On two-fold generalizations of Cauchy's lemma

Emma Bravo

The University of Montana

1939

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ON TWO-FOLD GENERALIZATIONS OF CAUCHY'S LEMMA

by

Emma Bravo

B.A., Montana State University, 1933

Presented in partial fulfillment of the requirement for the degree of Master of Arts

Montana State University

1939

Approved:

[Signatures]

Chairman of Board of Examiners

W. E. Bateman

Chairman of Committee on Graduate Study
ACKNOWLEDGMENT

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INTRODUCTION

The problem is to find the conditions on \( a \) and \( b \) under which there exist integral solutions \( > -k \) of the pair of equations

\[
\begin{align*}
(1) \quad a &= \sum_{i \in \mathbb{Z}} c_i x_i^s, \\
(2) \quad b &= \sum_{i \in \mathbb{Z}} c_i x_i
\end{align*}
\]

where \( k \) is an integer and each \( c_i \) is a given positive integer. Necessary conditions for solutions are

\[
(2) \quad a, b \text{ integers, } b \geq t(1-k), \quad a \equiv b \pmod{2}
\]

Cauchy investigated the conditions on \( a \) and \( b \) when each \( c_i = 1 \). In his paper, "Two-fold Generalizations of Cauchy's Lemma," L. E. Dickson treated all the cases in which \( \sum_{i \in \mathbb{Z}} c_i = 8 \). H. Chatland (in his Master's thesis) considered the cases in which \( \sum_{i \in \mathbb{Z}} c_i's = 9 \). In this paper, cases in which \( \sum_{i \in \mathbb{Z}} c_i's = 10 \) are considered.

**THEOREM:** Let \( (c_1, \ldots, c_s) \) be one of the two sets below. Let (2) hold and

\[
(3) \quad ta \geq b^s, \quad (t-1)a < b^s + 2bk + tk^s.
\]

Then there exist integral solutions \( > -k \) of (1) with \( s = 4 \) if the following further conditions hold.

---

(1,2,3,4), 10a - b^4 \neq 4n+3, 8n+2, 4^{k(16n+10)}, and others.
(1,1,1,7), 10a - b^4 \neq 5^{2r}(5n+2), 5^{2r}(5n+3), and others.

The same general method may be applied in all cases. A ternary quadratic form of genus G is regular if and only if it represents every integer represented by every form of G.

The problem of finding the conditions on a and b has been reduced by L. E. Dickson, in the paper mentioned, to one of finding the representations of a ternary quadratic form. He arrives at the identity

\[(4) \ (c_3+c_4)(ta-b^2) = (c_1+c_2)v^2 + c_1(c_3+c_4)(2vd+gd^2) + c_3c_4tw^2 \]

where
\[ g = c_2 + c_3 + c_4 \]
\[ t = c_1 + c_2 + c_3 + c_4 \]
\[ w = x_3 - x_4 \]
\[ d = x_1 - x_2 \]
\[ v = (c_3+c_4)x_3 - c_3x_2 - c_4x_4 \]

We shall use this identity transformed into one involving only squares. We shall then show the x_i's to be integers and find a form equal to (ta-b^2). The representations of this form will give us the further conditions of the theorem.
NOTATIONS

\( v = p^{t+1} \) = any proposed modulus

\( H = \) hessian of the form \( ax^a+by^b+cz^c+2ryz+2xsx+2txy \)

\( \Omega = \) the greatest common divisor of the literal coefficients of the adjoint of the form.

\( H = \Omega^2 \Delta \)

\( \Omega = \Omega' \Omega'', \quad \Delta = \Delta' \Delta'' \) where \( \Omega'' \), \( \Delta'' \) are the highest powers of 2 occurring in \( \Omega \) and \( \Delta \) respectively.

\( p = \) an odd prime factor of \( H \).

\( t = \) the highest power of \( p \) in \( H \).

\( t_1 = \) the highest power of \( p \) in \( \Omega \).

