Concerning peano spaces

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CONCERNING PEANO SPACES

by

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INTRODUCTION

A metric space which is compact, connected and locally connected plays an important role in the field of analysis. Such spaces are called Peano spaces. It is the intent of this thesis to investigate some of the more fundamental aspects of the structure of Peano spaces.

Giuseppe Peano (1858-1932) proved that a segment can be mapped onto a square. Peano's theorem is actually a special case of the first theorem which we shall prove. It is the famous theorem of Hahn and Mazurkiewicz that the closed unit interval \([0,1]\) can be mapped continuously onto a metric space \(T\) if and only if \(T\) is compact connected and locally connected.

Secondly, we shall introduce the concept of cut points and show that a compact connected set of which all but at most two points are cut points is a simple arc. With this and the aid of the Brouwer Reduction Theorem we shall be able to give a proof due to J. L. Kelley of the Arcwise Connectedness Theorem.

Thirdly, we shall find that Sierpinski's property \(S\) enables us to prove that every Peano space has a basis, each non-empty element of which is connected; has property \(S\); and has a Peano space as its closure.

Finally, we will discuss cyclic element theory and leaning heavily on the results of property \(S\), we shall prove the cyclic connectivity theorem which states that a Peano space \(T\) is cyclic if and only if for every two points \(a\) and \(b\) belonging to \(T\) there
is a simple closed curve in $T$ containing $a$ and $b$.

In many of our theorems and definitions, the spaces concerned need not be metric spaces for the statement to hold true. However, since our primary interest is in Peano spaces which are of necessity metric spaces, all spaces shall be considered to be metric unless otherwise indicated.

Furthermore, when sets are discussed without mention of their containing space, it shall be understood that they are subsets of the general metric space $S$. 
CHAPTER I

INTRODUCTORY TOPOLOGY

It is the purpose of this chapter to list definitions and theorems which will be needed in later sections. Proofs will be omitted. In general, Newman's Elements of the Topology of Plane Sets of Points may be used as a reference and in such cases no reference shall be given.

Throughout this paper capital letters shall denote spaces and sets and small letters shall denote points or elements.

As a matter of notation the symbol $\in$ will mean "belongs to" or "is a member of" and $\notin$ will mean "does not belong to" or "is not a member of." For example, if a point $x$ belongs to the set $A$, we will write $x \in A$ and if $y$ does not belong to $A$ we have $y \notin A$. The symbol $\subseteq$ will mean "is contained in" or "is a subset of" and $\not\subseteq$ will mean "is not contained in" or "is not a subset of." That is, if the set $A$ is a subset of the set $B$ we have $A \subseteq B$ and if $A$ is not a subset of $B$ we have $A \not\subseteq B$. A set $A$ is termed a proper subset of $B$ if $A \subseteq B$ but $A \neq B$.

1.1 A set is a collection or aggregate of objects or elements called points. Suppose $E$ is a subset of $S$. The complement $\mathcal{C}E$ of $E$ is the set of all points of $S$ not belonging to $E$. It is always true that $\mathcal{C}(\mathcal{C}E) = E$, and if $E \subseteq F$ then $\mathcal{C}F \subseteq \mathcal{C}E$. The empty or null set, to be denoted by $\emptyset$, is defined as being that set which has no points. A set is termed non-degenerate if it has at least two points.
1.2 On any set a metric may be defined by associating with every pair of points \( x \) and \( y \) a non-negative number \( \rho(x,y) \) called the distance between them, such that the following conditions are satisfied:

(i) \( \rho(x,y) = \rho(y,x) \)

(ii) \( \rho(x,y) = 0 \) if and only if \( x = y \)

(iii) \( \rho(x,z) \leq \rho(x,y) + \rho(y,z) \) (triangle inequality)

A non-empty set on which a metric has been defined is called a metric space.

1.3 Suppose \( A \subseteq S \). The diameter \( d(A) \) of \( A \) is defined as being the least upper bound of \( \rho(p,q) \) where the least upper bound is taken with respect to all pairs of points \( p \) and \( q \) in \( A \). If \( d(A) \) is finite, \( A \) is bounded. If \( A \) consists of a single point, \( d(A) = 0. \)

1.4 Suppose \( A \) and \( B \) are sets. The union of \( A \) and \( B \), written \( A \cup B \) and read "\( A \ union \ B \)", is the set of all points which belong to either \( A \) or \( B \). The intersection of \( A \) and \( B \), written \( A \cap B \) and read "\( A \ intersection \ B \)", is the set of all points which belong to both \( A \) and \( B \). If the sets \( A \) and \( B \) have no points in common, they are said to be disjoint and we write \( A \cap B = \emptyset \). As a matter of abbreviation, we shall often use \( AB \) in place of \( A \cap B \) and \( A-B \) in place of \( A \setminus B \).

Some identical relations involving \( \cup \) and \( \cap \) are:

(1) \( A \cup (B \cup C) = (A \cup B) \cup C \), \( A \cap (B \cap C) = (A \cap B) \cap C \)

(2) \( A \cup B = B \cup A \), \( A \cap B = B \cap A \)

(3) \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)

(4) \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)
(5) \[ A \cup A = A, \quad A \cap A = A \]

(6) \[ AC \cup B, \quad A \cap B \cap A \]

(7) If \( AC \subseteq C \) and \( BC \subseteq C \) then \( (A \cup B) \subseteq C \)

(8) If \( CC \subseteq A \) and \( CC \subseteq B \) then \( C \subseteq (A \cap B) \)

(9) If \( \mathcal{E} \) is any collection of sets \( E \) then \( \mathcal{C}(\bigcup E) = \bigcap_{E \in \mathcal{E}} (CE) \) and \( \mathcal{C}(\bigcap E) = \bigcup_{E \in \mathcal{E}} (CE) \).

1.5 Suppose \( p \) is a point and \( \varepsilon \) is a positive number. By an \( \varepsilon \)-neighborhood (or merely neighborhood) of the point \( p \), we mean the set of all points whose distance from \( p \) is less than \( \varepsilon \), and we write \( N(p, \varepsilon) \). Similarly, an \( \varepsilon \)-neighborhood \( N(E, \varepsilon) \) of a set \( E \) denotes the set of all points whose distance from \( E \) is less than \( \varepsilon \).

1.6 A subset \( A \) of \( S \) is said to be open if for every point \( p \) of \( A \) there is a neighborhood of \( p \) contained in \( A \). The following statements hold.

(1) The empty set is open and \( S \) itself is open.

(2) Any neighborhood is an open set.

(3) Any union of open sets is open.

(4) The intersection of a finite number of open sets is open.

1.7 A sequence \( x_1, x_2, \ldots, x_n, \ldots \) or \( \{x_n\} \) of points of a space is determined by assigning a point \( x_n \) to every positive integer \( n \). Thus, it should be noted that a sequence is a set of points indexed by the positive integers and is not merely any set of points. A sequence \( \{x_n\} \) is said to converge to a point \( a \), and we write \( x_n \to a \), if for every \( \varepsilon > 0 \), \( \rho(x_n, a) < \varepsilon \) for all but a finite number of values of \( n \). Equivalent symbolism for convergence is

\[ \lim_{n \to \infty} \rho(x_n, a) = 0. \]
A sequence can converge to only one point. As a consequence of this if \( x_n \rightarrow a, y_n \rightarrow b, \) and \( x_n = y_n \) for all \( n \) then \( a = b. \)

A sequence \( \{A_n\} \) of sets is said to be monotone decreasing if and only if \( A_{n+1} \subseteq A_n \) for each \( n \). The sequence \( \{A_n\} \) is monotone increasing if and only if \( A_n \supseteq A_{n+1} \) for each \( n \). (The analogy with sequences of points is obvious.)

1.8 The sequence \( \{y_n\} \) is a subsequence of \( \{x_n\} \) if \( y_n = x_{r_n} \) where \( r_n \) are positive integers such that \( r_1 < r_2 < r_3 < \ldots \). If \( x_n \rightarrow a \), then every subsequence of \( \{x_n\} \) converges to \( a. \)

1.9 In terms of neighborhoods, a point \( q \) is a limit point of a set \( B \) if every neighborhood of \( q \) contains an infinite number of points of \( B \). In fact, it can be shown that \( q \) is a limit point of \( B \) if and only if every neighborhood of \( q \) contains a point \( p \) of \( B \) distinct from \( q. \)

In terms of sequences, \( q \) is a limit point of \( B \) if and only if there is a sequence \( \{x_n\} \) of distinct points of \( B \) such that \( x_n \rightarrow q. \)

1.10 A set \( B \) is said to be closed if and only if its complement \( S-B \) is open. The following statements hold.

(1) The empty set is closed and \( S \) itself is closed.
(2) A set is closed if it contains all of its limit points.
(3) Any intersection of closed sets is closed.
(4) The union of a finite number of closed sets is closed.
(5) Every finite set is closed.

1.11 The closure \( \overline{E} \) of a set \( E \) is the smallest closed set containing \( E \). More explicitly, \( \overline{E} \) is the intersection of all closed sets containing \( E \). The following statements hold.

(1) \( \overline{E} \) is closed.
(2) \( E = \overline{E} \) if and only if \( E \) is closed.

(3) Always \( \overline{E} = \overline{E} \).

(4) If a single point \( p \) is considered as a set then \( \overline{p} = p \).

(5) If a point \( x \in \overline{E} \) then any neighborhood of \( x \) contains a point of \( E \). Thus, for every \( \alpha > 0 \) there is a point \( y \) of \( E \) such that \( \rho(x, y) < \alpha \).

(6) If \( E \) is bounded, \( \overline{E} \) is bounded and \( d(E) = d(\overline{E}) \).

(7) If sets \( E_1 \) and \( E_2 \) are such that \( E_1 \subseteq E_2 \) then \( \overline{E_1} \subseteq \overline{E_2} \).

(8) If \( E_1, \ldots, E_k \) is a finite collection of sets then

\[
\overline{\bigcup_{i=1}^{k} E_i} = \bigcup_{i=1}^{k} \overline{E_i}.
\]

(9) For any collection of sets \( E_1, \ldots, E_n, \ldots \),

\[
\bigcap_{n=1}^{\infty} E_n \supseteq \bigcap_{n=1}^{\infty} \overline{E_n}.
\]

An important fact concerning metric spaces is that given any point \( p \) of a metric space \( S \) and any open set \( H \) of \( S \) such that \( p \in H \), then there exists an open set \( G \) containing \( p \) and \( p \in \overline{G} \subset H \).

1.12 The interior \( i(E) \) of a set \( E \) is the largest open set contained in \( E \). More explicitly, \( i(E) \) is the union of all open sets contained in \( E \). It follows that if \( E \) is open then \( i(E) = E \).

Any point belonging to \( i(E) \) is an interior point of \( E \). The frontier \( fr(E) \) of \( E \) is by definition \( \overline{E} - i(E) \), and if \( E \) is open \( fr(E) = \overline{E} - E \).

The following statements hold.

(1) \( fr(\emptyset) = \emptyset \) and \( fr(S) = \emptyset \).

(2) Always \( E \cup fr(E) = \overline{E} \).

1.13 We also have the condition of a set \( A \) being open or closed in relation to another set \( M \) containing \( A \). That is, suppose \( A \subseteq M \subseteq S \). We say that \( A \) is open in \( M \) if and only if there is an open set \( G \) such that \( A = M \cap G \). Similarly, \( A \) is closed in \( M \) if and
only if there is a closed set $F$ such that $A = M \cap F$. The following statements hold.

(1) If $A$ is open (closed) then $A$ is open (closed) in $M$.

(2) If $M$ is open (closed) and $A$ is open (closed) in $M$ then $A$ is open (closed).

(3) $A$ is closed in $M$ if $A = M \cap \overline{A}$.

1.14 Given sets $S$ and $T$, a rule $f$ is called a mapping (transformation) of $S$ into $T$ and we write $f:S \rightarrow T$, if with every point $a \in S$ we associate a point $f(a) \in T$. $f(a)$ is called the image of the point $a$ under the mapping $f$. A mapping $f$ is said to map $S$ onto $T$ if $f(S) = T$. In mathematics it is grammatically correct to state that a mapping $f:S \rightarrow T$ is onto. Notice if $f$ is an onto mapping a point $b \in T$ if and only if there is at least one point $a \in S$ such that $f(a) = b$. It is easily verified that every onto mapping is itself an into mapping though the converse does not necessarily hold.

1.15 A mapping $f:S \rightarrow T$ is said to be a one-to-one mapping of $S$ onto $T$ (or a one-to-one correspondence between $S$ and $T$) if and only if each point $b$ of $T$ is the image of precisely one point $f^{-1}(b)$ of $S$. The point $f^{-1}(b)$ is called the inverse image of $b$. The relation is symmetrical and $f^{-1}$ is a one-to-one mapping of $T$ onto $S$.

1.16 Suppose there is a mapping $f:S \rightarrow T$. Let $F$ be any collection of subsets $E$ of $S$ and let $H$ be any collection of subsets $G$ of $T$. The following statements hold.

(1) $f(\emptyset) = \emptyset$ and $f^{-1}(\emptyset) = \emptyset$.

(2) If $E_1 \subseteq E_2$ then $f(E_1) \subseteq f(E_2)$.
(3) If $G_1 \subseteq G_2$ then $f^{-1}(G_1) \subseteq f^{-1}(G_2)$.

(4) Always $E_1 \subseteq f^{-1}(E_1)$ but $E_1 = f^{-1}(E_1)$ if and only if $f$ is one-to-one.

(5) Always $f^{-1}(G_1) \subseteq G_1$ but $f^{-1}(G_1) = G_1$ if and only if $f$ is onto.

(6) $Cx(E_1) \subseteq f(GE_1)$ if $f$ is onto but $Cx(E_1) = f(GE_1)$ if and only if $f$ is onto and one-to-one.

(7) Always $f^{-1}(CG_1) = Cx^{-1}(G_1)$.

(8) Always $f(\bigcup E) = \bigcup f(E)$ and $f(\bigcap E) \subseteq \bigcap f(E)$

(9) Always $f^{-1}(\bigcup G) = \bigcup f^{-1}(G)$ and $f^{-1}(\bigcap G) \supseteq \bigcap f^{-1}(G)$.

1.17 A set which is either finite or can be placed in a one-to-one correspondence with the set of all positive integers is said to be enumerable (countable). Every subset of a countable set is countable.

1.18 Suppose $\mathcal{A}$ is a collection of open sets in the space $S$. $\mathcal{A}$ is said to form a base or basis for $S$ if every open set contained in $S$ can be expressed as the union of some of the sets in the collection.

We shall also speak of a basis at a point. Suppose $p$ is a point of the space $S$ and let $\mathcal{U}$ be a collection of open sets in $S$ each of which contains $p$. We shall say that $\mathcal{U}$ is a base at the point $p$ if and only if given any open set $E$ of $S$ such that $p \in E$, then there exists a set $F \subset \mathcal{U}$ such that $F \subseteq E$. It can be easily verified that in metric spaces every point $p$ has a countable basis at $p$.

1.19 A set $A$ is dense in a set $B$ if $A \subseteq B$ and every neighborhood containing a point of $B$ also contains a point of $A$. If $A$
is dense in $S$ where $S$ is the whole space, then $\overline{A} = S$. A set is **separable** if it has an enumerable dense subset. It can be easily shown that a metric space is separable if and only if it has an enumerable base.

1.20 **By a covering of a subset $E$ of $S$ we mean a collection of sets $\mathcal{B}$ contained in $S$ such that $E \subseteq \bigcup_{F \in \mathcal{B}} F$. Thus, if $\mathcal{B}$ covers the whole space $S$ then $\bigcup_{F \in \mathcal{B}} F = S$. If the sets of a covering are open, closed or finite in number, the covering is termed open, closed or finite respectively. It is an $\epsilon$-covering if each of the sets of $\mathcal{B}$ has diameter not exceeding $\epsilon$. If $\mathcal{B}^*$ is a subfamily of $\mathcal{B}$ and also covers $E$ then $\mathcal{B}^*$ is called a subcovering of $E$.**

1.21 A set $H$ is **compact** if and only if from every open covering of $H$ a finite subcovering can be selected. It is easily verified that any closed interval $[a,b]$ is compact. Thus, in particular the closed unit interval $[0,1]$ is compact. The open interval $(0,1)$ is not compact, however, since no finite subcovering can be selected from the covering $\mathcal{B} = \bigcup_{n=1}^{\infty} \left( \frac{1}{n}, 1 \right)$. The following statements hold.

1. A set $H$ is compact if every sequence in $H$ has a subsequence which converges to a point of $H$.

2. If $H$ is compact then $H$ is closed.

3. If $H$ is a closed subset of a compact space $S$ then $H$ is compact.

4. All compact sets are separable, thus, in view of paragraph 1.19 every compact set has an enumerable dense base.

5. A decreasing sequence of non-empty compact sets has a non-empty common part.
A subset $H$ of $S$ admits of a separation if there exists sets $A$ and $B$ in $S$ such that $H = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. If this is the case, we write $H = A/B$. $H$ is termed disconnected if it admits of a separation. Otherwise, $H$ is connected. For example, suppose $H$ is the set of all non-zero real numbers. By letting $A$ be the set of all positive reals and $B$ be the set of all negative reals it is readily seen that $H = A/B$.

The following statements hold.

1. If a set $E$ consists of a single point then $E$ is connected.

2. Every interval is connected and in fact the set of all real numbers $\mathbb{R}$ is connected.

3. Suppose $E$ is a connected subset of $\mathbb{R}$. If points $a$ and $b$ belong to $E$ then $[a, b] \subseteq E$.

4. The set $H$ is disconnected if and only if it is the union of two non-empty disjoint sets, each of which is open in $H$. In particular, $S$ is disconnected if and only if it is the union of two non-empty disjoint open sets (open may be replaced by closed throughout).

5. If $G_1$ and $G_2$ are non-empty disjoint open sets, then $G_1 \cup G_2 = G_1/G_2$.

6. Suppose $E \subseteq S$ and $E = C/D$. If $E$ is closed, then $C$ and $D$ are closed.

7. If $H = A/B$ and there is a connected set $E$ contained in $H$ then either $E \subseteq A$ or $E \subseteq B$. Suppose $G$ and $H$ are subsets of $S$, $G$ and $H$ are closed, $G \cup H$ is connected and $G \cap H$ contains at most two points. Then $G$ is connected or $H$ is connected.
(8) If $A$ is connected and $\mathcal{H}$ is any collection of connected sets such that if $B \in \mathcal{H}$ implies $A \cap B \neq \emptyset$ then $A \cup \bigcup_{B \in \mathcal{H}} B$ is connected.

(9) If $E$ is connected and $E \subset H \subset \overline{E}$, then $H$ is connected. Thus, we see that if $E$ is connected then $\overline{E}$ is connected.

(10) If $E$ is a non-empty subset of a space $S$ and $x$ is a point of $S$ such that $E \cup \{x\}$ is connected, then $x \in \overline{E}$.

(11) If $S$ is connected and has more than one point, then $S$ is non-countable.

1.23 A compact connected set with at least two points is called a continuum. It is clear from part (11) of 1.22 that a continuum is non-countable.

1.24 A set $G$ is a component of a set $E$ if and only if the following conditions hold: (i) $G \neq \emptyset$, (ii) $G \subset E$, (iii) $G$ is connected, and (iv) if $G \subset F \subset E$ and $F$ is connected, then $G = F$. A set $E$ is totally disconnected if all of its components are single points. The following statements hold.

(1) Any set is the union of its components.

(2) If $G$ and $H$ are components of $E$, then either $G = H$ or $G \cap H = \emptyset$.

(3) If $G$ is a component of $E$ and $B$ is any connected subset of $E$ then either $B \subset G$ or $B \cap G = \emptyset$.

(4) If $E$ is connected and $E \neq \emptyset$ then there is only one component of $E$, namely, $E$ itself.

(5) If a set $E$ is disconnected then there are at least two components of $E$.

(6) If $E$ is closed and $A$ is a component of $E$ then $A$ is closed.
(7) If \( p \) is a point of a set \( E \) then there is exactly one component of \( E \) containing \( p \).

1.25 A set \( H \) is said to be **locally connected** if and only if components of open sets (open in \( H ! \)) are open in \( H \). The following statements hold.

1. If all neighborhoods of a set \( H \) are connected then \( H \) is locally connected.

2. A set \( H \) is locally connected if and only if for every point \( a \in H \) and for every positive number \( \varepsilon \), there is a positive number \( \delta \) such that if a point \( b \in N(a, \delta) \subseteq H \) then there exists a connected set \( C \) such that \( a \in C \), \( b \in C \) and \( C \subseteq N(a, \varepsilon) \subseteq H \).

3. If for every point \( p \) of a set \( H \) and every positive number \( \varepsilon \) there is a positive number \( \delta \leq \varepsilon \) and \( N(p, \delta) \subseteq H \) and is connected then \( H \) is locally connected.

4. A set \( H \) is locally connected if and only if the existence of an open set \( G \) (open in \( H \)) contained in \( H \) and containing a point \( p \) implies there is a subset \( E \) of \( G \) such that \( E \) is open in \( G \), \( E \) is connected, and \( p \in E \).

5. Any open set is a locally connected space is itself locally connected.

1.26 A **Peano space** is a compact, connected, locally connected metric space. As an example, any closed interval \([a, b]\) in \( \mathbb{R} \) is a Peano space. We have seen that \([a, b]\) is both compact and connected. Suppose \( G \) is an open set contained in \([a, b]\) and containing a point \( p \). Thus, there is a \( \delta > 0 \) such that \( N(p, \delta) \subseteq G \). \( N(p, \delta) \) is open and being an interval it is connected. Certainly \( p \in N(p, \delta) \) hence by part (4) of 1.25, \([a, b]\) is locally connected.
To define a metric on \([a, b]\) we use the standard formula for distance in \(\mathbb{R}_1\), \(\rho(x, y) = |x-y|\). Therefore, \([a, b]\) and, in particular, \([0, 1]\) are Peano spaces.

Some other simple examples of Peano spaces are: a single point, the boundary of a rectangle or circle, the boundary of a rectangle or circle plus all of their interior points.

1.27 A mapping \(f: S \to T\) is said to be continuous at a point \(a\) of \(S\) if for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that if \(x \in N(a, \delta)\) in \(S\) then \(f(x) \in N(f(a), \varepsilon)\) in \(T\), i.e., \(f(N(a, \delta)) \subseteq N(f(a), \varepsilon)\). The mapping \(f\) is continuous on \(S\) if it is continuous at every point of \(S\). The following statements hold:

(1) A mapping \(f: S \to T\) is continuous at a point \(p\) if and only if for every open set \(G \subseteq T\) such that \(f(p) \in G\), there is an open set \(H \subseteq S\) such that \(p \in H\) and \(f(H) \subseteq G\).

(2) A mapping \(f: S \to T\) is continuous on \(S\) if and only if for every open set \(G \subseteq T\), \(f^{-1}(G)\) is an open subset of \(S\).

(3) A mapping \(f: S \to T\) is continuous on \(S\) if and only if for every closed set \(H \subseteq T\), \(f^{-1}(H)\) is a closed subset of \(S\).

(4) Suppose a mapping \(f: S \to T\) is continuous at a point \(p \in S\) and there is a sequence \(\{x_n\}\) of \(S\) such that \(x_n \to p\), then \(f(x_n) \to f(p)\).

(5) Suppose there is a mapping \(f: S \to T\) and a point \(p \in S\). If every sequence \(\{x_n\}\) of \(S\) is such that \(x_n \to p\) implies \(f(x_n) \to f(p)\) then \(f\) is continuous at \(p\).

(6) Suppose a mapping \(f: S \to T\) is continuous on \(S\). If \(E\) is compact subset of \(S\) then \(f(E)\) is compact.

(7) Suppose a mapping \(f: S \to T\) is continuous on \(S\) and \(S\) is compact. If \(E\) is a closed subset of \(S\) then \(f(E)\) is closed.
(8) Suppose a mapping \( f : S \to T \) is continuous on \( S \). If \( E \) is a connected subset of \( S \) then \( f(E) \) is connected.

(9) Suppose there is a continuous mapping \( f \) of \( S \) onto \( T \). If \( S \) is compact and locally connected then \( T \) is compact and locally connected.

Thus, we see that if \( f \) is a continuous mapping of a Peano space \( S \) onto a metric space \( T \), then \( T \) is a Peano space.

1.28 Let \( f \) be a one-to-one mapping of \( S \) onto \( T \). Recall that for any point \( p \) of \( T \) there is exactly one point \( q \) of \( S \) such that \( f(q) = p \). Let \( g(p) = q \). This defines a mapping \( g : T \to S \) such that \( f(g(p)) = f(q) = p \). The mapping \( g \) is called the inverse mapping of \( f \). It is easily verified that \( g \) is one-to-one and maps \( T \) onto \( S \). The following statements hold:

(1) If \( f : S \to T \) is a one-to-one continuous mapping of \( S \) onto \( T \) then there exists \( g : T \to S \) which is a one-to-one mapping of \( T \) onto \( S \).

(2) If \( f : S \to T \) is a one-to-one continuous mapping of \( S \) onto \( T \) and \( S \) is compact then the inverse mapping \( g \) is continuous on \( T \).

If \( f : S \to T \) is a one-to-one continuous mapping of \( S \) onto \( T \) and if the inverse mapping \( g \), in addition to being one-to-one and onto, is continuous on \( T \) then \( f \) is called a topological mapping of \( S \) onto \( T \). The reader may easily verify that if \( f \) is a topological mapping of \( S \) onto \( T \) then open sets in \( S \) correspond to open sets in \( T \) and vice versa (open may be replaced by closed throughout).
CHAPTER II

THE HAHN-MAZURKIEWICZ THEOREM

We have seen in section 1.27 that if a continuous mapping $f$ maps a Peano space $S$ onto a metric space $T$, then $T$ is a Peano space. Indeed, this proves the necessary part of the Hahn-Mazurkiewicz Theorem that a space $S$ is a Peano space if and only if the closed unit interval $[0,1]$ can be mapped continuously onto $S$. In this chapter, we shall prove the sufficiency part of the theorem after suitable introduction.

Definition 2.1 A chain of sets is a finite succession of sets $E_1, E_2, \ldots, E_k$ such that $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots, k-1$.

Theorem 2.2 Suppose $Q$ is a connected set and $F$ is the collection of sets $F_1, F_2, \ldots, F_n$ which form a finite closed covering of $Q$ such that $Q \cap F_i \neq \emptyset$ for each $i$. Then any two sets $F_j$ and $F_k$ of $F$ can be joined by a chain of sets in $F$.

Proof. Let $F_k$ be an arbitrary set of the collection $F$. Let $H_1$ be the union of those sets which are joined to $F_k$ by a chain of sets and let $H_2$ be the union of all the rest of the sets in $F$. Clearly $H_1 \cap H_2 = \emptyset$ and since $F_k \subseteq H_1$, $H_1 \neq \emptyset$. $H_1$ and $H_2$ are each a finite union of closed sets, hence are both closed. Now $Q \subseteq H_1 \cup H_2$, thus $Q = Q \setminus (H_1 \cup H_2) = (Q \cap H_1) \cup (Q \cap H_2)$. Since $Q$ is connected, we cannot have $Q = Q \cap H_1 / Q \cap H_2$, but notice:
\[(Q \cap H_1) \cap (Q \cap H_2) \subset Q \cap H_1 \cap Q \cap H_2 \]
\[\subset H_1 \cap H_2 = H_1 \cap H_2 = \emptyset.\]

Similarly, \((Q \cap H_1) \cap (Q \cap H_2) = \emptyset\) and
\[(Q \cap H_1) \cap (Q \cap H_2) = Q \cap H_1 \cap H_2 = \emptyset\]
and as we have seen \(H_1 \cap Q \neq \emptyset\). It follows that we must have \(H_2 \cap Q = \emptyset\) in order to preserve the connectedness of \(Q\). Suppose \(H_2 \neq \emptyset\), i.e., there is some set \(F_m\) of \(F\) such that \(F_m \subset H_2\). By hypothesis \(F_m \cap Q \neq \emptyset\), hence \(H_2 \cap Q \neq \emptyset\) which is impossible. Therefore, the assumption that \(H_2 \neq \emptyset\) led to a contradiction, hence \(H_2 = \emptyset\), i.e., every set \(F_1\) of \(F\) may be joined to \(F_k\) by a chain of sets in \(F\). Q.E.D.

Corollary. Suppose \(Q\) is a connected set and \(F\) is the collection of sets \(F_1, F_2, \ldots, F_n\) which form a finite closed covering of \(Q\) such that \(Q \cap F_i \neq \emptyset\) for each \(i\). Given any two sets \(F_j\) and \(F_k\) there is a chain of sets beginning with \(F_j\) and ending with \(F_k\) including all of the sets of \(F\). (Repetitions are allowed.)

Proof. Let the sets of \(F\) be arranged so that
\[F_j = F_1, F_2, \ldots, F_n = F_k.\]
By theorem 2.2, \(F_r\) and \(F_{r+1}\) can be joined by a chain of sets for each \(r\). Insert these chains between each of the original consecutive pairs and we have the desired chain. Q.E.D.

Theorem 2.3 If \(S\) is a compact locally connected space and \(\varepsilon > 0\), then there is a finite \(\varepsilon\)-covering of \(S\) by compact connected sets.
Proof. Consider all \( \frac{1}{2} \epsilon \)-neighborhoods of all points \( p \) of \( S \). Each neighborhood is an open subset of the locally connected space \( S \), hence their components form an open \( \epsilon \)-covering of \( S \). Due to the compactness of \( S \) we may select a finite number of these components \( C_1, C_2, \ldots, C_k \) which also form a \( \epsilon \)-covering of \( S \). Their closures \( \overline{C}_1, \overline{C}_2, \ldots, \overline{C}_k \) are connected, have diameters less than \( \epsilon \) (see 1.11), and are closed subsets of \( S \), hence are compact. Thus, the required covering is obtained. Q.E.D.

Theorem 2.4 (Hahn-Mazurkiewicz Theorem) The closed unit interval \([0,1]\) can be mapped continuously onto a metric space \( S \) if and only if \( S \) is compact, connected, and locally connected.

Proof. The necessary part follows immediately from section 1.27 and the fact that \([0,1]\) is a Peano space.

