u$. We would like to define $I$ on functions that are not in $L_u$. This we do in two steps.

DEFINITION 1.19. Suppose that $f$ is an extended real-valued function on $X$. Define the upper integral of $f$ by $\bar{I}(f) = \inf_{g \geq f} I(g)$ where $\inf (\emptyset) = \infty$. Define the lower integral of $f$ by $\underline{I}(f) = -\bar{I}(-f)$.

The following lemma includes some very useful properties of $\bar{I}$ that are derived easily from the definition.

LEMMA 1.20. Suppose that $f$ and $g$ are two extended real-valued functions on $X$. Then
(1) \( \overline{1}(f + g) \leq \overline{1}(f) + \overline{1}(g) \).

(2) \( \overline{1}(cf) = c\overline{1}(f) \) if \( c > 0 \).

(3) \( f \leq g \) implies \( \overline{1}(f) \leq \overline{1}(g) \) and \( \underline{1}(f) \leq \underline{1}(g) \).

(4) \( \underline{1}(f) \leq \overline{1}(f) \).

(5) \( f \) in \( L_u \) implies \( \overline{1}(f) = \overline{1}(f) = \underline{1}(f) \).

**PROOF.** (1) \( \overline{1}(f + g) = \inf_{h \geq f + g} \overline{1}(h) \leq \inf_{h \in L_u} (\overline{1}(s) + \overline{1}(t)) \) where \( s \geq f \) and \( t \geq g \) for \( s, t \in L_u \), and \( h \in L_u \).

\[
= \inf_{s \geq f} \overline{1}(s) + \inf_{t \geq g} \overline{1}(t) = \overline{1}(f) + \overline{1}(g).
\]

(2) if \( c > 0 \),

\[
\overline{1}(cf) = \inf_{h \geq cf} \overline{1}(h) = \inf_{h \in L_u} c \overline{1}(h/c) = c \inf_{h \in L_u} \overline{1}(h/c) = c\overline{1}(f).
\]

(3) \( f \leq g \) implies that if \( h \) is in \( L_u \) and \( h \geq g \) then \( h \geq f \). Hence, \( \overline{1}(f) \leq \overline{1}(g) \). Also, \( f \leq g \) implies that \( -f \geq -g \) so \( -\overline{1}(-f) \leq -\overline{1}(-g) \). Thus, \( \overline{1}(f) \leq \underline{1}(g) \).

(4) \( 0 = \overline{1}(0) = \overline{1}(f + (-f)) \leq \overline{1}(f) + \overline{1}(-f) \). Hence, \( -\overline{1}(-f) \leq \overline{1}(f) \). Thus, \( \underline{1}(f) \leq \overline{1}(f) \).

(5) By definition, if \( f \) is in \( L_u \) then \( \overline{1}(f) \geq \overline{1}(f) \).

But \( f \) and \( h \) in \( L_u \) with \( f \leq h \) implies \( \overline{1}(f) \leq \overline{1}(h) \). Therefore \( \overline{1}(f) \leq \inf_{h \geq f} \overline{1}(h) = \overline{1}(f) \). Hence, \( \overline{1}(f) = \overline{1}(f) \).

Suppose that \( h \) is in \( L \). Then \( -h \) is in \( L \) which is contained in \( L_u \). Hence, \( \overline{1}(-h) = I(-h) = -I(h) \) which implies that \( \overline{1}(h) = I(h) \) if \( h \) is in \( L \). Now suppose that \( f \) is in \( L_u \).
Then there exists a sequence \( \{h_n\} \) of functions in \( L \) such that \( h_n \uparrow f \) and \( f \geq h_n \) for each \( n \). Hence,
\[
I(f) \geq I(h_n) = I(h_n) \quad \text{for all } n.
\]
Therefore,
\[
I(f) \geq \lim_{n \to \infty} I(h_n) = I(f).
\]
But by (3),
\[
I(f) \leq \overline{I}(f) = I(f).
\]
Hence,
\[
I(f) = I(f) = \overline{I}(f).
\]

**Lemma 1.21.** Suppose that \( \{f_n\} \) is a sequence of non-negative extended real-valued functions on \( X \) and that
\[
f = \sum_n f_n.
\]
Then
\[
\overline{I}(f) \leq \sum_n \overline{I}(f_n).
\]

**Proof.** Suppose that for some \( n \), \( \overline{I}(f_n) = \infty \).
Then \( \sum_n \overline{I}(f_n) = \infty \). Now suppose that \( \overline{I}(f_n) \neq \infty \) for all \( n \). Then given \( \epsilon > 0 \), there exists a \( g_n \in L_u \) such that \( f_n \leq g_n \) and with \( I(g_n) \leq \overline{I}(f_n) + \epsilon/2^n \). Then
\[
g_n \geq f_n \geq 0 \text{ for all } n \quad \text{and hence, } \quad g = \sum_n g_n \text{ is in } L_u
\]
by 1.17, and \( I(g) = \sum_n I(g_n) \). But
\[
I(g) = \sum_n I(g_n) \leq \sum_n (\overline{I}(f_n) + \epsilon/2^n) = \sum_n \overline{I}(f_n) + \epsilon.
\]
g_n \geq f_n implies that \( g = \sum_n g_n \geq \sum_n f_n = f \). Therefore,
\[
I(g) \geq \overline{I}(f).
\]
Hence,
\[
\overline{I}(f) \leq I(g) \leq \sum_n \overline{I}(f_n) + \epsilon.
\]
But \( \epsilon > 0 \) was arbitrary. Hence,
\[
\overline{I}(f) \leq \sum_n \overline{I}(f_n).
\]
Now we are in position for the second step.

**DEFINITION 1.22.** Suppose \( f \) is a finite real-valued function on \( X \). Then \( f \) is said to be \( I \) integrable iff \( \overline{I}(f) = \underline{I}(f) \) and is finite. We denote the class of all \( I \) integrable functions by \( L_1 \). If \( f \) is in \( L_1 \) we will write \( I(f) \) for \( \overline{I}(f) \).

**THEOREM 1.23.** The set \( L_1 \) is a vector lattice of functions containing \( L \). \( I \) is a positive linear functional on \( L_1 \) which extends the functional \( I \) on \( L \).

**PROOF.** (1) We show \( f \) in \( L_1 \) implies that \( cf \) is in \( L_1 \) for all real \( c \). Suppose that \( f \) is in \( L_1 \), \( c \geq 0 \). Then \( \overline{I}(cf) = c\overline{I}(f) = c\underline{I}(f) = \underline{I}(cf). \)

Now suppose that \( c \leq 0 \). Then

\[
\overline{I}(cf) = \overline{I}(-|c|f) = |c|\overline{I}(-f) \\
= -|c|\overline{I}(f) = -|c|\overline{I}(f) \\
= -\overline{I}(|c|f) \\
= \underline{I}(-|c|f) = \underline{I}(cf).
\]

Hence, \( cf \) is in \( L_1 \).

(2) We show that \( f \) and \( g \) in \( L_1 \) implies that \( f + g \) is in \( L_1 \). Suppose that \( f \) and \( g \) are in \( L_1 \). Then

\[
\overline{I}(f + g) \leq \overline{I}(f) + \overline{I}(g) \quad \text{by 1.20.}
\]

Also, \( -f \) and \( -g \) are in \( L_1 \) by (1), hence

\[
\overline{I}(-f - g) \leq \overline{I}(-f) + \overline{I}(-g) = -\overline{I}(f) - \overline{I}(g).
\]

Thus

\[
\overline{I}(f + g) = -\overline{I}(-f - g) \geq \overline{I}(f) + \overline{I}(g) \geq \overline{I}(f + g).
\]

But

\[
\underline{I}(f + g) \leq \overline{I}(f + g).
\]
Hence \( I(f + g) = I(f) + I(g) = \overline{I}(f + g) \)
and so \( f + g \) is in \( L_1 \).