\( t_s = t - 2t_1 \)

fNa denotes: \( f \) represents no integer of the form \( a \).
I. CASE (1, 2, 3, 4)

Take \( c_1 = 1, c_2 = 3, c_3 = 2, c_4 = 4 \). We have by (4)
\[
6(10a-b^2) = 4v^2 + 12vd + 54d^2 + 80w^2
\]
\[
6(10a-b^2) = (2v+3d)^2 + 45d^2 + 80w^2
\]

Set \( \xi = 2v+3d \)
\[
f = 6 \varphi = \xi^2 + 45d^2 + 80w^2
\]

We have
\[
b = x_1 + 3x_3 + 2x_4 + 4x_5
\]
\[
v = 6x_3 - 2x_4 - 4x_5
\]
\[
d = x_1 - x_3
\]
\[
w = x_3 - x_4
\]

Solving these we find
\[
10x_1 = b + v + 9d
\]
\[
10x_3 = b + v - d
\]
\[
30x_1 = 3b - 2v - 3d + 20w
\]
\[
30x_3 = 3b - 2v - 3d - 10w
\]

Since \( a \equiv b \pmod{2}, \xi^2 + 45d^2 \equiv 0 \pmod{2} \), whence \( \xi^2 + d^2 \equiv 0 \pmod{2} \) and \( \xi \equiv d \pmod{2} \). Therefore \( v \) is an integer. From \( 6 \varphi \) we have \( \xi^2 \equiv 6 \varphi \pmod{5} \). \( \xi^2 \equiv -6b^2 \equiv 4b^2 \pmod{5} \). Then \( \xi \equiv \pm 2b \pmod{5} \). By choice of signs \( \xi \equiv -2b \pmod{5} \). Then \( 2v+2b+3d \equiv 0 \pmod{5} \). Therefore \( 2v+2b+18d \equiv 0 \pmod{5} \), i.e., \( 2(v+b+9d) \equiv 0 \pmod{5} \). Therefore \( v+b+9d \equiv 0 \pmod{5} \). Also \( v+b-d \equiv 0 \pmod{5} \) and \( x_1 \) and
The $x_i$ are integers. Set $b+v-d = \mu$. Then $3\mu \equiv m \pmod{5}$. Also $-3\mu + m + 20w = 5(4w-v)$. By $6\varphi$, $80w^2 + 4v^2 \equiv 0 \pmod{6}$. Therefore $4w - v \equiv 0 \pmod{6}$ or $4w + v \equiv 0 \pmod{6}$. Choose the sign of $v$ so that $4w - v \equiv 0 \pmod{6}$. Then $5(4w - v) \equiv 0 \pmod{30}$. Also $3\mu - m + 10w = 5(2w + v)$. Since $4w - v \equiv 0 \pmod{6}$, $v - 4w \equiv v + 2w \equiv 0 \pmod{6}$. Therefore $5(2w + v) \equiv 0 \pmod{30}$ and the $x_i$ are all integers.

We have

\[
6\varphi = 4v^2 + 12vd + 54d^2 + 80w^2
\]
\[
3\varphi = 2v^2 + 6vd + 27d^2 + 40w^2
\]

We may write

\[
v = z - 3x, \quad d = x - y, \quad w = z
\]

Then

\[
3\varphi = 27x^2 + 27y^2 + 42z^2 - 6yz - 6xz - 36xy
\]
\[
\varphi = 9x^2 + 9y^2 + 14z^2 - 2yz - 2xz - 12xy
\]

This form is reduced and of hessian 600. The hessian is $abc + 2rst - ar^2 - bs^2 - ct^2$.

All reduced primitive positive ternary quadratic forms of hessian 600 are given in the table on the following page.
REDUCED PRIMITIVE POSITIVE TERNARY QUADRATIC FORMS OF HESSIAN 600

\[ f = ax^2 + by^2 + cz^2 + 2xyz + 2xzt + 2txy \]

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We seek the representation of

\[ \Phi = 9x^2 + 9y^2 + 14z^2 - 2yz - 2xz - 12xy \]

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The following lemma, (Smith p. 461-2), is useful.

