Approximation theorems for valuations on commutative rings

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APPROXIMATION THEOREMS FOR VALUATIONS ON
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By

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TABLE OF CONTENTS

Introduction .................................................. iii

I. Valuations and Valuation Pairs ............................ 1

II. Dominance .................................................. 12

III. Extensions .................................................. 18

IV. The Inverse Property and Approximation Theorems .... 23

References ..................................................... 40
INTRODUCTION

This paper is the result of an investigation of the approximation theorems developed by M. Manis in Chapter III of his doctoral thesis ([2]). The results obtained in [2] were those needed for the author's development of Galois theory for rings. This study was made in an attempt to discover additional and more general cases in which these results apply. Particular emphasis was put on the so-called "inverse property" which can be considered the weakest form of an approximation theorem.

Sections I and II are adapted from Chapters I and II of [2] and contain the definitions and background material necessary for Sections III and IV. The arguments used are all taken from [2] or from lecture notes of a seminar given by M. Manis during the school year of 1966 and 1967.

In Section III, we introduce the concept of extending a valuation on a ring to an extension of the ring. Except for those dealing with the inverse property, the theorems of Section IV are limited to these extensions. Propositions 3.6 and 3.7 are included to show that every valuation on a ring can be extended to any integral extension of the ring; and hence, that these extensions occur with sufficient frequency to merit the consideration given them. The arguments in this section are taken from the same sources as
those in Sections I and II with the exception of 3.6 which was adapted from a more general theorem on page 255 of [1] and was simplified to its present form by M. Manis in the course of this writing.

Part I of Section IV outlines the approximation theorems obtainable for valuations on a field and indicates the results desired for valuations on a ring.

Part II of Section IV considers the inverse property which somewhat replaces the multiplicative inverses inherent in a field. Propositions 4.9 and 4.10 and Examples 1, 2, and 3 are the result of an attempt to correlate the inverse property for two valuations with the relationship between the sets of elements from the ring for which the valuations assume the value zero.

Part III of Section IV shows that the conditions assumed for Part IV hold in the case of an integral extension.

Propositions 4.17 through 4.20 are the approximation theorems of Chapter III of [2]. These theorems are limited to sets of extensions of a single valuation. Proposition 4.21 concerns approximation properties 4.18 and 4.19 for sets of extensions of more than one valuation; that is, given conditions which make the previous theorems apply in the special case of extensions of a single valuation, then the inverse property and 4.18 "extend" to finite sets of distinct pairwise-independent extensions.
The material covered in this paper is but a beginning of a complete approximation theory for rings. Many cases and situations are still open to investigation.
SECTION I
VALUATIONS AND VALUATION PAIRS

Throughout this paper, we will use the following conventions: "Ring" will mean "commutative ring with identity", and subrings will always contain that identity. Ring homomorphisms will always take identity to identity. Prime ideals are always proper. The identity of a ring will be denoted by 1 and that of a group by e. Once it is introduced, notation will be assumed as standard wherever it does not cause ambiguity.

**Definition 1.1.** By a valuation semigroup G, we mean an abelian (multiplicative) group with a zero adjoined, linearly ordered by a relation "≤" satisfying:
1) \( a < b \Rightarrow ac < bc \) for all \( a, b, c \) in \( G \), \( c \neq 0 \),
2) \( 0 \cdot a = a \cdot 0 = 0 \leq b \) for all \( a, b \) in \( G \).

**Definition 1.2.** A valuation \( V \) on a ring \( R \) is a homomorphism of the multiplicative semigroup of \( R \) onto a valuation semigroup satisfying:
\[
V(x+y) \leq \max \{V(x), V(y)\} \quad \text{for all } x, y \text{ in } R.
\]

We note that \( V(1) = e \) and \( V(0) = 0 \) for all valuations. If \( R \) is a field and \( t \) a non-zero element of \( R \), then \( 0 \neq e = V(1) = V(t)V(t^{-1}) \), so \( V^{-1}(\overline{0}) = \overline{0} \). For this reason, in studying fields, one works with ordered groups rather than semigroups. The condition of 1.2 is the
non-Archimedean condition in a field.

**Proposition 1.3.** Let $V$ be a valuation on a ring $R$, and set

$$A_V = \{x \in R \mid V(x) \leq \epsilon\},$$

$$P_V = \{x \in R \mid V(x) < \epsilon\},$$

and

$$N_V = \{x \in R \mid V(x) = 0\}.$$

Then $A_V$ is a subring of $R$, $P_V$ is a prime ideal of $A_V$, and $N_V$ is a prime ideal of $R$. Further, if $J$ is an ideal of $R$, $J \subseteq A_V$, and $A_V \neq R$, then $J \subseteq N_V$.

**Proof:** Note that $V(-1) = \epsilon$ since $G$ is linearly ordered, $V$ is a homomorphism, and $(-1)(-1) = 1$. Thus, $V(-x) = V(-1)V(x) = V(x)$ for all $x \in R$. Thus we have $A_V = -A_V$, $P_V = -P_V$, and $N_V = -N_V$. The condition of 1.2 gives $(A_V + A_V) \subseteq A_V$, $(P_V + P_V) \subseteq P_V$, and $(N_V + N_V) \subseteq N_V$. If $x$ is in $A_V$ and $y$ in $P_V$, then $V(x) \leq \epsilon$ and $V(y) < \epsilon$, so $V(xy) = V(x)V(y) < V(x)\epsilon = V(x) \leq \epsilon$; thus $A_VP_V \subseteq P_V$, and $P_V$ is an ideal of $A_V$, a subring of $R$. If $x$ is in $R$ and $y$ in $N_V$, then $V(xy) = V(x)V(y) = V(x)0 = 0$; so $RN_V \subseteq N_V$, and $N_V$ is an ideal of $R$. If $ab$ is in $P_V$, then $e > V(ab) = V(a)V(b)$, so either $e > V(a)$ or $e > V(b)$, so $P_V$ is a prime ideal of $A_V$ ($V(1) = \epsilon$ so $1$ is not in $P_V$). If $ab$ is in $N_V$, then $0 = V(ab) = V(a)V(b)$, so $V(a) = 0$ or $V(b) = 0$, so $N_V$ is a prime ideal of $R$.

Finally, suppose $A_V \neq R$ and $J$ is an ideal of $R$. If $J \notin N_V$, then $V(a) \neq 0$ for some $a$ in $J$; but then
$V(a) = V(b)^{-1}$ for some $b$ in $R$, and $V(c) > e$ for some $c$ in $R$ since $A_v \neq R$. But then $abc$ is in $J$ while $V(abc) = V(a)V(b)V(c) = eV(c) = V(c) > e$, so $J \neq A_v$.

**Definition 1.4.** By a valuation pair of a ring $R$, we mean a pair $(A, P)$, where $A$ is a subring of $R$ and $P$ is a prime ideal of $A$, such that $x$ in $R \setminus A$ implies $xy$ in $A \setminus P$ for some $y$ in $P$.

**Proposition 1.5.** $(A, P)$ is a valuation pair of $R$ iff there is a valuation $V$ on $R$ with $A = A_v$ and $P = P_v$. Furthermore, if $V'$ is another valuation on $R$ with $P = P_v$, and $A = A_v, R$, then there is an order-preserving isomorphism $\phi: G_v \rightarrow G_v$ with $\phi \circ V' = V$.

**Proof:** Let $V$ be a valuation on $R$ with $A = A_v$ and $P = P_v$. If $x$ in $R \setminus A$, then $V(x) > e$, and $V(y) = V(x)^{-1}$ for some $y$ in $R$. $e = V(x)V(x)^{-1} > V(x)^{-1} = V(x)^{-1}$ so $y$ is in $P$. Now $V(xy) = V(x)V(y) = V(x)V(x)^{-1} = e$ so $xy$ is in $A \setminus P$. Thus by 1.3, $(A, P)$ is a valuation pair of $R$.

Conversely, let $(A, P)$ be a valuation pair of $R$. For $x$ in $R$, define $V(x) = [z$ in $R / xz$ in $F]$, and let $G_v = G = [V(x) / x$ in $R]$. 

**Claim 1.** $V(x) = V(1)$ iff $x$ in $A \setminus P$.

**Subproof 1:** If $x$ in $A \setminus P$, then $xP \subseteq P$ so $P \subseteq V(x)$. 
$V(x) \cap (A \setminus P) = \emptyset$ since $P$ is a prime ideal of $A$. If $y$ is not in $A$, then there is a $p$ in $P$ with $yp$ in $A \setminus P$. $x(yp) = (xy)p$ is in $A \setminus P$ so $xy$ is not in $A$ since $P$ is an ideal of $A$. Thus, $xy$ is not in $P$, so $y$ is not in $V(x)$. Therefore, $V(x) \subset P$ so $V(x) = P = V(l)$.

Suppose $V(x) = V(l) = P$. If $x$ is in $P$, then $x \cdot 1$ is in $P$ so $1$ is in $V(x) = V(l)$ so $1 \cdot 1$ is in $P$, a contradiction. If $x$ is not in $A$, then $xp$ is in $A \setminus P$ for some $p$ in $P$ so $p$ is not in $V(x) = P$, a contradiction. Thus, $x$ is in $A \setminus P$.

Claim 2. Let $V(x)V(y) = V(xy)$. Then this is a well-defined multiplication for $G$ and makes $G$ into an abelian group with zero ($= V(0)$) adjoined.

Subproof 2: Let $V(x) = V(a)$ and $V(y) = V(b)$. Then, $t$ is in $V(xy)$ iff $txy$ is in $P$ iff $tx$ is in $V(y)$ iff $tx$ is in $V(b)$ iff $txb$ is in $P$ iff $tb$ is in $V(x)$ iff $tb$ is in $V(a)$ iff $tba$ is in $P$ iff $t$ is in $V(ab)$. Thus, $V(xy) = V(ab)$ so $V(x)V(y) = V(a)V(b)$ and multiplication is well-defined. Furthermore, it is associative and commutative since multiplication in $R$ is; $V(l)$ is clearly an identity and $V(0)$ a zero, and $V(l) \neq V(0)$ since $1$ is in $V(0)$ but $1$ is not in $V(l)$.

Finally, if $V(x) \neq V(0) = R$, then there is a $y$ in $R$ such that $xy$ is not in $P$. If $xy$ is in $A \setminus P$, then $V(xy) = V(1) = V(x)V(y)$ so $V(x)^{-1} = V(y)$. Otherwise, $xy$ is not in $A$, so $xyp$ is in $A \setminus P$ for some $p$ in $P$; hence, $V(xyp) = V(1) = V(x)V(yp)$, and $V(x)^{-1} = V(yp)$. Thus, $G \setminus [V(0)]$
is an abelian group.