(3) Suppose that \( f \) and \( g \) are in \( L_1 \), and that \( a \) and \( b \) are real numbers. Then \( af + bg \) is in \( L_1 \) so
\[
I(af + bg) = I(af) + I(bg) = aI(f) + bI(g).
\]
Hence \( L_1 \) is a vector space and \( I \) is a linear functional.
Further by 1.20, part 3, we see that \( I \) is positive.

(4) By 1.20, part 5, if \( f \) is in \( L \) then
\[
\overline{I}(f) = I(f) = \overline{I}(f).
\]
Also, \( f \) in \( L \) implies that \( I(f) < \infty \). Hence \( f \) is in \( L_1 \).
Thus \( L_1 \) is an extension of \( L \).

(5) To show that \( L_1 \) is a vector lattice we must show that if \( f \) is in \( L_1 \) then \( f \lor 0 \) is in \( L_1 \). Suppose that \( f \) is in \( L_1 \) and \( h \) is in \( L_u \) with \( h \geq f \). Then
\[
h \lor 0 \geq f \lor 0, \quad \text{and} \quad h \land 0 \geq f \land 0
\]
so
\[
\overline{I}(f \lor 0) + \overline{I}(f \land 0) \leq \overline{I}(h \lor 0) + \overline{I}(h \land 0) = I(h).
\]
This is true for all \( h \) in \( L_u \) such that \( h \geq f \). But this implies that
\[
\overline{I}(f \lor 0) + \overline{I}(f \land 0) \leq I(f).
\]
If we replace \( f \) by \( -f \) in the above inequality, we must replace \( f \lor 0 \) by \( -(f \land 0) \) and \( f \land 0 \) by \( -(f \lor 0) \). Hence we see that
\[
\overline{I}(-(f \land 0)) + \overline{I}(-(f \lor 0)) \leq I(-f).
\]
Thus
\[
-I(f \land 0) = I(f \lor 0) \leq I(-f) = -I(f).
\]
Therefore, \( \overline{I}(f \lor 0) + \overline{I}(f \land 0) \leq I(f) \leq \overline{I}(f \land 0) + \overline{I}(f \lor 0) \)
which implies that
\[
\overline{I}(f \lor 0) - I(f \lor 0) + \overline{I}(f \land 0) - I(f \land 0) \leq 0.
\]
But \( \overline{I}(g) - I(g) \geq 0 \) for all functions \( g \).
Hence \( \overline{I}(f \vee 0) - \underline{I}(f \vee 0) \equiv 0 \).

Thus \( f \vee 0 \) is in \( L_1 \) and \( L_1 \) is a vector lattice by 1.4.

Now that we have extended \( L \) to \( L_1 \) we may ask the following questions. When is a function in \( L_1 \)? What types of functions are in \( L_1 \)? Can we characterize a function in \( L_1 \) in terms of functions in \( L \)?

**THEOREM 1.24.** Suppose that \( \{f_n\} \) is a sequence of functions in \( L_1 \) such that \( f_n \not\rightarrow f \). Then \( f \) is in \( L_1 \) iff \( \lim I(f_n) \) is finite. If \( f \) is in \( L_1 \), then

\[ I(f) = \lim I(f_n). \]

**PROOF.** only if: Suppose that \( \lim I(f_n) = \infty \).

Then since \( f_n \not\rightarrow f \), we know that \( f \nless f_n \) for all \( n \) and so

\[ I(f) \nless \lim I(f_n) = \infty. \]

This implies that \( f \) is not in \( L_1 \).

if: Since \( f \nless f_n \) we know \( -f \nless -f_n \) for all \( n \). Hence

\[ \overline{I}(-f) \nless \overline{I}(-f_n) \quad \text{for all } n. \]

Thus

\[ \underline{I}(f) \nless \overline{I}(f_n) = I(f_n) \quad \text{for all } n \]

which implies that

\[ \underline{I}(f) \nless \lim I(f_n). \]

Given \( \varepsilon > 0 \) choose a sequence \( \{g_n\} \) of functions from \( L_u \) such that \( g_1 \nless f_1, \ g_n \nless f_n - f_{n-1} \) for \( n \geq 2 \) and with

\[ I(g_1) < I(f_1) + \varepsilon/2 \quad \text{and} \quad I(g_n) < I(f_n - f_{n-1}) + \varepsilon/2^n \]

for \( n \geq 2 \). Then if \( n \geq 2, \ g_n \nless f_n - f_{n-1} \geq 0. \)

Now define \( h_n = \sum_{i=1}^{n} g_i \) for all \( n \). Then \( h_n \) is in \( L_u \) for all \( n \) and the sequence \( \{h_n\} \) is monotone increasing, which
implies that \( \lim h_n \) is in \( L_u \) and \( I(\lim h_n) = \lim I(h_n) \).

Then
\[
h_n = \sum_{i=1}^{n} g_i = \sum_{i=2}^{n} (f_i - f_{i-1}) + f_1 = f_n
\]

which implies that \( \lim h_n \geq \lim f_n = f \)

so that
\[
I(f) \leq \lim I(h_n).
\]

But
\[
I(h_n) = \sum_{i=1}^{n} I(g_i) \leq \sum_{i=2}^{n} I(f_i - f_{i-1}) + I(f_1) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} = I(f_n) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i}.
\]

Thus
\[
\lim I(h_n) \leq \lim I(f_n) + \varepsilon.
\]

Hence
\[
I(f) \leq \lim I(h_n) \leq \lim I(f_n) + \varepsilon \leq I(f) + \varepsilon.
\]

But \( \varepsilon > 0 \) was arbitrary

so
\[
I(f) \leq I(f).
\]

Thus
\[
I(f) = \lim I(f_n) = I(f) = I(f)
\]

and is finite by assumption. Hence \( f \) is in \( L_1 \).

**LEMMA 1.25.** Suppose \( \{f_s\} \) is a sequence of non-negative functions in \( L_1 \). Then \( \inf f_s \) is in \( L_1 \).

Further, if \( \lim I(f_s) \) is finite then \( \lim f_s \) is in \( L_1 \) and in this case
\[
I(\lim f_s) \leq \lim I(f_s).
\]

**PROOF.** (1) Let \( g_n = f_1 \land f_2 \land \ldots \land f_n \). Then the sequence \( \{g_n\} \) is a decreasing sequence of non-negative functions in \( L_1 \). At any point \( p \) of \( X \),
\[
g_n(p) = \text{glb} \ f_i(p), \quad 1 \leq i \leq n
\]

Hence \( g = \lim g_n = \inf f_s \) yields \( -g_n \uparrow -g \).

Since \( g_n \geq 0 \) we know \( -g_n \leq 0 \) so \( I(-g_n) \leq 0 \) for all \( n \).
Thus \( \lim I(-g_n) \leq 0 \) and is finite, so by 1.24, \(-g\) is in \( L_1 \).

But by 1.23, this implies that \( g = \inf f_s \) is in \( L_1 \).

(2) Define the sequence \( \{h_n\} \) by \( h_n = \inf_{s \geq n} f_s \).

Since \( f_s \geq 0 \) we know that \( h_n \geq 0 \) and \( h_n \) is in \( L_1 \) for all \( n \) by above. But by definition \( \lim h_n = \lim_{s \to \infty} f_s \), and \( h_n \leq f_s \) if \( n \leq s \), thus

\[
\lim I(h_n) \leq \lim I(f_s) \quad \text{and is finite.}
\]

Hence by 1.24, \( \lim h_n = \lim f_s \) is in \( L_1 \),
and \( I(\lim f_s) = I(\lim h_n) = \lim I(h_n) \leq \lim I(f_s) \).

**Lemma 1.26.** Suppose \( \{f_n\} \) is a sequence of functions in \( L_1 \) and suppose that there exists a function \( g \) in \( L_1 \) such that \( |f_n| \leq g \) for all \( n \). Then if \( f = \lim f_n \), \( I(f) = \lim I(f_n) \).