To prove the converse suppose \( S \) is a compact, connected, locally connected, metric space. Let \( a \) and \( b \) be any point of \( S \). Without loss of generality assume \( d(s) \leq 1 \). We shall define a series of integers \( N_m \) and a series of \( 2^{-m} \)-coverings of \( S \) by chains of connected compact sets \( K^1_m \) for \( i = 1, 2, \ldots, N_m \). Such coverings and chains exist by virtue of theorem 2.5 and the corollary to theorem 2.2. Recall that since \( S \) is the whole space any covering \( \mathcal{A} \) of \( S \) is such that \( \bigcup_{F \in \mathcal{A}} F = S \).

For \( m = 0 \) we set \( N_0 = 1 \) and \( K^1_0 = S \). For \( m = 1 \) we have the \( \frac{1}{2} \)-covering by the chain \( K^1_1, K^2_1, \ldots, K^{N_1}_1 \). Let \( a \in K^1_1 \) and \( b \in K^{N_1}_1 \).

Suppose we have completed the definition up to and including \( m \) with \( a \in K^1_m \) and \( b \in K^{N_m}_m \). Choose a point \( a_i \) in \( K^i_m \cap K^{i+1}_m \) for

\[ i = 1, 2, \ldots, N_m - 1, \]

letting \( a = a_0 \) and \( b = a_{N_m} \).
There is also a finite $2^{-(m+1)}$-covering of $S$ by connected compact sets $F_1, F_2, \ldots, F_k$. Considering all $i$, $r$ and $m$ retain the sets $F_r \cap K^i_m$ which are non-empty. Clearly, these sets are finite in number, closed and also cover $S$, hence cover $K^i_m$. Therefore, for each $m$ the sets $F_r \cap K^i_m$ can be arranged (with repetitions) as a chain of sets beginning with a set containing $a_{i-1}$ and ending with one containing $a_i$. Suppose the greatest number of sets in any chain is $n_m$. By repeating the last set of a chain a sufficient number of times, we may assume all chains to contain exactly $n_m$ sets. We now have chains of sets associated with $K^1_m, K^2_m, \ldots, K^{N_m}_m$, placed consecutively in this order. The union of these chains of course covers $S$, hence may be arranged to form a single chain of connected closed sets. We will number the sets of this single chain as $K^i_{m+1}$ where $i = 1, 2, \ldots, n_m N_m$ and $n_m N_m = N_{m+1}$. This completes our definition of $K^i_m$.

If integers $j$ and $i$ are such that $0 < j \leq N_m$ and

\begin{equation}
(1) \quad n_m(j-1) < i \leq n_m j
\end{equation}

then clearly $K^j_m \cap K^i_{m+1} \neq \emptyset$ and since $d(K^i_{m+1}) \leq 2^{-(m+1)}$ it follows that $K^i_{m+1} \subseteq N(K^j_m, 2^{-(m+1)})$. Thus, letting $A^i_m$ denote $N(K^i_m, 2^{-m})$ we have $A^i_{m+1} \subseteq A^j_m$ whenever $j$ and $i$ satisfy (1).

For every number $0 \leq \xi < 1$ there must be an integer $j$ such that $0 < j \leq N_m$ and

\begin{equation}
(2) \quad \frac{(j-1)}{N_m} \leq \xi < \frac{j}{N_m}
\end{equation}

Let $A^j_m(\xi)$ denote the set $A^j_m$ for this particular value of $j$. $A^j_m(1)$ will be $A^{N_m}_m$ by definition. Suppose for this same $\xi$, $A^j_{m+1}(\xi) = A_{m+1}$, i.e.,
(3) \[ \frac{(i-1)}{N_{m+1}} \leq \xi < \frac{i}{N_{m+1}} \]

Since \( N_{m+1} = n_m n_m \), we then have

\[ \frac{(i-1)}{N_{m+1}} \leq \xi < \frac{i}{N_{m+1}} = \frac{n_m j}{N_{m+1}} \]

and

\[ \frac{n_m (j-1)}{N_{m+1}} = \frac{(j-1)}{N_m} \leq \xi < \frac{i}{N_{m+1}} \]

hence, \( i-1 < n_m j \), i.e., \( i \leq n_m j \) and \( n_m (j-1) < i \), i.e., \( n_m (j-1) \leq i-1 \).

It now follows from (2) and (3) that

\[ \frac{(j-1)}{N_m} \leq \frac{(i-1)}{N_{m+1}} \leq \xi < \frac{i}{N_{m+1}} \leq \frac{i}{N_m} \]

Multiplying through by \( N_{m+1} \), we notice that both \( i \) and \( j \) satisfy the relations (1). Thus, \( A_{m+1}(\xi) \subseteq A_m(\xi) \) if \( 0 \leq \xi < 1 \), and if \( \xi = 1 \) we also have \( A_{m+1}(1) \subseteq A_m(1) \).

For any \( \xi \) consider the sets \( A_m(\xi) \), \( m = 1, 2, \ldots \). These sets are closed, hence compact and since \( \{A_{m+1}(\xi) \subseteq A_m(\xi) \} \) for all \( m \) they all have at least one point in common. However, since \( d(A_m(\xi)) \to 0 \) as \( m \to \infty \) they have exactly one point in common. We shall define this point as \( f(\xi) \), i.e., \( f(\xi) = \bigcap_{m=1}^{\infty} A_m(\xi) \). Similarly, since \( a \in A_m^0 = A_m(0) \) and \( b \in A_m^1 = A_m(1) \), we have \( f(0) = \bigcap_{m=1}^{\infty} A_m(0) = a \) and \( f(1) = \bigcap_{m=1}^{\infty} A_m(1) = b \) respectively. Thus, we have defined a mapping \( f: [0, 1] \to S \). To complete the proof we will show that \( f \) maps \( [0, 1] \) continuously onto \( S \).

For every \( \varepsilon > 0 \) there is an integer \( m \) such that \( \varepsilon > \frac{\varepsilon}{2^m} > 0 \).

Supposing \( |\xi - \gamma| < \frac{1}{2^m} \) we have two cases to consider:

1. If \( \xi \) and \( \gamma \) belong to the same set \( A_m^j \), then \( A_m(\xi) = A_m(\gamma) = A_m^j \) and \( f(\xi) \) and \( f(\gamma) \) both belong to \( A_m(\xi) = A_m(\gamma) \).

   Therefore, \( \rho(f(\xi), f(\gamma)) \leq \frac{\varepsilon}{2^m} < \varepsilon \).
(ii) Suppose \( \xi \) and \( \eta \) belong to consecutive sets. Without loss of generality assume \( \xi < \eta \). Therefore, \( f(\xi) \in A_m(\xi) = A_m^j \) and \( f(\eta) \in A_m(\eta) = A_m^{j+1} \). Since \( K_m^i \cap K_m^{i+1} \neq \emptyset \), \( A_m^j \cap A_m^{j+1} \neq \emptyset \). Suppose \( q \in A_m^j \cap A_m^{j+1} \). Then \( \rho(f(\xi),q) \leq \frac{3}{2m}, \rho(q,f(\eta)) \leq \frac{2}{2m}, \) and by the triangle inequality \( \rho(f(\xi),f(\eta)) \leq \rho(f(\xi),q) + \rho(q,f(\eta)) \leq \frac{3}{2m} + \frac{2}{2m} = \frac{6}{2m} < \epsilon. \)

Thus, in either case we have \( \rho(f(\xi),f(\eta)) < \epsilon \) if \( |\xi-\eta| < \frac{1}{N_m} \), hence \( f \) is continuous.

Suppose a point \( x \in A_m^{j+1} \). Since \( f(\frac{1}{N_m}) \in A_m(\frac{1}{N_m}) = A_m^{j+1} \), we have \( \rho(x,f(\frac{1}{N_m})) \leq \frac{3}{2m} \). That is, for any neighborhood \( N(x,\epsilon) \) of \( x \) there is an integer \( m \) such that \( \frac{3}{2m} < \epsilon \) and \( f(\frac{1}{N_m}) \in N(x,\epsilon) \). Thus, the set of points \( f(\frac{1}{N_m}) \) for all \( m \) and all \( j = N_m \) is dense in \( S \) and since \( S \) is the whole space, we have \( f(\frac{1}{N_m}) = S \). But \( \frac{1}{N_m} \subset [0,1] \) for all \( j \) and \( m \), hence \( f(\frac{1}{N_m}) \subset f[0,1] \) and \( f(\frac{1}{N_m}) \subset f[0,1] \). It follows that \( S = f(\frac{1}{N_m}) \subset f[0,1] \). Thus, \( f[0,1] = S \). But since \( [0,1] \) is closed, \( f[0,1] \) is closed. Finally, then we have \( f[0,1] = f[0,1] = S \), and \( f \) is an onto mapping. Q.E.D.

A similar discussion of the Hahn-Mazurkiewicz Theorem may be found in Newman's Elements of the Topology of Plane Sets of Points.
CHAPTER III

SIMPLE ARCS AND ARCWISE CONNECTEDNESS

Definition 3.1 A point \( x \) of a connected set \( E \) is termed a **cut point** of \( E \) if \( E - x \) is disconnected. Otherwise \( x \) is called a **non-cut point**.

Theorem 3.2 If \( S \) is connected but \( S - a = H_1 \cup H_2 \) then \( \overline{H_1} = H_1 \cup a \) and \( \overline{H_2} = H_2 \cup a \).

Proof. \( S = H_1 \cup H_2 \cup a, \overline{H_1} \cap H_2 \cap a = \emptyset \) and since \( \overline{H_1} \cap H_2 = \emptyset \) we have \( \overline{H_1} \subset \overline{H_2} = H_1 \cup a \) in \( S \). Thus, \( \overline{H_1} \subset \overline{H_1} \subset H_1 \cup a \) and either \( \overline{H_1} = H_1 \) or \( \overline{H_1} = H_1 \cup a \).

Suppose \( \overline{H_1} = H_1 \). Now \( H_1 \cap (H_2 \cup a) = \emptyset, \overline{H_1} \cap (H_2 \cup a) = \emptyset \) and \( H_1 \not\subset \emptyset \) and \( (H_2 \cup a) \not\subset \emptyset \). Therefore, \( S = H_1 / H_2 \cup a \) a contrary to the assumption that \( S \) is connected, hence \( \overline{H_1} \neq H_1 \), i.e., \( \overline{H_1} = H_1 \cup a \).

Similarly, \( \overline{H_2} = H_2 \cup a \). (It follows since \( H_1 = \overline{H_2} \) and \( H_2 = \overline{H_1} \) that \( H_1 \) and \( H_2 \) are open sets in \( S \).) Q.E.D.

Definition 3.3. A space \( T \) is a **simple arc** if and only if there is a topological mapping \( f \) of the closed interval \([0,1]\) onto \( T \). Thus, in view of 1.28 if \( T \) is a simple arc then open sets of \( T \) correspond to open sets of \([0,1]\) and vice versa (open may be replaced by closed throughout).

If \( A \) is a simple arc with end points \( a \) and \( b \), for convenience we may often denote \( A \) by \( ab \). Suppose points \( u \) and \( v \) lie on the simple arc \( ab \). If \( E \) is the set consisting of \( u, v \) and all points
x of ab which lie between u and v then the reader can easily verify
that \( E(=uv) \) is a simple arc and that \( E \subset ab \). When such a situation
exists we say that \( E \) is a subarc of \( ab \).

In view of theorem 2.4 it is clear from definition 3.3 that
a simple arc is a Peano space.

**Theorem 3.4** A continuum, \( X \), of which all but at most two
points are cut points is a simple arc.

**Proof.** Let \( X \) be the given continuum and \( A \) the set of all
non-cut points in \( X \). Therefore, \( A \) has 0, 1 or 2 members and is not
identical with \( X \) (see 1.23). Let \( x_0 \) be a point of \( X \) not in \( A \) such
that \( X-x_0 = P/Q \).

1. We have seen in theorem 3.2 that \( \overline{P}=P \cup x_0, \overline{Q}=Q \cup x_0 \)
and \( P \) and \( Q \) are open sets.

2. \( \overline{P} \) and \( \overline{Q} \) are connected. For if not let \( P = H_1/H_2 \) and
let \( x_0 \in H_1 \). Now \( X = H_1 \cup \overline{Q} \cup H_2, H_1 \cup \overline{Q} \neq \emptyset, H_2 \neq \emptyset, \)
\( H_2 \cap (H_1 \cup \overline{Q}) = (H_2 \cap H_1) \cup (H_2 \cap \overline{Q}) = \emptyset, \)
\( H_2 \cap (H_1 \cup \overline{Q}) = (H_2 \cap H_1) \cup (H_2 \cap \overline{Q}) = \emptyset, \)
and
\( H_2 \cap (H_1 \cup \overline{Q}) = H_2 \cap (H_1 \cup \overline{Q}) = (H_2 \cap H_1) \cup (H_2 \cap \overline{Q}) = \emptyset. \)
Thus, \( X = H_1/H_2 \cup \overline{Q} \) contrary to the assumption that \( X \) is connected.
Hence, \( \overline{P} \) is connected and, similarly, \( \overline{Q} \) is connected.

3. If \( y \in P \) and \( P_1/Q_1 \) is an arbitrary partition of \( X-y, \)
either \( P_1 \) or \( Q_1 \) but not both is contained in \( P \). Since \( y \in P, \)
\( \overline{Q} \subset X-y = P_1/Q_1 \). Now \( \overline{Q} \) is connected, hence by paragraph 1.22,
either \( \overline{Q} \subset P_1 \) or \( \overline{Q} \subset Q_1 \). If \( \overline{Q} \subset P_1 \), we have \( \overline{Q} \cap Q_1 = \emptyset \) and since
\( x_0 \in \overline{Q}, x_0 \notin Q_1 \). Therefore, \( Q_1 \subset X-x_0 = Q \cup P \) and we must have \( Q_1 \subset P \).
Suppose also that \( P_1 \subset P \). Thus, \( P_1 \cup Q_1 \subset P \) and \( X = P_1 \cup Q_1 \cup y \subset P \).
which is contrary to the assumption that \( X-x_0 = P/Q, \) hence \( P_1 \notin P. \)

Similarly, the assumption that \( Q \subseteq Q_1 \) gives \( P_1 \subseteq P \) and \( Q_1 \notin P. \)

(4) \( P \) contains at least one non-cut point. For if not then \( P = P \cup x_0 \) also contains no non-cut points. Since \( P \) is closed, it is compact and hence contains an enumerable dense set of points \( x_1, x_2, \ldots, x_n, \ldots \) (see 1.21) each different from \( x_0. \) Therefore, \( x_1, x_2, \ldots, x_n, \ldots \) is an enumerable dense subset of \( P. \) Let \( X-x_1 = P_1/Q_1 \) and assume that \( P_1 \) is the part contained in \( P. \) We now inductively define a series of integers \( n_r \) and of sets \( P_r \) such that for all \( r, \) \( X-x_n = P_r/Q_r \) and \( P \subseteq P_{r+1} \subseteq \cdots \subseteq P. \) Each \( P_r \) is a non-empty open (see part (1) above) subset of \( P \) and, therefore, contains a neighborhood which itself contains at least one of the points \( x_n. \) Let \( n_{r+1} \) be the least integer such that \( x_n \in P_{r+1}. \) Since \( P \subseteq P, \) \( x_n \) is a cut point and thus \( X-x_n = P_{r+1}/Q_{r+1}, \) where again \( P_{r+1} \subseteq P_{r+1} \subseteq \cdots \subseteq P. \) It follows that \( x_n \in P_r \) but \( x_n \notin P_{r+1} \) and thus all of the integers \( n_r \) are distinct. This completes the definition.

We now have a decreasing sequence of compact sets \( \bar{P}, \bar{P}_1, \bar{P}_2, \ldots, \bar{P}_r, \ldots \) such that \( \bar{P}_{r+1} \subseteq \bar{P}_r \) for all \( r \) and the common part of all of them, \( P_0, \) is non-empty (see 1.21). Now

\[
\bar{P}_{s+1} = P_{s+1} \cup x_n \subseteq P_s
\]

and \( P_\infty \) is also the common part of all sets \( P_s. \) Let \( z \) be a point of \( P_\infty. \) Since \( z \in P, \) it is a cut point. Therefore, \( X-z = H_1/H_2 \) and every \( P_m \) contains either \( H_1 \) or \( H_2 \) (see part (3) above). Let \( H_1 \) be the part contained in \( P_m \) for all \( m. \) Hence, \( H_1 \subseteq P_\infty. \) By part (1) \( H_1 \) is open and again contains a neighborhood which itself contains some point \( x_k \) of the set \( x_1, x_2, \ldots, x_n, \ldots \). Letting \( n_{s+1} \)
be the first of the suffixes $n_2, n_3, \ldots$ that exceeds $k$ we have $x_k \in P_s$. But this implies that $n_{s+1}$ is not the least integer such that $x_{n_{s+1}} \in P_s$ which is contrary to our earlier definition in the previous paragraph. Thus, the assumption that $P$ contained no non-cut points led to a contradiction and we have at least one non-cut point in $P$. Similarly, $Q$ contains at least one non-cut point. We have, therefore, proved that $X$ has exactly two non-cut points. Denote them by $a$ and $b$.

(5) Suppose $x$ is a cut point. Then $X-x$ has two components, each containing one of the non-cut points. Let $X-x = P/Q$. We have seen above that $a$ and $b$ belong to $P$ and $Q$ respectively. We shall prove that $P$ and $Q$ are components of $X-x$. Clearly, $P \neq \emptyset$, $Q \neq \emptyset$ and $P$ and $Q$ are both contained in $X-x$. If there is a connected set $P$ such that $P \subset X-x$ then either $P \subset Q$ or $Q \subset P$. Suppose $P \subset Q$. Then $P \subset Q \subset Q$ which is impossible since $P \cap Q = \emptyset$. Thus, $P \subset Q$ and we have $P = P$. Similarly, the assumption that a connected set $Q$ contains $Q$ and is contained in $X-x$ implies $Q = Q$. We must yet show that $P$ and $Q$ are connected sets. Suppose $P = H_1/H_2$ and without loss of generality let $a \in H_1$. Then $a \notin H_2$ and since $b \in Q$, $b \notin H_2$. We will show that $X-x$ has the separation $H_2/(X-x)-H_2$. Clearly, $H_2 \not\subset \emptyset$ and $(X-x) \cap CH_2 \not\subset \emptyset$. Also, the reader can easily verify that $H_2 \cup [(X-x) \cap CH_2] = X-x$ and $H_2 \cap [(X-x) \cap CH_2] = \emptyset$. Furthermore,

\[
\begin{align*}
    \overline{H_2} \cap [(X-x) \cap CH_2] &= \overline{H_2} \cap [(H_1 \cup Q \cup H_2) \cap CH_2] \\
    &= \overline{H_2} \cap [(H_1 \cap CH_2) \cup (Q \cap CH_2)] \\
    &= \overline{H_2} \cap (H_1 \cap CH_2) \cup \overline{H_2} \cap Q \cap CH_2 \\
    &\subset \emptyset \cup [\overline{P} \cap Q] \quad (\text{since } H_2 \subset P) \\
    &= \emptyset,
\end{align*}
\]
and by similar reasoning $H_2 \cap \left[ \frac{(x-x) \cap CH_2}{(x-x) \setminus H_2} \right] = \emptyset$. Thus

$$X-x = H_2/(X-x) \setminus H_2,$$

but this is impossible since $H_2$ contains neither $a$ nor $b$, contrary to part (4). Therefore, the assumption that $P$ is disconnected is false. Similarly, $Q$ is connected.

Before continuing the proof of theorem 3.4, we shall set up an order in $X$. A relation $<$ will be termed a total ordering if it holds between certain elements of a set $E$ such that the following conditions are satisfied.

(i) for no $x$ is $x < x$;

(ii) if $x \neq y$ then either $x < y$ or $y < x$;

(iii) if $x < y$ and $y < z$ then $x < z$.

It is clear from (i) and (iii) that we cannot have both $x < y$ and $y < x$.

Consider any point $x$ of $X$. Define $L_x$ to be the component of $X-x$ containing $a$ if $x \neq a$. If $x = a$ let $L_x = \emptyset$. Define $R_x$ similarly with $b$ replacing $a$ throughout. No $L_x$ shall contain $b$ and no $R_x$ contains $a$. Therefore, for any point $x$, $X = L_x \cup x \cup R_x$.

(6) $x \in L_y$ if and only if $L_x \subseteq L_y$ but $L_x \neq L_y$. First suppose $x \in L_y$, i.e., $x$ is in the component of $X-x$ containing $a$. Since $x \notin L_x$ we have $L_x \neq L_y$ but since $L_x$ is a component of $X-x$ also containing $a$ we have $L_x \cap L_y \neq \emptyset$. The point $y$ cannot be a since $L_y \neq \emptyset$ and $L_a = \emptyset$. It is always true that $b \notin L_x$ and $R_b = \emptyset$, hence if $y = b$ we have $L_x \subseteq X-b = L_b = L_y$, i.e., $L_x \subseteq L_y$. Suppose $y \neq b$. Thus, since $X-x = L_x/R_x$ and $x \in L_y$, we see by part (3) that either $L_x \subseteq L_y$ or $R_x \subseteq L_y$, but $b$ belongs to $R_x$ and not to $L_y$, hence again we must have $L_x \subseteq L_y$. 

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To prove the converse suppose \( L_x \subseteq L_y \) but \( L_x \neq L_y \). Again the point \( y \) cannot be \( a \) for if it were then \( L_y \) would be empty and since \( L_x \subseteq L_y \), we would also have \( L_x = \emptyset \). Thus, \( L_x = L_y \) contrary to our hypothesis. If \( x = a \), \( x \in L_y \) by definition. Suppose \( x \neq a \). Since \( x-x = L_x/R_x \), it follows from part (1) that

\[
L_x \cup x = \overline{L_x} \subseteq \overline{L_y} = L_y \cup y,
\]
i.e., \( L_x \cup x \subseteq L_y \cup y \). But \( x \neq y \) (since \( L_x \neq L_y \)), hence \( x \in L_y \).

Similarly, \( x \in R_y \) if and only if \( R_x \subseteq R_y \) but \( R_x \neq R_y \).

For all points of \( X \) we shall define the relation \( x \prec y \) to mean "\( x \in L_y \)." Thus, if \( x \neq a \), \( a \prec x \) and if \( x \neq b \), \( x \prec b \) and for no \( x \) does \( x \prec a \) or \( b \prec x \). The symbol \( x \prec y \) may be read "\( x \) precedes \( y \)"

and we shall use such expressions as "first point", "between" and "successor" accordingly.

(7) The relation \( \prec \) is a total ordering. Condition (i) is obviously satisfied since \( x \notin L_x \) by definition. To verify that condition (ii) is satisfied suppose \( x \neq y \). We must show that either \( x \prec y \) or \( y \prec x \). Suppose \( x \prec y \), i.e., \( x \notin L_y \), hence

\[
x \in X-L_y = R_y \cup y.
\]
Since \( x \neq y \) we have \( x \in R_y \) and by virtue of part (6), this is equivalent to \( R_x \subseteq R_y \). Thus,

\[
L_y \cup y = CR_y \subseteq CR_x = L_x \cup x
\]
i.e., \( L_y \cup y \subseteq L_x \cup x \), and therefore, \( L_y \subseteq L_x \cup x \). But \( x \notin L_y \), hence \( L_y \subseteq L_x \). Again by part (6) this means \( y \in L_x \), i.e., \( y \prec x \). Similarly, the assumption that \( y \nprec x \) would lead us to conclude that \( x \prec y \).

Verification of condition (iii) follows immediately from part (6) since if \( x \prec y \) and \( y \prec z \) we have \( L_x \subseteq L_y \subseteq L_z \), i.e., \( L_x \subseteq L_z \) which implies that \( x \prec z \).
The set \([x<p]\) shall denote the set of all points \(x\) which precede a fixed point \(p\). Clearly, then \([x<p] = L_p\) and by part (1) \([x<p]\) is an open set. The set \([p<x]\) is the set of all points \(x\) which \(p\) precedes. \([p<x]\) is the complement of \([x<p]\) in \(X-p\), i.e., \([p<x] = R_p\) and, therefore, it is also open. Thus, the intersection \([p<x] \cap [p<x]\) of two such sets for points \(p\) and \(q\) is an open set. We will denote it by \(<p,q>\).

(8) **If** \(p<q\) **the set** \(<p,q>\) **is not empty.** Suppose \(<p,q>\) is empty. We will show that this assumption leads to the contradiction that \(X\) has the partition \(L_p U p / R_q U q\). Clearly, \(X = L_p U p U R_q U q, L_p U p \neq \emptyset\) and \(R_q U q \neq \emptyset\). \(L_p U p\) and \(R_q U q\) have no point in common since \(p<q\) implies \(p \neq q\), \(p \notin R_p\) and \(q \notin L_p\). Thus, if the sets had a common point it would have to belong to both \(L_p\) and \(R_q\) which contradicts condition (iii). Now \([p<x] = R_p\) which is open. But \(R_p = L_p U p\), hence \(L_p U p\) is closed. Similarly, \([x<q]\) is the open set \(L_q\) and since \(L_q = R_q U q\), we have \(R_q U q\) closed. It readily follows that

\[
(L_p U p) \cap (R_q U q) = \emptyset
\]

and

\[
(L_p U p) \cap (R_q U q) = \emptyset.
\]

We now have our partition, hence \(<p,q> \neq \emptyset\).

Since compact sets are separable (1.21) and \(X\) is compact, there is an enumerable dense set \(C = (y_1,y_2,...)\) in \(X\). We want \(C\) to contain neither \(a\) nor \(b\). To show this is possible, let \(p\) be an arbitrary point of \(X\) different from both \(a\) and \(b\). Since \(X\) is open there is an \(\varepsilon>0\) such that \(N(p,\varepsilon) \subset X\). If \(N(p,\varepsilon)\) contains neither \(a\) nor \(b\) then since \(C\) is dense in \(X\) there is a point \(y_r\) of \(C\), \(y_r \neq a\),...
Suppose \( N(p, \varepsilon) \) contains a and not b (if \( N(p, \varepsilon) \) contained b and not a an entirely similar discussion would follow). Thus, \( \rho(p, a) = \lambda \) where \( \lambda > 0 \) and \( a \not\in N(p, \lambda) \) and b \( \not\in N(p, \lambda) \). There is, therefore, a point \( y_j \) of C such that \( y_j \in N(p, \lambda) \), \( y_j \not\in a \) and \( y_j \not\in b \).

Suppose \( N(p, \varepsilon) \) contains both a and b, thus \( \rho(p, a) = \lambda \) and \( \rho(p, b) = \mu \) where \( \lambda > 0 \) and \( \mu > 0 \). Let \( \delta = \frac{\lambda + \mu}{2} \). Therefore, \( N(p, \delta) \) contains a point \( y_m \) of C where \( y_m \not\in a \) and \( y_m \not\in b \).

Suppose \( p = a \). (A similar discussion would follow if \( p = b \)). Since X is open, there is an \( \varepsilon > 0 \) such that \( N(a, \varepsilon) \subset X \). Suppose \( b \not\in N(a, \varepsilon) \). \( N(a, \varepsilon) \not\subset a \) since \( N(a, \varepsilon) \) is open and the single point a is closed. That is, if \( N(a, \varepsilon) = a \), \( N(a, \varepsilon) \) would be both open and closed, hence \( X - N(a, \varepsilon) \) would be both open and closed. Thus, X would be the union of two non-empty disjoint open sets \( N(a, \varepsilon) \) and \( X - N(a, \varepsilon) \). Therefore, \( X = N(a, \varepsilon)/X - N(a, \varepsilon) \) by 1.22 which is impossible since X is connected. Hence, there is a point \( d \) of \( N(a, \varepsilon) \) and \( d \not\in a \), i.e., \( \rho(d, a) = \kappa \) where \( \kappa > 0 \). Let \( \gamma = \frac{\lambda + \kappa}{2} \). Now \( N(d, \gamma) \subset N(a, \varepsilon) \), \( a \not\in N(d, \gamma) \) and there is a point \( y_n \) of C such that \( y_n \in N(d, \gamma) \), i.e., \( y_n \in N(a, \varepsilon) \) and \( y_n \not\in a \) and \( y_n \not\in b \). If \( b \in N(a, \varepsilon) \) then \( N(a, \varepsilon) \) contains a and b but \( N(a, \varepsilon) \not\subset a \cup b \), for if \( N(a, \varepsilon) = a \cup b \) we would, by the same reasoning as above, again have the connected set \( X = N(a, \varepsilon)/X - N(a, \varepsilon) \) which is impossible. Therefore, there is a point \( c \in N(a, \varepsilon) \), \( c \not\in a \), \( c \not\in b \), \( \rho(a, c) = \theta \) where \( \theta > 0 \), and \( \rho(b, c) = \beta \) where \( \beta > 0 \). Let \( \xi = \frac{\theta + \beta}{2} \). Now \( N(c, \xi) \subset N(a, \varepsilon) \), \( a \not\in N(c, \xi) \) and \( b \not\in N(c, \xi) \). Thus, there is a point \( y_i \) of C such that \( y_i \in N(c, \xi) \), \( y_i \not\in a \) and \( y_i \not\in b \). Thus, every neighborhood of an arbitrary point \( p \) contains a point of C different from a and b, i.e., \( C - (a \cup b) \) is...
dense in $X$. But $C - (a \cup b)$ is a subset of the enumerable set $C$, hence $C - (a \cup b)$ is also enumerable (see 1.18). Let $E_0 = C - (a \cup b)$ and let the points of $E_0$ be $x_1, x_2, \ldots$.

If $p < q$, the non-empty open set $(p, q)$ contains at least one point of $E_0$, thus, in particular, there is a point of $E_0$ between any two points of $E_0$ itself.

Let $\alpha_1, \alpha_2, \ldots$, be any enumeration of the rational points of the open interval $(0, 1)$. We now construct two sequences

$$y_1, y_2, \ldots$$ of points of $E_0$

$$\beta_1, \beta_2, \ldots$$ of rational points of $(0, 1)$,

as follows. Let $y_1 = x_1$, and $\beta_1 = \alpha_1$. Suppose that $y_1, y_2, \ldots, y_{n-1}$ and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ have been defined. If $n$ is even let $y_n$ be the point $x_k$ of lowest $k$ not already in the set $y_1, y_2, \ldots, y_{n-1}$, and let $\beta_n$ be the $\alpha_k$ of lowest $k$ having the same relations to $\beta_1, \beta_2, \ldots, \beta_{n-1}$, relative to $<$, as $y_n$ has to $y_1, y_2, \ldots, y_{n-1}$ relative to $\prec$. That is, if $y_n$ is such that

$$\cdots < y_g < y_n \prec y_h \prec \cdots$$

the $\beta_n$ is such that

$$\cdots < \beta_g \prec \beta_n \prec \beta_h \prec \cdots$$

If $n$ is odd we reverse the roles of $x$’s and $\alpha$’s, $y$’s and $\beta$’s: $\beta_n$ will be the rational number $\alpha_k$ of lowest $k$ not among the set $\beta_1, \beta_2, \ldots, \beta_{n-1}$, and $y_n$ the point $x_k$ of lowest $k$ with the appropriate $\prec$-relations. That is, if $\beta_n$ is such that

$$\cdots < \beta_e < \beta_n < \beta_f < \cdots$$

then $y_n$ is such that

$$\cdots < y_e < y_n < y_f < \cdots$$

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We are assured of such a point \( y_n \) existing since \( y_e < y_f \) implies \( \langle y_e, y_f \rangle \neq \emptyset \).