**Claim 3.** Define \( V(x) < V(y) \) if \( V(y) \not< V(x) \). Then "<" is a linear ordering on \( G \), and \( G \) is a valuation semigroup.

**Subproof 3:** Let \( x \) and \( y \) be in \( R \) and \( V(x) \not< V(y) \). Then there is an \( a \) in \( V(x) \setminus V(y) \); i.e., \( xa \) is in \( P \) and \( ya \) is not in \( P \). If \( b \) is in \( V(y) \setminus V(x) \), then \( yb \) is in \( P \) and \( xb \) is not in \( P \); so there are \( t \) and \( t' \) in \( A \) with \( txb \) in \( A \setminus P \) [i.e., \( t = 1 \) if \( xb \) is in \( A \), otherwise \( t \) is in \( P \) since \( (A,P) \) is a valuation pair] and \( t'ya \) in \( A \setminus P \). Then \( (txb)(t'ya) \) is in \( A \setminus P \) since \( P \) is a prime ideal of \( A \); but \( (txb)(t'ya) = (tt')(xa)(yb) \), \( tt' \) is in \( A \), and \( xa \) and \( yb \) are in \( P \), so \( (txb)(t'ya) \) is in \( P \), a contradiction. Thus, \( b \) in \( V(y) \) implies \( b \) is in \( V(x) \), so \( V(x) \not< V(y) \) implies \( V(y) \subset V(x) \); i.e., \( V(x) \not< V(y) \) implies \( V(x) \subset V(y) \) or \( V(y) < V(x) \).

Now if \( V(x) < V(y) \), \( z \) in \( R \), and \( V(z) \not< V(0) \), then \( V(y) \not< V(x) \). \( t \) in \( V(z)V(y) = V(zy) \Rightarrow tzy \) is in \( P \Rightarrow tz \) is in \( V(y) \subset V(x) \Rightarrow tzx \) is in \( P \Rightarrow t \) is in \( V(zx) = V(z)V(x) \), so \( V(z)V(y) \subset V(z)V(x) \). \( V(z) \not< V(0) \Rightarrow V(z)^{-1} = V(z') \) for some \( z' \) in \( R \), so \( V(zx) = V(zy) \Rightarrow V(x) = V(1)V(x) = V(zz')V(x) = V(z')V(zx) = V(z')V(zy) = V(z'z)V(y) = V(1)V(y) = V(y) \). Thus, \( V(y) \not< V(x) \Rightarrow V(z)V(y) \not< V(z)V(x) \) for all \( V(z) \not< V(0) \); i.e., \( V(z)V(x) \subset V(z)V(y) \). Thus condition i) of 1.1 is satisfied. \( 0 \cdot V(x) = V(0)V(x) = V(0 \cdot x) = V(0) \) for all \( x \) in \( R \), and \( V(0) = R \Rightarrow V(y) \subset V(0) \) for all \( y \) in \( R \) so \( V(0) \leq V(y) \) for
all \( y \) in \( R \). Thus, condition ii) is satisfied, and \( G \) is a valuation semigroup.

**Claim 4.** \( V \) is a valuation on \( R \).

**Subproof 4:** \( V \) is obviously a homomorphism from \( R \) onto \( G \) by the definition of multiplication in \( G \). Let \( V(x) = \max[\forall(x), V(y)] \). Then \( V(y) \leq V(x) \) so \( V(x) \leq V(y) \).

If \( t \) is in \( V(x) \), then \( tx \) and \( ty \) are in \( P \) so \( (tx + ty) = t(x+y) \) is in \( P \) so \( t \) is in \( V(x+y) \); i.e., \( V(x) \leq V(x+y) \) so \( V(x+y) \leq V(x) = \max[\forall(x), V(y)] \). Thus, \( V \) is a valuation on \( R \).

**Claim 5.** \( A = A_v \) and \( P = P_v \).

**Subproof 5:** If \( x \) is in \( P \), then \( P = V(1) \leq V(x) \).

By Claim 1, \( V(1) = V(x) \) iff \( x \) is in \( A \backslash P \), so \( V(1) \leq V(x) \) so \( P \subset P_v \). Let \( x \) not be in \( P \). Then \( x \) in \( A \backslash P \Rightarrow V(x) = V(1) \Rightarrow x \) is not in \( P_v \), or \( x \) not in \( A \Rightarrow \) there is a \( z \) in \( P \) with \( xz \) in \( A \backslash P \Rightarrow P \neq V(x) \Rightarrow V(x) \subset P = V(1) \Rightarrow V(1) \leq V(x) \Rightarrow x \) is not in \( P_v \). Thus, \( P_v \subset P \). Therefore, \( P = P_v \), and \( A_v = \{ x \text{ in } R \mid V(x) = V(1) \} \cup P_v = (A \backslash P) \cup P = A \). Thus, \( V \) is the valuation claimed in the proposition.

Now if \( V' \) is another valuation on \( R \) with \( A = A_v \), \( \neq R \) and \( P = P_v \), define \( \varnothing:G_v \rightarrow G \) by \( \varnothing(V'(x)) = V(x) \).

**Claim:** \( \varnothing \) is an order-preserving isomorphism.

**Subproof:** Note that by 1.3, \( N_v = N_v \), since \( N_v \subset A_v = A_v \), \( \neq R \) and \( N_v \subset A_v = A_v \neq R \). Thus, \( V'(x) = V'(0) = V'(y) \) iff \( V(x) = V(0) = V(y) \). If \( V'(x) = \)
\( V'(y) \neq 0 \), then there is a \( z \) in \( R \) with \( xz \) in \( A_{v} \backslash P_{v} \Rightarrow A_{v} \backslash P_{v} = A_{v} \backslash P_{v} \). Thus, \( V'(1) = V'(xz) = V'(x)V'(z) = V'(yz) \) so \( yz \) is in \( A_{v} \backslash P_{v} \Rightarrow A_{v} \backslash P_{v} \). Thus, \( V(xz) = V(1) = V(yz) \), and \( V(x) = V(x)V(1) = V(x)V(yz) = V(xz)V(y) = V(1)V(y) = V(y) \). Interchanging \( V \) and \( V' \) we obtain \( V(x) = V(y) \Rightarrow V'(x) = V'(y) \), so \( V'(x) = V'(y) \) iff \( V(x) = V(y) \). Thus, \( \phi \) is well-defined and "1-1". \( \phi \) is obviously a homomorphism and "onto" by the way it is defined, so \( \phi \) is an isomorphism.

Finally, \( V'(x) < V'(y) \Rightarrow V'(y) \neq V'(0) \) so that there is a \( z \) in \( R \) such that \( V'(yz) = V'(1) = e' \). \( V(yz) = \phi(e') = e \). Thus, \( V'(xz) = V'(x)V'(z) < V'(y)V'(z) = V'(yz) = e' \), so \( xz \) is in \( P_{v} \Rightarrow P_{v} \) and \( V(xz) < e \). Thus, \( V(x) = V(x)V(yz) \Rightarrow V(xz)V(y) < eV(y) = V(y) \), so \( \phi(V'(x)) < \phi(V'(y)) \) as claimed.

Thus, \( \phi \) is the order-preserving isomorphism claimed in the proposition; and henceforth, we will speak of the valuation determined by a valuation pair \((A,P)\).

**Corollary 1.6.** If \((A,P)\) is a valuation pair of \( R \), then

1) \( R \setminus A \) is closed under multiplication;

2) \( R \setminus P \) is closed under multiplication;

3) \( xy \) in \( A \Rightarrow x \) in \( A \) or \( y \) in \( P \);

4) \( x^{n} \) in \( A \Rightarrow x \) in \( A \);

5) \( x^{n} \) in \( A \setminus P \Rightarrow x \) in \( A \setminus P \);

6) \( A = \{ x \) in \( R \mid xP \subseteq P \} \); and

7) \( A = R \) or \( P = \{ x \) in \( A \mid xy \) in \( A \) for some \( y \) not in \( A \} \).
Proof: Let $V$ be the valuation associated with $(A,P)$ in 1.5. Translating, we have

i) $V(x)V(y) > e$ if $V(x) > e$ and $V(y) > e$;

ii) $V(x)V(y) \geq e$ if $V(x) \geq e$ and $V(y) \geq e$;

iii) $V(x)V(y) \leq e$ if $V(x) \leq e$ or $V(y) < e$;

iv) $V(x)^n \leq e \Rightarrow V(x) \leq e$;

v) $V(x)^n = e \Rightarrow V(x) = e$;

vi) $V(x) \leq e$ iff $V(x)V(y) < e$ for all $V(y) < e$;

vii) If $V(z) > e$ for some $z$ in $R$, then $V(x) < e$ iff $V(x)V(t) \leq e$ for some $V(t) > e$.

Proposition 1.7. Let $V$ be a valuation on a ring $R$, $a,b$ in $R$ with $V(a) \neq V(b)$. Then $V(a+b) = \max \left[ V(a), V(b) \right]$.

Proof: Without loss of generality, we may assume $V(a) > V(b)$. Then $V(a) = V(a+b-b) \leq \max \left[ V(a+b), V(b) \right] = V(a+b) \leq \max \left[ V(a), V(b) \right] = V(a)$, so $V(a) = V(a+b)$.

Corollary 1.8. Let $V$ be a valuation on a ring $R$ and $a_i$ in $R$ for $i = 1,2, \ldots, n$. If $V(\sum_{i=1}^{n} a_i) \leq \max V(a_i)$, then $V(a_j) = \max V(a_i) = V(a_k)$ for some $j \neq k$.

Proof: Let $V(a_j) = \max V(a_i)$. Then since $V(\sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} a_i + a_j \leq \max \left\{ V(\sum_{i=1}^{n} a_i), V(a_j) \right\}$, $V(\sum_{i=1}^{n} a_i) = V(a_j)$ by 1.7. But $V(\sum_{i=1}^{n} a_i) \leq \max V(a_i)$, so $\max V(a_i) \geq V(a_j) = \max V(a_i)$; that is, $V(a_j) = \max V(a_i) = V(a_k)$ for some $k \neq j$. 
Corollary 1.9. Let $V$ be a valuation on a ring $R$ and $a_i$ in $R$ for $i = 1, 2, \ldots, n, n+1, \ldots, k$ with $V(a_i) = 0$ for $n < i \leq k$. Then $V(\sum_{i=1}^{k} a_i) = V(\sum_{i=1}^{n} a_i)$.

