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On the finite intersection of maximal regular right ideals

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ON THE FINITE INTERSECTION OF
MAXIMAL REGULAR RIGHT IDEALS

by

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B. A., Montana State University, 1953

Presented in partial fulfillment
of the requirements for the degree of
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V. G. F.
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CHAPTER I

INTRODUCTION

The purpose of this paper is to investigate finite intersections of maximal regular right ideals to see if they are regular. Impetus for this investigation is derived from the fact that knowledge of the regular ideals leads to knowledge of units in a ring. Rings in general may or may not have an identity. The existence of an identity in a ring can be postulated, as is the usual procedure, or its existence must be demonstrated. The latter case may not always be as simple as it might appear. Reinhold Baer obtained criteria for the existence of an identity in the presence of chain conditions using total zero divisors. In our investigation, we determine conditions which insure the regularity of finite intersections of regular ideals from which elements which behave like identities can be concluded.

A reasonable understanding of modern algebra, particularly group theory, is pre-supposed on the part of the reader.

In this discussion, we shall restrict ourselves to a very general ring which will be defined as follows:

Definition: A ring is a module (an additive abelian group) which is closed under associative multiplication.

---1---

Baer [1], pp. 630-638

---1---

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such that this multiplication is distributive with respect to addition.

However, we shall impose additional postulates upon this ring throughout this exposition. In a manner analogous to that of defining a subgroup of a group, we make the following

**Definition:** A subring $S$ of a ring $R$ is a subset of $R$ which is itself a ring.

The algebraic system which is used as a tool in determining the structure of a ring in much the same manner as the normal subgroups provide us with a better understanding of groups is the ideal of the ring. We define the ideal as follows:

**Definition:** An ideal $I$ of a ring $R$ is a subring of $R$ which contains the elements $ar$ and $ra$ for all $r$ in $R$ and $a$ in $I$.

Again reflecting the similarity between group theory and ring theory, we shall define a ring which corresponds to the well-known factor group.

**Definition:** If $R$ is a ring and $I$ is an ideal of $R$, then $R/I$ is said to be a difference ring if the elements of $R/I$ are of the form $a_1 + I$, where $a_1$ are elements of $R$, such that

1) $(a_1 + I) + (a_2 + I) = (a_1 + a_2) + I$,
2) $(a_1 + I)(a_2 + I) = (a_1a_2) + I$,

and such that $I$ acts as the zero element of the ring.
CHAPTER II

STRUCTURE OF RINGS

In our discussion of rings, we shall first consider the ring of endomorphisms of a module since any ring with an identity may be obtained as a subring of the ring of endomorphisms of its module.¹ Also, as we shall see, any ring can be imbedded in a ring with an identity.

1. Rings of Endomorphisms

Let \( M \) be any module (an additive abelian group). An endomorphism of \( M \) is a homomorphism of \( M \) into itself. Let \( E(M) \) denote the set of all endomorphisms of \( M \). We define

\[
(x)(A+B) = (x)A + (x)B
\]

\[
(x)(AB) = ((x)A)B
\]

for all \( x \) in \( M \) and for all \( A \) and \( B \) in \( E(M) \). It is easy to see that \( E(M) \) is a ring with an identity.

Let \( N \) denote a subgroup of \( M \) and \( Q \) denote a subset of \( E(M) \). Then for \( A \) in \( Q \),

\[
(M)A \subseteq M
\]

\[
(N)A \subseteq M.
\]

**Definition:** \( N \) is said to be a \( Q \)-subgroup of \( M \) if \( N \) is mapped into \( N \) under all elements of \( Q \).

¹Jacobson [1], p. 1.
As an example, suppose $Q$ denotes the inner-auto-
morphisms of $M$. The $Q$-subgroups of $M$ are the normal sub-
groups, since each is mapped onto itself under $Q$. $M$ is
abelian so every subgroup of $M$ is a $Q$-subgroup.

2. **Ideals and Irreducible Modules**

With each element $a$ of $R$, we associate the mapping
of $x$ to $xa$ of the ring $R$ into itself. We denote this
mapping as $a_r$ and call it the right multiplication
determined by $a$. Notice that $a_r$ is an endomorphism. Now
define $R_r$ to be the set of all $a_r$ for $a$ in $R$. The $R_r$-
subgroups of $M$ are the right ideals $I_r$ of $R$, by definition.
That is, $I_r$ is a right ideal of $R$ if

1) $a-b$ is an element of $I_r$, if $a$ and $b$ are in $I_r$,
2) $ar$ is an element of $I_r$, if $r$ is in $R$ and $a$ is in $I_r$.

Likewise, we define $R_1$ to be the set of mappings of
$R$ which map $x$ onto $ax$ for all $a$ in $R$. The $R_1$-
subgroups of $M$ are the left ideals $I_1$ of $R$. That is, $I_1$ is a left
ideal of $R$ if

1) $a-b$ is an element of $I_1$, if $a$ and $b$ are in $I_1$,
2) $ra$ is an element of $I_1$, if $r$ is in $R$ and $a$ in $I_1$.

It is quite obvious that the $(R_1, R_r)$-subgroups of $M$
are the two-sided ideals of $R$. Henceforth, we shall refer
to two-sided ideals as ideals, and we shall denote them
by $I$.

Let the module $M$ and $QcE(M)$ be fixed and $N_1$ and $N_2$
be $Q$-subgroups of $M$. The union $N_1 \cup N_2$ of the two sub-
groups is the smallest subgroup containing both $N_1$ and $N_2$. 

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The intersection $N_1 \cap N_2$ is the smallest subgroup containing elements which are in both $N_1$ and $N_2$.

**Definition:** The module $M$ is said to be $Q$-irreducible (i.e., irreducible relative to $Q$) if $M$ is not the zero element, and the only $Q$-subgroups of $M$ are $M$ and $0$.