**Proof.** Since \( -f_n \leq |f_n| \leq g \) implies that \( f_n + g \geq 0 \) for all \( n \), \( f_n + g \) is in \( L_1 \) for all \( n \),

\[
f_n \leq |f_n| \leq g \quad \text{for all } n,
\]
and \( I(f_n) \leq I(g) \) for all \( n \).

Therefore, \( I(f_n + g) = I(f_n) + I(g) \leq 2I(g) \) for all \( n \).

Thus \( \lim I(f_n + g) \leq 2I(g) < \infty \).

Hence by 1.25, \( \lim (f_n + g) = \lim f_n + g = f + g \) is in \( L_1 \).

Thus \( (f + g) - g = f \) is in \( L_1 \).

Therefore,

\[
I(f) + I(g) = I(f + g) = I(\lim (f_n + g)) \leq \lim I(f_n + g) = \lim I(f_n) + I(g).
\]

Therefore, \( I(f) \leq \lim I(f_n) \).
But \( g - f_n \geq 0 \) implies that
\[
I(g) - I(f) = I(g - f) = I(\lim (g - f_n)) \leq \lim I(g - f_n)
\]
\[
= I(g) + \lim I(-f_n) = I(g) + \lim (-I(f_n)) = I(g) - \lim I(f_n).
\]
Therefore, \( \lim I(f_n) \leq I(f) \leq \lim I(f_n) \)
which implies that \( \lim I(f_n) \) exists and is \( I(f) \).

**LEMMA 1.27.** Suppose that \( f \) is a real-valued function on \( X \). Then \( f \) is in \( L_1 \) iff there exists a sequence \( \{f_n\} \) of functions in \( L \) such that \( \overline{I}(|f - f_n|) \) goes to zero as \( n \) goes to infinity.

**PROOF.** *if:* Suppose that \( \overline{I}(|f - f_n|) \to 0 \). Then
\[
\overline{I}(f) = \lim I(f_n) < \infty.
\]
But \( \overline{I}(f) = \lim I(f_n) \) implies that
\[
-I(f) = \overline{I}(-f) = \lim I(-f_n) = \lim (-I(f_n)) = \lim I(f_n) = -\overline{I}(f).
\]
Hence \( \overline{I}(f) = \overline{I}(f) \) and is finite so \( f \) is in \( L_1 \).

*only if:* Suppose that \( f \) is in \( L_1 \). Then there exists a sequence \( \{g_n\} \) of functions in \( L_u \) with \( g_n \neq f \) and such that
\[
I(f) \leq I(g_n) < I(f) + 1/n.
\]
But \( g_n \) in \( L_u \) implies that for each \( n \) there exists a sequence \( \{h_{n,m}\} \) of functions in \( L \) such that \( h_{n,m} \nearrow g_n \) and
\[
I(h_{n,m}) \searrow I(g_n).
\]
For each \( n \), we choose \( f_n \) in \( L \) by
\[
f_n = h_{n,m} \quad \text{for some} \quad m
\]
with
\[
I(g_n) \geq I(f_n) = I(h_{n,m}) > I(g_n) - 1/n.
\]
Then \( f_n \) is in \( L \) for all \( n \) and for each \( n \),

\[
I(f - f_n) = I(f) - I(f_n) \geq I(g_n) - 1/n - I(f_n) \geq I(g_n) - 1/n - I(g_n) = -1/n.
\]

\[
I(f - f_n) = I(f) - I(f_n) \leq I(g_n) - I(f_n) < I(g_n) - I(g_n) + 1/n = 1/n.
\]

Therefore, 

\[-1/n < I(f - f_n) < 1/n\]

so 

\[I(f - f_n) \to 0.\]

Consider \((f_n \vee f) - f\). It is in \( L_1 \). And \( f_n \leq g_n \) implies that 

\[f \leq f_n \vee f \leq g_n \vee f = g_n.\]

Hence 

\[0 \leq I((f_n \vee f) - f) \leq I(g_n) - I(f).\]

But we know 

\[I(g_n) \downarrow I(f)\]

and hence 

\[I(g_n) - I(f) \downarrow 0.\]

Thus 

\[I((f_n \vee f) - f) \downarrow 0.\]

Suppose \( p \) is in \( X \). Then 

\[((f_n \vee f) - f)(p) > 0 \iff (f_n)(p) > f(p),\]

and 

\[f_n(p) \leq f(p)\]

implies that 

\[(f_n \vee f)(p) = f(p)\]

and hence 

\[((f_n \vee f) - f)(p) = 0.\]

Thus 

\[(f_n \vee f) - f = (f_n - f)^+.\]

Now 

\[(f_n - f)^- = (f_n - f)^+ - (f_n - f) = (f_n - f)^+ + (f - f_n).\]

Hence 

\[I((f_n - f)^-) \to 0.\]

Thus, since 

\[|f - f_n| = |f_n - f| = (f_n - f)^+ + (f - f_n)^-;\]

we know that 

\[I(|f - f_n|) \to 0.\]

**Lemma 1.28.** Suppose that \( h \) is a real-valued function defined on \( X \). Suppose further that \( g \wedge h \) is in \( L \) for all functions \( g \) in \( L \). Then \( f \wedge h \) is in \( L_1 \) for all functions \( f \) in \( L_1 \).
PROOF. Suppose that \( f \) is in \( L_1 \). Then by 1.27, there exists a sequence \( \{ f_n \} \) of functions in \( L \) with 
\[
I(|f - f_n|) \to 0.
\]
But by hypothesis, \( f_n \) in \( L \) implies that \( f_n \wedge h \) is in \( L \), and by the inequality
\[
0 \leq |(f \wedge h) - (f_n \wedge h)| \leq |f - f_n|
\]
we know that 
\[
I(|(f \wedge h) - (f_n \wedge h)|) \to 0.
\]
Hence by 1.27, \( f \wedge h \) is in \( L_1 \).

THEOREM 1.29. The functional \( I \) is a Daniell integral on the vector lattice \( L_1 \).

PROOF. \( I \) is a positive linear functional and \( L_1 \) is a vector lattice by 1.23. So suppose that \( \{ f_n \} \) is an increasing sequence of functions in \( L_1 \), and that \( f \) is in \( L_1 \) with \( f \leq \lim f_n \). Let \( h_n = f_n - f_1 \) for all \( n \). Then
\[
I(f_n) = I(f_1) + I(h_n)
\]
so that
\[
\lim I(f_n) = I(f_1) + \lim I(h_n).
\]
Let
\[
h = f_1 + \lim h_n = \lim f_n.
\]
Since \( h_n \geq g \), 1.25 applies and \( \lim h_n \) is in \( L_1 \) and hence \( h \) is in \( L_1 \). Then
\[
f \leq \lim f_n = h
\]
so
\[
I(f) \leq I(h) = I(f_1) + \lim I(h_n) = \lim I(f_n).
\]
Thus by 1.9 \( I \) is a Daniell integral.
CHAPTER II

MEASURABILITY

In this chapter it is our objective to show that under certain conditions every Daniell integral gives rise to a unique measure.

DEFINITION 2.1. Suppose that $f$ is a non-negative real-valued function on $X$. Then $f$ is a measurable function with respect to $I$ iff $g \land f$ is in $L_1$ for all $g$ in $L_1$.

LEMMA 2.2. Suppose $f$ and $g$ are non-negative measurable functions. Then $f \land g$ and $f \lor g$ are non-negative measurable functions. Suppose that $\{f_n\}$ is a sequence of non-negative measurable functions. Then $f = \lim f_n$ is a measurable function.

PROOF. Suppose that $f$ and $g$ are non-negative measurable functions and that $h$ is in $L_1$. Then $h \land (f \lor g) = (h \land f) \lor (h \land g)$ with $h \land f$ in $L_1$ and $h \land g$ in $L_1$. Hence $f \lor g$ is measurable. Also $h \land (f \land g) = (h \land f) \land g$ and $h \land f$ is in $L_1$ so $f \land g$ is measurable.