We shall now define a mapping \( f \). It is clear that every point \( x_i \) appears once and only once as a \( y_j \) and every rational number \( \alpha_i \) once and only once as a \( \beta_j \). Define \( f(y_i) \) to be \( \beta_i \). Since \( E_0 \) is an enumerable set and the rational numbers of \((0,1)\) are also enumerable we see that \( f \) is a one-to-one mapping of \( E_0 \) onto the rationals of \((0,1)\). In addition to this, we have so defined \( y_n \) and \( \beta_n \) so that \( f \) is an order preserving mapping, i.e., if \( y_r < y_s \) then \( f(y_r) < f(y_s) \).

Let us now define a subset \( \Lambda \) of \( E_0 \) to be a section if

1. \( \Lambda \) has no last point,
2. and if \( x \in \Lambda \) all predecessors of \( x \) in \( E_0 \) also belong to \( \Lambda \).

(Clearly, \( \Lambda \) may be either the empty set or the whole of \( E_0 \)). The analogy with sections of rational numbers in \((0,1)\) is obvious.

Let \( \Lambda \) be a section of \( E_0 \) and \( K \) the set of all points of \( X \) not followed by any point of \( \Lambda \). Then \( K \) has a first point. \( K \) cannot be empty since it contains \( b \). If \( K = X \) our required first point is \( a \). Therefore, suppose \( K \neq X \). Consider the set \( X-K \) and let \( x \) be one of its points. If \( x \neq a \) there is a point \( y \) of \( \Lambda \) following \( x \), and \( \langle a, y \rangle \) is an open set containing \( x \) and contained in \( X-K \). Thus, there is a \( \epsilon > 0 \) such that \( N(x, \epsilon) \subset \langle a, y \rangle \subset X-K \), i.e., \( X-K \) is open. Similarly, if \( x = a \) we may again use the point \( y \) of \( \Lambda \) and we have \( x \in L_y \subset X-K \). \( L_y \) is open, hence there is a \( \delta > 0 \) such that \( N(x, \delta) \subset L_y \subset X-K \). Thus, in either case \( X-K \) is an open set. Suppose \( K \) has no first point. We shall show that this
assumption leads to the contradiction that the connected set $X$ has
the partition $K/X-K$. Clearly, $K \neq \emptyset$, $X-K \neq \emptyset$ and $K \cap (X-K) = \emptyset$.
Since $K$ has no first point, $K$ is open for if $x \in K$ there is a point
$y$ of $K$ such that $y \prec x$ and $y$ follows all points of $\Lambda$. Now $\langle y, b \rangle$
is an open set contained in $K$ and containing $x$, hence there is an
$\epsilon_1 > 0$ such that

$$N(x, \epsilon_1) \subset \langle y, b \rangle \subset K.$$ 

Therefore, $K$ is open and $X$ is the union of two non-empty disjoint
open sets $K$ and $X-K$, hence $X = K/X-K$ (see 1.22).

We have now shown that for every section of $E_0$ there
 corresponds a unique point of $X$, namely the first point of $K$, which
will we call the point determined by the section. It follows
immediately that points determined by different sections are
different, for if $x \in \Lambda_1$ but $x \notin \Lambda_2$ then $x$ precedes the point
determined by $\Lambda_1$ but does not precede the point determined by
$\Lambda_2$. It is also true that every point $x$ of $X$ is determined by a
section and different points are determined by different sections.
To verify this let $\Lambda$ be the subset of $E_0$ consisting of all points
of $E_0$ which precede $x$. Again let $K$ be the set of points of $X$ not
followed by any point of $\Lambda$. Clearly, $x \notin K$ but we must show that
$x$ is the first point of $K$. Suppose $y \prec x$ and $y \in K$. There is a
point $z$ of $E_0$ such that $z \in \langle y, x \rangle$, i.e., $y \prec z \prec x$. Hence, $z \in \Lambda$
and $y \prec z$ which contradicts the assumption that $y \in K$. Thus, $x$ is
the first point of $K$ and is determined by $\Lambda$. Suppose $x$ and $y$ are
different points of $X$ and are determined by sections $\Lambda_x$ and $\Lambda_y$
respectively. Without loss of generality assume $y \prec x$. Again
there is a point $z$ of $E_0$ such that $z \in \langle y, x \rangle$, i.e., $y \prec z \prec x$. 

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Now $y < z$, hence $z \notin \Lambda_y$ but $z \prec x$ implies $z \in \Lambda_x$ and we have
\[ \Lambda_x \neq \Lambda_y. \]
It is now clear that we have set up a one-to-one correspondence between the sections of $E_0$ and the points of $X$.

This result enables us to extend the mapping $f$ of $E_0$ onto the rationals of $(0,1)$, to be a mapping of $X$ onto all points of $(0,1)$. Suppose $x$ is any point of $X$ and $\Lambda_x$ is the section determining it. Since $f$ is order preserving it clearly maps $\Lambda_x$ onto a section of the rationals of $(0,1)$. Define $f(x)$ to be the real number determined by this section. Letting $f(a) = 0$ and $f(b) = 1$ it follows that $f$ is a one-to-one mapping of $X$ onto $[0,1]$ and if $y \in E_0$ the new definition of $f(y)$ agrees with the old.

(10) The mapping $f$, so extended, is order preserving, i.e., if $x \prec y$ then $f(x) < f(y)$. Suppose $x$ and $y$ are points of $X$ and $x \prec y$. If both $x$ and $y$ belong to $E_0$ it follows from the original definition of $f$ that $f(x) < f(y)$. Suppose $x \in E_0$ but $y \notin E_0$. Since $x \prec y$, $x \in \Lambda_y$ and it is always true that $y \notin \Lambda_y$. Now $f(x) \in f(\Lambda_y)$ and there is a point $z$ of $E_0$ such that $z \in <x, y>$, i.e., $x \prec z \prec y$ and $z \in \Lambda_y$, hence $f(x) < f(z)$ and $f(z) \in f(\Lambda_y)$. By definition $f(y)$ is the first real number of $(0,1)$ not followed by any number in $f(\Lambda_y)$. Clearly then $f(y) \notin f(x)$ and we must have either $f(x) = f(y)$ or $f(x) < f(y)$. If $f(x) = f(y)$ it follows from $f(x) \in f(\Lambda_y)$ that $f(y) \in f(\Lambda_y)$ which is impossible since $y \notin \Lambda_y$. Therefore, $f(x) < f(y)$. Suppose $y \in E_0$ but $x \notin E_0$. Since $x \prec y$, $y \notin \Lambda_x$. Let $z$ be any point of $\Lambda_x$, i.e., $z \in E_0$ and $z \prec x \prec y$, hence $f(z) \in f(\Lambda_x)$ and $f(z) < f(y)$. Thus, $f(y)$ is greater than every number in $f(\Lambda_x)$. If $f(y) < f(x)$, $y \in E_0$ implies that $f(y) \in f(\Lambda_x)$, but this is impossible since $y \notin \Lambda_x$. 

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Therefore, $f(x) \leq f(y)$. There is a point $w$ of $E_o$ such that $w \in \langle x, y \rangle$, i.e., $x < w < y$. Now the only restrictions we have on $y$ is that $y \in X$, $x < y$ and $y \in E_o$. In view of the fact that $w$ also satisfies these restrictions, we have $f(x) \leq f(w)$. However, $f(w) < f(y)$ since both $w$ and $y$ belong to $E_o$. Hence $f(x) < f(y)$.

Suppose neither $x$ nor $y$ belongs to $E_o$. Again there is a point $z$ of $E_o$ and $x < z < y$, i.e., $z \in \Lambda_y$ but $z \notin \Lambda_x$. In a manner entirely similar to the previous case we have $f(x) < f(z)$. Also, $f(y) < f(z)$ for if it were, $f(z)$ would not belong to $f(\Lambda_y)$ which contradicts the fact that $z \in E_o$ and $z \in \Lambda_y$. Therefore, $f(z) \leq f(y)$, i.e., $f(x) < f(y)$.

\textbf{(11)} \textit{f is a topological mapping of X onto $[0,1]$.} We have seen that $f$ is a one-to-one mapping of $X$ onto $[0,1]$. To prove $f$ is continuous on $X$ we will show that if $G$ is any open set in $[0,1]$ then $f^{-1}(G)$ is an open set in $X$ (paragraph 1.27). Since $[0,1]$ is locally connected, components of $G$ are open sets. Now $G$ is equal to the union of its components, hence $f^{-1}(G)$ is equal to the union of the inverse images of the components of $G$. Thus, if the components of $G$ map onto open sets in $X$ it will follow that $f^{-1}(G)$ is the union of open sets hence is itself an open set (paragraph 1.6). To begin, it is clear that since components are connected sets, the components of $G$ must be open intervals (open in $[0,1]$).

The typical open intervals in $[0,1]$ are $(\gamma, \delta), (\gamma, 1], \text{ and } [0, \gamma)$.

Let $D = (\gamma, \delta)$. Since $f$ is one-to-one, there are points $t$ and $u$ of $X$ such that $f(t) = \gamma$ and $f(u) = \delta$. Clearly, $f(t) < f(u)$ and since $f$ is order preserving $t < u$. Recall that $\langle t, u \rangle$ is by
definition the open set \( L_t \cap R_t \) in \( X \) (part 7). We will show that
\[
f^{-1}(D) = L_t \cap R_t
\]
thus proving \( f^{-1}(D) \) is open in \( X \). Suppose \( p \) is any point of \( f^{-1}(D) \). There is a point \( q \) of \( D \) such that \( f(p) = q \). Thus
\[
f(t) = \forall q < \delta = f(u),
\]
i.e.,
\[
f(t) < f(p) < f(u)
\]
which implies that \( t < p < u \), hence \( p \in L_t \cap R_t \). Suppose a point \( r \) belongs to \( L_t \cap R_t \), i.e., \( t < r < u \), and hence
\[
\forall = f(t) < f(r) < f(u) = \delta
\]
Letting \( f(r) = s \) we have \( \forall s < \delta \), i.e., \( s \in D \) which implies
\[
f^{-1}(s) \in f^{-1}(D). \quad \text{But } f^{-1}(s) = f^{-1}(r) = r, \text{ hence } r \in f^{-1}(D) \) and we have \( f^{-1}(D) = L_t \cap R_t \).

Let \( E = (\forall, 1] \). Again \( f(t) = \forall \) where \( t \in X \). We will show \( f^{-1}(E) \) equals the open set \( R_t \) in \( X \). Suppose \( m \) is any point of \( f^{-1}(E) \). There is a point \( c \) of \( E \) such that \( f(m) = c \). Therefore,
\[
f(t) = \forall < c = f(m),
\]
i.e.,
\[
f(t) < f(m)
\]
which implies \( t < m \) hence \( m \in R_t \). Let \( n \) be any point of \( R_t \), i.e., \( t < n \). Therefore,
\[
\forall = f(t) < f(n)
\]
hence \( f(n) \in E \) and it follows that \( n \in f^{-1}(E) \).

Letting \( F = [0, \forall] \) it can be shown in a manner entirely similar to the previous case that \( f^{-1}(F) \) is equal to the open set \( L_t \) in \( X \). Thus, in very instance the inverse image of a component of \( G \) is an open set hence \( G \) is open and \( f \) is continuous on \( X \).
Letting $g$ be the mapping $f^{-1}$ and since $X$ is compact, we have in $g$ a continuous one-to-one mapping of $[0,1]$ onto $X$ (paragraph 1.28). Therefore, $f$ is a topological mapping of $X$ onto $[0,1]$. This result enables us to state that $X$ is a simple arc (paragraph 3.3) and the theorem is proved. Q.E.D.

**Corollary.** $a$ and $b$ are the end points.

**Theorem 3.5** Every continuum has at least two points that are non-cut points. (Proved in part 4 of 3.4)

For a similar discussion of theorems 3.2 and 3.4, the reader may refer to Newman's Elements of the Topology of Plane Sets of Points.

**Definition 3.6** A set of points $E$ is said to be irreducible with respect to a given property $P$ provided the set $E$ has property $P$ but no non-empty closed proper subset of $E$ has property $P$.

**Definition 3.7** A property $P$ is said to be inducible provided that when each set of a monotone decreasing sequence $A_1, A_2, \ldots$ of compact sets has property $P$ then so also does their intersection $A = \bigcap_{i=1}^{\infty} A_i$. As an example, the property of being non-empty is inducible for the intersection of a monotone decreasing sequence of non-empty compact sets is itself non-empty (1.21).

**Theorem 3.8** (Brouwer Reduction Theorem) If property $P$ is inducible and $K$ is a non-empty compact space having property $P$, then there is a non-empty compact subset $Q$ of $K$ such that $Q$ is irreducible with respect to property $P$.

**Proof.** (We shall assume throughout that $K$ itself is not irreducible with respect to property $P$.) Let $R_1, R_2, \ldots$ be an enumerable base for $K$ (1.21). Let $n_1$ be the least integer such
that $K$ contains a non-empty compact set $K_1$, $K_1 \neq K$, $K_1$ has property $P$ and $K_1 \cap R_{n_1} = \emptyset$. To assure us that such a set exists, recall that $K$ is not irreducible with respect to property $P$. This implies there exists a set $K_1$ such that $K_1 \neq \emptyset$, $K_1 \subseteq K$, $K_1 \neq K$ and $K_1$ is compact and has property $P$. Thus, there is a point $a_1$ of $K-K_1$, i.e., $a_1 \in \mathcal{G}K_1$. Since $K_1$ is closed, $\mathcal{G}K_1$ is open. Therefore, $\mathcal{G}K_1$ is the union of some of the sets $R_m$, $m = 1, 2, \ldots$, and $a_1$ belongs to at least one of these, say $R_s$. Now $a_1 \in R_s \subseteq \mathcal{G}K_1$, i.e., $K_1 \cap R_s = \emptyset$. We therefore have at least one set of our base which is disjoint from $K_1$ and we let $n_1$ be the least integer such that $K_1 \cap R_{n_1} = \emptyset$. Let $n_2$ be the least integer greater than $n_1$ such that $K_1$ contains a non-empty compact set $K_2$, $K_2 \neq K_1$, $K_2$ has property $P$ and $K_2 \cap R_{n_2} = \emptyset$. We assume there does exist a set $K_2$ such that $K_2 \neq \emptyset$, $K_2 \subseteq K_1$, $K_2 \neq K_1$ and $K_2$ is compact and has property $P$, for if not, then $K_1$ has property $P$ irreducibly and the theorem is proved. In addition, there is a point $a_2$ of $K_1-K_2$, i.e., $a_2$ belongs to the open set $\mathcal{G}K_2$. And, since $\mathcal{G}K_2$ is the union of some of the sets $R_m$, $m = 1, 2, \ldots$, there is at least one set $R_t$ contained in $\mathcal{G}K_2$ such that $a_2 \in R_t$, i.e., $a_2 \in R_t \subseteq \mathcal{G}K_2$ hence $K_2 \cap R_t = \emptyset$. Also, $t > n_1$.

To show this, notice that since $a_2 \in K_1$ and $a_2 \in R_t$ we have $K_1 \cap R_t \neq \emptyset$. If $t = n_1$ this implies $K_1 \cap R_{n_1} \neq \emptyset$ which is impossible since $K_1 \cap R_{n_1} = \emptyset$. Thus, $t \neq n_1$. Nor can $t$ be less than $n_1$ for recall that $K_2 \neq \emptyset$, $K_2$ is compact, $K_2$ has property $P$ and $K_2 \cap R_t = \emptyset$, and since $K_2 \subseteq K_1$ we have $K_2 \subseteq K$ and $K_2 \neq K$. That is, $K_2$ is a set satisfying all of the appropriate properties that $K_1$ satisfied, and if $t < n_1$ we see that $n_1$ was not the least integer such that some set satisfied these properties. Thus again we have a contradiction,
hence \( s > n_1 \). We, therefore, conclude that there is at least one integer (namely \( t \)) greater than \( n_1 \) such that \( K_2 \cap R_t = \emptyset \). Hence there is certainly a least integer \( n_2 \) greater than \( n_1 \) such that \( K_2 \cap R_{n_2} = \emptyset \).

In general let \( n_1 \) be the least integer greater than \( n_{i-1} \) such that \( K_{i-1} \) contains a non-empty compact set \( K_i \), \( K_i \not\subset K_{i-1} \), \( K_i \) has property \( P \) and \( K_i \cap R_{n_i} = \emptyset \). And, we assume there does exist a set \( K_i \) such that \( K_i \not= \emptyset \), \( K_i \subset K_{i-1} \), \( K_i \not\subset K_{i-1} \) and \( K_i \) is compact and has property \( P \), for if not, then \( K_{i-1} \) has property \( P \) irreducibly and the theorem is proved. Now by the same reasoning as before there is a point \( a_i \) of \( K_{i-1} - K_i \), i.e., \( a_i \) belongs to the open set \( \mathcal{G}K_i \) which in turn is the union of some of the sets \( R_m \), \( m = 1, 2, \ldots \).

Thus, there is at least one set \( R_u \) such that \( a_i \in R_u \subset \mathcal{G}K_i \), hence \( K_i \cap R_u = \emptyset \). Also, \( u > n_{i-1} \), for if not then there are three possibilities to consider.

(i) Suppose \( u = n_j \) where \( j \leq i-1 \). Following our general procedure \( K_{i-1} \subset K_j \) and \( a_i \in K_{i-1} \cap R_u \), hence \( K_{i-1} \cap R_u \not= \emptyset \). But with \( u = n_j \) we have

\[
K_{i-1} \cap R_u \subset K_j \cap R_u = K_j \cap R_{n_j},
\]

i.e., \( K_j \cap R_{n_j} \not= \emptyset \) which is impossible since we know that \( K_j \cap R_{n_j} = \emptyset \).

(ii) Suppose \( n_{j-1} < u < n_j \) where \( j \leq i-1 \). Recall that \( K_i \not= \emptyset \), \( K_i \) is compact, \( K_i \) has property \( P \) and \( K_i \cap R_u = \emptyset \). Since \( K_i \subset K_{i-1} \subset K_j \subset K_{j-1} \) we also have \( K_i \subset K_{j-1} \) and \( K_i \not\subset K_{j-1} \). That is, \( K_i \), in conjunction with \( u \) satisfies all of the appropriate properties satisfied by \( K_j \), in conjunction with \( n_j \), and since \( n_{j-1} < u < n_j \) we see that \( n_j \) is not the least integer greater than \( n_{j-1} \) such that some set has these properties. Thus again we have a contradiction.
(iii) Suppose \( u < n_1 \). By the same reasoning as in (ii), \( K_i \), in conjunction with \( u \) satisfies all of the appropriate properties satisfied by \( K_1 \), in conjunction with \( n_1 \), and since \( u < n_1 \), \( n_1 \) is not the least integer such that some set satisfies these properties. Therefore, \( u < n_1 \).

Thus in each case we have a contradiction, hence \( u > n_{i-1} \) and we conclude that there is at least one integer (namely \( u \)) greater than \( n_{i-1} \) such that \( K_i \cap R_u = \emptyset \). Hence there certainly is a least integer \( n_i \) such that \( K_i \cap R_{n_i} = \emptyset \).

We now have a sequence \( \{K_i\} \) of sets such that for all \( i; \)
\( K_i \neq \emptyset, K_i \subseteq K_{i-1} \subseteq \ldots \subseteq K, K_i \neq K_{i-1} \), \( K_i \) is compact, has property \( P \) and \( K_i \cap R_{n_i} = \emptyset \). Let \( Q = \bigcap_{i=1}^{\infty} K_i \). Each \( K_i \) is compact, hence closed. Thus \( Q \) is a closed subset of the compact space \( K \), hence \( Q \) is compact.

The sequence \( \{K_i\} \) is a decreasing sequence of compact sets. Therefore, their common part \( Q \) is non-empty (1.21). Since \( P \) is inducible and each set \( K_i \) has property \( P \) so also does their intersection \( Q \).

Now if \( Q \) does not contain a non-empty closed proper subset which also has property \( P \), then \( Q \) is irreducible relative to property \( P \) and hence is our required set. To show this is the case suppose there is a subset \( S \) of \( Q \) such that \( S \neq \emptyset, S \neq Q \), \( S \) is closed and has property \( P \). There is a point \( p \) of \( Q-S \), i.e., \( p \in \mathcal{E}S \) which is open, hence is the union of some of the sets \( R_m, m = 1, 2, \ldots \). In particular, there is a set \( R_v \) of the basis such that \( p \in R_v \subseteq \mathcal{E}S \), thus \( Q \cap R_v \neq \emptyset \) and \( S \cap R_v = \emptyset \). There are four possibilities for \( v \).

(\( o_i \)) If \( v > n_i \) for all \( i \) then there clearly is a least integer \( v_i > n_i \) for all \( i \). Thus \( S \) satisfies all of the appropriate properties required in order to belong to the sequence \( \{K_i\} \).
Therefore, \( Q \subseteq S \) and since \( S \subseteq Q \) we have \( S = Q \) which is contrary to our assumption that \( S \neq Q \).

\((o_2)\) If \( v = n_j \) where \( j \leq i \) then \( S \cap R_{n_j} = \emptyset \) and \( Q \cap R_{n_j} \neq \emptyset \). But \( Q \subseteq K_j \) and \( K_j \cap R_{n_j} = \emptyset \), hence \( Q \cap R_{n_j} = \emptyset \). This is impossible, hence \( v \neq n_j \) where \( j \leq i \).

\((o_3)\) Suppose \( n_{j-1} < v < n_j \) where \( j \leq i \). Recall that \( S \neq \emptyset \), \( S \) is compact, \( S \) has property \( P \) and \( S \cap R_v = \emptyset \). Since \( S \subseteq Q \), we have \( S \subseteq K_{j-1} \) and \( S \neq K_{j-1} \). Thus, \( S \), in conjunction with \( v \) satisfies all of the properties that \( K_j \) satisfies, in conjunction with \( n_j \), and since \( n_{j-1} < v < n_j \) we see that \( n_j \) is not the least integer greater than \( n_{j-1} \) such that some set has these properties. This, of course, contradicts the way \( n_j \) was set up.

\((o_4)\) Suppose \( v < n_1 \). By the same reasoning as in \((o_3)\), \( S \), in conjunction with \( v \) satisfies all of the appropriate properties satisfied by \( K_1 \), in conjunction with \( n_1 \). But \( v < n_1 \), hence \( n_1 \) is not the least integer such that some set has these properties.

In each case we have arrived at a contradiction, hence no such set \( S \) exists and \( Q \) is irreducible with respect to property \( P \). Therefore, \( Q \) is our required set. Q.E.D.

The name arcwise connected is given to a set \( H \) provided every two points of \( H \) can be joined by a simple arc lying entirely in \( H \). Leaning heavily on the Hahn-Mazurkiewicz Theorem \((2,4)\) we are now able to prove the Arcwise Connectedness Theorem.

**Theorem 3.9 (Arcwise Connectedness)** Every two points \( a \) and \( b \) of a compact, connected, locally connected space \( S \) can be joined in \( S \) by a simple arc.
Proof. Let $I = [0,1]$. In view of theorem 2.4 there is a continuous mapping $f$ of $I$ onto $S$ such that $f(0) = a$ and $f(1) = b$. We shall say that a closed subset $F$ of $I$ has property $P$ provided $a \cup b \subset f(F)$ and if $(xy)$ is any maximal segment of $I-F$, then $f(x) = f(y)$. Clearly, $I$ has property $P$.

We will show that property $P$ is inducible. Let $F_1, F_2, \ldots$ be a monotone decreasing sequence of sets in $I$, i.e., $F_{n+1} \subset F_n$ for all $n$, such that each $F_n$ is compact and has property $P$. Let $F = \bigcap_{n=1}^{\infty} F_n$ and let $(xy)$ be a maximal open segment in $I-F$. Now $a \cup b \subset f(F_n)$ for all $n$, hence $a \cup b \subset \bigcap_{n=1}^{\infty} f(F_n)$. But

$$\bigcap_{n=1}^{\infty} f(F_n) \subset f(\bigcap_{n=1}^{\infty} F_n) = f(F),$$

for since $a \in \bigcap_{n=1}^{\infty} f(F_n)$ there is a point $x_n$ of $F_n$ such that $f(x_n) = a$ for all $n$. Thus we have a sequence of points $\{x_n\}$ in the compact set $I$. Hence there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $x_{k_n} \to z_0$ where $z_0 \in I$. Since $\{F_n\}$ is a monotone decreasing sequence of sets, if $m \geq n$ we have $x_{k_m} \in F_m \subset \subset F_n \subset F_n$ and since $x_{k_n} \to z_0$, $z_0 \in \overline{F_n}$ (1.9). But each $F_n$ is closed, hence $z_0 \in F_n$. By letting $m$ be $\geq n$ for each $n$ we have $z_0 \in F_n$ for all $n$, i.e., $z_0 \in \bigcap_{n=1}^{\infty} F_n = F$.

Now since $f$ is continuous and $x_{k_n} \to z_0$, $f(x_{k_n}) \to f(z_0)$. But $f(x_n) = a$ for all $n$, thus the sequence $\{f(x_n)\}$ converges to $a$. And since $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$, $\{f(x_{k_n})\}$ is a subsequence of $\{f(x_n)\}$ hence $f(x_{k_n}) \to a$ (1.8). Therefore, $f(z_0) = a$ (1.7) and since $z_0 \in F$, $f(z_0) \in f(F)$, i.e., $a \in f(F)$. Similarly, since $b \in \bigcap_{n=1}^{\infty} f(F_n)$, $b \in f(F)$ and thus $a \cup b \subset f(F)$. Now $\overline{f(F)} = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} C F_n$, i.e., $I-F = \bigcup_{n=1}^{\infty} (I-F_n)$. Therefore, for any point $d$ of $(xy)$ there is a $k$ such that $d \in I-F_k$. $F_k$ is closed, hence $I-F_k$ is open and there is an $\epsilon > 0$ such that
Let $x_k$ be the first point of $F_k$ going from $d$ to $x$ and let $y_k$ be the first point of $F_k$ going from $d$ to $y$. Thus $(x_k, y_k)$ is a maximal segment of $I-F_k$, $x_k \leq d - \epsilon$, $y_k \geq d + \epsilon$ and since $F_k$ has property $P$, $f(x_k) = f(y_k)$. Similarly, if $x_{k+1}$ and $y_{k+1}$ are the first points of $F_{k+1}$ going from $d$ to $x$ and $y$ respectively, $(x_{k+1}, y_{k+1})$ is a maximal segment of $I-F_{k+1}$, $f(x_{k+1}) = f(y_{k+1})$ and

$$x \leq x_{k+1} \leq \cdots \leq x_k < y_k \leq \cdots \leq y_{k+1} \leq y.$$

Continuing this process we obtain $(x_{k+j}, y_{k+j})$ as a maximal segment of $I-F_{k+j}$, $f(x_{k+j}) = f(y_{k+j})$, and

$$x \leq x_{k+j} \leq \cdots \leq x_k < y_k \leq \cdots \leq y_{k+j} \leq y.$$

Clearly, $x$ is a lower bound for the sequence $\{x_{k+j}\}$ and $y$ is an upper bound for the sequence $\{y_{k+j}\}$. We shall show that these sequences converge to $x$ and $y$ respectively. Suppose not, i.e., suppose $\{x_{k+j}\}$ converges to a point $q$ of $q > x$. Since $q \in (xy)$, $q \in I-F = \bigcap_{n=1}^{\infty} (I-F_n)$, thus there is an $h$ such that $q \in I-F_h$. $I-F_h$ is open, hence there is a $\lambda > 0$ such that $N(q, \lambda) \subseteq I-F_h$. Let $\lambda$ be less than the minimum of $\lambda_1$ and $y_h - q$. Thus $N(q, \lambda) \subseteq I-F_h$. Since the sequence $x_k, x_{k+1}, \ldots$ converges to $q$, there is an $m$ such that if $s \geq m$ then $q \leq x_s < q + \lambda$. Let $r$ equal the maximum of $h$ and $m$, i.e., $r \geq h$ and $r \geq m$. Therefore, $q \leq x_r < q + \lambda$ and $(x_r, y_r)$ is a maximal segment of $I-F_r$. In addition to this since $r \geq h$ we have $\lambda < y_r - q$, thus $N(q, \lambda) \subseteq I-F_r$. Now $(x_r, y_r)$ is a non-empty interval, hence is a non-empty connected set. Therefore, there is only one component of $(x_r, y_r)$, namely $(x_r, y_r)$ itself (1.24). It is easily verified that

$$(q - \lambda, y_r) = (q - \lambda, q + \lambda) \cup (x_r, y_r),$$

and since $q \leq x_r$, this implies that $(x_r, y_r)$ is a proper subset of $(q - \lambda, y_r)$. But recall $N(q, \lambda) \subseteq I-F_r$ hence $(q - \lambda, y_r) \subseteq I-F_r$ and since
(q-λ,y_r) is an interval, it is a connected set. This is impossible, for (x_r,y_r) is a component of I-F_r hence cannot be the proper subset of a connected set of I-F_r. Thus we have a contradiction, i.e., \{x_k+j\} does not converge to a point q>x. Therefore, x_k+j→x.
Similarly y_k+j→y and since f is continuous we have f(x_k+j)→f(x) and f(y_k+j)→f(y). But f(x_k+j) = f(y_k+j) for all k+j, hence f(x) = f(y) (1.7) and property P is inducible.

Now I is a non-empty compact set having property P, thus by the Brouwer Reduction Theorem (3.8) there is a non-empty compact subset A of I which is irreducible with respect to property P. Let f(A) = T. We shall show that T is our required simple arc in S joining a and b.