**Definition:** A ring $R$ is said to be simple if $R$ has no proper ideals. That is, if the only ideals of $R$ are $R$ and $0$.

**Theorem 1:** If $(R_1, R_2) = Q$, then $R$ considered as a module $M$ is $Q$-irreducible if and only if $R$ is simple.

**Proof:** 1) Suppose $M$ is $Q$-irreducible. Then the only $Q$-subgroups of $M$ are $M$ and $0$. But this means that the only ideals of $R$ are $M$ and $0$. However, $M = R$ as a module, so the only ideals of $R$ are $R$ and $0$.

2) If $R$ is simple, $R$ has no proper ideals, i.e., $M$ and $0$ are the only ideals of $R$. Hence $M$ is $(R_1, R_2)$-irreducible. Q.E.D.

**Definition:** A minimal right ideal $I_\pi$ is an $R_\pi$-irreducible subgroup of $M$.

**Definition:** A maximal right ideal $I_\pi$ is an ideal that is contained in no other ideal but the ring itself.

We say that $L$ is a direct sum of the $Q$-subgroups $L_i$, $i = 1, 2, \ldots, h$ if

$L = L_1 + L_2 + \cdots + L_h$

where $L_1 + L_2 + \cdots + L_h$ is the set of all finite sums of elements in $L_1$ and $L_i$, and if

$L_1 \cap (L_1 + \cdots + L_{i-1} + L_{i+1} + \cdots + L_h) = 0$

for all $i$. The decomposition is proper if all $L_i \neq 0$. 

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If no proper decomposition exists other than \( L = I \), \( I \) is indecomposable. We use the notation

\[
L = L_1 \oplus L_2 \oplus \cdots \oplus L_n
\]

for direct sum. Since the \( L_i \) are invariant,

\[
L_i + L_j = L_j + L_i,
\]

and we may permute the \( G \)-subgroups in the direct sum. If \( a \) is in \( L_i \) and \( b \) is in \( L_j \), \( j \neq i \), then the commutator 

\[-a - b + a + b \text{ is in } L_i \cap L_j = 0. \]

Hence \( a + b = b + a \) and any elements of \( L_i \) commute with any elements of \( L_j \).

**Definition:** A ring is said to be completely reducible if it can be expressed as the direct sum of minimal right (left) ideals.

In our original definition of a ring, the existence of an identity came as an additional postulate, so that abstractly, a ring may or may not possess an identity. However, if a given ring does not possess an identity, it can be imbedded in a ring with an identity. In order to show this, we shall show that any ring \( R \) is isomorphic to a subring \( R_1 \) of the ring \( T \) which has an identity. Since \( T \) is isomorphic to a ring of endomorphisms, it will follow that \( R_1 \) and hence \( R \) is isomorphic to a ring of endomorphisms. The ring \( T \) is called an extension of \( R \).\(^2\)

We define \( T \) as a ring of pairs \((n, r)\), where \( n \) is an integer and \( r \) is in \( R \). Then define:

\[\text{[2] Jacobson [3], pp. 84-86.}\]
(n,r)+(m,s) = (n+m,r+s) and (n,r)(m,s) = (nm,mr+ns+rs)
for m and n integers and r and s in the ring R. We see that
T is a module, the 0-element being (0,0) and -(n,r)=(-n,-r).
The commutative law of addition and the associative laws in
the integers and in the ring R yield the associative law of
multiplication in T. The distributive laws are trivially
valid. (1,0) acts as the identity of T, i.e.,

(1,0)(n,r) = (n,r+0+0) = (n,r).

Next consider the subset R₁ of T of elements of the
form (0,r). Since

(0,r)+(0,s) = (0,r+s), -(0,a) = (0,-a), and
(0,a)(0,b) = (0,ab),

R₁ is a subring of T. Now set a¹ = (0,a), then the cor-
respondence a onto a¹ is an isomorphism of R onto R₁.
Thus R is imbedded in T, a ring with an identity. It is
also clear that R is an ideal in T.

3. The Radical of a Ring satisfying Minimum Conditions

Definition: We say that a ring R satisfies the minimum
condition if every non-empty collection of right ideals of
R contains a minimal right ideal.

Definition: If for a decreasing sequence of right ideals
I₀>₁>... of R, there exists an integer n such that Iₙ=Iₙ₊₁
and so on for all integers m greater than n, then we say
that the ring R satisfies descending chain conditions.
The descending chain condition is equivalent to the minimum condition, for assume the descending chain condition and the axiom of choice. Let \([I_j]\) be a non-empty collection of ideals. Select \(I_1\) from the collection. Either \(I_1\) is minimal or there is an \(I_2\) in \([I_j]\) such that \(I_2 \subset I_1\). Then either \(I_2\) is minimal or there is an \(I_3\) in \([I_j]\) such that \(I_3 \subset I_2\). This process leads to a minimal ideal in a finite number of steps, for otherwise we obtain an infinite chain, contrary to our assumption. Conversely, suppose the minimum condition holds, and consider an infinite decreasing sequence of ideals. Let \(I_0\) be a minimal element in \([I_j]\). Then certainly we have \(I_0 = I_{n+1} = \ldots\).

**Definition:** An element \(a\) in \(R\) is said to be nilpotent if there exists an integer \(n\) such that \(a^n = 0\).

**Definition:** A nil ideal is an ideal in which every element is nilpotent.

**Definition:** An ideal \(I\) is said to be a nilpotent ideal if there exists an integer \(m\) such that \(I^m = 0\). That is, the product of any \(m\) elements of \(I\) equals zero.

A nilpotent ideal is a nil ideal since \(I^m\) may be the product of \(m\) elements which are identical.

**Definition:** The radical \(N^*\) of a ring \(R\) which satisfies the descending chain condition (minimum condition) is the union of all nil right ideals of \(R\).