Suppose now that $g$ is in $L_1$ and $\{f_n\}$ is as given in the hypotheses. Then $f_n \land g$ is in $L_1$ for all $n$ since
each $f_n$ is measurable. Also
\[
\lim (f_n \land g) = g \land \lim f_n = g \land f.
\]
But
\[
|f_n \land g| \leq |g|,
\]
hence $f \land g$ is in $L_1$ by 1.26, so $f$ is measurable.

DEFINITION 2.3. Suppose that $A$ is a subset of a set $X$. Then the characteristic function of the set $A$, which we denote by $C(A)$, is defined by $C(A)x$ is 1 if $x$ is in $A$ and is 0 if $x$ is not in $A$.

DEFINITION 2.4. Any subset $A$ of $X$ is said to be measurable with respect to $\mathcal{I}$ if its characteristic function is measurable with respect to $\mathcal{I}$. The subset $A$ of $X$ is called integrable with respect to $\mathcal{I}$ if its characteristic function is $\mathcal{I}$ integrable.

Now we wish to show that the class of measurable sets forms a $\sigma$-algebra.

LEMMA 2.5. Suppose that $A$ and $B$ are two measurable sets with respect to $\mathcal{I}$. Then $A \cup B$, $A \cap B$ and $A \ast B$, where $A \ast B$ is $A - (A \cap B)$, are measurable sets with respect to $\mathcal{I}$. Further, if $\{A_n\}$ is a sequence of measurable sets with respect to $\mathcal{I}$, then $\bigcup A_n$ and $\bigcap A_n$ are measurable sets.

PROOF. It is easily seen that
\[
C(A \cup B) = C(A) \lor C(B)
\]
and that
\[
C(A \cap B) = C(A) \land C(B).
\]
But A and B being measurable sets implies that \( C(A) \) and \( C(B) \) are measurable functions. Hence by 2.2, \( C(A \cup B) \) and \( C(A \cap B) \) are measurable functions which implies that \( A \cup B \) and \( A \cap B \) are measurable sets. Also we can see that
\[
C(A \ast B) = C(A - (A \cap B)) = C(A) - C(A \cap B)
\]
so that \( A \ast B \) is a measurable set. Now consider \( \{A_n\} \), a sequence of measurable sets.
\[
C(A_1 \cup A_2 \cup \ldots \cup A_n) = C(A_1) \lor C(A_2) \lor \ldots \lor C(A_n)
\]
by an extension of the above. Hence
\[
C(\bigcup_n A_n) = \lim_n (C(A_1) \lor \ldots \lor C(A_n)).
\]
Thus \( C(\bigcup_n A_n) \) is a measurable function so \( \bigcup_n A_n \) is a measurable set with respect to \( I \). Similarly,
\[
C(\bigcap_n A_n) = \lim_n (C(A_1) \land \ldots \land C(A_n)).
\]
Therefore, \( \bigcap_n A_n \) is a measurable set with respect to \( I \).

**DEFINITION 2.6.** Let \( G \) be a collection of subsets of some set \( X \). Then \( G \) is called a \( \sigma \)-algebra of subsets of \( X \) if:

(i) \( \emptyset \) is in \( G \),

(ii) \( A \) in \( G \) implies that \( A' \) is in \( G \), and

(iii) \( A_n \) in \( G \) for all \( n \) implies that \( \bigcup_n A_n \) is in \( G \).

The following are several properties of \( \sigma \)-algebras

(i) If \( A_n \) is in \( G \) for all \( n \) then \( \bigcap_n A_n \) is in \( G \).
(ii) The intersection of $\sigma$-algebras is a $\sigma$-algebra. Hence, given a set $E$ the $\sigma$-algebra generated by $E$ is the $\sigma$-algebra obtained by forming the intersection of all $\sigma$-algebras containing $E$.

(iii) The set of all subsets of a set is a $\sigma$-algebra.

We shall denote by $1$ the function which is identically 1 for all values in its domain.

**Theorem 2.7.** Suppose that the function $1$ is measurable. Then the class $G$ of measurable sets is a $\sigma$-algebra.

**Proof.** If $1$ is a measurable function, then $1(X)$ is a measurable function which implies that $X$ is a measurable set. Suppose that $A$ is in $G$. Then $X \ast A$ is in $G$ by 2.5. But $X \ast A$ is $A'$ which implies that $A'$ is in $G$. And in particular $X' = \emptyset$ is in $G$. Also by 2.5, if $A_n$ is in $G$ for all $n$ then $\bigcup_n A_n$ is in $G$. Hence $G$ is a $\sigma$-algebra.

**Lemma 2.8.** Suppose that the function $1$ is a measurable function and that $f$ is a non-negative integrable function. Then if $a$ is a real number, the set $E = \{ x : f(x) > a \}$ is a measurable set.

**Proof.** Suppose that $a < 0$. Then since $f$ is non-negative, $E = X$ and so is in $G$. Now suppose that $a \geq 0$. Define functions $g_n = (n(f - (f \wedge a))) \wedge 1$, for all $n$,
and suppose that \( p \) is in \( E \). Then \( f(p) > a \) implies that \( g_n(p) = n(f(p) - a) \wedge 1 \). But there exists a positive integer \( N \) such that \( n \geq N \) implies that \( n(f(p) - a) > 1 \) which implies that \( nf(p) - na > 1 \). Therefore,
\[
\lim g_n(p) = 1.
\]

Now suppose that \( q \) is in \( X \ast E \). Then \( 0 \leq f(q) \leq a \) which implies that \( (f \wedge a)(q) = f(q) \) so that \( g_n(q) = 0 \).

Therefore,
\[
\lim g_n(q) = 0.
\]

Hence
\[
\lim g_n = C(E).
\]

But \( g_n \) is in \( L_1 \) for all \( n \) by 2.2, so \( C(E) \) is a measurable function and hence \( E \) is a measurable set.

**DEFINITION 2.9.** Suppose that \( H \) is a \( \sigma \)-algebra of subsets of a set \( X \). Suppose that \( u \) is a non-negative function defined on \( H \) such that \( u \) satisfies the following

(a) \( u(\emptyset) = 0 \) and

(b) if \( \{E_n\} \) is a sequence of pair-wise disjoint sets of \( H \), then \( u(\bigcup_n E_n) = \sum_n u(E_n) \).

Then \( u \) is a measure on \( H \).

We now proceed to define a measure on \( G \), the \( \sigma \)-algebra of sets measurable with respect to \( I \), which is induced by \( I \).

**LEMMA 2.10.** Suppose that the function \( I \) is measurable. Then \( G \) is a \( \sigma \)-algebra. Define a function \( u \) on \( G \)
by \( u_E = I(C(E)) \) if \( C(E) \) is integrable or \( u_E = \sup \{ u_A : A \text{ is contained in } E \text{ and } C(A) \text{ is integrable} \} \) otherwise. Then \( u \) is a measure on \( G \).

**PROOF.** By 2.7, \( G \) is a \( \sigma \)-algebra. \( u\emptyset = I(0) = 0 \).

Now suppose that \( A \) and \( B \) are integrable sets with \( A \) contained in \( B \). Then \( C(A) \leq C(B) \) so \( I(C(A)) \leq I(C(B)) \) which implies that \( uA \leq uB \). Thus \( u \) is monotone for integrable sets and so \( u \) is additive for integrable sets.

Now suppose that \( A \) and \( B \) are measurable sets with \( A \) contained in \( B \). Suppose that \( E \) is an integrable set contained in \( A \). Then \( E \) is contained in \( B \). Hence \( u(A) = \sup \{ u_E : E \text{ is an integrable set contained in } A \} \leq \sup \{ u_E : E \text{ is an integrable set contained in } B \} = uB \).

Thus \( uA \leq uB \).

Hence \( u \) is monotone for measurable sets. Suppose that \( \{ E_n \} \) is a sequence of pair-wise disjoint measurable sets with \( E = \bigcup E_n \). Suppose that \( A \) is an integrable subset of \( E \). Let \( A_i = A \cap E_i \) for all \( i \).