T is compact (1.27). T is also connected, for if not then T has a separation E_1/E_2. Since T is closed \( \overline{T} = T \) and
\[
\overline{E_1} = \overline{E_1} \cap T = \overline{E_1} \cap (E_1 \cup E_2) = (\overline{E_1} \cap E_1) \cup (\overline{E_1} \cap E_2) = E_1,
\]
i.e., E_1 is closed. Similarly, E_2 is closed and by the continuity of f, both \( A \cap f^{-1}(E_1) \) and \( A \cap f^{-1}(E_2) \) are closed sets. They are also disjoint since
\[
f^{-1}(E_1) \cap f^{-1}(E_2) = f^{-1}(E_1 \cap E_2) = f^{-1}(\emptyset) = \emptyset.
\]
Furthermore, \( A = (A \cap f^{-1}(E_1)) \cup (A \cap f^{-1}(E_2)) \) for if c is a point of A then \( f(c) \in T = E_1 \cup E_2 \). Without loss of generality let \( f(c) \in E_1 \). Therefore, \( c \in f^{-1}(E_1) \), i.e., \( c \in (A \cap f^{-1}(E_1)) \cup (A \cap f^{-1}(E_2)) \), and clearly, if d is a point of \( (A \cap f^{-1}(E_1)) \cup (A \cap f^{-1}(E_2)) \), then \( d \in A \), hence we have the equality. Thus, there is a maximal segment (xy) in I-A such that \( x \in f^{-1}(E_1) \) and \( y \in f^{-1}(E_2) \). But this implies that
f(x) ∈ E₁ and f(y) ∈ E₂ which is impossible since E₁ ∩ E₂ = ∅ but A has property P, i.e., in particular, f(x) = f(y).

Let p ∈ T-(a ∪ b) and P' = A ∩ f⁻¹(p). P' ≠ ∅ since p ∈ f(A) implies there is a point t of A such that f(t) = p, i.e., t ∈ f⁻¹(p), thus t ∈ A ∩ f⁻¹(p). Similarly, A ∩ f⁻¹(a) and A ∩ f⁻¹(b) are non-empty sets. Let a₀ be the first point of [0, 1] in A. If a₀ = 0 then f(a₀) = f(0) = a. If a₀ > 0 then [0, a₀) is a maximal segment of I-A and again f(a₀) = f(0) = a. Thus in either case f(a₀) = a, i.e., a₀ ∈ f⁻¹(a) hence a₀ ∈ A ∩ f⁻¹(a). Similarly, if b₀ is the last point of [0, 1] in A we have f(b₀) = b and b₀ ∈ f⁻¹(b) hence b₀ ∈ A ∩ f⁻¹(b). It is readily seen that A ⊆ [a₀, b₀] and a₀ ∉ P' and b₀ ∉ P'.

Let p₁ be the first point of P' in [a₀, b₀] and let p₂ be the last point of P' in [a₀, b₀]. If p₁ = p₂ then P' = p₁ = p₂. If p₁ ≠ p₂ then p₁ < p₂. Suppose there is a point p₃ of A such that p₁ < p₃ < p₂. Let

D = A ∩ ([a₀, p₁] ∪ [p₂, b₀]).

D is a proper subset of A since p₃ ∈ A but p₃ ∉ D, and (a₀ ∪ b₀) ⊆ D thus (a₀ ∪ b₀) ⊆ f(D). Let (xy) be a maximal segment of I-D. If (xy) is [0, a₀) then f(0) = a = f(a₀), i.e., f(x) = f(y). Similarly, if (xy) is (b₀, 1], f(b₀) = b = f(1) and f(x) = f(y). If (xy) is (p₁, p₂) then since p₁ and p₂ belong to P', p₁ = f⁻¹(p) and p₂ = f⁻¹(p), i.e., f(p₁) = p = f(p₂) and again f(x) = f(y). Suppose a₀ ≤ x < y ≤ p₁ and let w be an arbitrary point of (x, y). Then w ∈ I-D and w ∈ [a₀, p₁] ∪ [p₂, b₀]. But w ∈ D, hence w ∈ A, i.e., w ∈ I-A. Thus (x, y) ⊆ I-A and since (xy) is a maximal segment of I-D, x and y belong to D which implies that x and y belong to A. Therefore,
(xy) is a maximal segment of I-A and \( f(x) = f(y) \). Similarly, if \( p_2 \leq x < y \leq b_0 \), \( f(x) = f(y) \). Now in every case \( f(x) = f(y) \), hence \( D \) has property \( P \). This is impossible since \( D \) is a proper subset of \( A \) and \( A \) has property \( P \) irreducibly. Thus, the assumption that there is a point \( p_3 \) of \( A \) such that \( p_1 < p_3 < p_2 \) is false and we see that \( P' = p_1 \cup p_2 \) where \( p_1 \) may equal \( p_2 \).

Let \( A_1 = A \cap [a_0, p_1) \) and \( A_2 = A \cap (p_2, b_0] \). Now
\[
A_1 \cup P' = (A \cap [a_0, p_1]) \cup p_2
\]
and
\[
A_2 \cup P' = (A \cap [p_2, b_0]) \cup p_1.
\]
Let \( T_1 = f(A_1) \) and \( T_2 = f(A_2) \). We shall show that \( T \cdot p = T_1 / T_2 \)
thus showing that any arbitrary point \( p \) of \( T \cdot (a \cup b) \) is a cut point of \( T \). First it is easily verified that
\[
[(A \cap [a_0, p_1]) \cup p_2] \cup [(A \cap [p_2, b_0]) \cup p_1] = A,
\]
i.e., \( A = A_1 \cup A_2 \cup P' \). Therefore, \( T = f(A) = f(A_1) \cup f(A_2) \cup f(P') = T_1 \cup T_2 \cup P \). Now, \( p \notin T_1 \) for if it did then \( p \in f(A_1) \) hence there is a point \( s \) of \( A_1 \) such that \( f(s) = p \). Now \( p_1 \) and \( p_2 \) do not belong to \( A_1 \) hence \( s \neq p_1 \) and \( s \neq p_2 \). Therefore, \( s \notin P' \), i.e.,
\[
s \in f^{-1}(p) = (A \cap f^{-1}(p)) = A \cap f^{-1}(p).
\]
Hence \( s \in \partial A \). But since \( s \in A_1 \), \( s \notin A \) and we have a contradiction. Thus, \( p \notin T_1 \). Similarly, \( p \notin T_2 \) and since \( T = T_1 \cup T_2 \cup P \) we have
\[
T \cdot p = T_1 \cup T_2.
\]
Clearly \( T_1 \neq \emptyset \) and \( T_2 \neq \emptyset \). To show that \( T_1 \cap T_2 = \emptyset \) suppose \( q \in T_1 \cap T_2 \). Thus, \( q \in f(A_1) \) and \( q \in f(A_2) \), and there are points \( q_1 \) of \( A_1 \) and \( q_2 \) of \( A_2 \) such that \( f(q_1) = q = f(q_2) \) where \( q_1 \neq p_1 \) and \( q_2 \neq p_2 \) since \( p_1 \notin A_1 \) and \( p_2 \notin A_2 \). Let \( E = A \cap ([a_0, q_1) \cup [q_2, b_0]) \). Now the identical reasoning by which we showed that \( D \) was a proper
subset of A and had property P, may be used to conclude that E also is a proper subset of A and has property P. Again this is impossible, hence the assumption that $T_1 \cap T_2 \neq \emptyset$ is false, i.e., $T_1 \cap T_2 = \emptyset$.

Clearly, $A_1 \cup P'$ is closed, hence $f(A_1 \cup P')$ is closed (1.27). But $f(A_1 \cup P') = f(A_1) \cup f(P') = T_1 \cup p$, thus $T_1 \cup p$ is closed. Since $T_1 \subset T_1 \cup p$ we have

$$\overline{T_1} \subset \overline{T_1 \cup p} = T_1 \cup p,$$

but $p \notin T_2$ hence $\overline{T_1} \cap T_2 = \emptyset$. Similarly, $T_1 \cap \overline{T_2} = \emptyset$ and we have shown that any point $p$ of $T-(a \cup b)$ is a cut point of $T$. In summary, $a \cup b \subset T$, $T \subset S$ and we have just shown that $T$ is a continuum of which all but at most two points (namely $a$ and $b$) are cut points. Thus by theorem 3.4 $T$ is a simple arc. Q.E.D.

A similar presentation of the Brouwer Reduction and Arcwise Connectedness Theorems may be found in Whyburn's Analytic Topology.

Since every Peano space satisfies the hypotheses of theorem 3.9 we see that every Peano space is arcwise connected, and with the aid of 1.22, part (6), the reader may easily verify the following theorem.

Theorem 3.10 Every arcwise connected space is connected.
CHAPTER IV

SIMPLE CLOSED CURVES AND PROPERTY S

The orientation and presentation of the definitions and theorems of this chapter are similar to those found in Hall and Spencer's Elementary Topology.

Definition 4.1 The unit circle is the subspace of $\mathbb{R}^2$ consisting of all points $(x,y)$ that satisfy the equation $x^2 + y^2 = 1$. (Throughout this chapter we shall let $J$ denote the unit circle.)

We have seen that $J$ is a Peano space (1.26). In addition, it is easily verified that every point of $J$ is a non-cut point and if $a$ and $b$ are any two points of $J$ then $J - (a \cup b)$ is not connected.

Definition 4.2 A space $C$ is a simple closed curve if and only if there is a topological mapping $f$ of the unit circle $J$ onto $C$. Thus in view of paragraph 1.28 if $C$ is a simple closed curve then open sets of $C$ correspond to open sets of $J$ and vice versa (open may be replaced by closed throughout).

We now see that just as the simple arc is related to the unit interval $[0,1]$ in $\mathbb{R}_1$, so is the simple closed curve related to the unit circle in $\mathbb{R}_2$.

Definition 4.3 Suppose $a$ and $b$ are distinct points of $S$ and $F_1$ and $F_2$ are simple arcs in $S$, each having $a$ and $b$ as its end points. Then we say that $F_1$ and $F_2$ are independent arcs from $a$ to $b$ if and only if $F_1 \cap F_2 = a \cup b$.

Theorem 4.4 Suppose $C$ is a non-degenerate space. Then $C$ is
a simple closed curve if and only if for every two distinct points \( a \) and \( b \) of \( C \), the space \( C \) may be expressed as the union of two independent arcs from \( a \) to \( b \).

**Proof.** First suppose \( C \) is a simple closed curve and let \( a \) and \( b \) be distinct points of \( C \). There is a topological mapping \( f \) of the unit circle \( J \) onto \( C \). Let \( a' = f^{-1}(a) \) and \( b' = f^{-1}(b) \). Therefore, \( a' \) and \( b' \) belong to \( J \) and since \( f \) is one-to-one, \( a' \neq b' \).

Clearly, there exists two independent arcs \( E_1 \) and \( E_2 \) in \( J \) joining \( a' \) and \( b' \), i.e., \( J = E_1 \cup E_2 \) and \( E_1 \cap E_2 = a' \cup b' \). Let \( f(E_1) = F_1 \) and \( f(E_2) = F_2 \). Now

\[
f(E_1) \cap f(E_2) = f(E_1 \cap E_2) = f(a' \cup b') = f(a') \cup f(b') = a \cup b,\]

i.e., \( F_1 \cap F_2 = a \cup b \) and obviously \( F_1 \cup F_2 = C \).

To show the converse let \( a \) and \( b \) be distinct points of \( C \) and suppose \( C \) is the union of two independent arcs \( F_1 \) and \( F_2 \) in \( C \) joining \( a \) and \( b \), i.e., \( C = F_1 \cup F_2 \) and \( F_1 \cap F_2 = a \cup b \). We must show that \( C \) is a simple closed curve. Let \( I = [0,1] \). Since \( F_1 \) and \( F_2 \) are simple arcs there exist topological mappings \( f_1 \) of \( F_1 \) onto \( I \) and \( f_2 \) of \( F_2 \) onto \( I \) such that \( f_1(a) = 0 \), \( f_2(a) = 0 \), \( f_1(b) = 1 \) and \( f_2(b) = 1 \). Clearly, \( J \) can be expressed as the union of two independent arcs \( J_1 \) and \( J_2 \) joining the points \((-1,0)\) and \((1,0)\).

Therefore, there exist topological mappings \( g_1 \) of \( I \) onto \( J_1 \) and \( g_2 \) of \( I \) onto \( J_2 \) such that \( g_1[(1,0)] = (1,0) \), \( g_2[(1,0)] = (1,0) \), \( g_1[(0,0)] = (-1,0) \) and \( g_2[(0,0)] = (-1,0) \). We shall now define a
mapping $f$ of $C$ onto $J$. If $x \in F_1$ let $f(x) = g_1[f_1(x)]$ and if $x \in F_2$ let $f(x) = g_2[f_2(x)]$. It readily follows that $f$ is a topological mapping of $C$ onto $J$, hence $C$ is a simple closed curve. Q.E.D.

Corollary. A space $C$ is a simple closed curve if $C$ contains two distinct points $a$ and $b$ and two independent arcs $F_1$ and $F_2$ from $a$ to $b$ such that $C = F_1 \cup F_2$.

Theorem 4.5 A space $C$ is a simple closed curve if and only if $C$ is a compact connected space such that $C-(a \cup b)$ is not connected.

Proof. Suppose $C$ is a simple closed curve. Then there exists a topological mapping $f$ of the unit circle $J$ onto $C$, thus $f(J) = C$ is compact and connected (1.27). Suppose $a$ and $b$ are points of $C$. We must show that $C-(a \cup b)$ is not connected. Since $f$ is a topological mapping there is a continuous, one-to-one mapping $g$ of $C$ onto $J$, hence there are points $u$ and $v$ of $J$ such that $g(a) = u$ and $g(b) = v$. Suppose $C-(a \cup b)$ is connected. Therefore, $g[C-(a \cup b)]$ is connected, but notice

$$g[C-(a \cup b)] = g[C \cap C(a \cup b)]$$

$$= g(C) \cap g(C(a \cup b))$$

$$= J \cap g(C(a \cup b))$$

$$= J \cap g(g(a) \cup g(b))$$

$$= J \cap g(u \cup v)$$

$$= J-(u \cup v),$$

i.e., $g[C-(a \cup b)] = J-(u \cup v)$ but $J-(u \cup v)$ is not connected (4.1). Thus, the assumption that $C-(a \cup b)$ is connected is false, i.e., $C-(a \cup b)$ is not connected.

To show the converse suppose $C$ is a compact connected space such that for every pair of points $a$ and $b$ of $C$ the set $C-(a \cup b)$ is
disconnected. First, we shall show that every point of \( C \) is a non-cut point of \( C \). Suppose otherwise, i.e., suppose there is a point \( z \) of \( C \) such that \( C-z \) is disconnected. Thus \( C-z \) has some separation \( P/Q \). Now \( \overline{P} = P \cup z \), \( \overline{Q} = Q \cup z \) and \( \overline{P} \) and \( \overline{Q} \) are connected sets (3.4). Since they are closed subsets of the compact space \( C \), \( \overline{P} \) and \( \overline{Q} \) are also compact. Thus each set \( \overline{P} \) and \( \overline{Q} \) has at least two non-cut points (3.5). Therefore, there exists points \( e \) and \( d \) such that \( e \not\in z \), \( d \not\in z \), \( e \in \overline{P}-z \), \( d \in \overline{Q}-z \) and both of the sets \( \overline{P}-e \) and \( \overline{Q}-d \) are connected. It follows that

\[
C-(e\cup d) = (\overline{P}\cup \overline{Q}) \cap C(e\cup d)
\]

\[
= [\overline{P} \cap C(e\cup d)] \cup [\overline{Q} \cap C(e\cup d)]
\]

\[
= [\overline{P} \cap C(e,d)] \cup [\overline{Q} \cap C(e,d)]
\]

\[
= (\overline{P} \cap C(e,d)) \cup (\overline{Q} \cap C(e,d))
\]

But \( \overline{P}-e \) and \( \overline{Q}-d \) clearly have the point \( z \) in common, hence

\[
(\overline{P}-e) \cup (\overline{Q}-d) = C-(e\cup d)
\]

is a connected set which is a contradiction since \( C-(e\cup d) \) is disconnected. Thus every point of \( C \) is a non-cut point.

Recall that \( a \) and \( b \) are arbitrary points of \( C \) and that there exists a separation \( C-(a\cup b) = A/B \). By a proof similar to that of theorem 3.4 it is easily seen that \( A = A\cup a\cup b \) and \( B = B\cup a\cup b \).

Now \( \overline{A} \) and \( \overline{B} \) are closed, \( \overline{A}\cup \overline{B} = C \), a connected set, and \( \overline{A}\cap \overline{B} = a\cup b \) hence by paragraph 1.22 \( \overline{A} \) or \( \overline{B} \) is connected. Without loss of generality let \( \overline{A} \) be the connected set. We shall show that \( \overline{B} \) is also connected, for if not then there exists a separation \( \overline{B} = B_1/B_2 \). There are two distinct cases to consider.
First, if \( a \in B_1 \) and \( b \in B_1 \) the reader can easily verify that \( C \) has the separation \((A \cup B_1)/B_2\). But this is impossible since \( C \) is connected. Similarly, if \( a \in B_2 \) and \( b \in B_2 \) we arrive at the contradiction that \( C \) is disconnected.

Therefore, suppose \( a \in B_1 \) and \( b \in B_2 \). Since we have \( B \neq \emptyset \), \( a, b \in B \), \( \overline{B} \) must contain at least three points. Let \( B_2 \) be the set with at least two points (a similar discussion would follow if \( B_1 \) were the set with at least two points). It can now be easily verified that

\[
C - b = (\overline{A} - b) \cup B_1/B_2 - b.
\]

This is, of course, impossible since no point of \( C \) is a cut point. Similarly, if \( a \in B_2 \) and \( b \in B_1 \) we arrive at a contradiction.

Therefore, both sets \( \overline{A} \) and \( \overline{B} \) are connected. Furthermore, since they are closed subsets of the compact space \( C \), they are compact. We shall show that each of these sets is an independent arc from \( a \) to \( b \). First, suppose that neither set is an arc from \( a \) to \( b \). Since each set \( \overline{A} \) and \( \overline{B} \) is a continuum, they both have at least two non-cut points (3.5). Thus, if neither \( a \) nor \( b \), or if just one of the points \( a \) or \( b \) is a non-cut point of \( \overline{A} \) then there is a point \( r \in \overline{A} - (a \cup b) \) such that \( \overline{A} - r \) is connected. Now if both \( a \) and \( b \) are non-cut points of \( \overline{A} \) the non-cut point \( r \) of \( \overline{A} - (a \cup b) \) still exists. For if not then by theorem 3.4 \( \overline{A} \) is a simple arc, and by the corollary to theorem 3.4, \( a \) and \( b \) are its end points which contradicts our assumption that \( \overline{A} \) was not an arc from \( a \) to \( b \). Thus we exclude this possibility and let \( r \) of \( \overline{A} - (a \cup b) \) be such that \( \overline{A} - r \) is connected. Similarly, there is a point \( s \in \overline{B} - (a \cup b) \) such that \( \overline{B} - s \) is connected. Since \( \overline{A} - r \) and \( \overline{B} - s \) have the common points \( a \) and \( b \),
(A-r) U (B-s) is a connected set. But notice

\[(A-r) U (B-s) = (A \cup B) \cap (C U \overline{B}) \cap (A \cup C) \cap (C U \overline{C})\]

\[= C \cap C \cap C \cap C\]

\[= C -(r \cup s),\]

i.e., C-(r U s) is connected which is impossible. Thus, one of the sets A or B is an arc from a to b. Without loss of generality let A be this set. Therefore, a and b are the only non-cut points of A. Let w be any point of A-(a U b). Now w is a cut point, hence A-w consists of exactly two components \(A_1\) and \(A_2\) containing a and b respectively (3.4, part (5)). Suppose \(B\) is not an arc from a to b. Then as before, there is a point s of \(B-(a U b)\) such that \(B-s\) is connected and \((a U b) \subseteq B-s\). Since \(b \in A_2\) and \(b \in (B-s)\), \(A_2 \cup (B-s)\) is the union of two connected sets which meet, thus is itself a connected set. Also \(a \in A_1\) and \(a \in (A_2 \cup (B-s))\) hence \(A_1 \cup A_2 \cup (B-s)\) is the union of two connected sets, thus is itself connected. But \(A_1 \cup A_2 \cup (B-s) = (A-w) \cup (B-s)\) and as we have seen above

\[(A-w) \cup (B-s) = C-(w \cup s).\]

This is impossible since \((A-w) \cup (B-s)\) is connected and \(C-(w \cup s)\) is disconnected. Finally, then both \(A\) and \(B\) are arcs from a to b and since \(A \cap B = a U b\), they are independent arcs. Also \(C = A \cup B\) and it follows from the corollary to theorem 4.4 that \(C\) is a simple closed curve. Q.E.D.

We have seen that the closed unit interval \([0,1]\) is a Peano space. Consider all neighborhoods of every point in \([0,1]\). Since these neighborhoods are intervals, they form a basis of connected sets for \([0,1]\) such that the closure of every non-empty element of this basis is itself a Peano space. In view of the Hahn-Mazurkiewicz

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Theorem, we are inclined to believe that every Peano space has a basis with such properties. Indeed every Peano space has a basis of connected sets such that the closure of every non-empty element of the basis is compact and connected, but the closures in such a basis are not necessarily locally connected. However, with the aid of Seirpinski's property $S$ we shall prove that in every Peano space there does exist a basis with these properties.

Definition 4.6 Suppose $H$ is a subset of the space $T$. We shall say that $H$ has property $S$ if and only if, for every $\varepsilon > 0$, $H$ can be expressed as the union of a finite number of connected sets, each having diameter less than $\varepsilon$. Clearly, then if $H$ has property $S$, $H$ is bounded.

Definition 4.7 Suppose $H$ is a subset of the space $T$. $H$ is termed totally bounded if and only if, for every $\varepsilon > 0$, there exists a set of points $x_1, x_2, \ldots, x_n$ of $T$ which are finite in number and such that $H \subseteq \bigcup_{i=1}^{n} N(x_i, \varepsilon)$.

Theorem 4.8 If a subset $H$ of the space $T$ has property $S$ then $H$ is totally bounded.

Proof. Suppose $H \subseteq T$ and $H$ has property $S$. Letting $\varepsilon > 0$ we have $H = \bigcup_{i=1}^{n} C_i$ where each set $C_i$ is connected and has diameter less than $\varepsilon$. Now, there is a point $x_i$ of $C_i$ for $i = 1, 2, \ldots, n$ and $C_i \subseteq N(x_i, \varepsilon)$ for all $i$. Thus, $H \subseteq \bigcup_{i=1}^{n} N(x_i, \varepsilon)$, i.e., $H$ is totally bounded. Q.E.D.

Theorem 4.9 Suppose $T$ is a space. If $A$ is the union of a finite number of subsets of $T$ each having property $S$ then $A$ has property $S$. 

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Proof. Suppose $A$ is the union of a finite number of subsets $K_i$, $i = 1, 2, \ldots, n$ of $T$ such that each $K_i$ has property $S$. Thus $A = \bigcup_{i=1}^{n} K_i$ and for every $\epsilon > 0$ and every $i$, $K_i = \bigcup_{j=1}^{m} E_{ij}^i$ where for each $i$ and $j$, $E_{ij}^i$ is connected, $d(E_{ij}^i) < \epsilon$ and $\bigcup_{j=1}^{m} E_{ij}^i$ is finite for $i = 1, 2, \ldots, n$. Therefore,
\[ A = \bigcup_{i=1}^{n} K_i = \bigcup_{j=1}^{m} \bigcup_{i=1}^{n} E_{ij}^i \]
and since the union of a finite set of finite sets if finite, $A$ has property $S$. Q.E.D.

Theorem 4.10 A space $T$ is locally connected if and only if for every point $x$ of $T$ and every open set $G$ containing $x$ there is an open set $H$ containing $x$ such that $H$ is contained in a single component of $G$.

Proof. Suppose $T$ is locally connected, $x$ is any point of $T$ and $G$ is an open set containing $x$. By virtue of paragraph 1.25 there is an open connected set $H$ containing $x$ and contained in $G$. Now $x \in G$, hence $x$ belongs to some component $K$ of $G$. Therefore, $H \cap K \neq \emptyset$ which implies $H \subseteq K$ (1.24). It follows since components are either equal or disjoint that $K$ is the only component of $G$ which contains $H$.

To show the converse let $G$ be an open set of $T$ and let $K_n$ be any component of $G$. If $x$ is any point of $K_n$ then $x \in T$, hence by supposition there is an open set $H$ containing $x$ and contained in a single component $K_m$ of $G$. Since $H$ is open, there is an $\epsilon > 0$ such that $N(x, \epsilon) \subseteq H$, hence $N(x, \epsilon) \subseteq H \subseteq K_m$.

Now $x \in K_m$ and $x \in K_n$, i.e., $K_n$ and $K_m$ are two components with a
common point, hence $K_n = K_m$. Thus $N(x, \varepsilon) \subseteq K_n$ and we see that for
any point $x$ of $K_n$ there is a neighborhood of $x$ contained in $K_n$, i.e.,
$K_n$ is an open set. Since $K_n$ is an arbitrary component of an arbi-
trary component of an arbitrary open set $G$ we conclude that com-
ponents of open sets are open, hence $T$ is locally connected (1.25).
Q.E.D.

**Theorem 4.11** Suppose $K$ is a subset of the space $T$. If $K$
has property $S$, then $K$ is locally connected.

**Proof.** Let $x$ be any point of $K$ and $\varepsilon > 0$. To show that $K$
is locally connected it shall be sufficient to show there is a $\delta > 0$ such
that if $y \in N(x, \delta) \cap K$ then there exists a connected set $C$ such that
$x \cup y \subseteq C$ and $C \subseteq N(x, \varepsilon) \cap K$ (1.25).

$K$ has property $S$, hence $K = \bigcup_{i=1}^{n} G_i$ where each $G_i$ is connected
and has diameter less than $\frac{1}{2} \varepsilon$. Since $x \in K$, there is some set $G_r$ of
$\bigcup_{i=1}^{n} G_i$ such that $x \in G_r$ and, therefore, $x \in \overline{G_r}$. Let

(1) $x \in \overline{G_i}$ for $i = 1, \ldots, j$

(2) $x \not\in \overline{G_i}$ for $i = j+1, \ldots, n$.

If $j = n$ then $x \in \overline{G_i}$ for all $i$. Let $\delta = \frac{1}{2} \varepsilon$ and let $y$ be a
point of $N(x, \delta) \cap K$. Thus $y \in K$, hence $y \in G_k$ for some $k$, and $y \in \overline{G_k}$.
But $x \in \overline{G_i}$ for all $i$, hence $x \in \overline{G_k}$. Now then $x \cup y \subseteq \overline{G_k}$ and $G_k$ is
connected, hence $\overline{G_k}$ is connected. Since $G_k \subseteq G_k \cup x \subseteq \overline{G_k}$ we see that
$G_k \cup x$ is connected and contains both $x$ and $y$. Furthermore,

$$d(G_k \cup x) = d(\overline{G_k}) = d(G_k) < \frac{1}{2} \varepsilon.$$ 

To show that $G_k \cup x \subseteq N(x, \varepsilon) \cap K$, let $z \in G_k \cup x$. Thus $z \in K$ and

$$\rho(z, x) \leq d(G_k \cup x) < \frac{1}{2} \varepsilon,$$

i.e., $z \in N(x, \frac{1}{2} \varepsilon)$ and we have

$$z \in N(x, \frac{1}{2} \varepsilon) \cap K \subseteq N(x, \varepsilon) \cap K.$$
Therefore, \( G_k \cup x \subset N(x, \varepsilon) \cap K \). Hence if \( j = n \), \( K \) is locally connected.

If \( j \neq n \) then (1) and (2) imply \( x \in G_{i+1} \) for \( i = j+1, \ldots, n \).

But each set \( G_{i+1} \) is open, hence there is a \( \delta_i \) for \( i = j+1, \ldots, n \), such that \( N(x, \delta_i) \subset G_{i+1} \), i.e., \( N(x, \delta_i) \cap G_i = \emptyset \), for \( i = j+1, \ldots, n \).

Let \( \delta = \delta_{j+1}, \ldots, \delta_n \) and \( y \in N(x, \delta) \cap K \). Thus, \( y \in G_{i+1} \) for \( i = j+1, \ldots, n \) hence \( y \) belongs to some \( G_s \) where \( 1 \leq s \leq j \). But \( x \notin G_{i+1} \) for \( i = 1, \ldots, n \) hence \( x \cup y \subset G_s \). As before, \( G_s \) is connected and since \( G_s \subset G_s \cup x \subset G_s \), \( G_s \cup x \) is connected. Also \( d(G_s \cup x) > \frac{1}{2} \varepsilon \), and if we let \( z \in G_s \) then \( z \in K \) and \( \rho(z, x) < \frac{1}{2} \varepsilon \), i.e., \( z \in N(x, \frac{1}{2} \varepsilon) \).

Hence we have \( z \in N(x, \varepsilon) \cap K \), i.e., \( G_s \cup x \subset N(x, \varepsilon) \cap K \) and again \( K \) is locally connected. Q.E.D.

**Corollary.** If a space \( T \) has property \( S \), then \( T \) is locally connected.

**Theorem 4.12** Suppose \( K \) is a compact subset of a space \( T \). Then \( K \) has property \( S \) if and only if \( K \) is locally connected.

**Proof.** By theorem 4.11 if \( K \) has property \( S \) then \( K \) is locally connected. Thus, suppose \( K \) is locally connected. Let \( x \) be a point of \( K \) and \( \varepsilon > 0 \). Consider \( \frac{\varepsilon}{2} \). The local connectedness of \( K \) implies there is a \( \delta > 0 \) such that if \( y \in N(x, \delta) \cap K \) then there exists a connected set \( C_y \) containing \( x \) and \( y \) and \( C_y \subset N(x, \frac{\varepsilon}{2}) \cap K \).