Proof: $V\left(\sum_{i=1}^{k} a_i\right) = V\left(\sum_{i=1}^{n} a_i + \sum_{i=n+1}^{k} a_i\right) \leq \max\left\{V(\sum_{i=1}^{n} a_i), V(\sum_{i=n+1}^{k} a_i)\right\} = V(\sum_{i=1}^{n} a_i)$. The last equality holds since $V(\sum_{i=1}^{n} a_i) = 0$ by 1.3. $V(\sum_{i=1}^{n} a_i) < V(\sum_{i=n+1}^{k} a_i)$ implies $V(\sum_{i=1}^{k} a_i) = V(\sum_{i=n+1}^{k} a_i) = 0$ by 1.7, but this contradicts the fact that zero is the least element of $G_V$, so the claimed equality holds.

Definition 1.10. For $R$ a ring, let $T = T(R) = \left\{ (A, Q) \mid A \text{ is a subring of } R \text{ and } Q \text{ is a prime ideal of } A \right\}$. For $(A, Q)$ and $(B, S)$ in $T$ define $(A, Q) \preceq (B, S)$ if $A \subseteq B$ and $Q = A \cap S$.

"\preceq" is clearly an inductive partial order on $T$, so by Zorn's Lemma, $T$ has maximal elements. We call maximal elements of $T$ maximal pairs. Note that if $(A, Q)$ is in $T$, then there is a maximal pair $(B, S)$ with $(B, S) \succeq (A, Q)$.

Proposition 1.11. $(A, Q)$ is a maximal pair of $R$ iff it is a valuation pair of $R$.

Proof: If $(A, Q)$ is a valuation pair and $(A, Q) \preceq (B, S)$,
and if \( x \) is in \( B \setminus A \), then \( xp \) is in \( A \setminus Q \) for some \( p \) in \( Q \subseteq S \); but \( x \) in \( B \) and \( p \) in \( S \) imply that \( xp \) is in \( S \) so that \( xp \) is in \( (S \cap A) \setminus Q \) contradicting \( (A,Q) \leq (B,S) \). Thus, \( B \setminus A = \emptyset \), so \( B = A \) and \( S = Q \); i.e., \( (A,Q) \) is a maximal pair.

Conversely, let \( (A,Q) \) be a maximal pair of \( R \), \( x \) in \( R \setminus A \), \( B = A[x] \), and \( S = BQ \). Then \( S \) is an ideal of \( B \) with \( Q \subseteq (S \cap A) \). If \( Q = A \cap S \), then \( A \setminus Q \) is a multiplicatively closed subset of \( B \) with \( (A \setminus Q) \cap S = \emptyset \). Then by Krull's Lemma (see [1] page 253), there is a prime ideal \( S' \) of \( B \) with \( S \subseteq S' \) and \( (A \setminus Q) \cap S' = \emptyset \). That is, \( Q = S \cap A \) and \( (B,S') \supseteq (A,Q) \).

But since \( A \neq B \), this is a contradiction; hence, \( Q \nsubseteq (S \cap A) \).

Thus, there are \( p_1 \) in \( Q \) and \( a' \) in \( A \setminus Q \) with \( \sum_{i=0}^{n} p_i x^i = a' \),

so (*) \( \sum_{i=1}^{n} p_i x^i = a' - p_0 = a \) is in \( A \setminus Q \). We can assume \( n \) is minimal for an expression of this form.

If \( n = 1 \), we are done: \( p_1 x \) is in \( A \setminus Q \).

Suppose \( n > 1 \). Let \( y = \sum_{i=1}^{n} p_i x^{i-1} \). Then \( xy = a \) is in \( A \setminus Q \).

If \( y \) is in \( A \setminus Q \), then \( ya \) is in \( A \setminus Q \) and \( ya = \sum_{i=1}^{n} (p_i x y) x^{i-1} = \sum_{i=1}^{n} p_i x^{i-1} \), an expression of form (*) with degree \( n-1 \leq n \), a contradiction of the minimality of \( n \).

Thus, \( y \) is not in \( A \setminus Q \).

If \( y \) is not in \( A \), then the same argument used for \( x \)
gives \( q_i \) in \( Q \) and \( b \) in \( A \setminus Q \) with (**) \( \sum_{i=1}^{m} q_i y^i = b \).

Again, we can assume that \( m \) is minimal for an expression of this type. Now either 1) \( n \geq m \) or 2) \( m > n \).

Case 1) If \( n \geq m \), then \( p_n bx^n = \sum_{i=1}^{m} p_i q_i (xy)^i x^{n-i} \).

\[ a, b \text{ in } A \setminus Q \Rightarrow ab \text{ in } A \setminus Q, \text{ and } ab = \sum_{i=1}^{n-1} p_i b x^i + p_n b x^n = \sum_{i=1}^{n-1} p_i b x^i + \sum_{i=1}^{m} p_n q_i (xy)^i x^{n-i} = \sum_{i=1}^{n-1} p_i b x^i + \sum_{j=n-m}^{n-1} p_n q_j (xy)^{n-j} x^j = \sum_{i=1}^{n-1} q_i x^i + (xy)^m \text{ if } n=m, \text{ but then } (ab-q_0') \text{ is in } A \setminus Q]. \]

This is of form (**) and degree \( n-1 < n \), a contradiction; therefore, \( m > n \).

Case 2) Using \( q_m ay^m = \sum_{i=1}^{n} p_i q_m (xy)^i y^{m-i} \), we obtain

\[ ab = \sum_{i=1}^{m-1} p_i y^i \text{ contradicting the minimality of } m. \text{ Therefore, } y \text{ is in } A \text{ and } y \text{ is not in } A \setminus Q, \text{ so } y \text{ is in } Q; \text{ thus, } n=1 \text{ and } (A, Q) \text{ is a valuation pair of } R. \]
SECTION II

DOMINANCE

Definition 2.1. If $V$ and $V'$ are valuations on a ring $R$, we say $V'$ dominates $V$ and write $V' \geq V$ if there is an order homomorphism $\phi$ of $G'_v \to G'_v$ with $V' = \phi \circ V$. We say $V' = V$ if $\phi$ is an isomorphism.

Proposition 2.2. Let $V$ and $V'$ be valuations on $R$. Then $V' \geq V$ iff $N_v \subseteq P_v \subseteq P_v \subseteq A_v \subseteq A_v$.

Proof: Let $V' \geq V$.

1) If $V(a) \leq e$, then $V'(a) = \phi(V(a)) \leq \phi(e) = e'$ since $\phi$ preserves order; i.e., $A_v \subseteq A_v$.

2) If $V'(a) < e'$, then $\phi(V(a)) = V'(a) < e' = \phi(e)$ so $V(a) \leq e$ but $V(a) = e \Rightarrow \phi(V(a)) = \phi(e) = e'$, so $V(a) < e$; i.e., $P_v \subseteq P_v$.

3) If $V(a) = 0$, then $V'(a) = \phi(V(a)) = \phi(0) = 0$ so $N_v \subseteq N_v \subseteq P_v$.

Conversely, let $N_v \subseteq P_v \subseteq P_v \subseteq A_v \subseteq A_v$. Note: $N_v = N_v$ by 1.3. Let $\phi(V(a)) = V'(a)$.

Claim 1. $\phi$ is well-defined.

Subproof 1: Let $V(a) = V(b)$.

i) If $V(a) = V(b) = 0$, then $a, b$ are in $N_v = N_v$, so $V'(a) = V'(b) = 0$.

ii) If $V(a) = V(b) \neq 0$, then there is a $z$ in $R$
such that \( V(az) = e = V(a)V(z) = V(b)V(z) = V(bz) \), i.e., \( az, bz \) in \( (A_v \setminus P_v) \subseteq (A_v \setminus P_v') \); but then \( e' = V'(az) = V'(bz) \) so \( V'(a) = V'(a)V'(bz) = V'(az)V'(b) = V'(b) \).

Thus \( V(a) = V(b) \Rightarrow V'(a) = V'(b) \), and \( \emptyset \) is well-defined and clearly a homomorphism.

**Claim 2.** \( \emptyset \) is order-preserving.

**Subproof 2:** Let \( V(a) \leq V(b) \). If \( V(a) = 0 \), then \( V'(a) = 0 \leq V'(b) \) since \( N_v = N_v' \). If \( V(a) \neq 0 \), then \( V(b) \neq 0 \) so there is a \( z \) in \( R \) with \( V(bz) = e \).

\( V(az) \leq V(bz) = e \) so \( az \) is in \( A_v \subseteq A_v' \); and thus, \( V'(az) \leq e' = \emptyset(e) = \emptyset(V(bz)) = V'(bz) \). Therefore, \( V'(a) = V'(a)e' = V'(a)V'(bz) = V'(az)V'(b) \leq e'V'(b) = V'(b) \). Thus, \( \emptyset \) is the order-homomorphism required in 2.1.

Note that \( P_{v'} \) is a prime ideal of \( A_v \) since \( P_{v'} \subseteq A_v' \subseteq A_v \), and \( P_v' \) is a prime ideal of \( A_v' \).

**Proposition 2.3.** If \( P \) and \( P' \) are prime ideals of \( A_v \), \( N_v \subseteq P \subseteq P_v', \) and \( N_v \subseteq P' \subseteq P_{v'} \), then \( P \subseteq P' \) or \( P' \subseteq P \).

**Proof:** Let \( x \) be in \( P \setminus P' \) and \( y \) be in \( P' \setminus P \), then \( V(x) \neq 0 \) and \( V(y) \neq 0 \) since \( N_v \subseteq P \cap P' \) so there are \( x', y' \) in \( R \) with \( V(xx') = e = V(yy') \). Now \( V(x) \leq V(y) \) or \( V(y) \leq V(x) \).

Case 1) \( V(x) \leq V(y) \) gives \( V(xy') \leq V(yy') = e \) so \( xy' \) is in \( A_v \). Now \( y \) is in \( P' \) so \( yxy' \) is in \( P' \); but then, \( x \) in \( A_v \) and \( yy' \) in \( (A_v \setminus P_v) \subseteq (A_v \setminus P_v') \) imply that \( x \) is in \( P' \) since \( P' \) is a prime ideal of \( A_v \), which is a


contradiction of $x$ in $P \setminus P'$. Thus, $V(x) \neq V(y)$.

Case 2) $V(y) \leq V(x)$. Interchanging $x$ and $y$, $x'$ and $y'$, $P$ and $P'$ in the above argument, we obtain $y$ in $P \cap (P' \setminus P)$, a contradiction. Thus, $V(y) \neq V(x) \neq V(y)$ which contradicts the linear order on $G_v$.

Thus $(P \setminus P') = \emptyset$ or $(P' \setminus P) = \emptyset$; i.e., $P' \subseteq P$ or $P \subseteq P'$.

Henceforth, we will use the sign $\# \#$ for a contradiction.

**Proposition 2.4.** If $V$, $V'$, and $V''$ are valuations on $R$, $V' \geq V$, and $V'' \geq V$, then $V' \geq V''$ or $V'' \geq V$.