Then \( A = \bigcup A_i \) and by definition \( A_i \) is an integrable set for all \( i \). Thus \( C(A_i) \) is a sequence of functions in \( L_1 \). Further,

\[
C(A) = \lim (C(A_1) \vee \ldots \vee C(A_n))
\]

which implies that
\[ u_A = I(C(A)) = I(\lim (C(A_1) \vee \ldots \vee C(A_n))) = \lim (I(C(A_1)) + \ldots + I(C(A_n))) = \sum_{n=1}^{\infty} I(C(A_n)) = \sum_{i}^{\infty} u_{A_i} \leq \sum_{i}^{\infty} u_{E_i}. \]

This is true for all integrable subsets \( A \) of \( E \). Thus,
\[ u_E \leq \sum_{i}^{\infty} u_{E_i}. \]

Hence, if \( u_E = \infty \),
\[ u_E = \sum_{i}^{\infty} u_{E_i}. \]

Therefore, suppose that \( u_E \neq \infty \) and that \( \varepsilon > 0 \) is given. Then \( E_i \) contained in \( E \) for all \( i \) implies that \( u_{E_i} \leq u_E \) which implies that \( u_{E_i} < \infty \) for all \( i \). Therefore, there exists an integrable subset \( A_n \) of \( E_n \) for each \( n \) with \( u_{A_n} > u_{E_n} - \varepsilon / 2^n \). But then \( \bigcup_{n} A_n \) is contained in \( \bigcup_{n} E_n = E \). Hence
\[ u_E \geq u(\bigcup_{n} A_n) = \sum_{n}^{\infty} u_{A_n} > \sum_{n}^{\infty} (u_{E_n} - \varepsilon / 2^n) = \sum_{n}^{\infty} u_{E_n} - \varepsilon. \]

But \( \varepsilon > 0 \) was arbitrary. Hence,
\[ u_E \geq \sum_{n}^{\infty} u_{E_n}. \]

Thus
\[ u_E = \sum_{n}^{\infty} u_{E_n} \]

and \( u \) is a measure by 2.9.

We shall assume that the reader is familiar with the notion of integrability with respect to a measure. This notion is discussed in Royden [7 pp. 200].

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THEOREM 2.11. (STONE) Suppose that $L$ is a vector lattice of real-valued functions on a set $X$. Suppose that $f$ in $L$ implies that $1 \land f$ is in $L$, and that $I$ is a Daniell integral on $L$. Then there exists a $\sigma$-algebra $G$ of subsets of $X$ and a measure $\mu$ on $G$ such that a function $f$ on $X$ is $I$ integrable iff $f$ is also integrable with respect to $\mu$. In this case $I(f) = \int f \, d\mu$.

PROOF. (1) Since $1 \land f$ is in $L$ for all $f$ in $L$, Lemma 1.28 assures us that the function $1$ is measurable. Hence, the class $G$ of sets which are measurable with respect to $I$ is a $\sigma$-algebra by 2.7. By 2.8, the set $E$ where $E = \{ x : f(x) > 0, f \text{ non-negative, } I \text{ integrable function} \}$, is measurable with respect to $I$. Hence every non-negative $I$ integrable function is measurable with respect to $I$. But any $I$ integrable function can be written as the difference of two non-negative functions that are $I$ integrable. Hence every $I$ integrable function is measurable with respect to $I$. Let $\mu$ be the measure defined as in 2.10.

(2) only if: Suppose that $f$ is a non-negative $I$ integrable function on $X$. Let $E_{k,n} = \{ x : f(x) > k/n \}$ for all integers $k$ and $n$. Then by 2.8, $E_{k,n}$ is measurable with respect to $I$ for all $k$ and $n$. Suppose that $p$ is in $E_{k,n}$. Then $C(E_{k,n}) p = 1$ and $(n/k)f(p) > 1$. 

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so that \((C(E_k, n) \land (n/k)f)(p) = 1\).

Suppose that \(p\) is in \(X \ast E_{k,n}\). Then
\[
C(E_k, n)p = 0 \quad \text{and} \quad 0 \leq (n/k)f(p) \leq 1.
\]
Therefore
\[(C(E_k, n) \land (n/k)f)(p) = 0.\]
Hence
\[C(E_k, n) = C(E_k, n) \land (n/k)f\]
and we know that \((n/k)f\) is in \(L_1\), and \(C(E_k, n)\) is measurable so by definition \(C(E_k, n)\) is in \(L_1\). Further,
\[
u E_{k,n} = I(C(E_k, n)) < \infty.
\]

Now let
\[g_n = (1/n) \sum_{k=1}^{n^2} C(E_k, n)\].

Then \(g_n\) is in \(L_1\) for all \(n\). Now suppose that \(p\) is in \(X\).
Then given \(n\) there exists an integer \(m\) such that
\[
m/n \leq f(p) < (m + 1)/n.
\]
Then if \(m \geq n^2\), \(p\) is in \(E_{k,n}\) for \(1 \leq k \leq n^2\), which implies that \(g_n(p) = n\). If \(m < n^2\), then \(p\) is in \(E_{k,n}\) for \(1 \leq k \leq m\) and \(p\) is not in \(E_{k,n}\) for \(m + 1 \leq k \leq n^2\). Therefore,
\[g_n(p) = m/n.
\]
Hence for large \(n\), \(g_n(p) = m/n\) which implies that
\[0 \leq f(p) - m/n < 1/n.\]
Therefore,
\[
|f - g_n|(p) \leq 1/n
\]
so
\[f = \lim g_n.\]
But
\[g_n \leq g_{n+1}\]
which implies that \(g_n \not\rightarrow f\) and by 1.24,
\[I(f) = \lim I(g_n).\]
But \( I(g_n) = (1/n) \sum_{k=1}^{n^2} I(C(E_{k,n})) = (1/n) \sum_{k=1}^{n^2} uE_{k,n} = \int g_n \, du \).

Therefore, \( \int f \, du = \lim \int g_n \, du \) by the monotone convergence theorems for measurable functions. Therefore,

\[
I(f) = \int f \, du
\]

and \( f \) is integrable with respect to \( u \). Now suppose that \( f \) is any \( L \) integrable function on \( X \). Then there exist two non-negative integrable functions such that \( f \) is the difference of the two. Then

\[
I(f) = \int f \, du \quad \text{in this case.}
\]

(3) \textbf{If}: Suppose that \( f \) is a non-negative function on \( X \) which is integrable with respect to \( u \). Then we can construct \( E_{k,n} \) and \( g_n \) as before. Since \( f \) is integrable with respect to \( u \), \( \int f \, du < \infty \). Therefore, \( uE_{k,n} < \infty \) for all \( n \) and \( k \). Thus \( C(E_{k,n}) \) is measurable, so that \( C(E_{k,n}) \) and \( g_n \) are \( L \) integrable. As before \( g_n \gg f \), and \( \lim I(g_n) = \int f \, du < \infty \) implies that \( \lim g_n = f \) is in \( L_1 \) by 1.24. But as before, this can be extended to any function integrable with respect to \( u \).

So we now have a measure induced by the Daniell integral \( I \). We now wish to show that it is unique. Recall that \( L_u \) was the class of all those extended real-valued
functions on $X$ each of which is a limit of a monotone increasing sequence of functions in the vector lattice $L$.

**DEFINITION 2.12.** Denote by $L_0$ the class of extended real-valued functions on $X$ each of which is the limit of a decreasing sequence $\{f_n\}$ of functions in $L_0$ such that $I(f_n) < \infty$ and $\lim I(f_n) > -\infty$.

**LEMMA 2.13.** The class $L_0$ is contained in $L_1$.

**PROOF.** Suppose that $f$ is in $L_0$. Then there exists a sequence $\{f_n\}$ of functions in $L_0$ with $f_n \downarrow f$, $I(f_n) < \infty$, and $\lim I(f_n) > -\infty$. Since $I(f_n) < \infty$, $f_n$ is in $L_1$ which implies that $-f_n$ is in $L_1$ and hence $-f_n \uparrow f$.

$$\lim I(-f_n) = \lim -I(f_n) = -\lim I(f_n)$$

which is finite. Hence by 1.24, $-f$ and hence $f$ is in $L_1$.