Let \( D_x = \bigcup C_y \) where \( \bigcup C_y \) is taken with respect to all points \( y \) of \( N(x, \delta) \cap K \). Now each set \( C_y \) is connected and contains the point \( x \) hence \( \bigcup C_y \) is connected, i.e., \( D_x \) is connected. Furthermore, since each \( C_y \subset N(x, \delta) \cap K \), \( D_x \subset N(x, \frac{\varepsilon}{2}) \cap K \), thus \( D_x \subset K \) and clearly, \( d(D_x) \leq \frac{2}{3} \varepsilon < \varepsilon \). Also \( y \in N(x, \delta) \cap K \) implies \( y \in \bigcup C_y = D_x \). Thus we have \( N(x, \delta) \cap K \subset D_x \) for any \( x \) of \( K \). Since \( T \) is open every point \( x \)
of K has a neighborhood \( N(x, \varepsilon_x) \) contained in T. Therefore, 
\[ K \subseteq \bigcup_{x \in K} N(x, \varepsilon_x), \text{ i.e., } \bigcup N(x, \varepsilon_x) \text{ for all } x \text{ of } K \]
is a covering of K. But K is compact, hence 
\[ K \subseteq \bigcup_{i=1}^{n} N(x_i, \varepsilon_{x_i}). \]

Now then 
\[ K = \left[ \bigcup_{i=1}^{n} N(x_i, \varepsilon_{x_i}) \right] \cap K \]
\[ = \bigcup_{i=1}^{n} [N(x_i, \varepsilon_{x_i}) \cap K] \]
\[ \subseteq \bigcup_{i=1}^{n} D_{x_i}, \]
i.e., \( K \subseteq \bigcup_{i=1}^{n} D_{x_i} \) but \( D_{x_i} \subseteq K \) for each \( x \) of K, hence \( \bigcup_{i=1}^{n} D_{x_i} \subseteq K \) and we have \( K = \bigcup_{i=1}^{n} D_{x_i} \). Furthermore, we have seen above that each \( D_{x_i} \) is connected and has diameter less than \( \varepsilon \) hence K has property S.

Q.E.D.

Corollary. If a space T is compact then T has property S if and only if T is locally connected.

In view of this corollary, the following theorem is immediate.

Theorem 4.13 Every Peano Space has property S.

Theorem 4.14 Suppose K is a subset of the metric space T and there is a set H of T such that \( K \subseteq H \subseteq \overline{K} \). Then if K has property S, H has property S.

Proof. Let \( \varepsilon > 0 \). Since K has property S, \( K = \bigcup_{i=1}^{n} G_i \) where each set \( G_i \) is connected and has diameter less than \( \varepsilon \). Now 
\[ \overline{K} = \bigcup_{i=1}^{n} \overline{G_i} = \bigcup_{i=1}^{n} G_i \text{ and since each } G_i \text{ is connected each } \overline{G_i} \text{ is connected. Also } d(\overline{G_i}) = d(G_i) < \varepsilon. \]
(Thus we have proved that if K has property S then \( \overline{K} \) has property S). Since \( K \subseteq H \subseteq \overline{K} \) we have
Furthermore, \( G_i \subseteq \overline{G}_i \) and \( G_i \subseteq K \subseteq H \) for each \( i \) hence \( G_i \subseteq H \cap \overline{G}_i \subseteq \overline{G}_i \).

Therefore, \( H \cap \overline{G}_i \) is a connected set for each \( i \) (1.22) and since 
\[ d(H \cap \overline{G}_i) \leq d(\overline{G}_i) < \epsilon \]
we see that \( H \) is the union of a finite number of connected sets \( H \cap \overline{G}_i, \ i = 1, \ldots, n \), each having diameter less than \( \epsilon \) hence \( H \) has property S. Q.E.D.

By virtue of theorems 4.12 and 4.14, and since \( K \subseteq \overline{K} \subseteq K \), the following theorem is now immediate.

**Theorem 4.15.** Let \( K \) be a subset of a metric space \( T \). If \( K \) has property S then \( \overline{K} \) has property S. Hence if \( K \) has property S, \( \overline{K} \) is locally connected.

Recall that we are attempting to show that every Peano space \( P \) has a basis of connected sets such that the closure of every non-empty element of this basis is itself a Peano space. We have seen that the difficulty is not in finding a basis of non-empty connected sets whose closures are compact and connected, but in finding such a basis of sets whose closures are also locally connected. In view of theorem 4.15, we are now able to overcome this difficulty if we can show that there is a basis of connected sets each non-empty element of which has property S.

**Definition 4.16** Suppose \( H \) and \( K \) are subsets of the metric space \( T \) and \( \epsilon > 0 \). We shall say that \( K \) is an \( \epsilon \)-growth of \( H \) if and only if the following conditions hold:

(a) If \( x \in K \) then there is a connected subset \( A \) of \( K \), such that \( x \in A, A \cap H \neq \emptyset \) and \( d(A) < \epsilon \).
There is a $\delta > 0$, such that, if $B$ is any connected subset of $T$ satisfying $d(B) < \delta$ and $B \cap H \neq \emptyset$, then $B \subset K$.

Theorem 4.17 Suppose $H$ and $K$ are subsets of a metric space $T$ and there is an $\epsilon > 0$ such that $K$ is an $\epsilon$ growth of $H$. Then

(i) $H \subset K$.

(ii) If $H$ is connected then $K$ is connected.

(iii) $K$ is contained in the $\epsilon$ neighborhood of the set $H$ and thus $d(K) \leq d(H) + 2\epsilon$.

(iv) If $H = \emptyset$ then $K = \emptyset$.

Proof. To prove (i) suppose $H \notin K$. Therefore, there is a point $x$ of $H$ such that $x \notin K$. Now the single point $x$ is a connected subset of $T$ and $d(x) = 0$. Thus $d(x)$ is less than any positive number, in particular, $d(x) < \delta$ where $\delta$ is the $\delta$ of condition (b) in 4.16. Since $x \cap H \neq \emptyset$ we should, by condition (b) of 4.16, have $x \subset K$. But $x \notin K$ hence we have a contradiction, Thus $H \subset K$.

To prove (ii) suppose $H$ is connected but $K = E/F$. Since $H$ is connected and $H \subset K$ by (i), $H$ must be contained in either $E$ or $F$ (1.22). Without loss of generality let $H \subset E$. Now $F \neq \emptyset$ and $E \cap F = \emptyset$, hence there is a point $y \in F \subset K$ such that $y \notin E$. Since $y \in K$, by condition (a) of 4.16 above, there is a connected subset $A$ of $K$ such that $y \in A$, $A \cap H \neq \emptyset$ and $d(A) < \epsilon$. But $A$, being a connected subset of $K$ must be contained in either $E$ or $F$ (1.22). Thus, it cannot meet both $E$ and $F$ at the same time, hence it cannot contain $y$ and meet $H$ at the same time. Therefore, we have a contradiction of (b) of 4.16, hence $K$ is connected.

To prove (iii) let $u$ and $v$ be any points of $K$. By part (a) of 4.16 there is a connected subset $A$ of $K$ such that $u \in A$, $A \cap H \neq \emptyset$
and $d(A) < \varepsilon$. Similarly there is a connected subset $B$ of $K$ such that $v \in B$, $B \cap H \neq \emptyset$ and $d(B) < \varepsilon$. Let $p$ and $q$ be any points of $A \cap H$ and $B \cap H$ respectively. Thus, we have $\rho(x,p) \leq d(A)$, $\rho(q,y) \leq d(B)$ and $\rho(p,q) \leq d(H)$. Hence

$$\rho(x,y) \leq \rho(x,p) + \rho(p,q) + \rho(q,y) \leq d(A) + d(H) + d(B) \leq d(H) + 2\varepsilon.$$ 

But this is true for any points $x$ and $y$ of $K$ hence $d(K) \leq d(H) + 2\varepsilon$.

To prove (iv) suppose $H = \emptyset$ but $K \neq \emptyset$. Thus, there is a point $x$ of $K$. Then by condition (a) of 4.16 there is a connected subset $A$ of $K$ such that $A \cap H \neq \emptyset$. But since $H = \emptyset$, $H$ does not meet any set. Thus we have our contradiction, hence $K = \emptyset$. Q.E.D.

Theorem 4.18 Suppose $H$ is a subset of a metric space $T$. Then given any $\varepsilon > 0$ there is a subset $K$ of $T$ such that $K$ is an $\varepsilon$ growth of $H$.

Proof. If $H = \emptyset$ it is clear from 4.16 that $K = \emptyset$ is an $\varepsilon$ growth of $H$. Suppose $H \neq \emptyset$. If a point $p$ belongs to $H$ let $W(p)$ be the union of all connected subsets $C$ of $T$ such that $p \in C$ and $d(C) < \varepsilon$. Let $K = \bigcup_{p \in H} W(p)$. Now if $x \in K$, then $x \in W(p)$ for some point $p$ of $H$. Thus $x$ belongs to some set $C$ of $W(p)$ where $C$ is connected, $d(C) < \varepsilon$ and $p \in C$ hence $C \cap H \neq \emptyset$. Also, $C \subseteq W(p) \subseteq K$ hence $C \subseteq K$ and we see that $K$ satisfies condition (a) of 4.16. To show that $K$ satisfies condition (b) let $\varepsilon = \varepsilon$. Suppose there is a connected set $D$, $D \cap H \neq \emptyset$ and $d(D) < \varepsilon (= \varepsilon)$. Thus there is a point $p \in D \cap H$, i.e., $p \in D$ and $p \in H$. It follows that since $d(D) < \varepsilon$, $D \subseteq W(p) \subseteq K$ thus $D \subseteq K$. Since $H$ and $K$ satisfy both conditions (a) and (b) of definition 4.16, $K$ is an $\varepsilon$ growth of $H$. Q.E.D.
Definition 4.19 Suppose $H$ and $K$ are subsets of a metric space $T$ and $\varepsilon > 0$. We shall say that $K$ is an $\varepsilon$-sequential growth of $H$ if and only if there exists a sequence of positive numbers $\{\varepsilon_i\}$ and a sequence of subsets $\{H_i\}$ of $T$ such that

(a) $H_1$ is an $\varepsilon_1$ growth of $H$, $H_2$ is an $\varepsilon_2$ growth of $H_1$, and in general $H_{i+1}$ is an $\varepsilon_{i+1}$ growth of $H_i$ for $i = 1, 2, \ldots$.

(b) $K = \bigcup_{i=1}^{\infty} H_i$.

(c) $\sum_{i=1}^{\infty} \varepsilon_i \leq \varepsilon$.

Theorem 4.20 Suppose $H$ and $K$ are subsets of a metric space $T$ and $\varepsilon > 0$. Then

(i) There exists and $\varepsilon$-sequential growth of $H$.

(ii) If $K$ is an $\varepsilon$-sequential growth of $H$, then

(01) $K \subseteq K$

(02) If $H$ is connected, then $K$ is connected

(03) $K \subseteq N(H, \varepsilon)$, hence $d(K) < d(H) + 2\varepsilon$

(04) If $H = \emptyset$, then $K = \emptyset$.

Proof. Let $\varepsilon_i = \frac{1}{2^i} \varepsilon$, $i = 1, 2, \ldots$. First if $H = \emptyset$ we shall see that $K = \emptyset$ is an $\varepsilon$-sequential growth of $H$. By theorems 4.17 and 4.18, $H_1 = \emptyset$ is clearly an $\varepsilon_1$ growth of $H$, $H_2 = \emptyset$ is an $\varepsilon_2$ growth of $H_1$ and in general $H_{i+1} = \emptyset$ is an $\varepsilon_{i+1}$ growth of $H_i$. Thus we have a sequence $\{\varepsilon_i\}$ of positive numbers and a sequence of subsets $\{H_i\}$ of $T$ such that (a) of definition 4.19 is satisfied. Letting $F = \emptyset$, we have $F = \bigcup_{i=1}^{\infty} H_i = \emptyset$ and since $\sum_{i=1}^{\infty} \varepsilon_i \leq \varepsilon$ we see that $F$ is an $\varepsilon$-sequential growth of $H$ thus (i) is satisfied. To show (ii) suppose $K^*$ is an $\varepsilon$-sequential growth of $H = \emptyset$. Then there is a sequence of positive numbers $\{\varepsilon_i\}$ and a sequence of
subsets \{H_i\} of \(T\) such that (a), (b) and (c) of definition 4.19 are satisfied. If we can verify \((0^\ast)\) first, then \((0_1), (0_2),\) and \((0_3)\) will readily follow. Since \(H = \emptyset\) we see from theorem 4.17 that \(H_1 = \emptyset\). Similarly, since \(H_1 = \emptyset\) we have \(H_2 = \emptyset\). In general then \(H_i = \emptyset\) for each \(i = 1, 2, \ldots\), thus \(\bigcup_{i=1}^{f} H_i = \emptyset\). But \(K^* = \bigcup_{i=1}^{f} H_i\) by definition 4.16, therefore, \(K^*\) must be the empty set, hence \((0_1), (0_2),\) and \((0_3)\) follow immediately.

Suppose \(H \neq \emptyset\) and recall \(\epsilon_i = \frac{1}{2^i} \epsilon\) for \(i = 1, 2, \ldots\). Since \(H \subset T\), theorem 4.18 implies there is a subset \(H_1\) of \(T\) such that \(H_1\) is an \(\epsilon_1\) growth of \(H\). Similarly, there is a subset \(H_2\) of \(T\) such that \(H_2\) is an \(\epsilon_2\) growth of \(H_1\) and, in general, there is a subset \(H_{i+1}\) of \(T\) such that \(H_{i+1}\) is an \(\epsilon_{i+1}\) growth of \(H_i\). Thus we have a sequence of positive numbers \(\{\epsilon_i\}\) and a sequence of subsets \(\{H_i\}\) of \(T\) such that (a) of 4.19 is satisfied. Also, \(\sum_{i=1}^{f} \epsilon_i = \frac{1}{2} \epsilon + \frac{1}{4} \epsilon + \ldots\) hence \(\sum_{i=1}^{f} \epsilon_i \leq \epsilon\). Letting \(G = \bigcup_{i=1}^{f} H_i\) we then see that \(G\) is an \(\epsilon\)-sequential growth of \(H\) and we have proved (i). To show (ii) suppose \(K\) is any \(\epsilon\)-sequential growth of \(H\). Thus, there is a sequence of positive numbers \(\{\epsilon_i\}\) and a sequence of subsets \(\{H_i\}\) of \(T\) such that (a), (b) and (c) of 4.19 are satisfied. Suppose \(H \not\subseteq K\), thus \(H \not\subseteq \bigcup_{i=1}^{f} H_i = K\). But since \(H_1\) is an \(\epsilon\) growth of \(H\), theorem 4.14 implies \(H \subset H_1\) hence \(H \subset \bigcup_{i=1}^{f} H_i = K\). Therefore, we have a contradiction, hence \(H \subseteq K\) and \((0_1)\) is verified. To verify \((0_2)\) suppose \(H\) is connected but \(K = E/F\). Now \(H\) is contained in either \(E\) or \(F\), hence without loss of generality let \(H \subseteq E\). Clearly, \(H_1 \subseteq K\) and theorem 4.17 implies \(H_1\) is connected. Therefore, \(H_1\) is contained in either \(E\) or \(F\) but since \(H \subset H_1\), \(H_1\) must be contained in \(E\). Continuing this process, we see that since each set \(H_i\) is connected
and \( H_i \subseteq H_{i+1} \) for all \( i \) we must have each set \( H_i \) contained in \( E \).

Thus \( \bigcup_{i=1}^{\infty} H_i \subseteq E \) hence \( K \subseteq E \) which contradicts the assumption that 
\( K = E/F \). Therefore, \( K \) is connected. Now \( H_1 \subseteq N(H, \varepsilon_1) \) and similarly 
\( H_2 \subseteq N(H_1, \varepsilon_2) \subseteq N[N(H, \varepsilon_1), \varepsilon_2] \) (4.17). In general,
\[
H_i \subseteq N(H, \sum_{i=1}^{\infty} \varepsilon_i) \subseteq \cdots \subseteq N(H, \sum_{i=1}^{\infty} \varepsilon_i).
\]
Since this is true for all \( i \), we have 
\( K = \bigcup_{i=1}^{\infty} H_i \subseteq N(H, \sum_{i=1}^{\infty} \varepsilon_i) \subseteq N(H, \varepsilon) \) since \( \sum_{i=1}^{\infty} \varepsilon_i \leq \varepsilon \), i.e., \( K \subseteq N(H, \varepsilon) \). It readily follows that 
\( d(K) < d(H) + 2\varepsilon \) and we have verified \((O_2)\). Since we have shown 
earlier that if \( H = \emptyset \) then \( K = \emptyset \), the theorem is therefore proved.

Q.E.D.

**Theorem 4.21** Suppose \( T \) is a metric space with property \( S \),
\( H \) and \( K \) are subsets of \( T \) and \( \varepsilon > 0 \). If \( K \) is an \( \varepsilon \)-sequential growth 
of \( H \), then \( K \) has property \( S \) and is open in \( T \).

**Proof.** Suppose \( K \) is a \( \varepsilon \)-sequential growth of \( H \). Then there 
are sequences \( \{\varepsilon_i\} \) and \( \{H_i\} \) with the properties given in definition 
4.19. Let \( x \) be any point in \( K \). Since \( K = \bigcup_{i=1}^{\infty} H_i \) there is a set \( H_j \) 
such that \( x \in H_j \). Now \( T \) has property \( S \), hence \( T \) is locally connected 
by the corollary to theorem 4.11. \( H_{j+1} \) is an \( \varepsilon_{j+1} \)-growth of \( H_j \).

Let \( \delta_{j+1} > 0 \) where \( \delta_{j+1} \) is the \( \delta \) given in definition 4.16, and con-
sider the open neighborhood \( N(x, \frac{\delta_{j+1}}{2}) \) in \( T \). Let \( V \) be the component 
of \( N(x, \frac{\delta_{j+1}}{2}) \) containing \( x \). Thus \( V \) is a component of an open set 
which is contained in the locally connected space \( T \), hence \( V \) is 
open. Since \( V \subseteq N(x, \frac{\delta_{j+1}}{2}) \), \( d(V) \leq \frac{2\delta_{j+1}}{2} < \delta_{j+1} \). Also \( V \cap H_j \neq \emptyset \) 
since both sets contain the point \( x \). Thus by definition 4.16 
\( x \in V \subseteq H_{j+1} \subseteq K \). But \( V \) is open, hence there is a \( \lambda > 0 \) such that 
\( N(x, \lambda) \subseteq V \), i.e., \( x \in N(x, \lambda) \subseteq V \subseteq K \). And we see that for any point \( x \)
of \( K \) there is a neighborhood of \( x \) contained in \( K \), therefore, \( K \) is open.

We shall now show that \( K \) has property \( S \). To do this it will be sufficient to show that for any \( \beta > 0 \), \( K \) can be expressed as the union of a finite number of connected sets each having diameter less than \( \beta \). Clearly, there is an integer \( k \) such that \( \sum_{i=k}^{\infty} \varepsilon_i < \frac{\beta}{4} \) and by definition 4.16 there is a \( \delta_{k+1} > 0 \) such that if \( B \) is a connected subset of \( T \), \( d(B) < \delta_{k+1} \) and \( B \cap H_k \neq \emptyset \) then \( B \subseteq H_{k+1} \).

Since \( T \) has property \( S \), \( T = \bigcup C_i \) where \( \bigcup C_i \) is a finite collection of sets such that each set \( C_i \) is connected and has diameter less than the minimum of \( \frac{\beta}{4} \) and \( \delta_{k+1} \). \( H_k \subseteq T \) hence let \( G_1, G_2, \ldots, G_n \) be those sets which intersect \( H_k \). Now each of these sets \( G_1, \ldots, G_n \) is connected, contained in \( T \), meets \( H_k \) and has diameter less than \( \delta_{k+1} \) hence \( \bigcup_{i=1}^{n} G_i \subseteq H_{k+1} \) (4.16). But \( H_k \subseteq T = \bigcup_{i=1}^{n} G_i \) and if \( x \in H_k \) then \( x \) belongs to some of the sets \( G_1, \ldots, G_n \). Therefore, \( H_k \subseteq \bigcup_{i=1}^{n} G_i \subseteq H_{k+1} \). We shall now define a set of points \( W_i \) for each \( i = 1, 2, \ldots, n \), in the following manner. A point \( y \in W_i \) if and only if \( y \in K \), \( y \in C \) where \( C \) is a connected subset of \( K \), \( d(C) < \frac{\beta}{4} \) and \( C \cap G_j \neq \emptyset \). Now \( d(W_j) < \beta \) for \( i \leq j \leq n \) because if \( z \in W_j \) then \( z \in K \), \( z \in D \) where \( D \) is a connected subset of \( K \), \( d(D) < \frac{\beta}{4} \) and \( D \cap G_j \neq \emptyset \). Assuming \( y \in W_j \) as above let \( u \) and \( v \) be points of \( C \cap G_j \) and \( D \cap G_j \) respectively. Then \( \rho(y, u) \leq d(C) < \frac{\beta}{4} \), \( \rho(u, v) \leq d(G_j) < \frac{\beta}{4} \) and \( \rho(v, z) \leq d(D) < \frac{\beta}{4} \). Therefore, applying the triangle inequality twice we clearly have \( \rho(y, z) \leq \frac{3\beta}{4} < \beta \). Since \( y \) and \( z \) are arbitrary points of \( W_j \) we have \( d(W_j) < \beta \), for \( 1 \leq j \leq n \). Also \( G_j \subseteq W_j \) for \( 1 \leq j \leq n \), for if \( r \in G_j \) then \( r \in K \), \( r \in G_j \) where \( G_j \) is a connected subset of \( K \), \( d(G_j) < \frac{\beta}{4} \) and \( G_j \cap G_j \neq \emptyset \), thus by our
definition of $W_j$, $x \in W_j$, i.e., $G_j \subseteq W_j$. Furthermore, each $W_j$ is connected for $1 \leq j \leq n$. To show this suppose $x \in W_j$, thus $x \in C_x$ where $C_x$ is a connected subset of $K$, $d(C_x) < \frac{\rho}{4}$ and $C_x \cap G_j \neq \emptyset$.

Clearly, $W_j = \bigcup_{x \in W_j} C_x$ and since $G_j \subseteq W_j$, $W_j = \left( \bigcup_{x \in W_j} C_x \right) \cup G_j$. Now each set $C_x$ is connected and meets the connected set $G_j$ hence $(\bigcup_{x \in W_j} C_x) \cup G_j = W_j$ is connected (1.22).

We shall see that $K = \bigcup_{i=1}^{n} W_i$. Since it is easily seen that $\bigcup_{i=1}^{n} W_i \subseteq K$, we need only show that $K \subseteq \bigcup_{i=1}^{n} W_i$. Let $x$ be any point of $K$. Thus $x \in H_i$ for some $i$. If $i \leq k$ (where $k$ is the same $k$ previously used) then $x \in H_i \subseteq C_i \subseteq \bigcup_{i=1}^{n} W_i$, i.e., $x \in \bigcup_{i=1}^{n} W_i$. Suppose $i > k$ and recall that $H_i$ is an $\varepsilon_i$ growth of $H_{i-1}$. Then by the definition of an $\varepsilon$ growth (4.16), $x \in H_i$ implies there is a connected set $L_i$ such that $L_i \subseteq H_i$, $x \in L_i$, $L_i \cap H_{i-1} \neq \emptyset$ and $d(L_i) < \varepsilon_i$. Let $x_{i-1} \in L_i \cap H_{i-1}$. Again $H_{i-1}$ is an $\varepsilon_{i-1}$ growth of $H_{i-2}$ hence $x_{i-1} \in H_{i-1}$ implies there is a connected set $L_{i-1}$, $L_{i-1} \subseteq H_{i-1}$, $x_{i-1} \in L_{i-1}$, $L_{i-1} \cap H_{i-2} \neq \emptyset$ and $d(L_{i-1}) < \varepsilon_{i-1}$. Let $x_{i-2} \in L_{i-1} \cap H_{i-2}$. Continuing this process we have a connected set $L_{k+1}$, $L_{k+1} \subseteq H_{k+1}$, $x_{k+1} \in L_{k+1}$, $d(L_{k+1}) < \varepsilon_{k+1}$ and $x_k \in L_{k+1} \cap H_k$. Let $L = \bigcup_{m=k+1}^{n} L_m$. Then $L_{k+1} \cap H_k \neq \emptyset$ implies $L \cap H_k \neq \emptyset$. Now recall from elementary point set theory that if we have two sets $A_1$ and $A_2$ such that $A_1 \cap A_2 \neq \emptyset$ then $d(A_1 \cup A_2) \leq d(A_1) + d(A_2)$. Furthermore, by induction, it can be shown that if sets $A_1, A_2, \ldots, A_n$ are such that $A_i \cap A_{i-1} \neq \emptyset$ for $i = 2, \ldots, n+1$ then $d(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} d(A_i)$. Applying this to our case, we see that we have sets $L_{k+1}, L_{k+2}, \ldots, L_i$ such that $L_{k+r} \cap L_{k+r+1} \neq \emptyset$ for each $r$, hence $d(L) = d(\bigcup_{m=k+1}^{i} L_m) \leq \sum_{m=k+1}^{i} d(L_m) = \varepsilon_{k+1} + \varepsilon_{k+2} + \cdots + \varepsilon_i = \sum_{m=k}^{i} \varepsilon_m < \frac{\varepsilon}{4}$, i.e., $d(L) < \frac{\varepsilon}{4}$. Since $x \in L_i$, $x \in L_i$. Now each $L_m$ is connected, and, in general, we have $x_{m-1} \in L_m \cap L_{m-1}$ hence $L = \bigcup_{m=k+1}^{n} L_m$. 

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is connected. Also \( L_n \subseteq H_n \) for \( m = k+1, k+2, \ldots, i \), thus
\[
\bigcup_{m=k+1}^{m=i} L_m \subseteq \bigcup_{m=k+1}^{m=i} H_m \subseteq K.
\]
Therefore, \( L \subseteq K \). Recall that \( H_k \subseteq \bigcup_{i=1}^{G_i} \) and that \( L \cap H_k \neq \emptyset \), hence there is some set \( G_s \), where \( 1 \leq s \leq n \), such that \( L \cap G_s \neq \emptyset \). In summary, then \( x \in L \) where \( L \) is a connected subset of \( K \), \( d(L) < \frac{\beta}{4n} \) and \( L \cap G_s \neq \emptyset \) for \( 1 \leq s \leq n \), thus \( x \in W_s \) and we have \( x \in \bigcup_{i=1}^{n} W_i \). We now see that any arbitrary point \( x \) of \( K \) is such that \( x \in \bigcup_{i=1}^{n} W_i \).

Therefore, \( K = \bigcup_{i=1}^{n} W_i \), i.e., \( K \) is the union of a finite number of connected sets each having diameter less than an arbitrary positive number \( \beta \) hence \( K \) has property \( S \). Q.E.D.

Theorem 4.22 Suppose \( T \) is a metric space having property \( S \). Then \( T \) has a basis, every element of which is an open connected set having property \( S \).

Proof. Let \( V_n(p), n = 1, 2, \ldots \) be a countable basis for an arbitrary point \( p \) of \( T \) (1.18). Since each \( V_n(p) \) is an open set containing \( p \) there is an \( \epsilon_n > 0 \) such that \( N(p, \epsilon_n) \subseteq V_n(p) \) for \( n = 1, 2, \ldots \). Consider the neighborhood \( N(p, \frac{1}{2}\epsilon_n) \). Clearly, \( N(p, \frac{1}{2}\epsilon_n) \subseteq N(p, \epsilon_n) \). Let \( W_n(p) \) be the component of \( N(p, \frac{1}{2}\epsilon_n) \) containing the point \( p \). Since \( T \) is locally connected (4.11) and \( N(p, \frac{1}{2}\epsilon_n) \) is open we see that \( W_n(p) \) is an open connected set such that
\[
W_n(p) \subseteq N(p, \frac{1}{2}\epsilon_n) \subseteq V_n(p),
\]
for \( n = 1, 2, \ldots \). In view of theorem 4.20 there is an \( \epsilon \)-sequential growth \( K_n(p) \) of \( W_n(p) \) for each \( n = 1, 2, \ldots \) and for \( \epsilon = \frac{1}{2}\epsilon_n \). Now since \( T \) has property \( S \) and each \( W_n(p) \) is connected we see by theorem 4.21 that each \( K_n(p) \) is an open connected subset of \( T \).
having property S. Also for each $n = 1, 2, \ldots$

$$W_n(p) \subseteq K_n(p) \subseteq N(W_n(p), \frac{1}{n} \epsilon_n)$$

by theorem 4.20, but

$$N(W_n(p), \frac{1}{n} \epsilon_n) \subseteq [N(p, \frac{1}{n} \epsilon_n), \frac{1}{n} \epsilon_n] \subseteq N(p, \epsilon_n) \subseteq V_n(p),$$

i.e., in particular, $K_n(p) \subseteq V_n(p)$ for $n = 1, 2, \ldots$. We readily see that the collection of sets $K_n(p)$, $n = 1, 2, \ldots$ is a basis at $p$, for if $H$ is any open subset of $T$ containing $p$ then there is a $V_j(p) \subseteq H$, but also $K_j(p) \subseteq V_j(p)$, i.e., there is a set $K_j(p)$, such that $K_j(p) \subseteq H$. Since $p$ was an arbitrary point of $T$, the collection of sets $K_n(p)$ for $n = 1, 2, \ldots$ and for all points $p$ of $T$ is a basis of the space $T$. Hence this is a basis of open connected sets each of which has property S and the theorem is proved. Q.E.D.

We have seen that every Peano space is a metric space having property S. Thus the preceding theorem implies that every Peano space has a basis of open connected sets each of which has property S. We shall state this in a more formal manner in theorem 4.27, but first we briefly examine how the ideas of property S, local arcwise connectedness and local compactness are related.

Definition 4.23 Let $T$ be a space. Then $T$ is said to be locally arcwise connected if and only if, for any point $p$ of $T$ and any open set $H$ of $p$ in $T$, there exists an open set $G$ of $p$ in $T$ such that if $x \in G$, $y \in G$ and $x \neq y$ then there exists an arc in $H$ joining $x$ and $y$.

With the aid of 1.25, part (2), the reader may easily verify that every locally arcwise connected space is locally connected.

Definition 4.24 Let $T$ be a space. Then $T$ is said to be locally compact if and only if, for any point $p$ of $T$ and any open
set $H$ of $p$, there is an open set $G$ of $p$ such that $G \subseteq H$ and $\overline{G}$ is compact.

It is readily seen that the definition of compactness 1.21 implies that every compact space is locally compact.

**Theorem 4.25** If $T$ is a locally compact metric space having property $S$ then $T$ is locally arcwise connected.

**Proof.** Suppose $T$ is a locally compact metric space having property $S$. Let $p$ be any point of $T$ and let $A$ be any open set containing $p$. Since $T$ has property $S$, $T$ is locally connected. Thus, if $W$ is the component of $A$ containing $p$, then $W$ is an open connected set.