**Lemma**

For any $\mathcal{V}, s$ and its reciprocal form $S$ are equivalent to a pair of forms and satisfying the congruences

$$\Theta \equiv \alpha x^2 + \beta \Omega y^2 + r \Omega \Delta z^2 \pmod{\mathcal{V}}$$

$$\Psi \equiv \beta r \Omega \Delta x^2 + \alpha r \Delta y^2 + \alpha \beta z^2 \pmod{\mathcal{V}}$$

$$\alpha \beta r \equiv 1 \pmod{\mathcal{V}}$$

Here we have

$$\varphi = 9x^2 + 9y^2 + 14z^2 - 2yz - 2xz - 12xy$$

Its adjoint is

$$\mathcal{J} = 125x^2 + 125y^2 + 45z^2 + 30yz + 30xz + 170xy$$

Its reciprocal is

$$\Phi = 25x^2 + 25y^2 + 9z^2 + 6yz + 6xz + 34xy$$

$$\Phi = \mathcal{J}/\Omega$$, where $\Omega = 5$ = the greatest common divisor of the coefficients of $\mathcal{J}$.

Since the representation of a form is not changed under a linear transformation of determinant unity set

$x = -z, y = -y, z = -x$. The forms $\varphi$ and $\Phi$ become

$$\varphi' = 14x^2 + 9y^2 + 9z^2 - 12yz - 2xz - 2xy$$

$$\Phi' = 9x^2 + 25y^2 + 25z^2 + 34yz + 6xz + 6xy$$

Smith's method may now be applied. $H = \Omega \Delta = 600$. $\Omega = 5$, $\Delta = 24$. $\mathcal{V} = p^{t+1} = 3^s = 9$, where $p$ is an odd prime factor of $H$, and $t$ is the highest power of $p$ in $H$. $\mathcal{V}' \equiv \mathcal{V} \Omega^s \Delta = 5400$. $\varphi'$ and $\Phi'$ are properly primitive. $A'' = 13$, is prime to $\mathcal{V}'$ and also $A'' \equiv (\mod 4)$, i. e., $13 \equiv 5(\mod 4)$.
We must select values of $\alpha, \beta, r$ such that $\alpha \beta r \equiv 1 \pmod{\nabla}$, $A'' r \equiv 1 \pmod{\nabla}$, and $\alpha \beta \equiv A'' \pmod{\nabla'}$.

$13r \equiv 1 \pmod{9}$ therefore let $r = 7$. Let $\alpha = 7$. $7\beta \equiv 13 \pmod{5400}$ therefore let $\beta = 3859$. With this choice of $\alpha, \beta, r$ we have $\alpha \beta r \equiv 1 \pmod{9}$ as required in the lemma. Then

$$\phi' \equiv \alpha x^2 + \beta y^2 + r z^2 \pmod{9}$$

$$\bar{\phi}' \equiv \beta r z^2 + \alpha r y^2 + \alpha \beta z^2 \pmod{9}$$

where $\alpha = 7$, $\beta = 3859$, $r = 7$, $A = 24$, $\gamma = 5$. We may now find the representation of $\phi'$ and of $\phi$ (Jones p. 102, Part A). $p$ is an odd prime factor of $H$ and hence is 3.

If $p$ is prime to $\gamma$, it must divide $\Delta$. $\Delta/p^t = \gamma' \neq 0 \pmod{p}$.

$24/3 = 8 \neq 0 \pmod{3}$.

$$\left( -\alpha \beta \gamma / p \right) \left( -135065 / 3 \right) \left( -2 / 3 \right) = 1$$

Therefore there are no progressions involving $p$.