**Definition:** A ring \(R\) is a semi-simple ring if its

\(^3\)Jacobson [2], p. 300.
The importance of the radical for the structure theory of rings satisfying minimum conditions is due to the facts that 1) the radical $H^*$ is a two-sided ideal whose difference ring $R/H^*$ is semi-simple, and 2) the structure of semi-simple rings with minimum conditions can be subjected to a thorough analysis that leads in many important cases to a complete classification. However, the concept of a nil ideal in an arbitrary ring has to be abandoned, since there has not been a satisfactory structure theory developed for general semi-simple rings. Therefore we shall define a radical for arbitrary rings that will not depend upon the idea of nil ideals, but one that will be equivalent to the radical defined above when the ring $R$ satisfies minimum conditions.

4. The Jacobson Radical

Let $R$ be an arbitrary ring such that $r$ and $s$ are elements of $R$. Then we define

$$r \circ s = r+s+rs$$

to be a circle product, or quasi-product.

If the ring $R$ has an identity $e$, we define

$$r \circ s = (e+r)(e+s)-e = r+s+rs$$

---

4 Jacobson [2], p. 300.
5 Ibid.
6 Ibid., pp. 301-302.
to be the circle product or quasi-product. This quasi-multiplication is not unnatural as we have seen it before when we imbedded an arbitrary ring in a ring with an identity. Recall \((1,r)(1,s) = (1,r+s+rs)\).

R is a semi-group with an identity with respect to quasi-multiplication, for \(R\) is closed and associative under the quasi-multiplication, since this quasi-multiplication is defined in terms of the operations in the ring. Also the identity of \(R\) with respect to quasi-multiplication is the zero element of \(R\) as a module.

**Definition:** Let \(x\) be an element of an arbitrary ring \(R\). Then \(x\) is said to be right quasi-regular if there exists \(a \in R\) such that \(x \circ z = 0\). We call \(z\) the right quasi-inverse of \(x\). If \(R\) has an identity \(e\), \(x\) is right quasi-regular if there exists \(y \in R\) such that \((e+x)(e+y) = e\). We can define left quasi-regular elements in an analogous manner.

**Theorem 2:** If \(G\) is the set of all right quasi-regular elements of \(R\), then \(G\) is a group with respect to quasi-multiplication.

**Proof:** Associativity is trivial. The identity is the zero element. If \(a\) and \(b\) are in \(G\), there exist elements \(z\) and \(w\) such that \(a \circ z = 0\) and \(b \circ w = 0\). \((a \circ b) \circ (w \circ z)\) equals zero. Hence we have closure. Every element has an inverse since \(z \circ (a \circ z) = z = (z \circ a) \circ z\) and \(z \circ a = 0\).

**Definition:** We say that an element is quasi-regular if it is both left and right quasi-regular.
Definition: A right ideal I is defined to be quasi-regular if every element of I is right quasi-regular.

Theorem 3: If x is an element of a right quasi-regular ideal I and if y is a right quasi-regular element of R, then x+y is right quasi-regular.

Proof: y o z = 0 for some z in R. Now consider

\[(x+y) o z = x+y+zx+yz = x+xz.\]

Since I is a right ideal of R, x+xz is an element of I. Hence there exists a w in R such that (x+xz) o w = 0. This means that \((x+y) o (z o w) = 0\), so x+y is right quasi-regular. Q.E.D.

We are now in position to define the Jacobson radical, the radical for arbitrary rings that does not depend on the idea of nil ideals.

Definition: The Jacobson radical H of an arbitrary ring is the union of all right quasi-regular ideals of R, where this union is defined to be the set of all finite sums of right quasi-regular elements of R.

It follows as a corollary to Theorem 3 that this union is a right quasi-regular ideal.

Although the Jacobson radical was defined unsymmetrically as the union of quasi-regular right ideals, its usefulness lies in the fact that it is a two-sided ideal. In order to show that H is a two-sided ideal, we shall prove the following lemmata.

Lemma 1: z is an element of H if and only if za+iz
is right quasi-regular for all integral $i$ and all $a$ in $R$.

**Proof:** 1) Suppose $z$ is an element of $H$. Then, since $H$ is an ideal, $za+iz$ is in $H$. Hence $za+iz$ is right quasi-regular.

2) Let $za+iz$ be right quasi-regular. Then $za+iz$ is an element of $H$ and consequently $za$ is in $H$. But $a$ is any element of the ring $R$, so $z$ must be an element in $H$.

The set of all elements of the form $za+iz$, where $a$ is in $R$ and $i$ is an integer, is a right ideal, so it is a right quasi-regular ideal.

**Lemma 2:** $za$ is right quasi-regular if and only if $az$ is right quasi-regular.

**Proof:** 1) Suppose $za$ is right quasi-regular. Then there exists a $w$ in $R$ such that $za \circ w = 0$. That is, $za+w+za+w = 0$. Then

$$az+(-az-awz)+az(-az-awz) = -a(w+za+w)z = 0.$$ 

Hence $az$ is right quasi-regular, and its right quasi-inverse is $(-az-awz)$.

2) Let $az$ be right quasi-regular. Then consider $(-za-zwa)$. An argument analogous to 1) will show $(-za-zwa)$ to be the right quasi-inverse of $za$. Q.E.D.

**Theorem 5:** The Jacobson radical $H$ is a two-sided ideal.

**Proof:** Let $z$ be an element of $H$. Then $az$ is right quasi-regular, by Lemma 2. We must show that $az$ belongs to a right quasi-regular ideal. Now if $i$ is an integer
and if \( b \) is in \( R \), then \((az)i+(az)b = a(z+zb)\) is right quasi-regular. The set of all \( azi+azb \) is a right ideal which contains \( az \). Q.E.D.