**LEMMA 2.14.** Suppose that $f$ is a real-valued function on $X$ with $\overline{I}(f)$ finite. Then there exists $g$ in $L_0$ such that $f \leq g$ and $\overline{I}(f) = I(g)$.

**PROOF.** Since $\overline{I}(f)$ is finite we can find a function $h_n$ in $L_0$ such that $f \leq h_n$ and $\overline{I}(h_n) \leq \overline{I}(f) + 1/n$ for each $n$. Define $g_n = h_1 \wedge h_2 \wedge \ldots \wedge h_n$ for each $n$. Thus $f \leq g_n \leq h_1$ for $1 \leq i \leq n$. But $\{g_n\}$ is a decreasing sequence of functions of $L_0$ so $g = \lim g_n$ is in $L_0$. Further, $\overline{I}(f) \leq I(g_n) \leq I(h_n) \leq \overline{I}(f) + 1/n$ so

$$\overline{I}(f) = \lim I(g_n) = I(g),$$

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and $f \leq g_n$ implies that $f \leq g$.

DEFINITION 2.15. Suppose that $f$ is a real-valued function on $X$. Then $f$ is called a null function if $f$ is in $L_1$ and $I(\|f\|) = 0$.

A direct consequence of the above definition is the following. Suppose that $f$ is a null function and that $\|g\| \leq f$. Then

$$0 \leq I(\|g\|) \leq I(\|g\|) \leq I(f) = 0$$

Hence $g$ is in $L_1$ and $g$ is a null function.

THEOREM 2.16. Suppose that $f$ is a real-valued function on $X$. Then $f$ is in $L_1$ iff $f = g - h$ where $g$ is in $L_0$ and $h$ is a null function. Further, $h$ is a null function iff there exists a function $k$ in $L_0$ such that $k$ is a null function and $\|h\| \leq k$.

PROOF. (1) $h$ a null function implies that $h$ is in $L_1$, and $g$ in $L_0$ implies that $g$ is in $L_1$. Hence $f = g - h$ is in $L_1$. Now suppose that $f$ is in $L_1$. Then by 2.14, there exists a function $g$ in $L_0$ with $f \leq g$ and such that $I(f) = I(g)$. Let $h = g - f$, then $h$ is in $L_1$ and $I(h) = I(g) - I(f) = 0$. Thus $h$ is a null function.

(2) $\text{if}$: This follows immediately from 2.15.

only $\text{if}$: If $h$ is a null function then there exists a function $k$ in $L_0$ with $\|h\| \leq k$ and such that $I(k) = I(h) = 0$ by 2.14. Thus $k$ is a null function.
THEOREM 2.17. Suppose that $I$ and $J$ are Daniell integrals on the vector lattices $L$ and $M$, respectively, of real-valued functions on $X$, such that $M$ contains $L$. Suppose $I(f) = J(f)$ for all $f$ in $L$, then $M_1$ contains $L_1$ and $I(f) = J(f)$ for all $f$ in $L_1$.

PROOF. Suppose that $f$ is in $L_0$. Then there exists a sequence of functions $\{f_n\}$ in $L_0$ with $f_n \downarrow f$, $I(f_n) < \infty$, and $\lim I(f_n)$ is finite. Also, for each $n$, there exists a sequence of functions $\{f_{n,i}\}$ with $\lim f_{n,i} = f_n$ and such that each $f_{n,i}$ is in $L$ and $\lim I(f_{n,i}) = I(f_n)$.

But $f_{n,i}$ in $L$ implies that $I(f_{n,i}) = J(f_{n,i})$ by hypothesis.

Therefore, $I(f_n) = \lim I(f_{n,i}) = \lim J(f_{n,i}) = J(f_n)$.

Hence $I(f) = \lim I(f_n) = \lim J(f_n) = J(f)$.

Thus $J(f)$ is finite and $f$ is in $M_1$. Hence $M_1$ contains $L_0$, and $I(f)$ equals $J(f)$ for all $f$ in $L_0$. Suppose $h$ is a null function with respect to $I$. Then there exists a function $k$ in $L_0$ with $|h| \leq k$ and such that $I(|k|) = 0$ by 2.16.

Thus $I(k) = J(k) = 0$ and $h$ is a null function with respect to $J$. Now suppose that $f$ is in $L_1$. Then there exists $g$ and $h$ by 2.16 such that $f = g - h$, $g$ in $L_0$ and $h$ a null function with respect to $I$. But $h$ is a null function with respect to $J$ also, so $h$ is in $M_1$. Now $g$ in $L_0$ implies that $g$ is in $M_1$, so that $f = g - h$ is in $M_1$. Thus $M_1$ contains $L_1$ and
\[ I(f) = I(g) - I(h) = J(g) - J(h) = J(f) \]
for all \( f \) in \( L_1 \).

**THEOREM 2.18.** Suppose that \( L \) is a vector lattice of real-valued functions on a set \( X \) and that \( 1 \) is in \( L \). Let \( H \) be the smallest \( \sigma \)-algebra of subsets of \( X \) such that each function in \( L \) is measurable with respect to the \( \sigma \)-algebra \( H \). Then for each Daniell integral \( I \) there exists a unique measure \( u \) on \( H \) such that for every \( f \) in \( L_1 \),
\[ I(f) = \int f \, du. \]

**PROOF.** Stone's theorem assures us of the existence of a \( \sigma \)-algebra \( H \) and a measure \( u \) such that
\[ I(f) = \int f \, du. \]
Therefore, we need only show the uniqueness of \( u \) on \( H \).
Let \( G \) be the \( \sigma \)-algebra of measurable sets described in 2.7. Suppose that \( f \) is in \( L \). Then by 2.8, \( f \) is measurable with respect to \( G \) which implies that \( G \) contains \( H \) since \( H \) is the smallest such \( \sigma \)-algebra. Since the function \( 1 \) is in \( L \), \( 1 \) is integrable so that if \( A \) is in \( G \) then \( C(A) \) is in \( L_1 \). Thus if \( A \) is in \( H \) then \( C(A) \) is in \( L_1 \). Let \( M \) be the set of functions on \( X \) which are measurable with respect to \( H \) and integrable with respect to \( u \) and let \( J(f) \) be \( \int f \, du \) if \( f \) is in \( M \). Then by 2.17, if \( f \) is in \( L_1 \cap M \) then \( J(f) = I(f) \). Therefore, suppose that \( A \) is in \( H \). Then \( C(A) \) is in \( L_1 \) and \( C(A) \) is in \( M \).
Thus \( C(A) \) is in \( L_1 \cap M \) so

\[ u_A = J(C(A)) = I(C(A)) \]

for all \( A \) in \( H \). Therefore, if any measure \( u \) on \( H \) is such that for every \( f \) in \( L_1 \) \( I(f) = \int f \, du \) for the Daniell integral \( I \), then \( u_A = I(C(A)) \) if \( A \) is in \( H \). Hence \( u \) is uniquely determined.
CHAPTER III
APPLICATION

DEFINITION 3.1. Suppose that $f$ is a real-valued function defined on a topological space $X$. Let

$$S = \{ x : f(x) \neq 0 \}.$$

The closure of the set $S$, i.e. $\overline{S}$, is called the support of $f$. If $\overline{S}$ is compact then $f$ is said to have compact support.

In this chapter we shall start with the vector lattice of continuous real-valued functions of a real variable with compact support. We shall denote this vector lattice by $C_0$, and let $X$ be the set of Real numbers.

LEMMA 3.2. Suppose that $f$ is in $C_0$. Then $f$ is Riemann integrable on $X$.