**Proof:** $P_v \subseteq P_{v''}$ or $P_{v''} \subseteq P_v$, by 2.3. Without loss of generality, we may assume $P_v \subseteq P_{v''}$. If $x$ is not in $A_{v''}$, then $x$ is not in $A_v$ so there is a $y$ in $P_v$ with $xy$ in $A_v \setminus P_v \subseteq A_{v''} \setminus P_{v''}$ and $xy$ in $A_v \setminus P_v \subset A_{v''} \setminus P_{v''}$ so $V(xy) = e$, $V'(xy) = e'$, and $V''(xy) = e''$. Now $V'(x) > e' \Rightarrow V'(y) < e'$, i.e. $y$ in $P_v \subseteq P_{v''} \Rightarrow V''(y) < e'' \Rightarrow V''(x) > e'' \Rightarrow x$ not in $A_{v''}$. Thus $A_v \subseteq A_{v''}$ so $A_{v''} \subseteq A_v$, and we have $N_{v''} = N_v \subseteq P_{v''} \subseteq P_v \subseteq A_{v''} \subseteq A_v$; i.e., $V' \geq V''$.

Thus $V'$ a valuation on $R$ and $V' \geq V$ for a fixed valuation $V$ is linearly ordered by $\leq$.

**Definition 2.5.** A subgroup $H$ of a valuation semigroup $G$ is said to be isolated if $0$ is not in $H$ and whenever $a, b, c$ are in $G$ with $a \leq b \leq c$ and $a, c$ in $H$ then $b$ is in $H$. 
Proposition 2.6. The isolated subgroups of a valuation semigroup \( G \) are linearly ordered by inclusion.

Proof: Let \( H \) and \( H' \) be isolated subgroups of \( G \) and suppose that \( a \) is in \( H \setminus H' \) and \( b \) is in \( H' \setminus H \). Then \( a, b \) in \( G \) implies that \( a \leq b \) or \( b \leq a \).

Case 1) \( a \leq b \).
   i) If \( e \leq a \), then \( e \leq a \leq b \), \( e, b \) in \( H' \) give \( a \) in \( H' \), \#.
   ii) If \( a \leq e \) and \( b \leq e \), then \( a \leq b \leq e \), \( a, e \) in \( H \) give \( b \) in \( H \), \#.
   iii) If \( a \leq e \leq b \), then \( b^{-1} \leq e \). If \( a \leq b^{-1} \leq e \),
   then \( a, e \) in \( H \) give \( b^{-1} \) in \( H \), \#. If \( b^{-1} \leq a \leq e \),
   then \( b^{-1}, e \) in \( H' \) give \( a \) in \( H' \), \#.

Interchanging \( a \) and \( b \), \( H \) and \( H' \), we likewise obtain
a contradiction for case 2); but case 1) or case 2) must
hold for \( a, b \) in \( G \), so \( H \subset H' \) or \( H' \subset H \).

Proposition 2.7. Let \( V \) be a valuation on \( R \) and \( G = G_v \) its valuation semigroup. Then there is a "1-1"
order-preserving correspondence between \( I(G) = I = \left\{ H \mid H \text{ is an isolated subgroup of } G \right\} \) and \( D(V) = D = \left\{ V' \mid V' \text{ a valuation on } R \text{ with } V' \geq V \right\} \).

Proof: For \( V' \) in \( D \), let \( f(V') = \phi^{-1}(e') \) where \( \phi \)
is the order-homomorphism in the definition of \( V' \geq V \).

Claim 1. \( f : D \rightarrow I \); i.e., \( \phi^{-1}(e') \) is in \( I \).
Subproof 1: If $a, b, c$ are in $G$, $a \leq b \leq c$, and $\phi(a) = \phi(c) = e'$, then $e' = \phi(a) \leq \phi(b) \leq \phi(c) = e'$ since $\phi$ is order-preserving. Also, $e' = \phi(aa^{-1}) = \phi(a)\phi(a^{-1}) = \phi(a^{-1})$ and $e' = \phi(a)\phi(c) = \phi(ac)$ so $b, a^{-1}, ac$ are in $\phi^{-1}(e')$ so $\phi^{-1}(e')$ is an isolated subgroup of $G$ and hence in $I$.

Also $f$ is obviously well-defined and "1-1" since $V' = V''$ implies $V'(x) = e'$ iff $V''(x) = e''$.

Claim 2. For $H$ in $I$, there is an order-homomorphism $\phi_H = \phi$ of $G$ onto a valuation semigroup $G_{\phi^v}$ with $\phi^{-1}(e) = H$.

Subproof 2: Set $\phi(a) = aH$ for all $a$ in $G$. Then since $G$ abelian implies $H$ normal in $G$, $\phi(G) = (G \setminus \{0\})/H \cup \{0\}$, with the usual coset multiplication, is an abelian group with zero adjoined and $H \neq 0^*H = 0$.

Define: $aH < bH$ if $aH \neq bH$ and $a < b$.

If $aH \neq bH$ and $a < b$ ( $\Rightarrow ab^{-1} \leq e$), then $ah' \geq bh''$ for some $h', h''$ in $H$ gives $e > ab^{-1} \geq h'^{-1}h''$ and $e, h'^{-1}h''$ in $H$ so $ab^{-1}$ in $H$ since $H$ isolated so $aH = bH$, $##$. Thus, $ah' < bh''$ for all $h', h''$ in $H$ so "<" is well-defined on $\phi(G)$ and linear since if $aH, bH$ are in $\phi(G)$ and $aH \neq bH$, then $a \neq b$ so $a < b$ or $b < a$.

It is easily checked that $\phi(G)$ with this definition of "<" satisfies conditions i) and ii) of 1.1. Thus, $\phi(G)$ is a valuation semigroup, and $\phi$ is obviously an order-homomorphism onto. $\phi(a) = e = H$ iff $a$ in $H$ so
Further, $\phi \circ V = V_H$ is clearly a valuation on $R$ with $V_H \geq V$. Thus, given $H$ in $I$, there is a valuation $V_H$ on $R$ with $f(V_H) = H$; i.e., $f$ is onto.

Claim 3. Let $V', V''$ be in $D$ with $V'' \geq V'$. Then $f(V') \subseteq f(V'')$.

Subproof 3: There are order-homomorphisms $\phi, \phi', \phi''$ such that $\phi: G \rightarrow G_{V'}$, $\phi'': G \rightarrow G_{V''}$, and $\phi: G_{V'} \rightarrow G_{V''}$.

$\phi \circ \phi'(V(x)) = \phi(V'(x)) = V''(x) = \phi''(V(x))$ for all $x$ in $R$, i.e. all $V(x)$ in $G$. Thus $\phi \circ \phi' = \phi''$ so $\phi''^{-1} = \phi'^{-1} \circ \phi^{-1}$ so $\phi''^{-1}(e'') = \phi'^{-1}(\phi^{-1}(e'')) \circ \phi^{-1}(e')$; i.e., $f(V'') \supseteq f(V')$.

Thus, $f$ is the claimed "1-1" order-preserving correspondence between $I$ and $D$. 
Throughout this section, let $V$ be a fixed valuation on a ring $K$, and let $R$ be an extension of $K$.

**Definition 3.1.** A valuation $W$ on $R$ is called an extension of $V$ to $R$ if there is an order-isomorphism $\varnothing$ of $G_v$ into $G_w$ with $\varnothing(V(x)) = W(x)$ for all $x$ in $K$.

**Proposition 3.2.** Let $W$ be a valuation on $R$. Then the following are equivalent.

i) $W$ is an extension of $V$ to $R$.

ii) $(A_w, P_w) \geq (A_v, P_v)$ and $N_v = N_w \cap K$.

iii) $(A_w, P_w) \geq (A_v, P_v)$ and $W|_K$ is a valuation on $K$.

**Proof:** i) $\implies$ ii) If $x$ is in $A_v$, $W(x) = \varnothing(V(x)) \leq \varnothing(V(e)) = e$ since $\varnothing$ is order-preserving, so $A_v \subset A_w$. If $x$ is in $K$, $W(x) = \varnothing(V(x)) < e$ iff $V(x) < e$ since $\varnothing$ is an order-isomorphism, so $P_v = P_w \cap K = P_w \cap A_v$. Thus, $(A_w, P_w) \geq (A_v, P_v)$. Also, $N_w \cap K$ is an ideal of $K$ and $(N_w \cap K) \subset (P_w \cap K) = P_v \subset A_v$ so $(N_w \cap K) \subset N_v$ by 1.3. If $V(x) = 0$, $W(x) = \varnothing(V(x)) = \varnothing(0) = 0$, so $N_v \subset (N_w \cap K)$; and hence, $N_w \cap K = N_v$.

ii) $\implies$ iii) If $x$ is in $K$ with $W(x) \neq 0$, then $x$ is not in $N_v$ so there is a $y$ in $K$ with $xy$ in $A_v \setminus P_v \subset A_w \setminus P_w$. $W(xy) = \ldots$
$e = W(x)W(y)$ so $W(y) = W(x)^{-1}$. Thus, $W(K)$ is a valuation semigroup contained in $G_w$; i.e., $W|_K$ is a valuation on $K$.

iii) $\Rightarrow$ i)

$$(A_w|_K, P_w|_K) = (A_w \cap K, P_w \cap K) \geq (A_v, P_v)$$

so $$(A_w \cap K, P_w \cap K) = (A_v, P_v)$$ since $(A_v, P_v)$ is a maximal element of $T(K)$. Thus, by 1.5, there is an order-isomorphism $\phi$ of $G_v$ onto $G_w|_K$ with $W|_K(x) = \phi(v(x))$ for all $x$ in $K$, and $G_w|_K \subseteq G_w$.

Henceforth, if $W$ is an extension of $V$ to $R$, we will identify $G_v$ with $G_w|_K$ and thus consider $G_v \subseteq G_w$.

**Proposition 3.3.** $V$ has extensions to $R$ iff

$$K \cap RN_v = N_v.$$ 

**Proof:** If $V$ has an extension $W$ to $R$, then $N_v \subseteq N_w$ so $(K \cap RN_v) \subseteq (K \cap RN_w) = K \cap N_w = N_v$. Thus, $K \cap RN_v = N_v$.