We could have defined the radical \( H' \) as the union of all quasi-regular left ideals. \( H' = H \) for if \( z \) is an element of \( H \), then there exists a \( z' \) such that \( z+z'+zz' = 0 \). \( z' = -z-zz' \) is in \( H \), since \( H \) is an ideal. This implies that there must be a \( z'' \) such that \( z' o z'' = 0 \). Hence \( z' \) is right quasi-regular. But

\[
z'' = z''+(z+z'+zz')+(z+z'+zz')z'' = z + (z'+z''+z'z'') + z(z'+z''+z'z'') = z
\]

so every element of \( H \) is in \( H' \). By symmetry, we see that every element in \( H' \) is also in \( H \).

We state the following well-known theorem without proof:

**Theorem 6:** The difference ring \( R/I \) of \( R \) relative to \( I \) is a homomorphic image of \( R \). Conversely, any homomorphic image of \( R \) is isomorphic to a difference ring, in fact, to the difference ring of \( R \) relative to the kernel of the homomorphism.

**Theorem 7:** If \( H \) is the radical of \( R \), an arbitrary ring, then \( \overline{R} = R/H \) is semi-simple.

**Proof:** Let \( \overline{z} \) be an element of the radical \( H(\overline{R}) \) of \( \overline{R} \). And let \( z \) be an element of the coset \( \overline{z} \). Then there exists an element \( z' \) such that \( z+z'+zz' = u \), where \( u \) is an element.
of \( H \). Also there exists a \( u' \) such that \( u + u' + uu' = 0 \). Hence

\[
0 = (z + z' + zz') + u' + (z + z' + zz')u' = z + (z' + u' + z'u') + z(z' + u' + z'u').
\]

Thus \( z \) is right quasi-regular. Now since that totality of elements \( z \) in the sets \( \overline{z} \) of \( H(\mathbb{R}) \) is an ideal, this totality is a quasi-regular ideal. Hence \( z \) is an element of \( H \) and \( \overline{z} = 0 \). Hence \( \mathbb{R} = R/H \) is semi-simple. Q.E.D.

In making a connection between the Jacobson radical and the classical radical \( (H^*) \), the following lemma is a key trick.

**Lemma 5 (Jacobson):** If \( I \) is an ideal, and if \( z \) is an element of \( R \) such that \(-z\) is right quasi-regular, and if \( x \) is an element of \( I \) such that \( xz = x \), then \( x = 0 \).

**Proof:** Let \( z' \) be the quasi-inverse of \(-z\), i.e., \(-z + z' - zz' = 0\). But \( xz - x = 0 \); so \((xz - x)z' = xzz' - xz' = 0\). Adding \( xz - x \) and \((xz - x)z'\), we obtain

\[
xz - x + xzz' - xz' = -xz' + xz + xzz' - x = x(z - z' + zz') - x = 0.
\]

Since \( z - z' + zz' \) is the additive inverse of \(-z + z' - zz'\), which is itself zero, \( z - z' + zz' = 0 \). Therefore \( x = 0 \).

If an element \( z \) is nilpotent, it is right quasi-regular. Recall that \( z \) is nilpotent if there exists an integer \( n \) so that \( z^n = 0 \). To show that \( z \) is right quasi-regular, we must exhibit a \( y \) such that \( z \circ y = 0 \). Take
Theorem 8: If $S$ is a subring of the radical, if $z$ is an element of $S$, and if there exists an integer $m$ such that $z^m S = z^{m+1} S$, then $z^{m+1} = 0$.

Proof: If $z^m S = z^{m+1} S$, then $z^{m+1} = z^{m+1} y$ for some $y$ in $S$. But $y$ must be in $H$ since $S$ is a subring of $H$. So $-y$ is right quasi-regular. Then, by Lemma 3, $z^{m+1} = 0$.

Corollary: The radical $H$ contains no idempotent elements not equal to zero.

Proof: Suppose $H$ does contain idempotent elements not equal to zero. Then let $S$ be the set of all idempotent elements $e$ of $H$. Then $S$ is a subring of $H$ and $e S = S$. Hence by Theorem 8, $e = 0$. Contradiction.

Suppose a ring $R$ satisfies minimum conditions. We shall now see that the radical $H$ defined for an arbitrary ring is equivalent to the radical $H^*$ defined for the ring with minimum condition.

Theorem 9: If $R$ is a ring that satisfies minimum conditions for right ideals, the the Jacobson radical $H$ of $R$ is nilpotent.

Proof: Let $N$ be a two-sided ideal contained in $H$ and suppose $N^2 = N$. Then if $N \neq 0$, by minimum conditions there exists a minimal right ideal $T$ of $R$ with the properties 1) $T \subseteq N$ and 2) $TN \neq 0$.

Let $b$ be an element of $T$ such that $bN \neq 0$. Then
(bN)N = bN ≠ 0. bN ⊆ T, so bN = T by minimality of T.

Since b is an element of T, there is an element y in N such that by = b. But since y has a quasi-inverse, b = 0 by Lemma 3 contrary to our assumption that bN ≠ 0. Hence by the properties imposed upon T, N = 0. Now using the minimum conditions, for the decreasing sequence \( H^2 \supseteq H^3 \supseteq \ldots \) there is an integer \( p \) such that \( H^p = H^{p+1} = \ldots \). Then for \( N = H^p \), we have \( N^2 = N \). Hence \( N = H^p = 0 \). Q.E.D.

Recall that M is \( \mathbb{Q} \)-irreducible if and only if \( M ≠ 0 \) and the only \( \mathbb{Q} \)-subgroups of M are \( M \) and \( 0 \).

**Definition:** If \( R \) is a ring of endomorphisms of \( M \) onto itself, then \( R \) is said to be irreducible if \( M \) is \( R \)-irreducible.