For the progressions involving 2 we find (Jones, pp. 104-105) that if $\gamma = \gamma' \gamma''$ and $\Delta = \Delta' \Delta''$ where $\gamma''$ and $\Delta''$ are the highest powers of 2 dividing $\gamma$ and $\Delta$ respectively, then $\gamma'' = 1$, $\Delta'' = 8$, and $\gamma' = 5$, $\Delta' = 3$. Take $\nabla = 64 = 8 \gamma' \gamma'' \Delta = 38400$. As above we may take $\alpha = 7$, $\beta = 16459$, $r = 5$ to satisfy the following.

$$13r \equiv 1 \pmod{64}$$

$$7\beta \equiv 13 \pmod{38400}$$

$$\alpha \beta r \equiv 1 \pmod{64}$$
Take $\Delta' = \Delta = 21$. \(\beta \Delta' \Omega' = \beta' = 246,885. \quad r \Omega' = r' = 25.\) Therefore $\alpha' + \beta' = 246,906 \equiv 2 (\text{mod } 8)$. Progressions involving 2 are $4n+63$, $8n+18$, $4^k(16n+42)$.

\[ \phi \quad N \quad 4n+3, 8n+2, 4^k(16n+10) \]

If we investigate $f = 6 \phi$ in like manner we find that it represents no integer of the form $4n+3$, $8n+2$, $4^k(16n+10)$.

Upon investigation we find that the form $x^2 + 8y^2 + 77z^2 - 4yz$ of hessian 600 does not represent integers of the form $4n+3$, $8n+2$, $4^k(16n+10)$.

\[ \phi = x^2 + 8y^2 + 77z^2 - 4yz \]

Its adjoint is

\[ J = 600x^2 + 77y^2 + 8z^2 + 4yz \]

Its reciprocal is

\[ \phi^* = 600x^2 + 77y^2 + 8z^2 + 4yz \]

Since the representation of a form is not changed under a linear transformation of determinant unity, set $x = -z$, $y = -y$, $z = -x$. The forms $\phi$ and $\phi^*$ become

\[ \phi' = 77x^2 + 8y^2 + z^2 - 4xy \]
\[ \phi'^* = 8x^2 + 77y^2 + 600z^2 + 4xy \]

\(\Omega = 1\), the greatest common divisor of the adjoint.

Applying Smith's method again, \(H = \phi^* \Delta = 600, \quad \phi^* = 1\), \(\Delta = 600\). \(\nabla = p^{t+1} = 9. \quad \nabla' = \phi^* \Omega^* \Delta = 5400. \quad A^n = 13,\) prime to $\nabla'$ and $\equiv \phi (\text{mod } 4)$, i.e., $13 \equiv 1 (\text{mod } 4)$.

Proceeding as above we seek values of $\alpha$, $\beta$, $\tau$ such that

\[ \alpha \beta \tau \equiv 1 (\text{mod } 9), \quad A^n \tau \equiv 1 (\text{mod } 9) \quad \text{and} \quad \alpha \beta \equiv 13 (\text{mod } 5400). \]
We find $\alpha = 7$, $\beta = 3859$, $r = 7$, $\Delta = 600$. $\Omega = 1$. We may now find representations of $\phi'$ and of $\phi$. (Jones, p. 102, Part A).

$p$ is an odd prime factor of $H$ and hence is $3$. If $p$ is prime to $\Omega$ it must divide $\Delta$. $\Delta/p^t = 600/3 = 200 \equiv 0 \pmod{3}$.

$$(-\alpha\beta\Omega/p) (-27013/3) (-1/3) = 1$$

Therefore there are no progressions involving $p$.