Conversely, suppose $K \cap RN_v = N_v$. Then let $Q = P_v + RN_v$ and $B = A_v + RN_v$. Now $Q$ is an ideal of $B$, and $A_v = B \cap K$ and $P_v = Q \cap K = Q \cap A_v$ so $Q \cap (A_v \setminus P_v) = \emptyset$. Thus, by Krull's Lemma, there is a prime ideal $Q'$ of $B$ with $Q \subseteq Q'$ and $Q' \cap (A_v \setminus P_v) = \emptyset$; i.e., $(B, Q') \geq (A_v, P_v)$. If $(A_w, P_w)$ is any valuation pair of $R$ with $(A_w, P_w) \geq (B, Q')$, then $(A_w, P_w) \geq (A_v, P_v)$. $N_v \subseteq RN_v \subseteq A_w$ so $RN_v \subseteq N_w$ by 1.3; i.e., $K \cap RN_v = N_v \subseteq (N_w \cap K)$. $(N_w \cap K) \subseteq (P_w \cap K) = P_v$ so $(N_w \cap K) \subseteq N_v$; hence, $N_w \cap K = N_v$, and $W$ is an extension of $V$ to $R$ by 3.2.
Definition 3.4. If $W$ extends $V$ to $R$, we write \[ \left( \frac{G_w}{\langle 0 \rangle} \right) \cap \left( \frac{G_v}{\langle 0 \rangle} \right) \cup \{0\} \] as $G_w/G_v$. We say that $G_w/G_v$ is torsion iff for each $a$ in $G_w$ there is an integer $n > 0$ with $a^n$ in $G_v$.

Proposition 3.5. Let $V$ and $V'$ be valuations on $K$ with $V' \geq V$. Then

i) $V$ has extensions to $R$ iff $V'$ has extensions to $R$.

ii) If $W$ is an extension of $V$ to $R$, then there is an extension $W'$ of $V'$ to $R$ with $W' \geq W$; and further, if $G_w/G_v$ is torsion, then the $W'$ is unique.

Proof: $N_v = N_{v'}$ by 2.2 so i) is clear by 3.3.

ii) Let $H$ be the isolated subgroup of $G_v$ corresponding to $V'$ (confer 2.7). Let $S = \left\{ a \in G_w \mid \text{there are } b, c \in H \text{ with } b \leq a \leq c \right\}$. Then $S$ is clearly an isolated subgroup of $G_w$ with $S \cap G_v = H$. Now let $W'$ be the valuation corresponding to $S$ so that $W' \geq W$. Note that in proving 2.7, we also proved that $G_v/H \cong G_v$, and $G_w/S \cong G_w'$. Define $\phi: G_v/H \rightarrow G_w/S$ by $\phi(aH) = aS$ for all $a$ in $G_v$. $aH = bH \Rightarrow ab^{-1} \in H \Rightarrow ab^{-1} \in S \Rightarrow aS = bS$ so $\phi$ is well-defined. If $a, b$ are in $G_v$ and $aS = bS$, then $ab^{-1}$ is in $S \cap G_v = H$ so $aH = bH$ so $\phi$ is "1-1". $\phi$ is onto $\phi(G_v/H)$ and clearly a homomorphism.
with the usual coset multiplication, so \( \emptyset \) is an isomorphism onto \( \emptyset(G_v/H) \). If \( aH < bH \), then \( aH \neq bH \) and \( a < b \) so \( aS \neq bS \) and \( a < b \); i.e., \( aS < bS \). Hence, \( \emptyset \) is order-preserving. Thus, \( G_v, \cong G_v/H \cong \emptyset(G_v/H) \subset G_w/S \cong G_w \), and the map required in 3.1 is the obvious one; so \( W' \) extends \( V' \).

**Claim 1.** If \( W' \) is an extension of \( V' \) to \( R \) with \( W' \supseteq W \) and \( S' \) is the isolated subgroup of \( G_w \) corresponding to \( W' \), then \( S \subseteq S' \).

**Subproof 1:** By the above argument, \( W' \) extends the valuation \( V' \) corresponding to \( S' \cap G_v \). By 3.2, 
\[
(A_{v''}, P_{v''}) = (A_{v''} \cap K, P_{w''} \cap t) = (A_v', P_v')
\] so by 1.5, 
\[
G_v \cong G_{v''}; \text{ that is, } V' = V''.
\] Thus, by 2.7, \( S' \cap G_v = H \).

Now let \( x \) be in \( S \). Then there are \( a, b \) in \( H = S' \cap G_v \) with \( a \leq x \leq b \), so \( x \) is in \( S' \) since \( S' \) is isolated.

**Claim 2.** If \( G_w/G_v \) is torsion, then \( S' \subseteq S \).

**Subproof 2:** Let \( x \) be in \( S' \); then \( x \) is in \( G_w \) so there is an integer \( n > 0 \) with \( x^n \) in \( G_v \); i.e., \( x^n \) is in \( S' \cap G_v = H \). 1) If \( x \geq e \), then \( x^n \geq x \geq e \) and \( x^n, e \) are in \( H \), so \( x \) is in \( S \). 2) If \( x \leq e \), then \( x^n \leq x \leq e \), so \( x \) is in \( S \).

Thus, if \( G_w/G_v \) is torsion, then \( S = S' \) so \( W'' = W' \) by 2.7; i.e., \( W' \) is unique.

**Proposition 3.6.** If \((A, P)\) is a valuation pair of a ring \( R \), then \( A \) is integrally closed in \( R \).

**Proof:** Let \( \overline{A} \) be the integral closure of \( A \) in \( R \).
Then \( A \subseteq \overline{A} \) clearly. Let \( c \) be in \( \overline{A} \), then there are \( a_i \) in \( A \) and \( n > 0 \) such that \( c^n = \sum_{i=0}^{n-1} a_i c^i \). Let \( V \) be the valuation on \( R \) associated with \( (A, P) \). Now if \( c \) is not in \( A \), then \( V(c) > e \) so \( V(a_i c^i) \leq V(c^i) < V(c^n) \) for \( i < n \). But then \( V(c^n) = V(\sum_{i=0}^{n-1} a_i c^i) \leq \max_{i<n} [V(a_i c^i)] < V(c^n), \) \( \#\). Thus, \( c \) is in \( A \) so \( \overline{A} \subseteq A \); and hence, \( A = \overline{A} \).

**Proposition 3.7.** Suppose that \( R \) is integral over \( K \).
If \( V \) is a valuation on \( K \) and \( W \) a valuation on \( R \) with \( (A^w, P^w) \supseteq (A^v, P^v) \), then \( W \) extends \( V \) to \( R \).

**Proof:** \( KNN_w \subseteq KNw = P_v \subseteq A_v \) and \( KNw \) is an ideal of \( K \), so \( KNw \subset Nv \) by 1.3.

Let \( t \) be in \( N_v \) and \( x \) in \( R \). Since \( R \) is integral over \( K \), there are \( a_i \) in \( K \) and \( n > 0 \) with \( x^n + \sum_{i=0}^{n-1} a_i x^i = 0 \).
Then \( t^n \cdot 0 = 0 = (tx)^n + \sum_{i=0}^{n-1} a_i t^{n-i} (tx)^i \). But for \( i < n \), \( a_i t^{n-i} \) is in \( N_v \subseteq A_w \); that is, \( tx \) is integral over \( A_w \).
Since \( A_w \) is integrally closed in \( R \), \( tx \) is in \( A_w \). Thus, \( RN_v \subseteq A_w \) so \( RN_v \subseteq N_w \) by 1.3. Therefore, \( N_v \subseteq RN_v \cap K \subseteq N_w \cap K \), and \( W \) extends \( V \) by 3.2.

Thus, every valuation \( V \) on \( K \) has extensions if \( R \) is integral over \( K \).
Proof: Choose $y_i$ in $F^*$ with $V_i(y_i) < V_i(x_i)$ and use 4.2.

Proposition 4.4. If $x_1, \ldots, x_n$ are in $F^*$, then there is an $a$ in $F$ with $V_i(a) = V_i(x_i)$.

Proof: The $a$ in 4.3 works since $V_i(a-x_i) < V_i(x_i) \leq \max[V_i(a), V_i(x_i)]$ implies $V_i(a) = V_i(x_i)$ by 1.7.

Proposition 4.5. Let $x_1, \ldots, x_n$ be in $F^*$; then there is an $a$ in $F$ with $V_i(a) \leq V_i(x_i)$.

Proof: Choose $y_i$ in $F^*$ with $V_i(y_i) \leq V_i(x_i)$ for each $i$ and use 4.4 on the $y_i$'s.

Proposition 4.6. Let $L$ be the set of all valuations on $F$ and let $x$ be in $F^*$, then there is a $y$ in $F$ with $V(xy) = e$ for all $V$ in $L$.

Proof: Let $y = x^{-1}$.

Although this last proposition is quite trivial in the case of fields, we experience considerable difficulty in obtaining a similar result for rings. Part II is directed to this problem.
Proof: Choose \( y_1 \) in \( F^* \) with \( V_1(y_1) \leq V_1(x_i) \) and use 4.2.

**Proposition 4.4.** If \( x_1, \ldots, x_n \) are in \( F^* \), then there is an \( a \) in \( F \) with \( V_1(a) = V_1(x_i) \).

**Proof:** The \( a \) in 4.3 works since \( V_1(a-x_i) \leq V_1(x_i) = \max[V_1(a), V_1(x_i)] \) implies \( V_1(a) = V_1(x_i) \) by 1.7.

**Proposition 4.5.** Let \( x_1, \ldots, x_n \) be in \( F^* \); then there is an \( a \) in \( F \) with \( V_1(a) \leq V_1(x_i) \).

**Proof:** Choose \( y_1 \) in \( F^* \) with \( V_1(y_1) \leq V_1(x_i) \) for each \( i \) and use 4.4 on the \( y_i \)'s.

**Proposition 4.6.** Let \( L \) be the set of all valuations on \( F \) and let \( x \) be in \( F^* \), then there is a \( y \) in \( F \) with \( V(xy) = e \) for all \( V \) in \( L \).

**Proof:** Let \( y = x^{-1} \).

Although this last proposition is quite trivial in the case of fields, we experience considerable difficulty in obtaining a similar result for rings. Part II is directed to this problem.
PART II

THE INVERSE PROPERTY

**Definition 4.7.** We say that a set \( L \) of valuations on a ring \( R \) has the inverse property if for every \( x \) in \( R \) there is an \( x' \) in \( R \) such that \( V(xx') = e \) whenever \( V \) is in \( L \) and \( V(x) \neq 0 \). \( L \) is said to have the strong inverse property if for every \( x \) in \( R \) there is an \( x' \) in \( R \) with \( V(xx'-1) < e \) whenever \( V \) is in \( L \) and \( V(x) \neq 0 \).

**Proposition 4.8.** Let \( L \) be a set of valuations on \( R \) which has the inverse property and \( L' \) a set of valuations on \( R \) such that for every \( V' \) in \( L' \) there is a \( V \) in \( L \) with \( V' \geq V \). Then \( L \cup L' \) has the inverse property; in particular, \( L' \) has the inverse property.