**Theorem 10:** If \( R \) is an irreducible ring of endomorphisms of \( M \), then \( R \) is semi-simple.

**Proof:** Let \( Z \) be the set of all \( z \) in \( M \) for which \( za = 0 \) where \( a \) is an element of \( R \). Assert \( Z \) is an \( R \)-subgroup of \( M \). Then either \( Z = 0 \) or \( Z = M \). If \( Z = M \), then \( xa = 0 \) for all \( x \) in \( M \) and all \( a \) in \( R \). In this case, \( R \) must equal zero. But since \( R \) is irreducible, \( R \) cannot be zero, so \( Z \) is equal to zero. Now choose \( x \) in \( M \) so that \( x ≠ 0 \).

Consider \( xR \). Since \( xR \) is an \( R \)-subgroup of \( M \), and \( xR \) is not zero, \( xR \) must equal \( M \), because \( R \) is irreducible. Now assert \( R \) is semi-simple. Suppose not, then \( H ≠ 0 \). \( xR = M \) implies that there must exist an \( a \) in \( R \) such that \( xa = y \) for any \( y \) in \( M \). Choose \( b ≠ 0 \) in \( H \). Then there exists an \( x ≠ 0 \) in \( M \) for which \( xb = t \), where \( t ≠ 0 \) is in \( M \). And there is an a
in $R$ such that $(xb)a = x$ by irreducibility, but $ba$ is in $H$, so $-ba$ is quasi-regular. Hence, using Lemma 3, $x = 0$, so $xb = 0$. But this contradicts the fact that $t \neq 0$. Hence $R$ is semi-simple. Q.E.D.

**Lemma 4:** If $I$ is a two-sided ideal of $R$, and if $R/I$ is semi-simple, then $H \subseteq I$.

**Proof:** Let $x$ be an element of $H$. Then

$$H(R/I) = x + I = 0.$$ 

Hence $x$ is in $I$. Therefore $H \subseteq I$. Q.E.D.

5. **The Structure of the Jacobson Radical**

Suppose $I$ is a maximal right ideal in $R$. Then if $a$ is an element of $R$, the right multiplication $x \to xa$ determined by $a$ induces an endomorphism $a'$ in the difference ring $R/I$. The mapping $a'$ sends the coset $x+I$ into $xa+I$. The totality of elements $a'$ is a subring $R'$ of the ring of endomorphisms of $R/I$ and the correspondence between $a$ and $a'$ is a homomorphism between $R$ and $R'$. The kernel of this homomorphism is a two-sided ideal $(I;R)$ which we shall call the quotient of $I$ relative to $R$. Now $R'$ is an irreducible ring of endomorphisms of $R/I$. Since $I$ is maximal and $R/I$ is irreducible, $R/I$ is semi-simple by Theorem 10. And by Theorem 6, $R'$ is isomorphic to $R/(I;R)$. $H \subseteq (I;R)$, since $R'$ is semi-simple. Now take $b$ in $(I;R)$. $b$ is mapped under the homomorphism onto $b' = 0$. Consider $x+I$ in $R'$. $(x+I)b = (xb+Ib) = 0 - I$. This means that
$xb + I = I = 0$, so $xb$ is in $I$. Then $(I : R)$ is the set of all $b$ for which $Rb \subseteq I$. Hence $R(I : R) \subseteq I$. If $R$ contains an identity, $(I : R)$ is the largest two-sided ideal of $R$ contained in $I$.

**Definition:** $Z^\circ$ is the right ideal consisting of all elements of the form $x + zx$ for all $x$ in $R$.

**Lemma 5:** If $I$ is a right ideal such that $z$ is an element of $I$ and $Z^\circ \subseteq I$, then $I = R$.

**Proof:** $z$ is an element of $I$ by hypothesis. Hence $zx$ is in $I$ for all $x$ in $R$. $x + zx$ is an element of $I$ because $Z^\circ \subseteq I$. Hence $x = (x + zx) - zx$ is in $I$ for all $x$ in $R$. Therefore $R \subseteq I$. But this implies that $R = I$ since $I$ is an ideal of $R$. Q.E.D.

**Theorem 11:** $Z^\circ = R$ if and only if $z$ is right quasi-regular.

**Proof:** 1) Suppose $R = Z^\circ$. Consider $z$ to be an element of $R = Z^\circ$. Then $z = a + za$ for some $a$ in $R$. Therefore $z - a - za = z o a - a = 0$, and $z$ is right quasi-regular.

2) Let $z$ be right quasi-regular. Then there exists an element $z'$ such that $z o z' = 0$. So $z = (-z' - zz')$. Hence $z$ is an element of $Z^\circ$ and $Z^\circ$ is a right ideal of $R$. Therefore by Lemma 5, $Z^\circ = R$.

We shall now define the ideal which will be most pertinent in the discussion that follows.

**Definition:** $I$ is said to be a regular ideal if there exists some $e$ in $R$ such that $ex - x$ is an element of $I$ for
all \( x \) in \( R \).

An immediate example of a regular ideal is \( Z^* \), for if, in the definition of \( Z^* \), we let \( z = -z \), and \( x = -x \) we would obtain a set of elements of the form \( zx-x \).

**Lemma 6:** Any right ideal containing a regular ideal will also be regular.

**Proof:** Let \( I_1 \) be a regular ideal such that \( I_1 \subseteq I_2 \), a right ideal. \( ex-x \) is an element of \( I_2 \) since it is in \( I_1 \).

**Lemma 7:** If \( I \) is a regular right ideal and also a maximal right ideal, then \( I \) contains \( (I:_R) \).

**Proof:** Suppose \( b \) is an element of \( (I:_R) \). Then \( xb \) is in \( I \). For \( x = e \), we have \( eb \) in \( I \). Since \( I \) is regular, \( eb-b \) is in \( I \). Hence \( -b \) is an element of \( I \), and \( b \) must then be in \( I \). Q.E.D.