For the progressions involving 2 we find (Jones, pp. 104-105) that if $\Omega = \Omega' \Omega''$ and $\Delta = \Delta' \Delta''$ where $\Omega''$ and $\Delta''$ are the highest powers of 2 dividing $\Omega$ and $\Delta$ respectively. Then $\Omega'' = 1$, $\Delta'' = 8$, and $\Omega' = 1$, $\Delta' = 75$. Take $\nabla = 8 \Omega'' \Delta'' = 64$. $\nabla' = \nabla \Omega' \Delta' = 38400$. Proceeding as before

$$13r \equiv 1 \pmod{64}$$

Let $r = 5$

$$7\beta \equiv 13 \pmod{38400}$$

Therefore $\beta = 16459$, $\alpha = 7$. Take $\alpha \Delta' = \alpha' = 525$. $\beta \Delta' \Omega = \beta' = 1234425$. $r \Omega' = r' = 5$. $\alpha' + \beta' = 1234950 \equiv 6 \pmod{8}$. Then the progressions involving 2 are $4n + 3\alpha \Delta'$, $9n + 2\Delta'$, or $4n+1575$, $8n+150$. Therefore

$\phi \ N \ 4n+3$, $8n+6$.

Also, upon investigating $x^2 + 5y^2 + 120z^2$ we find that it does not represent integers of the form $4n+3$, $8n+2$, $4^k(16n+10)$. Thus it is of the same genus as $9x^2+9y^2+14z^2$.
2yz - 2xz - 12xy. Also it represents 1 and the latter does not. Therefore 9x^2 + 9y^2 + 14z^2 - 2yz - 2xz - 12xy is not regular. Hence there are other integers that are not represented by this form.

II CASE (1,1,1,7)

Take \( c_1 = c_2 = c_3 = 1, c_4 = 7 \). Then by (4) we have

\[
4(10 - b^2) = v^2 + 8vd + 36d^2 + 35w^2
\]

\[= (v+4d)^2 + 20d^2 + 35w^2\]

where \( \xi = v+4d \). By our choice of \( c \)'s we have

\[
b = x_1 + x_2 + x_3 + 7x_4,
\]

\[
v = 8x_2 - x_3 - 7x_4,
\]

\[
d = x_2 - x_3,
\]

\[
w = x_3 - x_4.
\]

Solving these we find

\[
10x_1 = b + v + 9d,
\]

\[
10x_2 = b + v - d,
\]

\[
40x_3 = 4b - v - 4d + 35w,
\]

\[
40x_4 = 4b - v - 4d - 5w.
\]

From 4 \( \varphi \) we have \( \xi^2 \equiv -4b^2 \pmod{5} \), \( \xi^2 \equiv 6b^2 \pmod{5} \) or \( \xi^2 \equiv b^2 \pmod{5} \). Then \( \xi \equiv \pm b \pmod{5} \). Select \( \xi \equiv -b \pmod{5} \). \( v+4d+b \equiv 0 \pmod{5} \). Therefore \( b+v+9d \equiv 0 \pmod{5} \) and \( b+v+9d \equiv b+v-d \equiv 0 \pmod{5} \), so \( x_1 \) and \( x_3 \) are integers.
Set \( \beta + \nu - d = \mu \) and \( 4 \beta - \nu - 4d = m \). \( 4 \mu \equiv m (\text{mod} \ 5) \).

\[ 4 \mu = m + 35w = 5(\nu + 7w) \].

By \( 4 \mu \) we have \( \nu^2 + 35w^2 \equiv 0 (\text{mod} \ 4) \). Therefore \( \nu + 7w \equiv 0 (\text{mod} \ 4) \) and \( \nu - 7w \equiv 0 (\text{mod} \ 4) \). Select \( \nu + 7w \equiv 0 (\text{mod} \ 4) \). \( 5(\nu + 7w) \equiv 0 (\text{mod} \ 20) \). Also \( 4 \mu = m - 5w = 5(\nu - w) \).

Since \( \nu + 7w \equiv 0 (\text{mod} \ 4) \), \( \nu + 7w = \nu - w \equiv 0 (\text{mod} \ 4) \). Therefore \( 5(\nu - w) \equiv 0 (\text{mod} \ 20) \) and the \( x_i \)'s are all integers.