**Proof:** Let \( x, x' \) be in \( R \) with \( V(xx') = e \) whenever \( V \) is in \( L \) with \( V(x) \neq 0 \). Let \( V' \) be in \( L' \) and suppose \( V' \geq V, V \) in \( L \), and \( V'(x) \neq 0 \). Then \( V(x) \neq 0 \) by 2.2 so \( V(xx') = e \). Then \( xx' \) is in \( A_v \setminus P_v \subset A_v \setminus P_{v'} \), so \( V'(xx') = e \).

**Proposition 4.9.** Let \( V, V' \) be valuations on \( R \) with \( P_v \subset P_{v'} \). Then \( L = \left[ V, V' \right] \) satisfies the inverse property iff \( A_{v'} \subset A_v \cup N_{v'} \).

**Proof:** If \( A_{v'} \subset A_v \cup N_{v'} \), then \( A_v \setminus P_{v'} \subset \left( A_v \cup N_{v'} \right) \setminus P_{v'} = A_v \setminus P_{v'} \subset A_v \setminus P_v \) and \( N_v \subset N_{v'} \) by 1.3. If \( x \) is in \( R \) and
$V'(x) \neq 0$, then there is an $x'$ in $R$ with $xx'$ in $A_v \setminus P_v \subset A_v \setminus P_v$ so $V'(xx') = e$ and $V(xx') = e$.

If $L$ satisfies the inverse property and $x$ is in $(A_v \cup N_v)$, then $V'(x) \neq 0$ and $V(x) > e$ so there is an $x'$ in $P_v \subset P_v$ with $V(xx') = e$ and $V'(xx') = e$. $x'$ in $P_v \Rightarrow V'(x') < e \Rightarrow V'(x) > e \Rightarrow x$ in $A_v$. Thus, $(A_v \cup N_v) \subset A_v$, so $A_v \subset (A_v \cup N_v)$.

**Example 1.** Let $Q$ be the rational numbers and $Q_p = \left\{ \frac{ab^{-1}}{b \neq 0, (a,b) = 1, and (b,p) = 1} \right\}$. Let $R = Q[x]$, $A_v = Q_p[x]$, $P_v = pQ_p[x]$, $A_v' = Q_p + xR$, and $P_v' = pQ_p + xR$. Then $(A_v, P_v)$ and $(A_v', P_v')$ are valuation pairs of $R$, $P_v \subset P_v'$, and $N_v' = xR$, all of which the reader can check for himself. $t = (1 + xp^{-1})$ is in $A_v \setminus (A_v \cup xR)$ so $[V, V']$ does not satisfy the inverse property. Specifically, it is not satisfied for $t$ since $t$ in $A_v \setminus P_v \Rightarrow V'(t) = e$ and $t$ not in $A_v \Rightarrow V(t) > e$; and if $V(tt') = e$, then $V(t') < e \Rightarrow t'$ is in $P_v \subset P_v' \Rightarrow V'(t') < e \Rightarrow V'(tt') < e$.

Notice that in Example 1, "x" is in $N_v \setminus N_v'$ so that $N_v' \neq N_v$. This observation led to the conjecture that perhaps if $N_v = N_v'$, then $[V, V']$ satisfies the inverse property. This is not always true as Example 2 will show.

For $V$ a valuation on a ring $R$, $R/N_v$ is a domain and $\overline{V}(x + N_v) = V(x)$ defines a valuation on $R/N_v$ with $G_{\overline{V}} = G_V$. Letting $F$ be the quotient field of $R/N_v$ and $W$ defined by $W(ab^{-1}) = \overline{V}(ac)$ where $\overline{V}(bc) = e$, then $W$
is an extension of \( \overline{V} \) to \( F \) with \( G_w = G_{\overline{V}} = G_V \). The details of these statements are easily checked. Thus, if \( V \) and \( V' \) are valuations on \( R \) with \( N_V = N_{V'} \), then we can consider \( N_V = N_{V'} = [0] \) since \( R/N_V = R/N_{V'} \) and \( \overline{V}(x + N_V) = 0 \) iff \( x \) is in \( N_V = N_{V'} \).

**Proposition 4.10.** Let \( V \) and \( V' \) be valuations on a domain \( R \) with \( N_V = N_{V'} = [0] \) and \( F \) be the quotient field of \( R \). Then the following are equivalent:

i) \( L = \left[ V, V' \right] \) satisfies the inverse property.

ii) \( F = \left[ xy^{-1} \right] \) for some \( x, y \) in \( R \) and \( y \neq 0 \) imply that there is a \( z \) in \( R \) with \( V(yz) = e \) and \( V'(yz) = e \) by the inverse property.

Thus, \( t = xy^{-1} = xz(yz)^{-1} \), and \( xz \) is in \( R \) and \( yz \) is in \( S \).

**Proof:** i) \( \Rightarrow \) ii). Let \( L \) satisfy the inverse property and let \( t \) be in \( F \). Then \( t = xy^{-1} \) for some \( x, y \) in \( R \), \( y \neq 0 \). \( y \) in \( R \), \( y \neq 0 \) imply that there is a \( z \) in \( R \) with \( V(yz) = e \) and \( V'(yz) = e \) by the inverse property.

Thus, \( t = xy^{-1} = xz(yz)^{-1} \), and \( xz \) is in \( R \) and \( yz \) is in \( S \).

ii) \( \Rightarrow \) i). If \( x \) is in \( F \), then there are \( y, z \) in \( R \) with \( x = yz^{-1} \) and \( V(z) = e = V'(z) \). If \( W \) and \( W' \) are the extensions to \( F \) noted preceding the proposition, then \( W(x) = W(yz^{-1}) = V(y)V(z)^{-1} = V(y) \) and \( W'(x) = W'(yz^{-1}) = V'(y)V'(z)^{-1} = V'(y) \); i.e., if \( x \) is in \( F \), then there is a \( y \) in \( R \) with \( W(x) = V(y) \) and \( W'(x) = V'(y) \). Thus, if
t is in R, then \( t^{-1} \) is in F so there is a \( y \) in R with \( W(t^{-1}) = V(y) \) and \( \overline{W}'(t^{-1}) = \overline{V}'(y) \) so \( e = W(t)W(t^{-1}) = V(t)V(y) \) and \( e = \overline{W}'(t)\overline{W}'(t^{-1}) = \overline{V}'(t)\overline{V}'(y) \).

i) \( \Rightarrow \) iii). L has the inverse property implies that for \( x \) in R, \( x \neq 0 \), there is a \( y \) in R such that \( xy \) is in S. Thus, \( xR \cap S = \emptyset \) iff \( x = 0 \).

iii) \( \Rightarrow \) i). \( x \) in R, \( x \neq 0 \), \( xR \neq [0] \) \( \Rightarrow \) \( xR \cap S \neq \emptyset \); i.e., there is a \( y \) in R with \( xy \) in S.

Example 2. Let \( Z \) be the integers. Let \( R = Z[x, x^{-1}] \), \( A_v = Z[x] \), \( P_v = xZ[x] \), \( A_v' = Z[x^{-1}] \), and \( P_v' = x^{-1}Z[x^{-1}] \).

The reader can check that \((A_v, P_v)\) and \((A_v', P_v')\) are valuation pairs of \( R \), \( N_v = N_v' = [0] \), and \( (A_v \setminus P_v) \cap (A_v' \setminus P_v') = Z \). \((1 + x)R \cap 0 = \emptyset \) but \((1 + x)R \neq [0] \) so \([V, V']\) does not satisfy the inverse property. Also note that \( F \neq \left[z y^{-1}\right] \) \( z \) is in R and \( y \) is in \( Z \), since \( \frac{1}{1 + x} \) cannot be written as \( z y^{-1} \) where \( z \) is in R and \( y \) is in \( Z \). Also if \( t = 1 + x \), then \( t \) is in \( A_v \setminus P_v \) and \( t \) is not in \( A_v' \), so \( V(t) = e \) and \( V'(t) > e \). Therefore, if \( V'(tt') = e \), then \( V'(t') < e \), i.e., \( t' \) is in \( P_v' \); but then, \( t' \) is not in \( A_v \) so \( V(tt') = V(t') \) \( > e \).

Example 3. Let \( p \) and \( q \) be distinct prime integers. Let \( R = Z[x, x^{-1}] \), \( A_v = Z[x] + pR \), \( P_v = xA_v + pR \), \( A_v' = Z[x^{-1}] + qR \), and \( P_v' = x^{-1}A_v' + qR \). Then \( N_v = pR \) and \( N_v' = qR \); but \([V, V']\) satisfies the inverse property. If \( t \) is in \( R \setminus (N_v \cup N_v') \), then \( t = \sum_{i=0}^{n} a_i x^{i-k} \), \( a_j \) is not in
pZ for some j and a_r is not in qZ for some r. Let J = \min \{ j \mid a_j \text{ is not in } pZ \} \text{ and } M = \max \{ r \mid a_r \text{ is not in } qZ \}. Then \( t(qx^{k-J} + px^{k-M}) \) is in \((A_\nu \setminus P_\nu) \cap (A_\nu \setminus P_\nu')\). The details are left to the reader. Thus, the fact that \([V, V']\) satisfies the inverse property does not imply that \(N_\nu = N_\nu'\).

**PART III**

**ALGEBRAIC EXTENSIONS**

Throughout Part III, \( R \) is assumed to be an extension of a ring \( K \), \( V \) a valuation on \( K \) with extensions to \( R \), and \( L \) a set of valuations on \( R \) which extend \( V \).

**Proposition 4.11.** Let \( J \) be an ideal of \( R \) with \( J \subseteq \cap \{ N_w \mid W \text{ in } L \} \) and \( J \cap K = N_\nu \). If \( R/J \) is algebraic over \( K/N_\nu \), then \( L \) satisfies the inverse property.

**Proof:** Note that \( W(t) = 0 \) for all \( t \) in \( J \) and \( W \) in \( L \). If \( x + J \) is in \( R/J \), then there are \( a_\nu \) in \( K \) and \( t \) in \( J \) with \( a_r \) not in \( J \) \((V(a_\nu) \neq 0)\) and \( \sum_{i=0}^{r} a_i x_i = t \).

Let \( s = \min \{ i \mid V(a_i) \neq 0 \} \). Then for \( W \) in \( L \), \( 0 = W(t) = W(\sum_{i=0}^{r} a_i x_i) = W(\sum_{i=s}^{r} a_i x_i) = W(x^s)W(\sum_{i=s}^{r} a_i x_i - s) \). Thus, if \( W(x) \neq 0 \), then \( W(\sum_{i=s}^{r} a_i x_i - s) = 0 = W(\sum_{i=s+1}^{r} a_i x_i - s + a_s) \).
max \left[ W( \sum_{i=s+1}^{r} a_i x^{i-s}), W(a_s) \right] \), so by 1.7, \( W( \sum_{i=s+1}^{r} a_i x^{i-s}) = W(a_s) = W(x)W( \sum_{i=s+1}^{r} a_i x^{i-s-1}) \). Choose \( a' \) in \( K \) with

\[ V(a'a_s) = e. \]

Then with \( x' = a'( \sum_{i=s+1}^{r} a_i x^{i-s-1}) \), \( W(xx') = W(a'a_s) = e \) whenever \( W \) is in \( L \) with \( W(x) \neq 0 \).