We obtain the structure of the Jacobson radical in terms of the following ideals:

**Definition:** A right ideal that is both maximal and regular is said to be a maximal regular right ideal.

In establishing the structure of the Jacobson radical, we make use of Zorn's Maximum Principle in the form below which is included for the sake of completeness.

**Definition:** \( P \) is a partially ordered set if \( P \) is a set with a binary relation \( \leq \) defined between some of the elements of \( P \) such that \( \leq \) satisfies the reflexive, asymmetric, and transitive properties as follows:

1) \( a \leq a \),
2) if \( a \leq b \) and \( b \leq c \), then \( a \leq c \),
3) if \( a \leq b \) and \( b \leq a \), then \( a = b \)
for \( a, b, \) and \( c \) belonging to \( P \).

**Definition:** A linearly ordered set if \( L \) is a subset of \( P \) in which the relation \( \leq \) is defined between every pair of elements of \( L \).

**Definition:** An element \( p \) of the partially ordered set \( P \) is said to be an upper bound for the subset \( L \) of \( P \) if \( a \leq p \) for every \( a \) in \( L \).

**Zorn's Maximum Principle:** If every linearly ordered set \( L \) has an upper bound, then there exists at least one maximal element of \( P \).

**Definition:** If a ring \( R \) is equal to its radical \( \sqrt{R} \), then \( R \) is said to be a radical ring.

**Theorem 12:** If \( R \) is a radical ring, then \( R \) contains no maximal regular right ideals.

**Proof:** Deny. Suppose there is a maximal regular right ideal \( I \) in \( R \). And let \( e \) be an element of \( R \) such that \( ex-x \) is in \( I \) for all \( x \) in \( R \). Then \( -e \) is an element of \( \sqrt{R} \), for \( R = \sqrt{R} \). So there must exist an \( e' \) such that \( -e+e'-se' = 0 \). This means that \( ee' = -e+e' \). But since \( I \) is regular, \( e(e'x)-(e'x) \) is an element of \( I \). Because of associativity within the ideal \( I \), \( (ee')x-(e')x \) is in \( I \). Then \( -(e+e')x-e'x = -ex \) belongs to \( I \). Hence \( (ex-x)-ex = -x \) is in \( I \). Therefore \( I = R = \sqrt{R} \). So \( I \) is not a maximal ideal. Q.E.D.
We adopt the following convention:

A null intersection of ideals (an intersection of empty ideals) is the whole ring $R$.

**Theorem 15:** If $R$ contains no maximal regular right ideals, then $R = H$.

**Proof:** Suppose $R \not= H$. Then there is some element $z$ in $R$ which is not in $H$. This means that $Z^*$ is contained in $R$ and $Z^* \not= R$, by Theorem 11. $z$ is not in $Z^*$, by definition of $Z^*$. Now let $B$ be the set of all ideals $I$ in $R$ for which (1) $Z^* \subseteq I$ and (2) $z$ is not an element of $I$. $B$ is not empty since $Z^*$ is such an ideal. Let the set of all ideals $I$ in $R$ be ordered by inclusion. Let $L$ be a linearly ordered subcollection of $B$. Define $U$ to be the union of all $I$ which are contained in the collection $L$. $U$ is a right ideal which satisfies (1) and (2). Also $U$ is an upper bound of $L$. Then by Zorn's Maximum Principle, there exists a maximal right ideal $I_m$ in $B$ having properties (1) and (2). Assert $I_m$ is not only maximal in $B$, but also maximal in $R$. $I_m \not= R$ since $I_m$ is a maximal right ideal in $B$, and $z$ is not an element of $I_m$ by (2). Now suppose $I_m \subseteq I_m' \subseteq R$. By (1), $Z^*$ is contained in $I_m'$. Hence the only way for $I_m'$ not to be an element of $B$ is for $z$ to be an element of $I_m'$. This implies that $I_m' = R$. So $I_m$ is a maximal right ideal in $R$. By Lemma 6, $I_m$ is regular since it contains $Z^*$. Hence $I_m$ is a maximal regular right ideal of $R$. So we have proved the theorem by proving the contrapositive.
The following theorem will serve as a very important alternate definition of the Jacobson radical. This will be the prevalent form of the definition in the discussion that follows.

**Theorem 14:** The Jacobson radical $\mathcal{H}$ is the intersection of all maximal regular right ideals $I_j$ of a ring $R$.

**Proof:** 1) Suppose the intersection of all $I_j$ is the empty set.

1.a) If $R = \mathcal{H}$, then there are no maximal regular right ideals $I_j$ in $R$, by Theorem 12. Hence we obtain a null intersection which is the ring itself. So $\bigcap I_j = \mathcal{H}$.

1.b) If there are no maximal regular right ideals $I_j$, then $R = \mathcal{H}$, by Theorem 13. So again we have $\bigcap I_j = R = \mathcal{H}$.

2) Suppose $\bigcap I_j$ is not empty. $I_j \supseteq (I_j;R)$ for all $j$, by Lemma 7. Also $I_j \supseteq (I_j;R) \supseteq \mathcal{H}$ for all $j$ by the definition of $(I_j;R)$. To show $\bigcap I_j = \mathcal{H}$, then, we must show $\bigcap I_j \subseteq \mathcal{H}$, for $\mathcal{H} \subseteq \bigcap I_j$ since $\mathcal{H} \subseteq I_j$ for all $j$. Let $z$ be an element of $\bigcap I_j$. And suppose $z$ is not right quasi-regular. Then $z^* \not\in R$, but $z^* \subseteq R$. Since $z^*$ is a right ideal, there exists a maximal regular right ideal $I_0$ which contains $z^*$. But $z$ is an element of $\bigcap I_j$ which includes $I_0$, so $z$ is in $I_0$. This implies that $I_0 = R$. But this contradicts the maximality of $I_0$, so $z$ is right quasi-regular and is in $\mathcal{H}$. Thus we have shown $\bigcap I_j \subseteq \mathcal{H}$ and finally that $\bigcap I_j = \mathcal{H}$.
CHAPTER III

REGULAR IDEALS

In this chapter, we investigate finite intersections of maximal regular right ideals to see if they are regular.