PROOF. Since the function $f$ is in $C_0$ there exists a set $S$ contained in $X$ such that $f$ is identically zero for all of $X \setminus S$. If it exists, the Riemann integral is additive, hence

$$\int_X f(x) \, dx = \int_S f(x) \, dx + \int_{X \setminus S} f(x) \, dx,$$

provided they exist.
But $f(x) = 0$ for all $x$ in $X \times \overline{S}$, hence

$$\int_{X \times \overline{S}} f(x)\,dx = 0.$$ 

But $\overline{S}$ is closed, and $f$ continuous on $\overline{S}$ implies that $f$ is Riemann integrable on $\overline{S}$. Hence

$$\int_X f(x)\,dx = \int_{\overline{S}} f(x)\,dx$$

which exists.

Now we shall show that $I = \text{Riemann integral on } C_0$ is a Daniell integral. We then will show that when we complete $C_0$ to $C_1$ as we completed $L$ to $L_1$, we obtain the Lebesgue integral and Lebesgue measure. It will be assumed that the reader is familiar with the Lebesgue Theory.

LEMMA 3.3. Suppose that $f$ is in $C_0$. Then let $I(f)$ be the Riemann integral of $f$. Then $I$ is a Daniell integral on $C_0$.

PROOF. Clearly, $I$ is a positive linear functional. Therefore, suppose that $\{f_n\}$ is an increasing sequence of functions of $C_0$ and that $f$ in $C_0$ is such that

$$f(x) \leq \lim f_n(x)$$

for all $x$ in $X$. Then we must show that

$$I(f) \leq \lim I(f_n).$$

We may assume that $f$ and $f_n$ are all non-negative since if not we may subtract $(f \wedge f_n)$ and make them non-negative. Since $f$ is in $C_0$ the support $S$ of $f$ is compact. Now
given \( \varepsilon < 0 \) we can form the sets \( O_n = \{ x : (1-\varepsilon)f(x) < f_n(x) \} \) for all \( n \). But these open sets certainly form a covering of \( S \). Hence, there are a finite number of \( O_n \) which covers \( S \), say \( O_1 \) to \( O_m \), i.e. \( S \) is contained in \( \bigcup_{n=1}^{m} O_n \).

But \( f_n(x) \leq f_{n+1}(x) \) implies that \( O_n \) is contained in \( O_{n+1} \) so that \( S \) is contained in \( O_m \). Hence \( (1-\varepsilon)f(x) < f_m(x) \) for all \( x \) in \( S \) and thus \( (1-\varepsilon)f(x) \leq f_m(x) \) for all \( x \) in \( X \). Therefore,

\[
I((1-\varepsilon)f) = (1-\varepsilon)I(f) \leq I(f_m) \leq \lim I(f_n).
\]

But \( \varepsilon > 0 \) was arbitrary, hence

\[
I(f) \leq \lim I(f_n).
\]

Thus \( I \) is a Daniell integral by 1.9.

Now we can extend \( C_0 \) to \( C_1 \). Then by 1.23, \( C_1 \) is a vector lattice of real-valued functions on \( X \) which contains \( C_0 \). Further, by 1.29, \( I \) is a Daniell integral on the completed lattice \( C_1 \).

**Lemma 3.4.** The function \( 1 \) is a measurable function with respect to \( I \) on \( X \).

**Proof.** By 1.28, it suffices to show that \( 1 \wedge f \) is in \( C_0 \) for all \( f \) in \( C_0 \). Suppose that \( g \) is in \( C_0 \). Then there is a compact set \( S \) such that \( g(x) = 0 \) for all \( x \) in \( X \setminus S \). Consider the function \( g \wedge 1 \).

\((g \wedge 1)(x) = 1 \text{ if } g(x) \geq 1 \text{ and is } g(x) \text{ if } g(x) \leq 1.\)
Therefore, \((g \wedge 1)(x) = 0\) iff \(g(x) = 0\).
Hence \(g \wedge 1\) has compact support. Clearly \(g \wedge 1\) is continuous since it is the upper truncation of \(g\) at 1. Hence \(g \wedge 1\) is in \(C^0\) for all \(g\) in \(C^0\). Thus \(f \wedge 1\) is in \(C^1\) for all \(f\) in \(C^1\) by 1.28. Thus the function 1 is measurable.

Since 1 is a measurable function the class \(G\) of all measurable sets forms a \(\sigma\)-algebra by 2.7. Then by Stone's Theorem 2.11, there exists a measure \(u\) on \(G\) such that \(f\) is in \(C^1\) iff \(I(f) = \int f\, du\). Now we desire to know what is in \(G\). Hence

**THEOREM 3.5.** Any Borel set is a measurable set.

**PROOF.** Suppose that \(A\) is the closed interval \([a, b]\). We want to show that \(A\) is in \(G\). Define a sequence of functions \(\{g_n\}\) by:

1. \(g_n(x) = 1\) if \(x\) is in \(A\).
2. \(g_n(x) = (1/n)(x - a + 1/n)\) if \(x\) is in the closed interval \([a - 1/n, a]\).
3. \(g_n(x) = (1/n)(b - x + 1/n)\) if \(x\) is in the closed interval \([b, b + 1/n]\).
4. \(g_n(x) = 0\) otherwise.

Then \(\lim n g_n = 0(A)\).

But for each \(n\), \(g_n\) is a real-valued continuous function with compact support. Therefore, \(g_n\) is in \(C^0\) for all \(n\). Hence \(I(g_n)\) exists and is equal to \(b - a + 1/n\). Thus the \(\lim I(g_n) = b - a\) and is finite. Therefore,
\( C(A) = \lim g_n \) is in \( C_1 \) by 1.24, and \( I(C(A)) = b - a \).

Since \( C(A) \) is in \( C_1 \), \( f \land C(A) \) is in \( C_1 \) for all \( f \) in \( C_1 \) since \( C_1 \) is a vector lattice. Hence \( A \) is in \( G \) with \( uA = b - a \). But the smallest \( \sigma \)-algebra containing all closed intervals is the \( \sigma \)-algebra of Borel sets. Hence any Borel set is in \( G \) and so is measurable.

**Lemma 3.6.** The function \( x^{-2} \land 1 \) is in \( C_1 \).

**Proof.** For each \( n \) let

\[
 f_n = ((x^{-2} \land 1) \land C([-n,n])) \lor (n^{-1}(x+n+1/n) \land C([-n^{-1}/n, -n])) \\
\lor (n^{-1}(n-x+1/n) \land C([n, n+1/n])).
\]

Then \( f_n(x) \) is

1. \( n^{-1}(x+n+1/n) \) if \( x \) is in \([-n^{-1}/n, -n]\).
2. \( 1/x^2 \) if \( x \) is in \([-n, -1]\), or in \([1, n]\).
3. \( 1 \) if \( x \) is in \([-1, 1]\).
4. \( n^{-1}(n-x+1/n) \) if \( x \) is in \([n, n+1/n]\).
5. \( 0 \) otherwise.

Then \( f_n \) is continuous with compact support, namely \([-n^{-1}/n, n+1/n]\).

Hence \( f_n \) is in \( C_0 \) for each \( n \) and \( I(f_n) \) exists and by evaluation we find that

\[
 I(f_n) = 2 - (2/n) + (2/3n^3).
\]

Therefore, the \( \lim I(f_n) \) exists and is 2. Also we can see that \( f_n \uparrow (x^{-2} \land 1) \). Hence by 1.24, \( \lim f_n = x^{-2} \land 1 \) is in \( C_1 \).
Now we shall denote by $m_E$ the Lebesgue measure of the Lebesgue measurable sets. We wish to show that $m$ and $u$ are the same. We know that the Lebesgue integral is a Daniell integral; this can be seen from the monotonicity of the integral and Fatou's lemma.

THEOREM 3.7. The vector lattice $C_1$ is contained in the vector lattice of all Lebesgue integrable functions and $I(f) = \text{the Lebesgue integral of } f \text{ for all } f \text{ in } C_1$.