**Proposition 4.12.** Let \( J = \bigcap \left[ N_w \middle| W \text{ in } L \right] \) and suppose \( R/J \) is algebraic over \( K/(K \cap J) = K/N_v \). Then \( G_w/G_v \) is torsion for all \( W \) in \( L \).

**Proof:** Let \( x \) be in \( R \) and \( W \) in \( L \). If \( W(x) = 0 \), there is nothing to show, so suppose \( W(x) \neq 0 \). Then there are \( a_1 \) in \( K \), \( t \) in \( J \), and \( a_r \) not in \( J \) such that \( \sum_{i=0}^{r} a_i x^i = t \). Since \( W(a_r x^r) \neq 0 \), we have \( 0 = W(t) = W(\sum_{i=0}^{r} a_i x^i) \leq \max \left[ W(a_i x^i) \right] \), so by 1.8, \( W(a_i x^i) = \max \left[ W(a_i x^i) \right] \), \( W(a_j x^j) \neq 0 \) for some \( i \neq j \).

Assume \( i > j \), and let \( W(x)^{-1} = W(x') \) and \( W(a_1)^{-1} = W(a') \); then \( W(x^{i-j}) = W(a_i x^i)W(x')^j W(a') = W(a_j x^j)W(x')^j W(a_1)^{-1} = W(a_j)W(a') \) is in \( G_v \).

**Proposition 4.13.** Let \( W \) be in \( L \), \( W' \supseteq W \), and \( V' = W'/K \). If \( G_w/G_v \) is torsion, then so is \( G_w'/G_{V'} \).

**Proof:** Let \( \phi: G_w \rightarrow G_w' \) be the homomorphism such that
$W' = \emptyset \circ W$. Then $V' = \emptyset \circ V$. If $\emptyset(x)$ is in $G_{w'}$, then $x^n$ is in $G_v$ for some $n > 0$ so $\emptyset(x^n)$ is in $G_{v'}$.

**Proposition 4.14.** If $W$ is in $L$ and $G_w/G_v$ is torsion, then $W(R) = [e, 0]$ iff $V(K) = [e, 0]$.

**Proof:** $V(K) \subset W(R)$ so "$\Rightarrow"$ is clear. $W(x^n)$ is in $[e, 0]$ for some $n > 0$ only if $W(x)$ is in $[e, 0]$ so "$\Leftarrow"$ is also clear.

**Note 4.15.** If $R$ is integral over $K$ and $J$ is any ideal of $R$, then $R/J$ is integral (and hence algebraic) over $K/(K/J)$. Clear.

**PART IV
APPROXIMATION THEOREMS**

In Part IV, we assume that $R$ is an extension of $K$, $V$ is a valuation on $K$, and $L$ is a set of extensions of $V$ to $R$ with the inverse property and such that $G_w/G_v$ is torsion for each $W$ in $L$. In some of the results, we also require $P_w \not\subset P_w'$ if $W, W'$ are in $L$ and $W \neq W'$. The following proposition indicates the effect of this additional restriction.

**Proposition 4.16.** Let $W$ and $W'$ be distinct elements of $L$ with $P_w \not\subset P_{w'}$. Then $P_v$ is an ideal of $K$, and $R$
is not integral over $K$.

**Proof**: If $P_v$ is an ideal of $K$, then $P_w$ and $P_{w'}$ are ideals of $R$ by 4.14. Then $A_w = A_{w'} = R$, and if $R$ were integral over $K$, we would also have $P_w = P_{w'}$ (see [4] page 259), contradicting $P_w$ and $P_{w'}$ distinct.

It remains only to show that if $P_v$ is not an ideal of $K$, then $P_w \neq P_{w'}$.

If $P_v$ is not an ideal of $K$, then $P_w$ and $P_{w'}$ are not ideals of $R$, so by 1.6, $A_w \neq A_{w'}$.

Case 1) $A_w \setminus A_{w'} \neq \emptyset$. Let $y$ be in $A_w \setminus A_{w'}$. Then $W(y) \leq e < W'(y)$. Since $G_w / G_v$ is torsion, there is an integer $n > 0$ and an $a$ in $K$ with $W'(y^n) = V(a)$. Then $W'(y) = W'(y^{n+1}a^*) > e$ while $W(y^{n+1}a^*) = W(y^{n+1})W(a^*) \leq e$ since $V(a^*) \leq e$. Thus, $y^{n+1}a^*$ is in $P_w \setminus P_{w'}$.

Case 2) $A_w \setminus A_{w'} = \emptyset$. By Case 1), there is a $y$ in $R$ with $W(y) > e \geq W'(y)$. Then $W(1 + y) = W(y) > e$ while $W'(1 + y) = W'(1) = e$, so $W((1+y)^*) \leq e$ while $W'((1+y)^*) = e$. Thus, $(1+y)^*$ is in $P_w \setminus P_{w'}$.

**Proposition 4.17.** Let $W_1, \ldots, W_n$ be distinct elements of $L$ with $P_{w_i} \neq P_{w_1}$ if $i \neq 1$. Then there is an $x$ in $R$ with $W_1(x) \leq e$ and $W_i(x) < e$ for $i \neq 1$. Further, if $P_v$ is not an ideal of $K$, one can require $W_1(x) > e$.

**Proof**: Case 1) $P_v$ an ideal of $K$. Then $P_{w_1}$ is
a prime ideal of $R$, $i = 1, \ldots, n$. Choose $x_i$ in $P_{w_1} \setminus P_{w_1}$, $i = 2, 3, \ldots, n$ and let $x = \prod_{i=2}^{n} x_i$.

Case 2) $P_v$ not an ideal of $K$. Proof by induction on $n$.

For $n = 2$, choose $y$ in $P_{w_2} \setminus P_{w_1}$. Then $W_1(y) \geq e > W_2(y)$.

Since $G_{w_2} / G_v$ is torsion and $G_v \neq [e, 0]$, there is an $n > 0$ and an $a$ in $K \setminus N_v$ with $e > W_2(a) > W_2(y^n)$. Then with $x = a'y^n$ we have $W_1(x) \geq W_1(a') > e$ while $W_2(aa') = e$.

Now assume 4.17 holds for $r = n - 1$, $n > 2$. For $i = 2, 3$, choose $y_i$ in $R$ with $W_1(y_i) \geq e$ and $W_j(y_i) < e$ if $j \neq 1$ and $j \neq i$. If $W_1(y_i) \leq e$, let $x_i = y_i$; otherwise let $x_i = (1+y_i)y_i$.

Claim. $W_1(x_i) \geq e$, $W_i(x_i) \leq e$, $W_j(x_i) < e$ if $i \neq j \neq 1$.

Subproof: This is automatic if $x_i = y_i$. Otherwise, $W_1(1+y_i) = W_1(y_i) > e$ and $W_1(x_i) = e$; $W_1(1+y_i) = W_1(y_i) > e$ and $W_1(x_i) = e$; $W_j(1+y_i) = W_j(1) = e$ and $W_j(x_i) = W_j(y_i) < e$.

Thus, we have $W_1(x_2x_3) \geq e$ and $W_i(x_2x_3) \leq e$ if $i \neq 1$.

Let $z = x_2x_3$. Again since $G_{w_1} / G_v$ is torsion and $G_v \neq [e, 0]$, there is an $n > 0$ and an $a$ in $K \setminus N_v$ with $e > W_1(a) > W_1(z^n)$ for all $i \neq 1$, and $x = a'z^n$ has $W_1(x) > e$ and $W_i(x) < e$ for all $i \neq 1$.

Proposition 4.18. Assume $P_v$ is not an ideal of $K$ and $W_1, \ldots, W_n$ in $L$ are pairwise independent. Then if
Proof: Since \( G_{w_i} / G_v \) is torsion for \( i = 2, 3, \ldots, n \), there are \( n_i > 0 \) with \( a_i^{n_i} \) in \( G_v \setminus \{0\} \). Let \( 0 < a = \min\{a_i, a_i^{-1}\} \) for \( i \neq 1 \). It suffices to show that there is an \( x \) in \( R \) with \( W_i(x) \geq e \) and \( W_i(x) \leq a \) for \( i = 2, \ldots, n \).

Let \( H = \left\{ a \in G_v \mid \text{there is an } x \text{ in } R \text{ with } W_i(x) \geq e \text{ and } W_i(x) \leq \min(a, a_i^{-1}) \right\} \). Then \( e \) is in \( H \) by 4.17, and it is easily checked that \( H \) is an isolated subgroup of \( G_v \). The proposition will be established if \( H = G_v \setminus \{0\} \), or equivalently, that if \( V' \) is the valuation determined by \( H \), then \( V'(K) = \left[ e, 0 \right] = G_v / H \).

Since \( V' \geq V \) and \( G_{w_i} / G_v \) is torsion for each \( i \), by 3.5 there is a unique \( W_i' \geq W_i \) which extends \( V' \), \( i = 1, \ldots, n \). Since the \( W_i \) are independent, either \( W_i'(R) = \left[ e, 0 \right] \) for some \( i \) so that \( V'(K) = \left[ e, 0 \right] \) by 4.14 and the proposition is established, or the \( W_i' \) are distinct.

Assume the \( W_i' \) are distinct. By 4.8 and 4.13, 4.17 applies to \( W_1', \ldots, W_n' \). Thus there is an \( x \) in \( R \) with \( W_i'(x) \geq e \) and \( W_i'(x) \leq e \), \( i = 2, \ldots, n \).

By 4.13, there is an integer \( r > 0 \) and a \( b \) in \( K \) with \( W_i'(x^r) \leq W_i'(b) = V'(b) \leq e \) for \( i = 2, 3, \ldots, n \). Let \( y = x^r \), then \( W_i(y)H \leq V(b)H \leq H \); so \( W_i(y) \leq V(b) \leq e \leq V(b)^{-1} \); so \( W_i(y) \leq \min\{V(b), V(b)^{-1}\} \) \( i = 2, 3, \ldots, n \). But \( W_i'(y) \geq e \) gives \( W_i(y) \geq H \); so \( W_i(y) \geq e \). This is a contradiction.
since then $V(b)$ is in $H$ so that $V'(b) = e$. Thus, $V'(K) = [e, 0]$.

**Proposition 4.19.** (Approximation Theorem) Suppose $P_{V}$ is not an ideal of $K$ and $W_{1}, \ldots, W_{n}$ are in $L$ and are pairwise independent. Then if $a_{i}$ is in $G_{W_{i}} \setminus [0]$, for $i=1, \ldots, n$, there is an $x$ in $R$ with $W_{i}(x) = a_{i}$ for $i=1, \ldots, n$.