If we can show that $W$ is arcwise connected we will have proved the theorem, for $W$ is now an open set containing $p$ and contained in $A$ where $A$ is any open set containing $p$. Furthermore, if $W$ is arcwise connected then any two points of $W$ may be joined by an arc contained in $W$, thus contained in $A$, hence by definition 4.23 $T$ will be locally arcwise connected. We shall now show that $W$ is indeed arcwise connected.

Let $z$ be any point in $W$. Since $T$ is locally compact there exists an open set $H$ such that $z \in H$, $H \subseteq W$ and $\overline{H}$ is compact. Since $T$ is a metric space there exists an open set $G$ such that

$$z \in G \subseteq \overline{G} \subseteq \overline{CH} \subseteq W \quad (1.11).$$

Thus, $\overline{G} \subseteq H$, i.e., $G$ is a closed subset of the compact set $H$, hence $G$ is compact. Now $T$ is a metric space having property $S$ hence theorem 4.22 implies $T$ has a basis every element of which is an open connected set having property $S$. Therefore, $G$ is the union of some of the elements of this basis and since $z \in G$ there is a
set $V(z)$ of the basis such that $z \in V \subseteq G$. Thus, in addition to being open, connected and having property $S$, the set $V(z)$ is such that

$$z \in V(z) \subseteq \overline{V(z)} \subseteq \overline{G} \subseteq W$$

and since $\overline{V(z)}$ is the closed subset of the compact set $G$, $\overline{V(z)}$ compact. Furthermore, $z$ was an arbitrary point of $W$ hence the collection of sets $\bigcup_{z \in W} V(z)$ is an open covering of $W$. Let $x$ and $y$ be distinct points of $W$. Let $P$ be the set of points of $W$ which can be joined to $x$ by a simple chain of sets in $\bigcup_{z \in W} V(z)$. Since $x \in P$, $P \neq \emptyset$. Let $Q$ be the set of points of $W$ which cannot be joined to $x$ by a simple chain of sets in $\bigcup_{z \in W} V(z)$. Suppose $P \neq W$, therefore $Q \neq \emptyset$ and clearly $W = P \cup Q$ and $P \cap Q = \emptyset$. Let $c$ be a point of $P \cap Q$. Thus, $c$ belongs to some set $V(c)$ of the open covering. Since $V(c)$ is open, there is a neighborhood of $c$ contained in $V(c)$ and since $c \in \overline{P}$ this neighborhood contains a point $d$ of $P$ (1.11). Thus $d \in P$ and $d \in V(c)$. Now $d$ can be joined to $x$ by a chain of sets in $\bigcup_{z \in W} V(z)$. Let $V(z_1), \ldots, V(z_r)$ be such a chain. Let $V(z)$ be the first set of this chain intersecting $V(c)$. Therefore, $V(z_1), \ldots, V(z_r), V(c)$ is a simple chain of sets of $\bigcup_{z \in W} V(z)$ joining $x$ and $c$. Hence $c \in P$, but we also had $c \in Q$. This is impossible since $P \cap Q = \emptyset$, and we see there is no point $c$ of $P \cap Q$, i.e., $P \cap Q = \emptyset$. Suppose $a \in P \cap \overline{Q}$. Thus, $a \in V(a)$ where $V(a)$ is some set of the covering $\bigcup_{z \in W} V(z)$. $V(a)$ is open, hence there is a neighborhood of $a$ contained in $V(a)$ but $a \in \overline{Q}$ hence this neighborhood contains a point $b$ of $Q$, i.e., $b \in Q$ and $b \in V(a)$. Since $a \in P$ we let $V(z_1), \ldots, V(z_m)$ be a simple chain joining $x$ and $a$. Let $V(z_j)$ be the first set of this chain intersecting $V(a)$. 

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Therefore, $V(z_k), \ldots, V(z_j), V(a)$ is a simple chain of sets of $z \in W$ joining $x$ and $b$. Thus $b \in P$. This is impossible since $b \notin Q$, hence $a \notin P \cap Q$, i.e., $P \cap Q = \emptyset$. Finally then we see that $W = P/Q$ which is a contradiction since $W$ is connected. Therefore, the assumption that $P \neq W$ is false, i.e., $P = W$ and there exists a simple chain $V(z_i), i = 1, \ldots, n$ joining $x$ and $y$. Let $M = \bigcup_{i=1}^{n} V(z_i)$. Recall that any simple chain $\bigcup C_i$ is such that $C_i \cap C_{i+1} \neq \emptyset$ but $C_i \cap C_j = \emptyset$ for $|i-j| > 1$. Thus since each set $V(z_i)$ of the chain joining $x$ and $y$ is connected, $M = \bigcup_{i=1}^{n} V(z_i)$ is connected (1.22). Furthermore, we have seen that each $V(z_i)$ has property $S$ hence $M = \bigcup_{i=1}^{n} V(z_i)$ has property $S$ (4.9). It follows that $M$ is connected and has property $S$ (4.15), thus $M$ is also locally connected (4.11). Clearly since each $\overline{V(z_i)}$ is compact $\bigcup_{i=1}^{n} \overline{V(z_i)}$ is compact. But $\overline{M} = \bigcup_{i=1}^{n} V(z_i) = \bigcup_{i=1}^{n} \overline{V(z_i)}$, i.e., $\overline{M}$ is compact. We now see that $\overline{M}$ is a Peano space contained in $W$. By theorem 3.9 there is a simple arc joining $x$ and $y$ in $\overline{M}$ hence in $W$. But $x$ and $y$ were arbitrary points of $W$ hence $W$ is arcwise connected and the theorem is proved. Q.E.D.

Definition 4.26 We shall denote a subset $G$ of a space $T$ as a region in $T$ if and only if $G$ is both connected and open in $T$.

In theorem 4.25 the only requirement we made of $W$ was that it be an open connected set containing an arbitrary point $p$ of $T$, thus the following corollary is immediate.

Corollary. Suppose $T$ is a locally compact metric space having property $S$. Then every region in $T$ is arcwise connected.

Since every compact space is locally compact and every Peano space is a compact metric space having property $S$, we see...
that theorem 4.25 applies to Peano spaces. We have now proved the following theorem.

**Theorem 4.27** Suppose $T$ is a Peano space. Then $T$ has a basis each non-empty element of which is connected, has property $S$, and has a Peano space as its closure. Also $T$ is arcwise connected (3.9), and locally arcwise connected (4.25) and every region in $T$ is arcwise connected.
CHAPTER V

CYCLIC ELEMENT THEORY

In this chapter we shall study the structure of Peano spaces with respect to their cut points. Thus, all spaces considered shall be Peano spaces, although this need not be the case for some of the theorems to hold.

Theorem 5.1 Suppose $a$, $b$ and $p$ are distinct points of a Peano space $T$. Then, $a$ and $b$ lie in different components of $T-p$ if and only if $T-p$ has a separation $G/H$ such that $a \in G$ and $b \in H$.

Proof. Suppose $T$ is a Peano space and $a$ and $b$ lie in different components of $T-p$. Let $G$ be the component of $T-p$ containing $a$. Since $T$ is locally connected and $G$ is a component of the open set $T-p$ of $T$, $G$ is open. Let $H = (T-p) \cap \overline{G}$. Now $b \in T-p$ and since $a \in G$, $b \notin G$. Therefore, $b \in H$. We shall show that $T-p = G/H$. Clearly, $T-p = G \cup H$, $G \cap H = \emptyset$, $G \neq \emptyset$ and $H \neq \emptyset$. Since $G$ is closed and $\overline{G}$ we have

$$\overline{H \cap GC \overline{G}} \cap G = GC \cap G = \emptyset$$

Furthermore, $G \subseteq \overline{G}$ and $G \subseteq T-p$, hence $G \subseteq \overline{G} \cap (T-p) \subseteq \overline{G}$. But $G$ is connected hence $\overline{G}$ is connected which implies that $\overline{G} \cap (T-p)$ is connected (1.22). Now $G$ is a component of $T-p$ and $\overline{G} \cap (T-p) \subseteq T-p$. Thus since $G \subseteq \overline{G} \cap (T-p)$, $G = \overline{G} \cap (T-p)$ (1.24). Finally then

$$H \cap \overline{G} = (T-p) \cap \overline{G} \cap \overline{G} = \overline{G} \cap G = \emptyset$$

and $T-p = G/H$.

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To prove the converse, suppose a Peano space T is such that \( T-p = C/D \) where points a and b of T-p belong to C and D respectively. Let E be the component of T-p containing a and let F be the component of T-p containing b. Clearly since E is connected and \( a \in E \cap C \) we must have \( E \subseteq C \). Similarly F is connected and \( a \in F \cap D \) hence \( F \subseteq D \). But \( C \cap D = \emptyset \) hence \( E \cap F = \emptyset \), i.e., a and b belong to different components of T-p. Q.E.D.

**Definition 5.2** Suppose a and b are points of a Peano space T. We shall say that a third point p of T cuts between a, b or separates a and b if a and b lie in different components of T-p.

Thus in view of theorem 5.1, a point p cuts between and b if and only if T-p has a separation G/H such that \( a \in G \) and \( b \in H \).

**Definition 5.3** A Peano space T is termed cyclic if T has no cut points.

**Definition 5.4** We shall say that a subset E of a Peano space T is semi-connected if and only if for every point x of T-E, the set E is a subset of some component of T-x. As a consequence of this definition the reader may easily verify the following statements.

(i) The empty set is semi-connected and T itself is semi-connected.

(ii) If a set E is connected, then E is semi-connected.

(iii) If E is semi-connected and \( \mathcal{A} \) is any collection of semi-connected subsets of T such that if \( A \in \mathcal{A} \) implies \( E \cap A \neq \emptyset \), then the set \( E \cup \bigcup_{A \in \mathcal{A}} A \) is also semi-connected.

(iv) If \( \mathcal{A} \) is a collection of semi-connected sets and H is the common part of some (or all) of the sets of \( \mathcal{A} \) then H is semi-connected.
(v) If $E$ is semi-connected and $E \subseteq F \subseteq \overline{E}$ then $F$ is semi-connected. Thus we see that if $E$ is semi-connected, then $\overline{E}$ is semi-connected.

Theorem 5.5 Suppose $p$ and $q$ are distinct points of a Peano space $T$ and $E$ is a semi-connected set containing $p$ and $q$. If a point $x$ cuts between $p$ and $q$, then $x \in E$.

Proof. Suppose $x \notin E$. Since $E$ is semi-connected $E$ lies in some component $B$ of $T-x$. Thus $p \cup q \subseteq B$, i.e., $p$ and $q$ lie in the same component. This is impossible since $x$ cuts between $p$ and $q$, hence $x \in E$. Q.E.D.

Definition 5.6 Suppose $p$ and $q$ are distinct points of a Peano space $T$. We shall let $K(p,q)$ be the set of all points of $T$ that cut between $p$ and $q$. Clearly $K(p,q)$ may be empty.

Theorem 5.7 The set $p \cup q \cup K(p,q)$ is semi-connected.

Proof. Let $x$ be any point such that $x \notin p \cup q \cup K(p,q)$. Thus $x \notin K(p,q)$ hence $x$ does not cut between $p$ and $q$. It follows that $p$ and $q$ are in the same component $C$ of $T-x$, and in view of (ii) of (5.4) that $C$ is semi-connected. Thus by 5.5, $p \cup q \cup K(p,q) \subseteq C$, that is, $p \cup q \cup K(p,q)$ is semi-connected.

Definition 5.8 Let $a$ and $b$ the points of a Peano space $T$. We shall say that $a$ is conjugate to $b$, written $a \sim b$, if and only if no point of $T$ cuts between $a$ and $b$. Clearly every point is conjugate to itself and if $a \sim b$ then $b \sim a$.

In view of definitions 5.6 and 5.8, the following theorem is now immediate.

Theorem 5.9 If $p$ and $q$ are two points of a Peano space $T$
such that \( p \circ q \) then \( p \cup q \) is a semi-connected set.

Definition 5.10 Suppose \( a_1, \ldots, a_n \) is a finite sequence of distinct points of \( T \). These points taken in the given order, will be said to constitute a \( \circ \)-chain if \( a_i \circ a_{i+1} \) for \( i = 1, \ldots, n-1 \). Also, if \( a_n \circ a_1 \), then we shall call the \( \circ \)-chain a simple closed \( \circ \)-chain. Otherwise, the \( \circ \)-chain will be termed open.

Theorem 5.11 If \( a_1, \ldots, a_n \) is a \( \circ \)-chain, then \( \bigcup_{i=1}^{n-1} a_i \) is semi-connected.

Proof. Since \( a_1 \circ a_2, a_1 \cup a_2 \) is semi-connected by 5.9. Suppose \( \bigcup_{i=1}^{n-1} a_i \) is semi-connected. Now \( a_{n-1} \circ a_n \), i.e., \( a_{n-1} \cup a_n \) is semi-connected and since \( (a_{n-1} \cup a_n) \cap \bigcup_{i=1}^{n-1} a_i \neq \emptyset \) we see that \( \bigcup_{i=1}^{n} a_i \) is semi-connected. Q.E.D.

Theorem 5.12 Suppose \( a_1, \ldots, a_n \) is a simple closed \( \circ \)-chain. Then any two of the points \( a_1, \ldots, a_n \) are conjugate.

Proof. Clearly if we consider a cyclic permutation of the points \( a_1, \ldots, a_n \), the new arrangement is again a simple closed \( \circ \)-chain. As examples, \( a_{n-1}, a_n, a_1, \ldots, a_{n-2} \) and \( a_1, \ldots, a_n, a_1, \ldots, a_{i-1} \) are each again simple closed \( \circ \)-chains. Thus, to prove the theorem it is sufficient to show that \( a_i \circ a_i \) for any \( i = 2, \ldots, n-1 \) since a similar proof would follow to show that \( a_j \circ a_i \) for any \( j \) and any \( i \).

Suppose the theorem is false, that is, there exists a point \( x \) which cuts between \( a_1 \) and \( a_i \). Now \( a_1, \ldots, a_i \) is a \( \circ \)-chain and by 5.11, \( \bigcup_{k=1}^{i} a_k \) is a semi-connected set containing \( a_1 \) and \( a_i \). In view of theorem 5.5 \( x \in \bigcup_{k=1}^{i} a_k \). Similarly, \( a_i, \ldots, a_n, a_1 \) is a \( \circ \)-chain and \( x \) belongs to the semi-connected set \( \bigcup_{k=1}^{i} a_k \). In summary then \( x \) belongs to the two sets

\[
\bigcup_{k=1}^{i} a_k = a_1 \cup a_2 \cup \ldots \cup a_i
\]
and
\[ \bigcup_{k=1}^{i} a_k = a_1 \cup a_{i+1} \cup \ldots \cup a_n \cup a_1. \]

But notice \((\bigcup_{k=1}^{i} a_k) \cap (\bigcup_{k=1}^{i} a_k) = a_1 \cup a_i\). Thus \(x \in a_1 \cup a_i\), i.e., \(x\) coincides with one of the points \(a_1\) or \(a_i\). This is impossible since we assumed that \(x\) cuts between \(a_1\) and \(a_i\). Therefore, no such point \(x\) exists and we have \(a_1 \cap a_i\). Q.E.D.

**Definition 5.15** Suppose a Peano space \(T\) contains a finite number of distinct points \(p_1, \ldots, p_n\) and an equal number of sets \(S_1, \ldots, S_n\) such that the following conditions hold,

(i) Each of the sets \(S_1, \ldots, S_n\) is semi-connected.

(ii) \(S_1 \cap S_2 = p_2, S_2 \cap S_3 = p_3, \ldots, S_{n-1} \cap S_n = p_n\) and 
\(S_n \cap S_1 = p_1\).

(iii) \(S_i \cap S_j = \emptyset\) if \(i < |i-j| < n-1\).

If this is the case we shall say that the points \(p_1, \ldots, p_n\) and the sets \(S_1, \ldots, S_n\) constitute a **simple closed \(O\)-polygon**. We shall call the points \(p_1, \ldots, p_n\) the **vertices** and the sets \(S_1, \ldots, S_n\) the sides of the \(O\)-polygon. We shall denote the \(O\)-polygon by 
\((p_1, \ldots, p_n; S_1, \ldots, S_n)\).

The reader can easily verify that if the set of points \(a_1, \ldots, a_n\) is a simple closed \(O\)-chain then we have the simple closed \(O\)-polygon \((a_1, \ldots, a_n; a_1 \cup a_2, a_2 \cup a_3, \ldots, a_{n-1} \cup a_n, a_n \cup a_1)\).

**Theorem 5.14** Any two vertices of a simple closed \(O\)-polygon are conjugate.

**Proof.** Let \((p_1, \ldots, p_2; S_1, \ldots, S_n)\) be a simple closed \(O\)-polygon. We see by theorem 5.12 that it shall be sufficient to show that the vertices \(p_1, \ldots, p_n\), in this order, form a simple
closed $O$-chain, i.e., that $p_i \circ p_{i+1}$ for $i = 1, \ldots, n-1$ and that $p_n \circ p_1$. Again we see that the situation is unaffected by a cyclic permutation, hence we need only show that $p_1 \circ p_2$.

Suppose otherwise, that is, there is a point $x$ which cuts between $p_1$ and $p_2$. Now $S_i$ is semi-connected and $p_1 \cup p_2 \subseteq S_i$. Thus by theorem 5.5, $x \in S_i$. Furthermore, $S_i$ is semi-connected for all $i$, hence by repeated use of 5.4, part (iii), $\bigcup_{i=2}^{n} S_i$ is semi-connected. Also, since $p_2 \in S_2$ and $p_1 \in S_n$ we have $p_1 \cup p_2 \subseteq \bigcup_{i=2}^{n} S_i$ and again by 5.5, $x \in \bigcup_{i=2}^{n} S_i$. But since $(\bigcup_{i=2}^{n} S_i) \cap S_1 \subseteq (S_1 \cap S_2) \cup (S_1 \cap S_n) = p_1 \cup p_2$, we must have $x$ identical with one of the points $p_1$ and $p_2$. This, of course, is impossible hence our assumption that there is a point $x$ which cuts between $p_1$ and $p_2$ is false. Thus $p_1 \circ p_2$. Q.E.D.

Definition 5.15 Suppose $C$ is a subset of a Peano space $T$. We shall say that $C$ is a **proper cyclic element** if and only if the following conditions hold.

(i) $C$ is non-degenerate.

(ii) Any two points of $C$ are conjugate.

(iii) If a point $x$ is conjugate to two distinct points of $C$ then $x \in C$.

Theorem 5.16 If $C$ is a proper cyclic element then $C$ is semi-connected.

**Proof.** Let $p_0$ be a point of $C$. Then $p_0 \circ p$ for all points $p$ of $C$. Thus, in view of theorem 5.9 $p_0 \cup p$ is a semi-connected set for all points $p$ of $C$. Hence $\bigcup (p_0 \cup p)$ for all $p$ of $C$ is semi-connected (5.4). But $C = \bigcup (p_0 \cup p)$ for all $p$ of $C$, hence $C$ is semi-connected. Q.E.D.
Theorem 5.17 Suppose $C_1$ and $C_2$ are proper cyclic elements. If $C_1 \cap C_2$ is non-degenerate then $C_1 = C_2$.

Proof. Suppose $C_1 \cap C_2 = a \cup b$ and let $x_1$ be any point of $C_1$. Then $x_1 \in a$ and $x_1 \in b$ since $a \cup b \subseteq C_1$. But $a \cup b \subseteq C_2$ also hence $x_1 \in C_2$. Thus, $C_1 \subseteq C_2$. Similarly $C_2 \subseteq C_1$, i.e., $C_1 = C_2$. Q.E.D.

Theorem 5.18 Suppose $p$ and $q$ are distinct points of a Peano space $T$ and $p \circ q$. Then there is exactly one proper cyclic element that contains both $p$ and $q$.

Proof. Let $F$ be the set consisting of all points $x$ of $T$ such that $x \circ p$ and $x \circ q$. We shall show that $F$ is a proper cyclic element containing $p$ and $q$ and, in fact, is the only one containing $p$ and $q$. Since $p \circ q$ we have $p \cup q \subseteq F$. Thus, $F$ is non-degenerate and contains $p$ and $q$. Also, any two points of $F$ are conjugate, for let $x_1$ and $x_2$ be any two points of $F$. Then, if $x_1 = x_2$ we have $x_1 \circ x_2$. Similarly, if $x_1$ is identical with either $p$ or $q$, or if $x_2$ is identical with either $p$ or $q$ we again have $x_1 \circ x_2$. Hence, we exclude these cases and assume that $p,q,x_1$, and $x_2$ are distinct points. Since $x_1$ and $x_2$ belong to $F$ we have $x_1 \circ p$, $x_1 \circ q$, $x_2 \circ p$ and $x_2 \circ q$. But the conjugacy relation is symmetric (5.8), hence we have the simple closed 0-chain $p \circ x_1 \circ q \circ x_2 \circ p$. Thus, in view of 5.12, $x_1 \circ x_2$, i.e., any two points of $F$ are conjugate.

Let $y$ be any point which is conjugate to two distinct points $x$ and $z$ of $F$. Thus

$$p \circ x \circ y \circ z \circ q$$

and $p \circ q$. We shall show that $p \circ y \circ q$, i.e., $y \in F$. In view of theorem 5.12 to show this it shall be sufficient to show that $y$
belongs to a simple closed 0-chain containing both p and q. Now, if y equals any one of the points p, q, x or z then clearly y \in F. Hence, we can assume that y is different from each of the points p, q, x and z. Furthermore, we know that p \neq q and x \neq z. There are seven possibilities to consider.

(i) If x = p and z = q, then from (1) we have the simple closed 0-chain pOyOq.

(ii) If x = p and z \neq q, then z \neq p since z \neq x and from (1) we have the simple closed 0-chain pOyOzOq.

(iii) If x = q and z = p, then from (1) we have the simple closed 0-chain qOyOq.

(iv) If x = q and z \neq p, then z \neq q since z \neq x and from (1) we have the simple closed 0-chain pOqOyOz.

(v) If x \neq p, q, and z = p, then from (1) we have the simple closed 0-chain xOyOqOq.

(vi) If x \neq p, q and z = q, then from (1) we have the simple closed 0-chain pOxOyOq.

(vii) If x \neq p, q and z \neq p, q, then (1) itself is a simple closed 0-chain.

Thus in every case, y belongs to a simple closed 0-chain containing p and q, hence pOyOq, i.e., y \in F. Therefore, F satisfies all the requirements of a proper cyclic element and F contains p and q. Furthermore, F is the only proper cyclic element containing p and q for by theorem 5.17 if there were another it would be identical to F. Q.E.D.

Theorem 5.19 Suppose x is a point of the Peano space T and E is the set of all points of T that are conjugate to x. Then E is closed.
Proof. Let $y$ be a point of $GE$. We shall show that $GE$ is open which implies that $E$ is closed. Now there is a point $z$ of $T$ such that $z$ cuts between $x$ and $y$. Thus, in view of 5.2, $x$ and $y$ are in different components $A$ and $B$ of $T-z$. Now the single point $z$ is closed, hence $T-z$ is an open subset of the locally connected space $T$. It follows that since $B \subset T-z$, $B$ is open (1.25). Clearly if $w$ is any point of $B$ then $z$ cuts between $w$ and $x$, i.e., $w$ is not conjugate to $x$, hence $w \notin E$ and we have $w \in GE$. Since for any point $w$ of $B$ we found that $w \in GE$ we see that $B \subset GE$. But $B$ is open and $y \in B$, hence there is an $\varepsilon > 0$ such that $N(y, \varepsilon) \subset B \subset GE$. That is, for any point $y$ of $GE$ there is a neighborhood of $y$ contained in $GE$, hence $GE$ is open. Therefore, $E$ is closed. Q.E.D.

Theorem 5.20 Suppose $C$ is a proper cyclic element of a Peano space $T$. Then $C$ is closed.

Proof. Let $p_1$ and $p_2$ be two distinct points of $C$. Let $E$ be the set of all points $x$ such that $p_1 \circ x \circ p_2$. Let $E_1$ be the set of all points $x$ such that $p_1 \circ x$. Let $E_2$ be the set of all points $x$ such that $p_2 \circ x$. In view of 5.15, $C = E$. But notice $E = E_1 \cap E_2$ and by theorem 5.19, $E_1$ and $E_2$ are closed sets. Thus, $E$ is closed, i.e., $C$ is closed. Q.E.D.

Theorem 5.21 Suppose $C$ is a proper cyclic element of a Peano space $T$. Then $C$ is arcwise connected. Furthermore, if $H$ is any simple arc in $T$ whose end points $p_1$ and $p_2$ are in $C$ then $H \subset C$.

Proof. $T$ is arcwise connected (3.9). Thus, it is clear since $CC T$ that any two points of $C$ can be joined by a simple arc in $T$. We must, therefore, only show that such an arc lies entirely in $C$ in order to prove the theorem.
Let $H$ be a simple arc whose end points $p_1$ and $p_2$ are in $C$. Suppose $H \not\subset C$. Then there exists a point $x$ such that $x \in H$ and $x \notin C$. Consider the subarc $xp_1$ of $H$. $C$ is closed (5.20), hence compact. Thus we have on $xp_1$ a first point $q_1$ of $C$. Let $H_1$ be the subarc of $H$ with end points $x$ and $q_1$. In a similar manner considering the subarc $xp_2$ of $H$ we have on $xp_2$ a first point $q_2$ of $C$. Let $H_2$ be the subarc of $H$ with end points $x$ and $q_2$. We shall show that

\[(q_1, x, q_2; H_1, H_2, C)\]

is a simple closed $0$-polygon. First $q_1$, $x$ and $q_2$ are distinct points for clearly $q_1 \neq q_2$ and if $x$ coincides with either $q_1$ or $q_2$ then $x \in C$ which would prove the theorem. Also, $C$ is a proper cyclic element, hence is semi-connected (5.16). Now $H_1$ and $H_2$ being arcs are connected, hence by 5.4 they are both semi-connected. Furthermore, $H_1 \cap H_2 = x$, $H_2 \cap C = q_2$ and $C \cap H_1 = q_1$. Therefore, by 5.13, (1) is a simple closed $0$-polygon and in view of 5.14, $xOq_1$ and $xOq_2$. That is, $x \in C$. This contradicts the assumption that $x \notin C$, hence $x \in C$ and we have every point of $H$ contained in $C$, i.e., $H \subset C$. Q.E.D.

**Theorem 5.22** Suppose $C$ is a proper cyclic element of a Peano space $T$ and $D$ is any open connected subset of $T$. Then $C \cap D$ is connected.

**Proof.** Clearly if $C \cap D = \emptyset$ or $C \cap D$ is a single point, the conclusion is proved. Therefore, suppose the set $C \cap D$ contains at least two points. Let $x_1$ and $x_2$ be points of $C \cap D$ such that $x_1 \neq x_2$. Since $D$ is a region in the Peano space $T$, $D$ is arcwise connected (4.27). Now $x_1 \cup x_2 \subset D$ so let $H$ be a simple arc in $D$ with
end points \( x_1 \) and \( x_2 \). But \( x_1 \cup x_2 \subset C \), hence theorem 5.21 implies \( H \subset C \). Thus, \( H \subset C \cap D \), i.e., any two distinct points of \( C \cap D \) are joined by a simple arc lying entirely in \( C \cap D \). Therefore, \( C \cap D \) is arcwise connected, hence connected (3.10). Q.E.D.

**Theorem 5.23** Suppose \( C \) is a proper cyclic element of a Peano space \( T \) and \( A \) is a component of \( T-C \). Then \( \overline{A-A} \) consists of exactly one point and this point belongs to \( C \) and is a cut point of \( T \).

**Proof.** \( T-C \) is open (5.20). Thus, \( A \) is a component of an open set \( T-C \) in a locally connected space \( T \), hence \( A \) is a region in \( T \). This implies that \( \overline{A} \neq A \) and we have \( \overline{A-A} \neq \emptyset \). Furthermore, \( \overline{A} \subset C \), for if not there is some point \( p \) of \( \overline{A} \) such that \( p \notin C \). Now then \( p \in \overline{A} \), \( p \in \overline{C} \) and \( p \notin C \), i.e., \( p \in \overline{T-C} \). Hence \( p \) must belong to some component \( Q \) of \( T-C \). If \( Q = A \), we would have \( p \in \overline{A} \) which is impossible since \( p \in \overline{C} \). Therefore, \( Q \neq A \). It follows that \( \overline{A} \cup p \subset \overline{A} \) which implies that \( A \cup p \) is connected (1.22). But \( A \cup p \subset T-C \) and since \( p \notin A \) we have \( A \neq A \cup p \). Thus, the component \( A \) of \( T-C \) is a proper subset of a connected set \( A \cup p \) in \( T-C \). Since this is contrary to our definition of a component (1.24), we see that our assumption that such a point \( p \) existed is false, i.e., \( \overline{A} \subset C \). In addition to this, if \( \overline{A} \) consists of exactly one point \( x \), then \( x \) is a cut point of \( T \). To show this notice that since \( x \notin A \) we have \( A \subset T-x \). \( A \) is open in \( T \), hence \( A \) is open in \( T-x \) (1.13). However, since \( \overline{A} = x \), it follows that \( \overline{A} = A \cup x \). Clearly then \( A = (T-x) \cap \overline{A} \), i.e., \( A \) is closed in \( T-x \) (1.13). Hence \( QA \) is open in \( T-x \) and we see that \( T-x \) is the union of two non-empty disjoint
sets \(A\) and \(\overline{A}\), each of which is open in \(T-x\). It follows from 1.22 that \(T-x\) is disconnected, i.e., \(x\) is a cut point of \(T\).

We shall now show that \(\overline{A-A}\) does consist of exactly one point thus proving the theorem. Suppose there are at least two distinct points \(p_1\) and \(p_2\) of \(\overline{A-A}\). Since \(\overline{A-A} \subset C\) we have \(p_1 \cup p_2 \subset C \cap (\overline{A-A})\).

Now \(p_1 \neq p_2\) hence \(\rho(p_1, p_2) = r\) where \(r > 0\). Consider the neighborhoods \(N(p_1, \frac{1}{2}r)\) and \(N(p_2, \frac{1}{2}r)\). These neighborhoods are clearly disjoint.