We have

\[ 4 \mu = \nu^2 + 8vd + 36d^2 + 35w^2 \]

We may write \( \nu = 4x+z, \ d = -y, \ w = -z \).

\[ 4 \mu = 16x^2 + 36y^2 + 36z^2 - 8yz + 8xz - 32xy \]

\[ \rho = 4x^2 + 9y^2 + 9z^2 - 2yz + 2xz - 8xy \]

This form is reduced and of hessian 175. The hessian is \( abc + 2rst - ar^2 - bs^2 - ct^2 \).

All reduced primitive positive ternary quadratic forms of hessian 175 are given in the following table.

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We have
\[ \Phi = 4x^2 + 9y^2 + 9z^2 - 2yz + 2xz - 8xy \]
and its reciprocal form
\[ \bar{\Phi} = 16x^2 + 7y^2 + 4z^2 - xz + 7xy \]
These are properly primitive.

We use Smith's lemma. \( \Omega = 5, \Delta = 7, \vartheta = 125, p = 5, t = 2, \vartheta' = \vartheta \Omega^t \Delta = 21875. A'' = 13. \) The values of \( \alpha, \beta, \gamma \) satisfying the congruences of the lemma are 9, 2432, 77. \( p = 5 \) and divides \( \Omega. t_1 = \) the highest power of \( p \) in \( \Omega \). \( t_1 = 1. \) By Jones, p. 103, we have \( \Omega \equiv 0(\mod p^{t_1}), t_1 > 0. 5 \equiv 0(\mod 5). \Omega/p^{t_1} = \Omega' = 1 \not\equiv 0(\mod p). \) We have \( \Delta \equiv 0(\mod p^{t-2t_1}). 7 \equiv 0(\mod 5^0) \) or \( 7 \equiv 0(\mod 1). \Delta/p^{t-2t_1} = \Delta' = 7/1 = 7 \not\equiv 0(\mod 5) \) where \( t-2t_1 \geq 0. \) If \( t = 2t_1 \) the progressions involving \( p \) are \( p^{2r}(pn + \alpha_{-1}) \) which in this case is \( 5^{2r}(5n + \alpha_{-1}). \) If \( t_1 \) is odd and
\[ (-\beta r \Delta / p) = (-\alpha \Delta / p) = -1, \ (1310848 / 5) = (-63 / 5) = (-3 / 5) = -1, \] then \( r = 0, 1, 2, 3, \ldots. \) \( (\alpha_{-1} / p) = -(\alpha / p). \) \( (\alpha_{-1} / 5) = -(\alpha / 5). \) \( (9 / 5) = -(4 / 5) = -1. \) Then \( \alpha_{-1} = 2 \) or 3. Therefore the progressions are \( 5^{2r}(5n+2), 5^{2r}(5n+3). \)

For the progressions involving 2, (Jones, p. 104, Part A) \( H = 175 \) and is odd. Hence we require the "simultaneous character" \( \psi (Smith, p.464). \) It is defined as follows:
If \( \Delta' \) and \( \Omega' \) are the greatest odd divisors of \( \Delta \) and \( \Omega \) respectively, and \( f \) and \( F \) are any odd numbers simultaneously represented by \( \phi \) and \( \Phi \), then

\[
\psi = (-1)^{1/2} (\Delta' f + 1)^{1/2} (\Omega p + 1).
\]

Here \( \Delta' = 7 \), \( \Omega' = 5 \).

Set \( x = 0 \), \( z = 0 \). Then \( f = 9 \), \( F = 7 \), whence \( \psi = 1 \). Then there are no progressions involving \( 2 \).

In this case also upon investigation the form \( x^2 + 5y^2 + 35z^2 \) does not represent integers of the form \( 5^{2r}(5n+2) \) and \( 5^{2r}(5n+3) \). Hence the form \( 4x^2 + 9y^2 + 9z^2 - 2yz + 2xz - 8xy \) is not regular.
BIBLIOGRAPHY