**Proof:** For each $i$, choose $z_{i}$ in $R$ with $W_{i}(z_{i}) = a_{i}$. Choose $x_{i}$ in $R$ with $W_{i}(x_{i}) > e$; and for $j \neq i$, $W_{j}(x_{i}) < \min [a_{j}W_{j}(z_{i})' , e]$ if $W_{j}(z_{i}) \neq 0$ and with $W_{j}(x_{i}) < e$ if $W_{j}(z_{i}) = 0$. (This can be done by 4.18.) Let $t_{i} = x_{i}(1+x_{i})'.

Then $W_{i}(t_{i}) = e$, and $W_{j}(t_{i}) = W_{j}(x_{i})$ if $i \neq j$.

Now $W_{i}(t_{i}z_{i}) = W_{i}(z_{i}) = a_{i}$, and if $i \neq j$, $W_{j}(t_{i}z_{i}) = W_{j}(t_{i})W_{j}(z_{i}) = 0$ if $W_{j}(z_{i}) = 1$,

$W_{j}(x_{i})W_{j}(z_{i}) < a_{j}$ if $W_{j}(z_{i}) \neq 0$.

Thus, $W_{j}(t_{i}z_{i}) = \max_{k} W_{j}(t_{k}z_{k})$ only if $i = j$, so by 1.8, $W_{j}(\sum_{i=1}^{n} t_{i}z_{i}) = W_{j}(t_{j}z_{j}) = a_{j}$ for $j = 1, 2, \ldots, n$.

**Proposition 4.20.** (Strong Approximation Theorem) Suppose $L$ has the strong inverse property and $W_{1}, \ldots, W_{n}$ in $L$ are pairwise independent. If $a_{i}$ in $R$ have $W_{i}(a_{i}) \neq 0$ $i = 1, 2, \ldots, n$, then there is an $x$ in $R$ with $W_{i}(x) = W_{i}(a_{i}) > W_{i}(x-a_{i})$ $i = 1, 2, \ldots, n$.

**Proof:** Case 1) $P_{V}$ an ideal of $K$. Then the $P_{W_{i}}$ are
maximal ideals of R so \( \mathbf{P}_{w_i} \not\subset \mathbf{P}_{w_j} \) if \( i \neq j \) and 4.17 applies.

For each \( i \), choose \( x_i \) in \( R \) with \( W_i(x_i) = e \) and \( W_j(x_i) = 0 \) if \( i \neq j \). Choose \( x_i' \) in \( \mathbf{A}_{w_i} \setminus \mathbf{P}_{w_i} \) with \( x_i x_i' = 1 + t_i \) for some \( t_i \) in \( \mathbf{P}_{w_i} \). Then \( W_j(x_i x_i' a_i) = 0 \) if \( i \neq j \) while \( W_i(x_i x_i' a_i - a_i) = W_i(a_i t_i) = 0 \) if \( W_j(a_i) = e \).

Let \( x = \sum_{i=1}^{n} x_i x_i' a_i \), then \( W_i(x - a_i) = W_i(x_i x_i' a_i - a_i + \sum_{j \neq i} x_j x_j' a_j) = 0. \)

Case 2) \( \mathbf{P}_v \) not an ideal of \( K \). Choose \( a_i' \) so that \( W_j(a_i a_i') = e \) whenever \( W_j(a_i) \neq 0 \). For each \( i \), choose \( x_i \) in \( R \) with \( W_i(x_i) > e \); \( W_j(x_i) \leq \min \{ W_j(a_j) W_j(a_i'), e \} \) if \( W_j(a_i) \neq 0 \), and \( W_j(x_i) \leq e \) if \( W_j(a_i) = 0 \). Choose \( y_i \) in \( R \) with \( W_j(y_i) = W_j(1 + x_i)^{-1} \) if \( W_j(1 + x_i) \neq 0 \) and so that \( W_i(y_i (1 + x_i) - 1) < e. \)

Then \( y_i (1 + x_i) = 1 + t_i \) where \( W_i(t_i) < e \); \( (x_i y_i - 1)(1 + x_i) = x_i y_i (1 + x_i) - 1 - x_i = x_i t_i - 1 \); so \( W_i(x_i y_i - 1) W_i(1 + x_i) \leq \max \{ W_i(x_i t_i), W_i(1) \} \leq W_i(x_i) = W_i(1 + x_i) \); so \( W_i(x_i y_i - 1) \leq e \) and \( W_i(x_i y_i a_i - a_i) \leq W_i(a_i) \).

Also if \( i \neq j \), \( W_j(y_i) = W_j(1 + x_i)^{-1} = W_j(1)^{-1} = e \) so \( W_j(x_i y_i a_i) = W_j(x_i) W_j(a_i) = W_j(a_j) \).

Now if \( x = \sum_{j=1}^{n} x_j y_j a_j \), we have \( W_i(x - a_i) = W_i((x_i y_i a_i - a_i) + \sum_{j \neq i} x_j y_j a_j) \leq \max \{ W_i(x_i y_i a_i - a_i), W_i(x_j y_j a_j) \ i \neq j \} \leq W_i(a_i). \)
**Proposition 4.21.** Let $R$ be an integral extension of $K$, $B$ a set of pairwise-independent valuations on $K$, and $E$ a set of valuations on $R$ with $W$ in $E \Rightarrow W^{*}|_{K}$ is in $B$. For each $V$ in $B$, suppose that $P_{V}$ is not an ideal of $K$. If finite subsets of $B$ satisfy the inverse property and the property of Proposition 4.18 (and hence 4.19) and if $W_{1}, \ldots, W_{m}$ are distinct, pairwise-independent elements of $E$, then $[W_{1}, \ldots, W_{m}]$ satisfies the inverse property and 4.18 (and hence 4.19).

**Proof:** Separate the $W_{i}$'s into classes $W_{11}, W_{21}, \ldots$, $W_{n1}; W_{12}, W_{22}, \ldots, W_{n2}; \ldots; W_{1r}, \ldots, W_{nr}$ such that

$W_{ij}|_{K} = W_{ks}|_{K}$ iff $j = s$. For $j = 1, \ldots, r$, let $W_{ij}|_{K} = V_{j}$. Note that $G_{W_{ij}}/G_{V_{j}}$ is torsion for all $i$ and $j$ by 4.12.

For each $j$ we have $[W_{1j}, \ldots, W_{nj}]$ satisfies the inverse property, 4.17, 4.18, and 4.19; so if $r = 1$, we are done.

Assume $r > 1$. If $x$ is in $R$, then by 4.11 there is a $y_{1}$ in $R$ with $W_{i1}(xy_{1}) = e$ for $i = 1, \ldots, n_{1}$. Let $t = xy_{1}$. By 4.12 there is an $n > 0$ with $W_{ij}(t^{n})$ in $G_{V_{j}}$ (let $n = \prod n_{ij}$ where $n_{ij}$ works for $ij$). By 4.19 there is a $z$ in $K$ with $V_{1}(z) = e$ and for $j \neq 1$, $V_{j}(z) < \min_{i=1}^{n_{j}} W_{ij}(t^{n})^{-1}$.
Thus, \( W_{ij}(zt^n) = e \); and for \( j \neq 1 \), \( W_{ij}(zt^n) = V_j(z)W_{ij}(t^n) < W_{ij}(t^n)^{-1}W_{ij}(t^n) = e \) when \( W_{ij}(t^n) \neq 0 \), and \( W_{ij}(zt^n) = 0 < e \) when \( W_{ij}(t^n) = 0 \). Therefore, letting \( t_1 = y_1^n x^{n-1} z \), we have \( W_{ij}(xt_1) = e \) if \( j = 1 \) and \( W_{ij}(xt_1) < e \) if \( j \neq 1 \). Thus, for each \( k = 1, \ldots, r \), there is a \( t_k \) in \( R \) with \( W_{ij}(xt_k) = e \) if \( k = j \) and \( W_{ij}(xt_k) < e \) if \( k \neq j \), so by 1.8

\( W_{ij}(x(\sum_{k=1}^{r} t_k)) = e \) for all \( i, j \); i.e., the inverse property is satisfied.

Let \( a_{ij} \) be in \( G_{W_{ij}} [0] \). By 4.18 there is an \( x \) in \( R \) with \( W_{11}(x) \geq e \) and \( W_{11}(x) < \min [e, a_{11}] \) for \( i = 2, 3, \ldots, n_1 \).

By the torsion property, there is an \( n > 0 \) with \( W_{ij}(x^n) \) in \( G_{V_j} \) and \( (a_{ij})^n \) in \( G_{V_j} [0] \) for all \( i \) and \( j \). By 4.19 there is a \( y \) in \( K \) with \( V_1(y) = e \) and for \( j \neq 1 \),

\[ V_j(y) < \min_{i=1}^{n_j} [W_{ij}(x^n)^{-1}(a_{ij})^n, W_{ij}(x^n)^{-1}] \] \( W_{ij}(x) \neq 0 \]. Thus if \( j \neq 1 \), \( W_{ij}(yx^n) = 0 < a_{ij} \) when \( W_{ij}(x) = 0 \); and when \( W_{ij}(x) \neq 0, W_{ij}(yx^n) = V_j(y)W_{ij}(x^n) < \min_{k=1}^{n_j} (a_{kj})^n \),

\[ W_{kj}(x^n)^{-1}W_{ij}(x^n) \leq \min [(a_{ij})^n, e] \] so that 1) if \( a_{ij} \leq e \),
then \((a_{ij})^n \leq a_{ij} \leq e\) and \(W_{ij}(yx^n) \leq \min [(a_{ij})^n, e] = (a_{ij})^n \leq a_{ij}\); or 2) if \(a_{ij} \geq e\), then \(W_{ij}(yx^n) \leq \min [(a_{ij})^n, e] = e \leq a_{ij}\). Hence if \(j \neq 1\), \(W_{ij}(yx^n) \leq a_{ij}\).

Now we have:

\[ W_{11}(yx^n) = V_1(y)W_{11}(x)^n = W_{11}(x)^n \geq e; \]

for \(i=2, 3, \ldots, n_1\),

\[ W_{i1}(yx^n) = V_1(y)W_{i1}(x)^n = W_{i1}(x)^n \leq W_{i1}(x) \leq \min [e, a_{i1}] \leq a_{i1}; \]

and for \(j \neq 1\),

\[ W_{ij}(yx^n) \leq a_{ij}. \] That is, 4.18 is satisfied.

Thus, \([W_1, \ldots, W_m]\) satisfies the inverse property and 4.18 (and hence 4.19).
REFERENCES