Recall the following:

Definition: An ideal I is maximal if for every ideal K such that I \( \subseteq \) K \( \subseteq \) R, K = R.

Definition: I is a regular ideal of R if there exists an element e in R such that ex - x is an element of I for all x in R.

That the problem which we are investigating is closely allied to the problem of determining whether a ring has an identity or not follows from the definition of regular ideals. The necessity of a finite intersection of maximal regular right ideals being again regular in the presence of a unit element is seen in the following

Theorem 13: If R is an arbitrary ring with an identity e and if \( I_a \) is a maximal regular right ideal of R for each integer a, then the finite intersection of the \( I_a \) is regular.

Proof: Since R has an identity e, ex must equal x. Therefore ex - x = 0 for all x in R. But zero is contained in every ideal, so zero is contained in a finite intersection of maximal regular right ideals. Hence ex - x is an element of \( \bigcap I_a \), and \( \bigcap I_a \) is regular. Q.E.D.

The converse of Theorem 13, namely the existence of an
identity in a ring requires considerably more than the regularity of a finite intersection of maximal regular right ideals. However, we have the following result:

**Theorem 16:** If $R$ is a ring in which finite intersections of maximal regular right ideals are regular, and if $R$ satisfies minimum conditions for right ideals, and if the element $e$ used in defining regular ideals is unique, then $R$ possesses an element which is an identity modulo the radical $H$ of $R$.

In order to prove this theorem, we shall need the following

**Lemma 8:** In a ring with descending chain conditions on right ideals, if the intersection of any arbitrary set of right ideals $[I]$ is the zero ideal, then the intersection of a finite set of right ideals is also the zero ideal.

**Proof:** Suppose the intersection of a finite number of right ideals is not zero. Then there is at least one ideal $I_1 \neq 0$. Now if $I_1 \cap I_2 = I_1$ for all $I_2$ in $[I]$, then $I_1 \subseteq I_2$ for all $I_2$. Hence $I_1 = \cap I$. However, this is contrary to our assumption that $\cap I = 0$. Therefore there exists some $I_2$ such that $I_1 \not\subseteq I_1 \cap I_2$. Similarly there exists an $I_3$ such that $I_1 \not\supset I_1 \cap I_2 \supset I_1 \cap I_2 \cap I_3 \supset \cdots$ and this contradicts our assumption of minimum conditions. Hence the intersection of a finite set of right ideals is the zero ideal. Q.E.D.

**Proof of Theorem 16:** If the ideals considered in
Lemma 8 are regular, then the zero ideal is regular. Now if ex - x is an element of the zero ideal, then ex - x = 0 for each x in R. Hence ex = x, so e is a left unit element for the ring modulo its radical. Now consider

\[(e+y-ye)x = ex+yx-yex = x+yx-yx = x.\]

Since e is unique by hypothesis, \((e+y-ye) = e\). For this to be true, y must equal ye. Hence e is also a right unit element of R modulo \(H\). Therefore, since e is both a right and a left unit element of R modulo \(H\), e must be an identity for a ring \(R\) modulo its radical \(H\). Q.E.D.

That our criteria of regularity is reasonable and the class of rings determined by the condition that the finite intersections of maximal regular right ideals be regular is not empty will be seen in the following:

**Theorem 17:** If \(R\) is a commutative ring, then a finite intersection of maximal regular ideals \(I_i\) of \(R\) is regular.

**Proof:** Consider \(I_1 \cap I_2\). Let \(e_1\) be an element of \(R\) such that \(e_1x-x\) is an element of \(I_1\) for all \(x\) in \(R\) and \(i = 1, 2\). Denote \(I_1 \cap I_2\) by \(I_3\). Define \(e_3 = e_1+e_2-e_1e_2\). Then

\[e_3x-x = (e_1+e_2-e_1e_2)x-x = e_1x+e_2x-e_1e_2x =
\]

\[= e_1(x-e_2x)-(x-e_2x)\]

is an element of \(I_1\). Also \(e_3x-x = e_2(x-e_1x)-(x-e_1x), an element of I_2\). Hence \(e_3x-x\) is an element of \(I_1 \cap I_2 = I_3\) for all \(x\) in \(R\). Likewise, define
\[ e_k = \sum_{i<k} a_i - \sum_{i,j<k} e_1 e_j + \cdots + (-1)^{k-1} e_1 e_2 \cdots e_{k-1} \]

for \( k \) finite, so that \( e_k x - x \) is in \( I_k = I_1 \cap I_2 \cap \cdots \cap I_{k-1} \)

for each \( x \) in \( R \). Q.E.D.

As an example of a commutative ring \( R \) in which a finite intersection of maximal regular ideals of \( R \) is regular, consider the ring of all even integers. Denote this ring by \( (2) \). Now consider ideals generated by \( 2r \), where \( r \) is a positive odd integer. Denote these ideals by \( (2r) \). Assert \( r+1 \) is the identity of the difference ring \( (2)/(2r) \).

\((r+1)^n \) is congruent to \((r+1) \mod (2r)\), because, since \( r \) is odd, \((r+1)^n-(r+1) = r^n + r \) is divisible by \( 2r \). Also \((r+1)2k \) is congruent to \( 2k \mod (2r) \) since \((r+1)2k-2k = 2k(r) = k(2r) \) is divisible by \( 2r \). Hence the elements \((r+1)x-x\) are in the ideal \((2r)\) for all \( x \) in \( R = (2) \).