Let \(G_1\) be the component of \(N(p_1, \frac{1}{2}r)\) containing \(p_1\) and let \(G_2\) be the component of \(N(p_2, \frac{1}{2}r)\) containing \(p_2\). Thus \(p_1 \in G_1\), \(p_2 \in G_2\) and \(G_1 \cap G_2 = \emptyset\). Now \(G_1\) and \(G_2\) are components of the open subsets \(N(p_1, \frac{1}{2}r)\) and \(N(p_2, \frac{1}{2}r)\) of the locally connected space \(T\) (1.6). Thus, \(G_1\) and \(G_2\) are open connected sets (1.25). Let \(G = G_1 \cup A \cup G_2\). \(G\) is open since \(G_1\), \(A\) and \(G_2\) are open. Also, \(G\) is connected. To show this notice that \(G_1\), \(A\) and \(G_2\) are each connected sets. Furthermore, since \(p_1 \in \overline{A-A}\), \(p_1 \in \overline{A}\). Thus \(A \cup p_1 \subset \overline{A}\), i.e., \(A \cup p_1\) is connected (1.22). Since \(p_1 \in G_1\) and \(p_1 \in A \cup p_1\), and both sets \(G_1\) and \(A \cup p_1\) are connected we have \(G_1 \cup (A \cup p_1)\) connected. But \(G_1 \cup (A \cup p_1) = G_1 \cup A\), i.e., \(G_1 \cup A\) is connected. Similarly, \(A \cup G_2\) is connected hence \(G_1 \cup A \cup G_2 = G\) is connected (1.22). In summary then \(G\) is an open connected subset of \(T\) hence theorem 5.22 implies that \(C \cap G\) is connected. This, however, is not the case for

\[
C \cap G = (C \cap G_1) \cup (C \cap A) \cup (C \cap G_2)
\]

since \(C \cap A = \emptyset\). Also \(C \cap G_1\) and \(C \cap G_2\) are both non-empty sets, and \(\overline{C \cap G_1} \subset G \subset \overline{C \cap G_2} \subset \overline{C \cap G_2}\), i.e., \((\overline{C \cap G_1}) \cap (C \cap G_2) = \emptyset\). Similarly, \((C \cap G_1) \cap (\overline{C \cap G_2}) = \emptyset\), and we see that \(C \cap G = C \cap G_1 / C \cap G_2\). Thus we
have a contradiction and our assumption that \( \overline{A} - A \) contained at least two distinct points is false. Therefore, \( \overline{A} - A \) consists of exactly one point \( x \) and as we have seen \( x \in C \) and \( T - x \) is disconnected. Q.E.D.

**Theorem 5.24** Suppose \( C \) is a proper cyclic element of a Peano space \( T \) and \( T - C \neq \emptyset \). Then \( T - C \) has at most a finite number of components \( A \) such that \( d(A) \) is greater than or equal to an assigned \( \delta > 0 \).

**Proof.** Suppose the conclusion of the theorem is false. Then there is a \( \delta > 0 \) such that infinitely many components \( S_1, \ldots, S_n, \ldots \) of \( T - C \) each have diameter greater than or equal to \( \delta \). Since each component is connected and has diameter greater than zero, it must contain a non-countable number of points (1.22). Thus for each \( n \) there are two points \( p_n \) and \( q_n \) of \( S_n \) such that

\[
\rho(p_n, q_n) > d(S_n) - \frac{1}{n} \geq \delta - \frac{1}{n}.
\]

Hence we have two sequences of points \( \{p_n\} \) and \( \{q_n\} \) for \( n = 1, 2, \ldots \). Since \( T \) is compact there is a subsequence \( \{p_{k_n}\} \) of \( \{p_n\} \) such that \( \{p_{k_n}\} \) converges (1.21). Now the subsequence \( \{q_{k_n}\} \) of \( \{q_n\} \) may not converge, but a subsequence \( \{q_{1_k_n}\} \) of \( \{q_{k_n}\} \) will converge (1.21). Let \( \{p_{1_k_n}\} \) be a subsequence of \( \{p_{k_n}\} \). Since \( \{p_{k_n}\} \) converges, \( \{p_{1_k_n}\} \) converges. Let \( y_n = p_{1_k_n} \) and \( z_n = q_{1_k_n} \). Thus we have two converging sequences \( \{y_n\} \) and \( \{z_n\} \) such that

\[
(1) \quad \rho(y_n, z_n) > \delta - \frac{1}{1_{k_n}} \geq \delta - \frac{1}{n}.
\]

If \( y_n \to p \) and \( z_n \to q \) we see that \( p \neq q \). For suppose \( 2\varepsilon < \delta - \frac{1}{n} \) where \( \varepsilon > 0 \). Thus for \( n \) sufficiently large \( \rho(y_n, p) < \varepsilon \) and \( \rho(z_n, q) < \varepsilon \).
But if $p = q$ we have
\[ \rho(y_n, z_n) \leq \rho(y_n, p) + \rho(p, z_n) \]
\[ = \rho(y_n, p) + \rho(q, z_n) \]
\[ < \epsilon + \epsilon, \]
i.e., $\rho(y_n, z_n) < 2\epsilon < \epsilon - \frac{1}{n}$ which is a contradiction of (1), hence $p \neq q$. Therefore, $\rho(p, q) = r$ where $r > 0$. Consider the neighborhoods $N(p, \frac{1}{2}r)$ and $N(q, \frac{1}{2}r)$. Clearly they are disjoint open sets (1.6). Let $P$ be the component of $N(p, \frac{1}{2}r)$ containing $p$ and let $Q$ be the component of $N(q, \frac{1}{2}r)$ containing $q$. Now $p \in P$, $q \in Q$ and $P$ and $Q$ are open connected sets since each is contained in an open set of the locally connected space $T$. Furthermore, $P \cap Q = \emptyset$.

Since $P$ is open there is a $\gamma > 0$ such that $N(p, \gamma) \subseteq P$. But $y_n \rightarrow p$, thus there is an $n_1$ such that if $n > n_1$ we will have all but a finite number of points of the sequence $\{y_n\}$ contained in $N(p, \gamma) \subseteq P$ (1.7). Therefore, $S_n \cap P \neq \emptyset$ for $n > n_1$. Similarly, there is an $n_2$ such that if $n > n_2$, $S_n \cap Q \neq \emptyset$. Let $n_0 = \overline{n_1, n_2}$. It follows that if $n > n_0$ we have $S_n \cap P \neq \emptyset$ and $S_n \cap Q \neq \emptyset$. Then since $P$ and $S_n$ are both connected, $P \cup S_n$ is connected. Now $S_n$ is a component of $T - C$ and $S_n \subseteq P \cup S_n$, hence we cannot have $P \cup S_n \subseteq T - C$ unless $P \subseteq S_n$ for $n > n_0$ (1.24). But the sets $S_1, \ldots, S_n, \ldots$ are disjoint, i.e., $P$ can be contained in $S_n$ for only one value of $n$.

Therefore, $P \cup S_n \not\subseteq T - C$, i.e., $(P \cup S_n) \cap C \neq \emptyset$ for $n > n_0$. However, $S_n \subseteq T - C$, i.e., $S_n \cap C = \emptyset$ hence we must have $P \cap C \neq \emptyset$. Similarly, we have $Q \cap C \neq \emptyset$. Let $D = P \cup S_n \cup Q$. Clearly, $D$ is open and if $n > n_0$, $P \cap S_n \neq \emptyset$ and $Q \cap S_n \neq \emptyset$ hence $D$ is connected (if $n > n_0$).

Now theorem 5.22 implies that $C \cap D$ is connected. But
\[ C \cap D = (C \cap P) \cup (C \cap S_n) \cup (C \cap Q) = (C \cap P) \cup (C \cap Q). \]
where \( C \cap P \neq \emptyset \) and \( C \cap Q \neq \emptyset \). Furthermore,

\[
(C \cap P) \subset \overline{P} \subset C \cap Q \subset C \cap (C \cap Q)
\]

hence \((C \cap P) \cap (C \cap Q) = \emptyset\). Similarly, \((C \cap P) \cap (C \cap Q) = \emptyset\) and we see that \( C \cap D = C \cap P / C \cap Q \). Therefore, we have a contradiction, hence our assumption that there existed a \( \delta > 0 \) such that infinitely many components of \( T - C \) each had diameter greater than or equal to \( \delta \) is false. Q.E.D.

Definition 5.25 A subset \( H \) of a space \( T \) is termed a retract of \( T \) if and only if there exists a continuous mapping \( f \) of \( T \) onto \( H \) such that \( f(x) = x \) for every point \( x \in H \). Such a mapping \( f \) is called a retraction of \( T \) onto \( H \).

Theorem 5.26 If \( C \) is a proper cyclic element of a Peano space \( T \), then \( C \) is a retract of \( T \).

Proof. If \( C = T \) then the identity mapping is clearly a retraction of \( T \) onto \( C \), hence we assume \( T - C \neq \emptyset \). We shall define a mapping \( f : T \to C \) in the following manner. Let \( f(x) = x \) if \( x \in C \). If \( y \in T - C \), let \( A_y \) be the component of \( T - C \) containing \( y \). By theorem 5.23, \( A_y - A_y \) is a single point \( z \in C \). Let \( f(y) = z \). Clearly then \( f(T) = C \), i.e., \( f \) maps \( T \) onto \( C \). To show that \( f \) is a retraction of \( T \) onto \( C \) we must yet show that \( f \) is continuous.

Suppose \( f \) is not continuous. Since \( T \) is compact and \( f \) is not continuous there is a sequence \( \{x_n\} \) and a point \( x_0 \) of \( T \) such that

\[
(1) \quad x_n \to x_0 \text{ but } f(x_n) \to y_0 \neq f(x_0).
\]

There are three cases to consider.

(i) Suppose \( x_0 \in T - C \). Then \( x_0 \in A_{x_0} \). \( A_{x_0} \) is contained in the open set \( T - C \), hence \( A_{x_0} \) is open (1.25). Thus, there is an
\( \epsilon > 0 \) such that \( N(x_0, \epsilon) \subseteq A_{x_0} \).
Furthermore, all but a finite number of the points \( x_n \) are contained in \( N(x_0, \epsilon) \), hence in \( A_{x_0} \). That is, there exists an \( m \) such that if \( n > m \), \( x_n \in A_{x_0} \). It follows that 
\[ A_{x_n} = A_{x_0} \text{ for } n > m. \]
Hence \( f(x_n) = A_{x_n} - A_{x_n} = A_{x_0} - A_{x_0} = f(x_0) \), i.e., \( f(x_n) = f(x_0). \)

(ii) Suppose \( x_0 \in C \) and \( x_n \in C \) for infinitely many values of \( n \). Then \( f(x_0) = x_0 \), and \( f(x_n) = x_n \) for the \( x_n \) in \( C \). Since \( x_n \in C \) for infinitely many values of \( n \), there is a subsequence \( \{x_k\} \) of \( \{x_n\} \) such that each point \( x_k \in C \), hence \( f(x_k) = x_k \) for each point \( x_k \). From (1) we see that \( x_n \to x_0 \) hence \( x_k \to x_0 \), i.e.,
\[ f(x_k) \to x_0. \]
But \( f(x_0) = x_0 \) hence \( f(x_k) \to f(x_0) \). Again from (1) we see that \( f(x_n) \to y_o \) and it follows that \( f(x_k) \to y_o \). Therefore, \( f(x_0) = y_o \) which contradicts (1).

(iii) Suppose \( x_0 \in C \) and \( x_n \in T - C \) for \( n \) greater than a certain \( m \). Then if \( n > m \), \( f(x_n) = A_{x_n} - A_{x_n} \). Again \( f(x_0) = x_0 \) and since from (1) \( f(x_0) \neq y_o \), we have \( x_0 \neq y_o \). We also know from (1) that \( x_n \to x_0 \) and \( f(x_n) \to y_o \). Thus if \( n \) is sufficiently large
\[ \rho(f(x_n), x_n) < \rho(x_0, y_0)/2. \]
This implies that \( d(A_{x_n}) < \rho(x_0, y_0)/2 \) since \( x_n \in A_{x_n} \subseteq A_{x_0} \) and \( f(x_n) = A_{x_n} - A_{x_n} \) for \( n \) sufficiently large and since
\[ d(A_{x_n}) = d(A_{x_0}). \]
In view of theorem 5.24 there must be a component \( A \) of \( T - C \) which occurs an infinite number of times in the sequence \( A_{x_1}, \ldots, A_{x_n}, \ldots. \)
Let \( p = A - A \). Thus \( x_n \in A \) for infinitely many values of \( n \), hence \( f(x_n) = p \) for infinitely many values of \( n \).

We now have \( x_n \to x_0 \), \( f(x_n) \to p \) and \( f(x_n) \to y_o \), i.e., \( y_o = p \). Since \( x_n \in A \) for infinitely many values of \( n \) there is a subsequence.
\{x_n\} of \{x_n\} such that \(x_n \in A\) for each \(n\). But \(x_n \to x_0\) implies that 
\(x_n \to x_0\), i.e., \(x_0 \in \overline{A}\) (1.9). But \(x_0 \in C\) by assumption and since 
\(C \cap A = \emptyset\), \(x_0 \in C\), we have \(x_0 \in \overline{A} - A = \emptyset\). Thus \(x_0 = p = y_0\). This is 
again a contradiction since \(x_0 \in C\) implies \(f(x_0) = x_0\) and (1) states 
that \(f(x_0) \neq y_0\), i.e., \(x_0 \neq y_0\). Therefore, in each case we found 
that the assumption that there existed a point \(x_0\) and a sequence 
\(\{x_n\}\) such that \(x_n \to x_0\) but \(f(x_n) \to y_0 \neq f(x_0)\) was false. Hence \(f\) is 
continuous. Q.E.D.

Theorem 5.27 Suppose \(C\) is a proper cyclic element of a 
Peano space \(T\) and \(A\) is a component of \(T-C\). According to theorem 
5.23, \(A-A\) is a single point \(p \in C\). Now let \(G\) be any connected set 
of \(T\) that intersects both \(A\) and \(C\). Then \(p \in G\).

Proof. Clearly, \(G = [A \cap G] \cup [(T-A) \cap G]\). Since \(G\) intersects 
both \(A\) and \(C\), and \(A \subset T-C\) we have \(A \cap G \neq \emptyset\) and \((T-A) \cap G \neq \emptyset\). Also \(A\) 
is open as we have seen in 5.23 hence \(T-A\) is closed, i.e., 
\(T-A = \overline{T-A}\) and since \(A \cap (T-A) = \emptyset\) we have \(A \cap (\overline{T-A}) = \emptyset\). Thus 
\[A \cap G] \cap [(T-A) \cap G] \subset A \cap G \cap (\overline{T-A}) \cap G = \emptyset, \]
hence also \([A \cap G] \cap [(T-A) \cap C] = \emptyset\). Now \(G\) is connected by hy-
pothesis, so we must have \([A \cap G] \cap [(T-A) \cap G] \neq \emptyset\) in order to pre-
serve this connectedness of \(G\). Furthermore, we found in theorem 
5.23 that \(\overline{A} = A \cup p\). Thus it follows that 
\(\emptyset \neq [A \cap G] \cap [(T-A) \cap G] \subset A \cap [(T-A) \cap G] \]
\[= [A \cup p] \cap [(T-A) \cap G] \]
\[= [A \cap (T-A) \cap G] \cup [p \cap (T-A) \cap G] \]
\[= p \cap G, \]
i.e., \(p \cap G \neq \emptyset\) hence \(p \in G\). Q.E.D.
For further reference, Rado's Length and Area contains a similar discussion of sections 5.2 through 5.27.

**Theorem 5.28** Suppose $C$ is a proper cyclic element of a Peano space $T$. Then

(i) If $H$ is any connected subset of $T$, then $H \cap C$ is connected.

(ii) $C$ is a cyclic Peano space.

**Proof.** First we notice that if either $H = T$ or $H \cap C = \emptyset$ the statement is immediate. So we can assume that $T - C \neq \emptyset$ and $H \cap C \neq \emptyset$. By theorem 5.26 there exists a retraction $f$ of $T$ onto $C$. We shall use the same retraction $f$ defined in that theorem. Since $f$ is continuous and $H$ is connected, the set $f(H)$ in $C$ is connected (1.27). We shall show that $f(H) = H \cap C$, thus proving the connectedness of $H \cap C$. Now $H \cap C \subseteq C$ and $f(x) = x$ for $x \in C$. Therefore,

$$H \cap C = f(H \cap C) \subseteq f(H).$$

To show that $f(H) \subseteq H \cap C$ let $w$ be any point in $f(H)$. Thus there is a point $r$ of $H$ such that $f(r) = w$. We now have $w \in f(H) \subseteq C$. There are two cases to consider.

(1) If $r \in C$ then $f(r) = r$, i.e., $r = w$ hence $w \in H$ and we have $w \in H \cap C$.

(2) If $r \in T - C$ then $r \in A_r$ where $A_r$ is the component of $T - C$ containing $r$. Recall that we defined $f$ so that $w = \overline{A_r} - A_r$. Now we have $H \cap C \neq \emptyset$ and since $r \in H$ and $r \in A_r$ we have $H \cap A_r \neq \emptyset$. Thus by theorem 5.27, $w \in H$, i.e., $w \in H \cap C$.

Therefore, in both cases $w \in f(H)$ implies $w \in H \cap C$, i.e.,

$f(H) \subseteq H \cap C$ and hence $f(H) = H \cap C$.  

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To prove (ii) notice that \( f \) maps \( T \) continuously onto \( C \) hence we see that \( C \) is a Peano space. Let \( p \) be any point of \( C \), and consider the set \( C-p \). We shall show that \( C-p \) is connected, thus in view of 5.3, \( C \) is cyclic. Let \( a \) be a fixed point of \( C-p \). Since \( T \) is cyclic, \( a \circ x \) in \( T \) for all points \( x \) of \( C-p \), hence \( a \) and \( x \) lie in the same component \( E \) of \( T-p \). Clearly, then \( p \not\in E \) and \( C-p \subset E \), and since \( C-p \subset C \) it follows that \( C-p = E \cap C \). But by virtue of (i) of this theorem, \( E \cap C \) is connected, i.e., \( C-p \) is connected. Q.E.D.

As a conclusion to this paper we are going to prove the cyclic connectivity theorem that any two points of a cyclic Peano space \( T \) lie on a simple closed curve in \( T \). We now have sufficient background to prove some introductory theorems leading to this result.

**Theorem 5.29** Suppose \( A \) and \( B \) are non-degenerate disjoint closed subsets of a cyclic Peano space \( T \). Then there exists two disjoint arcs \( P \) and \( Q \) in \( T \), such that each one has one end point in \( A \) and the other end point in \( B \) and no interior point in \( A \cup B \).

**Proof.** Let us define a subset \( E \) of \( T \) to consist of the set \( A \) and all points \( w \) of \( T-A \) such that

(i) There exists an arc \( F \) in \( T \) with one end point in \( A \), the other end point in \( B \), and no interior point in \( A \cup B \).

(ii) There is an arc \( G \) in \( T-F \) such that \( w \) is one end point of \( G \) and \( G \) intersects \( A \) in exactly one point, which is its other end point.

Clearly \( E \neq \emptyset \) since \( A \subset E \). If we can show that \( E \) is both open and closed in \( T \) then we must have \( E = T \) in order to preserve the connectedness of \( T \). For, otherwise, \( T \) will be the union of two
non-empty, disjoint open set $E$ and $\mathcal{G}E$, hence $T$ would be disconnected. It follows that if $E = T$ then $B \subset E$, thus clearly the required arcs exist.

First $E$ is open. For let $z$ be any point of $E$ and we have two cases to consider.

1. Suppose $z \in A$. Since $T$ is cyclic and $z$ is a closed set, $T - z$ is a region in $T$. Thus $T - z$ is arcwise connected (4.27). Furthermore, since $A \cap B = \emptyset$ and $z \in A$ we have $B \subset T - z$. Let $x$ and $y$ be any two points of $A$ and $B$ respectively such that $x \neq z$ and $y \neq z$. Then there exists an arc $F_1$ in $T - z$ joining $x$ and $y$. Clearly, $F_1$, $A$ and $B$ are each compact sets such that $F_1 \cap A \neq \emptyset$ and $F_1 \cap B \neq \emptyset$. Therefore, there are first points of $A$ and $B$ on $F_1$. Denoting these points by $p$ in $A$ and $q$ in $B$, we have an arc $F$ in $T - z$ joining $p$ and $q$, and $F$ satisfies (i) above. Since $F$ is closed and $F \subset T - z$, $z$ belongs to the open set $\mathcal{G}F$. Let $V$ be the component of $\mathcal{G}F$ containing $z$. $T$ is locally connected, hence $V$ is a region in $T$ (1.25). It follows that there is an $\varepsilon > 0$ such that $N(z, \varepsilon) \subset V$. If $V \subset A$ then $V \subset E$ hence for any point $z$ of $E$ such that $z \in A$ there is an $\varepsilon > 0$ such that $N(z, \varepsilon) \subset E$, i.e., $E$ is open. If $V \not\subset A$ then there is a point $w$ of $V - A$. Thus $z \cup w \subset V$ and since $V$ is a region in $T$, $V$ is arcwise connected. Let $G_1$ be an arc in $V$ joining $w$ and $z$. But $A$ and $G_1$ are compact, $w \in V - A$ and $z \in A$ hence there is a first point $r$ of $A$ in $G_1$. Let $G$ be the arc $wr$. Thus we have $G \subset V \subset \mathcal{G}F$, $G$ has one end point $r$ in $A$ and no other point in $A$ and the other end point is $w$. Hence $G$ satisfies (ii) and we have $w \in E$. Thus if $w \in V - A$ then $w \in E$ and it follows that whether $V \subset A$ or $V \not\subset A$ we have $V \subset E$ and again $N(z, \varepsilon) \subset V \subset E$, i.e., $E$ is open.
(2) Suppose \( z \notin A \), i.e., \( z \in E - A \). Now by the same reasoning as in (1) we find there are arcs \( F \) and \( G \) satisfying (i) and (ii) respectively. Thus \( G = tz \) where \( G \) intersects \( A \) in only one point \( t \) and \( z \in E - A \), i.e., \( z \in T - A \). Since \( G \subseteq T - F \), \( z \notin F \cup A \). Therefore, \( z \in C(F \cup A) \). Now \( F \) and \( A \) are both closed sets, hence \( F \cup A \) is closed, i.e., \( C(F \cup A) \) is open. Let \( M \) be the component of \( C(F \cup A) \) containing \( z \). Thus \( M \) is an open connected set (1.25). Let \( w \) be any point of \( M \). Since \( M \) is a region, \( M \) is arcwise connected hence there is an arc in \( M \) joining \( w \) and \( z \). Now \( z \in G \cap wz \) hence \( G \cap wz \neq \emptyset \).

Let \( s \) be the first point of \( G (=tz) \) on \( wz \) going in the order from \( w \) to \( z \). Thus we can obtain the arc \( tsw \). Clearly \( tsw \subseteq T - F \) and \( tsw \) has \( w \) as one end point and intersects \( A \) in only one point, namely, the other end point \( t \). Therefore, \( tsw \) satisfies (ii), hence \( w \in E \).

But \( w \) was an arbitrary point of \( M \) thus \( M \subseteq E \). Now \( z \in M \) and \( M \) is open hence there is an \( \epsilon > 0 \) such that \( N(z, \epsilon) \subseteq M \subseteq E \). That is, any arbitrary point \( z \) of \( E \) is in a neighborhood which is contained in \( E \). Thus \( E \) is open in either case (1) or (2).

We shall now show that \( E \) is also closed. (See Figure 1, page 93). Suppose \( E \) is not closed. Thus there exists some point \( w \) of \( T - E \) which is a limit point of \( E \), i.e., \( w \in \overline{E} \) but \( w \notin E \) (1.10).

Since the single point \( w \) is a closed set, and \( T \) is cyclic, we see that \( T - w \) is a region in \( T \). Therefore, \( T - w \) is arcwise connected.

Now \( A \) and \( B \) are non-degenerate sets by hypothesis, hence there is some arc \( F_1 \) in \( T - w \) such that \( F_1 \) has one end point in \( A \) and one end point in \( B \) and no other points in either \( A \) or \( B \), i.e., \( F_1 \) satisfies (i). Since \( A \subseteq E \) and \( w \in T - E \) we have \( w \notin A \). Also \( F_1 \subseteq T - w \) hence \( w \notin F_1 \), thus \( w \notin A \cup F_1 \), i.e., \( w \in C(A \cup F_1) \). \( C(A \cup F_1) \) is clearly an.
open set hence there is a $\delta > 0$ such that $N(w, \delta) \subseteq \overline{C(A \cup F_1)}$. Consider the neighborhood $N(w, \frac{1}{2}\delta)$. By theorem 4.27, $T$ has a basis, each non-empty element of which is open, connected and has a Peano space as its closure. Now $N(w, \frac{1}{2}\delta)$ is open hence can be expressed as the union of some of the elements of such a basis. Let $H$ be the element of the basis in $N(w, \frac{1}{2}\delta)$ such that $w \in H$. Then

$$w \in H \subseteq N(w, \frac{1}{2}\delta) \subseteq N(w, \frac{1}{2}\delta) \subseteq N(w, \delta) \subseteq \overline{C(A \cup F_1)}.$$ 

Figure 1

Furthermore, $\overline{H} \cap N(w, \frac{1}{2}\delta) \subseteq \overline{C(A \cup F_1)}$ hence $\overline{H} \cap (A \cup F_1) = \emptyset$. Since $w \in H$ and $w$ is a limit point of $E$, any neighborhood of $w$ will contain points of $E$. Therefore, clearly $H \cap E \neq \emptyset$. Let $z \in H \cap E$. Then $z \in E$ and since $\overline{H} \cap (A \cup F_1) = \emptyset$, $z \notin (A \cup F_1)$, thus $z \notin A$. It follows, by the same reasoning as used earlier in the theorem, that since $T$ is arcwise connected there exists an arc $G = vz$ satisfying (ii)., i.e., $G \cap A = v$, and there exists an arc $F_2$ satisfying (i). Now $z \in H$ hence $z \in \overline{H}$ thus $G \cap \overline{H} \neq \emptyset$. Let $y$ be the first point of $G$ in $\overline{H}$.
going in order from \( v \) to \( z \). Now \( y \in \mathbb{H} \) and \( w \in \mathbb{H} \). But \( \mathbb{H} \) is a Peano space hence we have an arc \( yw \subset \mathbb{H} \). Consider the arc \( vyw \). We readily see that \( vyw \cap F_1 \neq \emptyset \) and \( vyw \cap F_2 \neq \emptyset \). For if either of these intersections were the empty set then \( vyw \) would satisfy (ii) hence \( w \) would be a point of \( E \). This is impossible since by assumption \( w \notin E \), hence the intersections are both non-empty. On the other hand, we know that \( yw \cap F_1 = \emptyset \) since \( yw \subset \mathbb{H} \subset (A \cup F_1) \), so we must have \( vy \cap F_1 \neq \emptyset \). Similarly, \( vyw \cap F_2 \neq \emptyset \) implies \( yw \cap F_2 \neq \emptyset \) since \( vy \subset vz = G \subset T - F_2 \), i.e., \( vy \cap F_2 = \emptyset \). Now \( yw \subset \mathbb{H} \) hence the relation \( yw \cap F_2 \neq \emptyset \) implies \( \mathbb{H} \cap F_2 \neq \emptyset \). Let \( F_2 = ab \) where \( a \in A \), \( b \in B \) and \( ab \cap (A \cup B) = a \cup b \). Let \( x \) be the first point of \( F_2 (=ab) \) in \( \mathbb{H} \), going in order from \( a \) to \( b \). Thus we have the arc \( ax \). Since \( x \cup w \subset \mathbb{H} \) there is an arc \( xw \subset \mathbb{H} \) (since \( \mathbb{H} \) is arcwise connected). We see that the arc \( axw \) is such that \( axw \cap F_1 \neq \emptyset \). For if \( axw \cap F_1 = \emptyset \), then \( axw \) would satisfy (ii) and we would have the contradiction that \( w \in E \). But since \( xw \subset \mathbb{H} \subset (A \cup F_1) \) we have \( xw \cap F_1 = \emptyset \) thus \( ax \cap F_1 \neq \emptyset \). Recall that \( F_1 \) satisfies (i) so if we let \( F_1 = cd \) we have \( c \in A \), \( d \in B \) and \( cd \cap (A \cup B) = c \cup d \). Also recall that \( vy \cap F_1 \neq \emptyset \) and \( ax \cap F_1 \neq \emptyset \). Let \( q \) be the last point of \( F_1 (=cd) \) which lies on \( vy \cup ax \), going in order from \( c \) to \( d \). Thus \( q \in F_1 \) and \( q \in (vy \cup ax) \). But notice \( vy \cap ax = \emptyset \) since \( ax \subset ab = F_2 \), \( vy \subset vz = G \) and \( G \subset T - F_2 \), i.e., \( G \cap F_2 = \emptyset \). Thus we have two cases to consider, either \( q \in vy \) or else \( q \in ax \).