PROOF. We use Theorem 2.17. Surely $C_0$ is contained in the vector lattice of all real-valued functions that are continuous almost everywhere. It also is a well-known fact that the Lebesgue integral and the Riemann integral agree for continuous functions on compact sets. Hence, $I(f) = \text{the Lebesgue integral of } f \text{ for all } f \text{ in } C_0$. Therefore, by 2.17, the theorem follows.

COROLLARY 3.8. Suppose that $A$ is in $G$. Then $A$ is Lebesgue measurable.

PROOF. $A$ in $G$ implies that $C(A) \land f$ is in $C_1$ for all $f$ in $C_1$, which implies that $C(A) \land f$ is Lebesgue integrable for all $f$ in $C_1$. Let $f_0 = x^{-2} \land 1$. Then $f_0$ is in $C_1$ and the support of $f_0$ contains $A$. Further, $f_0$ is non-negative. Therefore, since $A = \{x \in X : (C(A) \land f_0)(x) > 0\}$, we know that $A$ is Lebesgue measurable.
We would like 3.7 and 3.8 to be iff statements. Therefore, we proceed as follows.

LEMMA 3.9. Suppose $K$ is an open set with $mK < \infty$. Then $K$ is in $G$ and $mK = uK$.

PROOF. Suppose that $K$ is open. Then $K$ is a countable union of disjoint open intervals of the form $(a_i, b_i)$ with $a_i < b_i$ and all the $a_i$'s and $b_i$'s are finite, since $mK < \infty$. But each open interval is a Borel set and hence is in $G$. Also $K$ is a Borel set so it is in $G$. We know that

$$u(a, b) = b - a = m(a, b),$$

$u$ and $m$ are both measures so they are countably additive (for references see Halmos [3] page 30) which implies that $uK = mK$.

LEMMA 3.10. Suppose that $K$ is a countable intersection of open sets $K_i$ such that $mK < \infty$. Then $K$ is in $G$ and $mK = uK$.

PROOF. $K$ is a Borel set and hence is in $G$ and is Lebesgue measurable. We know that $K = \bigcap_{i=1}^{\infty} K_i$ where $K_i$ is open. But there exists an open set $K_o$ containing $K$ such that $m(K_o - K) < 1$ which implies that $mK_o$ is finite. Then

$$K = K_o \cap \bigcap_{i=1}^{\infty} K_i = \bigcap_{i=0}^{\infty} K_i$$

which is still a countable intersection of open sets.
Since $K_0 \bigcap K_1 \bigcap \ldots \bigcap K_n$ is an open set we know that
\[ m(K_0 \bigcap \ldots \bigcap K_n) = u(K_0 \bigcap \ldots \bigcap K_n) \]
for each $n$ by 3.9. But \( \left\{ \bigcap_{i=0}^{n} K_1 \right\} \) is a decreasing sequence of \( u \)-measurable sets with \( u(K_0 \bigcap K_1) < \infty \) and for which
\[ \lim_{n \to \infty} \left( \bigcap_{i=0}^{n} K_1 \right) \]
is $K$ and is in $G$. Therefore,
\[ uK = u(\lim_{n \to \infty} \bigcap_{i=0}^{n} K_1) = \lim_{n \to \infty} u(\bigcap_{i=0}^{n} K_1) = \lim_{n \to \infty} m(\bigcap_{i=0}^{n} K_1) = m(\lim_{n \to \infty} \bigcap_{i=0}^{n} K_1) = mK. \]

**Lemma 3.11.** Suppose that $E$ is Lebesgue measurable with $mE = 0$. Then $E$ is in $G$ and $uE = 0$.

**Proof.** Since $E$ is Lebesgue measurable there exists a set $K$ which is a countable intersection of open sets such that $K$ contains $E$ and $m(K - E) = 0$. Thus,
\[ mK = mE = 0. \]
But by 3.10, $K$ is in $G$ and $uK = mK = 0$. Then
\[ 0 \leq I(C(E)) \leq \overline{I}(C(E)) \leq \overline{I}(C(K)) = I(C(K)) = uK = 0. \]
Therefore,
\[ I(C(E)) = \overline{I}(C(E)) = 0. \]
Hence $E$ is in $G$ and $uE = I(C(E)) = 0$.

**Theorem 3.12.** Suppose that $E$ is Lebesgue measurable. Then $E$ is $u$-measurable and $mE = uE$.

**Proof.** Suppose that $E$ is Lebesgue measurable with $mE < \infty$. Then there exists a set $K$ which is a countable intersection of open sets such that $K$ contains $E$ with $m(K - E) = 0$. But $K$ is in $G$ and $mK = uK$ by 3.10.
By 3.11, $K - E$ is in $G$ and $m(K - E) = u(K - E) = 0$. But $E = K - (K - E)$ and so is in $G$. $E$ and $K - E$ are disjoint and their union is $K$. Hence

$$uE = uK - u(K - E) = uK = mK = mE.$$ 

Now suppose that $E$ is Lebesgue measurable with $mE = \infty$. Then $E$ is the union of countably many, pair-wise disjoint sets of finite measure and $u$ and $m$ agree on each of these. But $m$ and $u$ are both measures and hence are countably additive and so $uE = mE = \infty$.

Hence by Corollary 3.8 and Theorem 3.12 we see that the $\sigma$-algebra of $u$ measurable sets is exactly the $\sigma$-algebra of Lebesgue measurable sets. Now we shall show that the two integrals agree.

**Lemma 3.13.** Suppose that $f$ is a simple function on $X$ that vanishes outside of a set $E$ of finite measure. Then $f$ is in $C_1$ and $I(f) = \int f$ (Lebesgue).

**Proof.** First let us consider a subset $A$ of $X$ such that $uA$ exists and is finite. Then we know that since $C([-n, \bar{n}])$ is in $C_1$, $g_n = C([-n, \bar{n}]) \cap C(A)$ is in $C_1$. But $g_n = C(A \cap [-n, \bar{n}])$ so

$$I(g_n) = I(C(A \cap [-n, \bar{n}])) = u(A \cap [-n, \bar{n}]) \leq \sup \{uB : B \text{ contains } A \} = uA.$$ 

Therefore, $\lim I(g_n) \leq uA < \infty$ and $g_n \uparrow C(A)$.

Hence by 1.24, $C(A)$ is in $C_1$ and $I(C(A)) = uA$. Hence if
a is a real number then $aC(A)$ is in $C_i$ and
\[ I(aC(A)) = aI(C(A)) = a \mu A. \]

Now suppose that $f$ is a simple function. Then $f$ has values $a_1, \ldots, a_n$ not zero, and measurable sets $A_1, \ldots, A_n$ such that $f$ is the sum of $a_1C(A_1)$ for $i$ between 1 and $n$, where $A_i = \{ x : f(x) = a_i \}$. Hence $f$ is in $C_i$ and
\[ I(f) = \sum_{i=1}^{n} I(a_iC(A_i)) = \sum_{i=1}^{n} a_i \mu A_i = \sum_{i=1}^{n} a_i m A_i = \int f. \]

THEOREM 3.14. Suppose that $f$ is a real-valued function on $X$. Then $f$ is Lebesgue integrable iff $f$ is in $C_i$ and in this case, the two integrals agree on $f$.

PROOF. if: Theorem 3.7.

only if: It is well known [7, cor. 11.3.1] that if $g$ is a non-negative function (with Lebesgue integral of $g$ finite) then there exists a sequence of simple functions $\{ g_n \}$ each of which vanishes outside of a set of finite measure and such that $g_n \nearrow g$. By 3.13, each $g_n$ is in $C_i$ and $g_n \leq g$ implies that $I(g_n) \leq \int g$ (Lebesgue) for all $n$. Thus, $\lim I(g_n) \leq \int g$ (Lebesgue) $< \infty$.

Hence by 1.24, $g$ is in $C_i$. Since $g$ is in $C_i$ and is Lebesgue integrable, then the two integrals agree on $g$ by Theorem 3.7. But then, given any Lebesgue integrable function $f$, $f = f^+ - (-f)^+$. But these are in $C_i$ and hence $f$ is in $C_i$. 

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REFERENCES