We define \((2r) \cap (2s)\) to be the least common multiple of the two ideals, for \( r \) and \( s \) positive odd integers. \((2rs)\) is certainly contained in \((2r) \cap (2s)\). Since \( rs \) is an odd integer, by the argument used above, \( rs+1 \) is the identity of \((2)/(2rs)\). Now \( rs+1 \) is congruent to \( r+1 \mod (2r) \), since \((rs+1)-(r+1) = rs-r = r(s-1) \) is divisible by \( 2r \) when \( s \) is odd. Also \( rs+1 \) is congruent to \( s+1 \) by the same argument. Hence \((rs+1)x-x\) is an element of \((2rs)\) for all \( x \) in \( R \), and by Lemma 6, \((2r) \cap (2s)\) is regular.

Let us examine a specific case of this example. Let \( r = 3 \). Then \( (2)/(6) = 0, 2, 4 \mod 6 \) has 4 as an identity. So \( 4x-x \) is in \((6)\) for all \( x \) in \((2)\). Now \( r = 5 \). Then
(2)/(10) = 0, 2, 4, 6, 8 (mod 10) has 6 for its identity. Now
(6) ∩ (10) = (30). The identity of (2)/(30) is 16. But 16
is congruent to 4 (mod 6) and also to 6 (mod 10). Hence
16x-x is an element of both (6) and (10), so it is an
element of (30). Consequently (30) is a regular ideal.

Although the cases we have mentioned here are inter­
sections of only two ideals, by inductive reasoning, we can
see that intersections of a finite number of these maximal
regular ideals will also be regular.

The hypothesis given in Theorem 17 can be replaced by
a somewhat weaker hypothesis as follows:

Theorem 18: If R is any ring with idempotent elements
such that these idempotent elements commute with one
another, then a finite intersection of maximal regular right
ideals is regular.

The proof of this theorem is the same as that for
Theorem 17.

The class of rings in which finite intersections of
maximal regular right ideals are not regular is not empty,
as we shall see in the following

Theorem 19: If R is a radical ring, then a finite
intersection of maximal regular right ideals of R is not
regular.

Proof: Since R is a radical ring, it contains no
maximal regular right ideals, by Theorem 12. Because an
intersection of empty sets is empty, it is not regular.
We state the following theorem without proof. However, the proof is available in several selected texts on modern algebra.¹

**Theorem 20**: If \( R \) is a ring satisfying minimum condition for right ideals, and if \( R \) is semi-simple, then \( R \) has an identity.

This theorem implies that the minimum condition for right ideals imposed on a ring is sufficient criteria for the finite intersection of maximal regular right ideals to be regular.

**Theorem 21**: If \( R \) is a ring satisfying the minimum condition for right ideals, then if a finite intersection of maximal regular right ideals \( I_a \) of \( R \) is a two-sided ideal, it is regular.

**Proof**: 1) Suppose the radical \( H \) of \( R \) equals zero. Then \( R \) has an identity by Theorem 20. By Theorem 15, a finite intersection of maximal regular right ideals is regular.

2) Suppose \( H \neq 0 \). Then \( H = \bigcap I_a \) by Theorem 14. So \( H \) is contained in a finite intersection of maximal regular right ideals. Consider the difference ring \( R/\bigcap I_a \). Here, \( \bigcap I_a \) is the zero element, and because \( \bigcap I_a \) contains \( H \), \( H(R/\bigcap I_a) \) must be zero. Hence \( R/\bigcap I_a \) is semi-simple and the conclusion follows from 1). Q.E.D.

The minimum conditions hypothesised in this theorem are certainly reasonable as can be seen from the following considerations:

¹Jacobson [1], p. 64; Van der Waerden [2], pp. 142-145.
A ring without minimum conditions on its ideals may contain an element that is both a zero-divisor and an inverse, or an element which possesses a right-inverse but no left-inverse, or an element that is neither a zero-divisor nor an inverse.\(^2\)

For example, let \( G \) be an infinite dimensional Hilbert space; and denote by \( R \) the ring of all the linear transformations of \( G \). Elements \( u, v, w \) in \( R \) are defined by

\[
\begin{align*}
(a_1,\ldots)^u &= (0,a_1,\ldots) \\
(a_1,\ldots)^v &= (a_1,0,\ldots) \\
(a_1,\ldots)^w &= (a_2,a_3,\ldots).
\end{align*}
\]

Hence

\[
\begin{align*}
uv &= 0 \\
uw &= 1 \\
wu &\neq 1.
\end{align*}
\]

Thus a left-inverse may be a left-zero-divisor.

Another interesting phenomenon that may occur when a ring does not satisfy minimum conditions is that an element may have more than one right inverse. If this happens, the element can have an infinite number of right inverses.\(^3\)

\(^2\)Baer [1], pp. 630-632.
\(^3\)Jacobson [4], p. 353.
If $v$ is one of the right inverses of the element $u$, then we have

$$uv = 1, \quad vu \neq 1.$$  

If $e_{ij} = v^{i-1}u^{j-1}v^{i}u^{j}$, then $ue_{ll} = u(l-vu) = u-u = 0$. Hence also $ue_{lk} = u(v^{0}u^{k-1}v^{k}) = u^{k}uvu^{k} = u^{k}u^{k} = 0$ for $k = 1, 2, 3, \ldots$. Assert $e_{lf} \neq e_{lk}$ for $k \neq f$. Otherwise $e_{lk}e_{kk} = e_{lf}e_{kk}$ and $e_{lk} = 0$. But $e_{lk}$ cannot be zero, for if $e_{lk} = 0$, $l-vu = 0$ and $vu = 1$, a contradiction of our assumption that $vu \neq 1$. Hence the $e_{lk}$ are all different and the elements $v_{k} = v+e_{lk}$ are all different. Evidently $uv_{k} = 1$, since $uv_{k} = 1+ue_{lk} = 1-0 = 1$. 

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