(1) Suppose \( q \in vy \). Thus \( q \notin ax \). Now \( q \neq v \), for if we had \( q = v \) then since \( v \in A \) we would have \( q \in A \). But we know \( q \in cd = F_1 \) where \( cd \cap A = c \). Therefore, we would have \( q = v = c \). Since \( q \) is the last point of \( cd \) which lies on \( vy \cup ax \) we see that

\[(qd - q) \cap (vy \cup ax) = \emptyset \]
hence \((qd-q) \cap ax = \emptyset\). But since \(q = c\) this implies \((cd-d) \cap ax = \emptyset\). Furthermore, \(q \notin ax\) hence \(c \notin ax\), i.e., \(ax \cap c = \emptyset\). It follows that \(cd \cap ax = \emptyset\), i.e., \(F_1 \cap ax = \emptyset\) which is impossible since we know that \(F_1 \cap ax \neq \emptyset\). Thus \(q \neq y\). By similar reasoning \(q \neq c\). Let \(D = vq \cup qd\).
Since \(d \in (vq \cup qd) \cap B\) we have \((vq \cup qd) \cap B \neq \emptyset\). Let \(t\) be the first point of \(D\) which lies on the compact set \(B\) and let \(F = vt\) and \(G^* = axw\). Clearly \(F\) is an arc joining \(v\) of \(A\) and \(t\) of \(B\) and \(F \cap (A \cup B) = v \cup t\). Thus \(F\) satisfies (i). Furthermore, \(G^*\) has one end point \(w\) in \(T \in C T\) and since \(\overline{H} \cap A = \emptyset\), we have
\[
G^* \cap A = axw \cap A = (ax \cup xw) \cap A
\]
\[
\subseteq (ab \cup \overline{H}) \cap A
\]
\[
= (ab \cap A) \cup (\overline{H} \cap A)
\]
\[
= a,
\]
i.e., \(G^* \cap A = a\). Therefore, if \(G^* \subsetneq T - F\) then \(G^*\) satisfies (ii), hence \(w \in E\), in contradiction with our assumption that \(w \notin E\). We shall show that this is the case. Since \(F_2\) and \(G\) satisfy (i) and (ii), \(F_2 \cap G = \emptyset\). But \(vq \subset G\) and \(ax \subset F_2\), hence \(vq \cap ax = \emptyset\). Also, since \(q\) is the last point of \(cd\) which lies in \(vy \cup ax\) and \(q \in vy\) we have \(qd \cap ax = \emptyset\). Thus \((vq \cup qd) \cap ax = \emptyset\) and since \(F \subset (vq \cup qd)\) we have \(F \cap ax = \emptyset\). Now \(q \in F_1\) and \(\overline{H} \cap (F_1 \cup A) = \emptyset\) implies \(\overline{H} \cap F_1 = \emptyset\). Thus \(q \notin \overline{H}\). But \(q \in vy\) where \(vy\) has only one point in \(\overline{H}\), namely \(y\), hence \(q \neq y\). Therefore, \(vq \cap \overline{H} = \emptyset\) and since \(xw \subset \overline{H}\) we have \(vq \cap xw = \emptyset\). Also \(qd \subset F_1\), hence \(qd \cap xw = \emptyset\), and we have \((vq \cup qd) \cap xw = \emptyset\). This implies that \(F \cap xw = \emptyset\). Finally then we have \(F \cap (ax \cup xw) = \emptyset\), i.e., \(F \cap G^* = \emptyset\), hence \(G^* \subsetneq T - F\).

(2) Suppose \(q \in ax\). (In the rest of the proof since we are assuming \(q \in ax\), we are considering a second possibility for \(F_1\) as
indicated by the dotted line in the figure). First $q \neq d$, for suppose otherwise, i.e., suppose $q = d$. Then $q \in F_2$ and since $d \notin B$ we have $q \in B$. Thus $q \in F_2 \cap B$. But $F_2 \cap B = b$, hence $q = b$. This is impossible since $a \subset x \subset a b$ and if $q = b$ clearly $x = b = q = d$, i.e., $x = d$. Then since $d \in F_1$, $x \in F_1$. But we know $x \in H$ hence $F_1 \cap H \neq \emptyset$ which contradicts the fact that $F_1 \cap H = \emptyset$. Therefore we can assume that $q \neq d$. Let $F^* = aq d$ and $G^{**} = vyw$. Clearly $F^*$ satisfies (i) and $G^{**}$ meets $A$ in exactly one point; namely $v$. Furthermore, $G^{**}$ has its other end point $w$ in $T - A$. Thus, if $G^{**} \subset T - F$, $G^{**}$ satisfies (ii) and again we will have $w \in E$.

Now $aq \subset F_2$, $vy \subset G$ and $F_2 \cap G = \emptyset$, hence $aq \cap vy = \emptyset$. Also $aq \subset ax$, $q \in F$, $x \in H$ and $F_1 \cap H = \emptyset$, hence $q \neq x$. Thus, $x \notin aq$ and we have $aq \cap H = \emptyset$. But since $yw \subset H$, it follows that $aq \cap yw = \emptyset$, thus $aq \cap vy w = \emptyset$, i.e., $aq \cap G^{**} = \emptyset$. Since we defined $q$ to be the last point of $F_1$ on $vy \cup ax$ we see that $(q d - q) \cap (vy \cup ax) = \emptyset$.

Thus $(q d - q) \cap vy = \emptyset$ and since $q \in ax$ implies $q \notin vy$, we have $qd \cap vy = \emptyset$. Also $qd F_1$ and $yw \in H$, hence $qd \cap yw = \emptyset$, thus $qd \cap yw = qd \cap G^{**} = \emptyset$. Finally then $F^* \cap G^{**} = aq d \cap G^{**} = \emptyset$, i.e., $G^{**} \subset T - F^*$ and $G^{**}$ satisfies (ii), i.e., $w \in E$.

Now in every case we found that $w \in E$. But the assumption that $E$ was not closed implied $w \subset T - E$, hence we have a contradiction, i.e., $E$ is closed. Q.E.D.

**Theorem 5.30** Suppose $x$ is any point of a Peano space $T$.

Then $x$ is a cut point of some Peano space $D$ contained in $T$ if and only if $x$ is an interior point of an arc in $T$.

**Proof.** Suppose $x$ is a cut point of a Peano space $D$ and $D \subset T$. Then $D - x$ has a separation $A/B$. Let $c$ be a point of $A$ and...
d be a point of B. Since D is arcwise connected and c and d are in D, there is an arc cd contained in D. Clearly c ≠ x, d ≠ x and cd is connected. If cd ⊆ D-x then either cd ⊆ A or else cd ⊆ B (1.22). But cd ∉ A since d ∈ B and A ∩ B = Ø. Similarly, cd ∉ B since c ∈ A. Thus cd ⊆ D-x. However, cd ⊆ D, hence x is an interior point of the arc cd and cd ⊆ D ⊆ T.

To show the converse suppose x is an interior point of an arc ab in T. Thus x ∈ ab, x ≠ a and x ≠ b. Since ab is a Peano space (3.3) and contains only two non-cut points, namely its end points a and b, we see that x must be a cut point of ab. Therefore, x is a cut point of a Peano space in T. Q.E.D.

Theorem 5.31 Suppose T is a non-degenerate Peano space, p is a non-cut point of T and H is an open set containing p. Then there is an open set G containing p such that p ∈ G ⊆ G ⊆ H and T-H is contained in a single component of T-G.

Proof. Let H be an open set containing p. Since T is non-degenerate there exists a point q of T such that p ≠ q. Thus ρ(p,q) = δ where δ > 0. H is open, hence there is an ε > 0 such that N(p,ε) ⊆ H. Let $\lambda = \frac{1}{2}\epsilon \frac{1}{2} \delta$ where $\lambda > 0$. Then p ∈ N(p,λ) and N(p,λ) is open. Therefore

$$ p \in N(p,\lambda) \subseteq N(p,\frac{1}{2}\epsilon \frac{1}{2} \delta) \subseteq N(p,\epsilon) \subseteq H $$

and clearly q ∈ N(p,λ). Let W = N(p,λ). Then W is open, W ⊆ H and q ∈ T-W hence T-W ≠ ∅. Since W ⊆ H we have T-H ⊆ T-W where T-W is an open set. Thus the components of T-W are open (1.25) and, furthermore, form an open covering of T-H. But T-H is closed, hence compact, and it follows from 1.21 that a finite number of these components $A_1, A_2, \ldots, A_n$ cover T-H. Then we have T-H ⊆ $\bigcup_{i=1}^{n} A_i$ where
each $A_i$ is non-empty, open, connected and contained in $T-W$. Since $p \notin W$, $p \notin T-W$ hence $p \notin T-H$. This implies that $p \notin A_i$ for $i = 1, \ldots, n$, so we have $A_i \subset T-p$ for $i = 1, \ldots, n$. Now $p$ is a non-cut point of $T$ and $p$ is also a closed set hence $T-p$ is a region in $T$. Therefore, $T-p$ is arcwise connected and there exists an arc $F_j$ for $j = 2, 3, \ldots, n$, joining a point $q_j$ of $A_j$ with a point $q_j$ of $A_j$. Clearly $F_j \subset T-p$ for $j = 2, \ldots, n$, and each $F_j$ is closed. Letting $F = \bigcup_{j=2}^{n} F_j$ we see that $F$ is closed (1.10). Also $F \subset T-p$, hence $p \notin F$. Since each $F_j$ is connected and has the common point $q_j$ it follows from 1.22 that $F$ is connected. We know that $p \notin W$ hence we now have $p \notin W-F$. Let $G = W-F$ and let $E = F \cup \bigcup_{i=1}^{n} A_i$. Clearly $q_j \in F \cap A_i$ for $i = 1, \ldots, n$ and since $F$ and each $A_i$ are connected sets we see that $E$ is connected (1.22). We shall show that $E \subset T-G$. Since $G \subset W-F$ implies $G \subset G-F = T-G$. Also $A_i \subset T-W$ for each $i$, but $T-W \subset T-G$ since $G \subset W$. Therefore, $\bigcup_{i=1}^{n} A_i \subset T-G$ and it follows that $E \subset T-G$. Recall that $T-H \subset \bigcup_{i=1}^{n} A_i$. Thus $T-H \subset E \subset T-G$. Finally, then since $E$ is connected, no component of $T-G$ can be a proper subset of $E$. And since components are disjoint or equal, $E$ must itself be a component of $T-G$ or else must lie in one component of $T-G$. In either case $T-H$ is contained in a single component of $T-G$. Furthermore, since $p \in G$ and $G \subset W$, we have $p \in G \subset G \subset W \subset H$ and since $G = W-F$, $G$ is open, hence the theorem is proved. Q.E.D.

**Theorem 5.32** Suppose $p$ is a point of a non-degenerate cyclic Peano space $T$ and $H$ is any open set containing $p$ such that $\bar{H}$ is a Peano space and $p$ is a non-cut point of $\bar{H}$. Then there is a proper cyclic element of $\bar{H}$ which contains $p$.

**Proof.** If $H = T$ the theorem is trivially true. Thus, we
assume $H$ is a proper subset of $T$. Furthermore, $p \notin H$, for if $p = H$ then since $p$ is closed, $H$ is closed. But $H$ is an open set by hypothesis hence $T$ would be the union of two non-empty disjoint open sets $H$ and $C_H$, i.e., $T$ would be disconnected which is impossible. Therefore, we have $p \notin H$ hence $H-p \not= \emptyset$, i.e., there is a point $q$ such that $q \notin p$ and $q \in H \cap H$. Thus $\overline{H}$ is a non-degenerate Peano space. Now if it can be shown that there exists a point $r$ of $\overline{H}$ such that $r \notin p$ and $r \cup p$ in $\overline{H}$ then by theorem 5.16 there will exist a proper cyclic element of $H$ containing $r$ and $p$, in particular containing $p$, and the theorem will be proved. We shall show that such a point does exist.

Since $T$ is a metric space and $H$ is an open set contained in $T$ and containing $p$ there is an open set $W$ containing $p$ such that $p \in W \cap H$ (1.11). Thus $W \cap H$, and since $H$ is open and $H \not= T$, we have $H \not\subseteq H$, hence $W \not\subseteq H$. Therefore, $\overline{H} \cap W \not= \emptyset$. Now $p$ is a non-cut point of the Peano space $\overline{H}$ and $W$ is an open set containing $p$ and contained in $\overline{H}$, hence by theorem 5.31 there is a set $G$ containing $p$ such that $G$ is open in $\overline{H}$, $G \subseteq W$ and $\overline{H} \cap W$ is contained in a single component $A$ of $\overline{H} \cap G$. Since $G$ is open in $\overline{H}$, $\overline{H} \cap G$ is closed in $\overline{H}$. Therefore, $A$ is closed in $\overline{H}$ (1.24). Let $t$ be a point in $A$. Clearly $p \not\in t$ since $A \subseteq \overline{H} \cap G$ and $p \in G$. Now both $p$ and $t$ belong to $\overline{H}$ and $\overline{H}$ is arcwise connected, hence there is an arc $pt$ in $\overline{H}$ and $pt \cap A \not= \emptyset$. Thus we may let $y$ be the first point of $pt$ in $A$. Clearly $p \not\in y$ and we have an arc $py$ such that $py \cap A = y$ and $y \in \overline{H}$. We shall show that $p \cup y$ in $\overline{H}$. Suppose otherwise, i.e., there exists a point $z$ of $\overline{H}$ such that $z$ cuts between $p$ and $y$. Therefore $p$ and $y$ lie in different components of $\overline{H} \cap z$.
This implies \( z \not\in p \) and \( z \not\in y \). Also \( z \in py \) for if not then since \( py \) is a connected set it must lie in one component of \( \overline{H} - z \) which is impossible since \( p \) and \( y \) lie in different components of \( \overline{H} - z \). It is clear that \( z \cap A \cap py \cap A = y \), i.e., \( z \cap A \subseteq y \). But \( z \not\in y \), hence \( z \cap A = \emptyset \), i.e., \( z \notin A \). But \( A \subseteq \overline{H} \) hence \( A \subseteq \overline{H} - z \). Now since \( A \) is connected and \( y \in A \), \( y \) and \( A \) must lie in the same component of \( \overline{H} - z \). Thus \( p \) and \( A \) lie in different components of \( \overline{H} - z \), i.e., \( z \) cuts between \( p \) and all of the points of \( A \). Let \( F \) be any arc in \( \overline{H} - z \) joining \( p \) to a point of \( A \). Suppose \( z \notin F \). Then \( p \cup F \cup A \) is clearly a connected set contained in \( \overline{H} - z \). Thus \( p \cup F \cup A \) must be contained in the same component of \( \overline{H} - z \). This is impossible since \( p \) and \( A \) lie in different components of \( \overline{H} - z \). Therefore, \( z \in F \). Recall that \( W \) is open, hence \( T - W \) is closed and since \( T \) is compact, \( T - W \) is compact. We have seen that \( \overline{H} - W \not\in \emptyset \) hence we may let \( b \) be a point of \( \overline{H} - W \). Clearly then \( \overline{H} \subseteq T \) implies \( \overline{H} - W \subseteq T - W \) and \( b \in T - W \). Furthermore, \( \overline{H} - W \subseteq A \), hence \( b \in A \) which implies \( b \not\in z \) since \( z \notin A \). Thus \( b \in T - z \). Now the single point \( z \) is closed and \( T \) is cyclic hence \( T - z \) is a region in \( T \). Therefore, \( T - z \) is arcwise connected so we have an arc \( pb \) in \( T - z \). Since \( b \in T - W \) we have \( pb \cap (T - W) \not\in \emptyset \). Let \( q \) be the first point of \( pb \) in the compact set \( T - W \). Therefore, \( pq \) is an arc such that \( pq \cap (T - W) = q \), i.e., \( q \in \overline{W} \). It follows that \( (pq - q) \cap (T - W) = \emptyset \), i.e., \( (pq - q) \subseteq W \). But

\[
pq = \overline{(pq - q)} \subseteq \overline{W} \subseteq \overline{C} \subseteq \overline{H},
\]

thus \( q \in \overline{H} \) hence \( q \in \overline{H} - W \subseteq A \), i.e., \( q \in A \). Finally then we have an arc \( pq \) joining the point \( p \) and a point \( q \) of \( A \). And since \( pq \subseteq \overline{H} - z \), \( z \notin pq \). But this is a contradiction since \( z \) cuts between \( p \) and all of the points of \( A \). It contradicts our assumption that \( p \) was not
conjugate to $y$. Hence $p \circ y$ where $y \in \overline{H}$ and $y \neq p$. Thus by theorem 5.18 there is a proper cyclic element of $\overline{H}$ containing $p$ and $y$, and in particular containing $p$. Q.E.D.

**Theorem 5.33** If $p$ is a point of a non-degenerate cyclic Peano space $T$, then $p$ is an interior point of some arc in $T$.

**Proof.** Let us assume to the contrary that no arc of $T$ has $p$ as an interior point. In view of theorem 5.30 this assumption is equivalent to assuming that $p$ is a non-cut point of every non-degenerate Peano subspace of $T$ which contains $p$. We shall arrive at our contradiction by showing that there is an arc in $T$ having $p$ as an interior point.

Since $T$ is a non-degenerate connected set, it contains a non-enumerable number of distinct points (1.22). We are going to define a series of arcs and sets inductively and we begin by letting $\varepsilon_n = \frac{1}{n}$ for $n = 1, 2, \ldots$. Let $a_0$ and $b_0$ be distinct points of $T - p$. Consider the open neighborhood $N(p, \varepsilon_1)$. Clearly, $T - (a_0 \cup b_0)$ is open hence $[N(p, \varepsilon_1) \cap T - (a_0 \cup b_0)]$ is an open set. Thus there is a $\delta > 0$ and an open neighborhood $N(p, \delta)$ such that $N(p, \delta) \subseteq [N(p, \varepsilon_1) \cap T - (a_0 \cup b_0)]$. Consider the open neighborhood $N(p, \frac{1}{2}\delta)$. Now $T$ is a Peano space, hence $T$ has a basis every non-empty element of which has a Peano space as its closure (4.27). Being an open set $N(p, \frac{1}{2}\delta)$ can be expressed as the union of some of the elements of this basis. Therefore, there is a set $H_1$ of the basis such that

$$p \in H_1 \subseteq N(p, \frac{1}{2}\delta) \subseteq N(p, \delta) \subseteq [N(p, \varepsilon_1) \cap T - (a_0 \cup b_0)].$$

Furthermore, $\overline{H_1}$ is a Peano space and

$$p \in \overline{H_1} \subseteq N(p, \frac{1}{2}\delta) \subseteq N(p, \delta) \subseteq [N(p, \varepsilon_1) \cap T - (a_0 \cup b_0)].$$
By assumption $p$ is a non-cut point of every non-degenerate Peano space containing it. Hence, $p$ is a non-cut point of $\overline{H}_1$. Thus applying theorem 5.32 there is a proper cyclic element $E_1$ of $\overline{H}_1$ such that $p \in E_1$. Furthermore, $E_1$ is closed in $\overline{H}_1$ (5.20). It follows from (1.13) that $E_1$ is closed in $T$. Since

$$E_1 \subseteq \overline{H}_1 \subseteq \left[ N(p, \epsilon_1) \right] \cap \left[ T - (a_0 \cup b_0) \right]$$

we have $E_1 \cap (a_0 \cup b_0) = \emptyset$, where $E_1$ and $a_0 \cup b_0$ are both non-degenerate closed subsets of the cyclic Peano space $T$. Thus by theorem 5.29 there exists two arcs $a_0 a_1$ and $b_0 b_1$ in $T$ such that $a_0 a_1 \cap E_1 = a_1$, $b_0 b_1 \cap E_1 = b_1$ and $a_0 a_1 \cap b_0 b_1 = \emptyset$, hence $a_1 \neq b_1$.

Since $E_1$ is a proper cyclic element of the Peano space $\overline{H}_1$ we see by theorem 5.28 that $E_1$ itself is a cyclic Peano space. Recall that $p \in E_1$. Suppose $p = a_1$. Since $E_1$ is a Peano space there is an arc $a_1 b_1$ in $E_1$ and $a_0 a_1 \cup a_1 b_1$ is clearly the union of two arcs whose only common point is $a_1$, hence $a_0 a_1 \cup a_1 b_1$ is itself an arc. Similarly $a_0 a_1 \cup a_1 b_1 \cup b_1 b_0$ is an arc, but more than that, it is an arc with $p = a_1$ as an interior point which is impossible by assumption. Thus $p \neq a_1$ and similarly, $p \neq b_1$.

By theorem 4.27 we see that $E_1$ has a basis every non-empty element of which has a Peano space as its closure. Consider the open neighborhood $N(p, \epsilon_2)$. Clearly $E_1 - (a_1 \cup b_1)$ is an open set in $E_1$, hence $N(p, \epsilon_2) \cap [E_1 - (a_1 \cup b_1)]$ is open in $E_1$. Thus there exists $\gamma > 0$ such that

$$N(p, \gamma) \subseteq N(p, \epsilon_2) \cap [E_1 - (a_1 \cup b_1)]$$

and $N(p, \gamma)$ is open in $E_1$. Consider $N(p, \frac{1}{2} \gamma)$. $N(p, \frac{1}{2} \gamma)$ is open in $E_1$ hence may be expressed as the union of some of the sets in the basis of $E_1$. Therefore, there is an element $H_2$ of this basis.
such that
\[ p \in \overline{H}_2 \subset N(p, \frac{1}{3}) \subset N(p, \gamma) \subset \left[ N(p, \varepsilon_2) \right] \cap [E_1 - (a_1 \cup b_1)] \]

where $\overline{H}_2$ is a Peano space and
\[ p \in \overline{H}_2 \subset N(p, \frac{1}{3}) \subset N(p, \gamma) \subset \left[ N(p, \varepsilon_2) \right] \cap [E_1 - (a_1 \cup b_1)]. \]

Furthermore, by assumption, $p$ is a non-cut point of $\overline{H}_2$ and clearly $H_2$ is open in $E_1$. Thus by theorem 5.32 there is a proper cyclic element $E_2$ of $\overline{H}_2$ such that $p \in E_2$ and $E_2$ is closed in $\overline{H}_2$ (5.20). Furthermore, $H_2$ is closed in $E_1$ and $E_2 \subset H_2 \subset E_1$, thus $E_2$ is closed in $E_1$ (1.13). Recall that $a_1 \cup b_1 \subset E_1$ and clearly the set $a_1 \cup b_1$ is closed in $E_1$. Since $E_2 \subset H_2 \subset \left[ N(p, \varepsilon_2) \right] \cap [E_1 - (a_1 \cup b_1)]$ we have $E_2 \cap (a_1 \cup b_1) = \emptyset$. Also we have seen that $E_1$ is itself a cyclic Peano space. Thus applying theorem 5.29 there exists two arcs $a_1 a_2$ and $b_1 b_2$ in $E_1$ such that $a_1 a_2 \cap E_2 = a_2$, $b_1 b_2 \cap E_2 = b_2$ and $a_1 a_2 \cap b_1 b_2 = \emptyset$. Since $E_2$ is a proper cyclic element of the Peano space $\overline{H}_2$ we see by theorem 5.28 that $E_2$ is itself a cyclic Peano space. Recall that $p \in E_2$. Suppose $p = a_2$. Clearly there is an arc $a_2 b_2$ in $E_2$ and $a_1 a_2 \cup a_2 b_2$ is also an arc since it is the union of two arcs whose only common point is $a_2$. Similarly, $a_1 a_2 \cup a_2 b_2 \cup b_2 b_1$ is an arc, but more than that, it is an arc with $p (= a_2)$ as an interior point which is impossible by assumption.

Thus $p \neq a_2$ and, similarly, $p \neq b_2$.

Now $a_0 a_1 \cup a_1 a_2$ is the union of two arcs whose only common point is an end point $a_1$, hence $a_0 a_1 \cup a_1 a_2$ is an arc and we denote it as $a_0 a_2$. Similarly, we obtain an arc $b_0 b_2$. Recall that $a_0 a_1 \cap E_1 = a_1$ and $E_2 \subset E_1 - (a_1 \cup b_1)$, hence $a_0 a_1 \cap E_2 = \emptyset$. Also $a_1 a_2 \cap E_2 = a_2$ hence we have $a_0 a_2 \subset E_2 = a_2$. Similarly, $b_0 b_2 \cap E_2 = b_2$ and clearly $a_0 a_1 \subset a_0 a_2$ and $b_0 b_1 \subset b_0 b_2$. 

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Continuing this process for \( n = 1, 2, \ldots \) we obtain a sequence of cyclic Peano spaces \( \{E_n\} \) and arcs \( a_{0n} \) and \( b_{0n} \) such that

\[
a_{0n} \cap E_n = a_n, \quad b_{0n} \cap E_n = b_n, \quad p \notin a_{0n}, \quad q \notin b_{0n}, \quad a_{0n} \subseteq a_{0n+1},
\]

\[
b_{0n} \subseteq b_{0n+1}, \quad a_{0n-1} \cup a_{n-1} = a_n \quad \text{and} \quad b_{0n-1} \cup b_{n-1} = b_n.
\]

Let

\[
A = \bigcup_{n=1}^{\infty} a_{n-1} = a_0 a_1 a_2 \cup \ldots
\]

and

\[
B = \bigcup_{n=1}^{\infty} b_{n-1} = b_0 b_1 b_2 \cup \ldots.
\]

\( A \) is connected since \( a_{n-1} a_n \) is connected for each \( n \) and

\[ a_{n-1} a_n \cap a_{n} a_{n+1} = a_n \neq \emptyset. \]

Similarly, \( B \) is connected. Thus \( \overline{A} \) and \( \overline{B} \) are each connected, and, furthermore, they are each compact.

Since \( a_{n-1} a_n \subseteq E_n \subseteq \overline{E}_n \subseteq C \subseteq N(p, \varepsilon_{n-1}) \) we have \( \rho(a_n, p) < \varepsilon_{n-1} = \frac{1}{n-1} \) for all \( n \). Clearly then \( \rho(a_n, p) \to 0 \) as \( n \to \infty \) and since \( a_n \in A \) for all \( n \) we have \( p \in \overline{A} \) (1.9). Similarly, \( p \in \overline{B} \) and it follows that

\( A \cup p \subseteq \overline{A} \) and \( B \cup p \subseteq \overline{B} \). We shall show that \( A \cup p = \overline{A} \) and \( B \cup p = \overline{B} \).

Let \( q \) be a point of \( \overline{A} \). Thus there is a sequence \( \{t_n\} \) of distinct points of \( A \) such that \( t_n \to q \) (1.9). If infinitely many points of \( \{t_n\} \) are in \( a_{0} a_{1} \), then since \( a_{0} a_{1} \) is compact there is a subsequence \( \{t_{n_h}\} \) of \( \{t_n\} \) which converges to some point \( r \) of \( a_{0} a_{1} \).

But since \( t_n \to q \) we have \( t_{n_h} \to q \) (1.8), i.e., \( q = r \), hence

\[ q \in a_{0} a_{1} \subseteq A, \quad \text{i.e.,} \quad q \in A. \]

Thus we can assume only a finite number of points of \( \{t_n\} \) are in \( a_{0} a_{1} \). Let \( \lambda > 0 \) and choose an integer \( k \) so that \( \frac{1}{k} < \lambda \). Now continuing this line of reasoning we can assume only a finite number of points of \( \{t_n\} \) are in \( a_{k-1} a_k \), i.e., there is a number \( n_0 \) such that if \( n > n_0 \) then \( t_n \notin a_{0} a_{1} \cup \ldots \cup a_{k-1} a_k \).

Thus if \( s > k \), and \( s-1 \geq k \),

\[
t_n \in a_{s-1} a_s \subseteq E_{s-1} \subseteq E_k \subseteq \overline{E}_k \subseteq C \subseteq N(p, \frac{1}{k}) \subseteq C \subseteq N(p, \lambda),
\]
therefore $t_n \in N(p, \lambda)$ for all $n > n_0$. But since $k$ can be made arbitrarily large and there still will exist an appropriate number $n_0$, we see that $\lambda$ can be made arbitrarily small. Thus as $n \to \infty$ we have $\rho(t_n, p) \to 0$, i.e., $t_n \to p$. However, since $t_n \to q$ this implies $p = q$, thus $q \in A \cup p$. Therefore, if $q$ is an arbitrary point of $\overline{A}$ we have $q \in A \cup p$, i.e., $\overline{A} \subseteq A \cup p$, hence $\overline{A} = A \cup p$. Similarly, $\overline{B} = B \cup p$.

Also $A \cap B = \emptyset$, for if not then there is a point $w$ of $A \cap B$. Therefore, there are specific numbers $m$ and $n$ such that $w \in a_{n-1}a_n$ and $w \in b_{m-1}b_m$. If $m = n$ then by the way we have defined our arcs we have $a_{n-1}a_n \cap b_{m-1}b_m = \emptyset$. This is impossible since

$$w \in a_{n-1}a_n \cap b_{m-1}b_m,$$

thus $m \neq n$. Suppose $m > n$. Then since $b_{m-1}b_m \subseteq F_{m-1} \subseteq F_n$ and $a_{n-1}a_n \cap F_n = a_n$ we must have $a_{n-1}a_n \cap b_{m-1}b_m \subseteq a_n$.

(i) If $m = n+1$, then $a_{n-1}a_n \cap b_{n+1}b_{n+1} \subseteq a_n$. But $a_n \notin b_nb_{n+1}$ since $a_{n+1} \cap b_nb_{n+1} = \emptyset$, hence $a_{n-1}a_n \cap b_{n+1}b_{n+1} = \emptyset$, i.e.,

$$a_{n-1}a_n \cap b_{m-1}b_m = \emptyset,$$

and we have a contradiction.

(ii) If $m > n+1$, i.e., $m-1 \geq n+1$, then

$$b_{m-1}b_m \subseteq F_{m-1} \subseteq F_{n+1} \subseteq [N(p, \epsilon_{n+1})] \cap [F_n - (a_n \cup b_n)],$$

i.e., $b_{m-1}b_m \subseteq F_n - (a_n \cup b_n)$, hence $a_n \notin b_{m-1}b_m$. Thus again we have the contradiction that $a_{n-1}a_n \cap b_{m-1}b_m = \emptyset$. In a similar manner, the reader may easily verify that the assumption that $m < n$ leads to a contradiction. Therefore, we conclude that no such point $w$ exists, hence $A \cap B = \emptyset$. Clearly then since $\overline{A} = A \cup p$ and $\overline{B} = B \cup p$ we have $\overline{A} \cap \overline{B} = p$.

Now since $\overline{A}$ and $\overline{B}$ are both compact connected sets, if it can be shown that each has at most two non-cut points then by theorem 3.5...
each set is a simple arc. We shall see that this is the case.

First $\overline{A} - p = (A \cup p) - p = A$ and $A$ is connected, hence $p$ is a non-cut point of $\overline{A}$. Similarly, $p$ is a non-cut point of $\overline{B}$. Clearly $\overline{A} - a_0$ may be expressed as follows,

$$\overline{A} - a_0 = (A \cup p) - a_0 = (a_0 a_1 - a_0) \cup a_1 a_2 \cup a_2 a_3 \cup \ldots \cup p$$

$$= (a_0 a_1 - a_0) \cup a_1 a_2 \cup a_1 a_3 \cup \ldots \cup a_{1n} \cup \ldots \cup p.$$

Now $a_0 a_1$ is an arc with $a_0$ and $a_1$ as its end points, hence $a_0$ is a non-cut point of $a_0 a_1$. Therefore, $a_0 a_1 - a_0$ is connected. Also, it is clear that $a_1 a_2 \cup a_1 a_3 \cup \ldots \cup a_{1n} \cup \ldots$ is a connected set meeting $a_0 a_1 - a_0$ in exactly one point; namely $a_1$, hence

$$(a_0 a_1 - a_0) \cup a_1 a_2 \cup \ldots \cup a_{1n} \cup \ldots$$

is a connected set. But

$$(a_0 a_1 - a_0) \cup a_1 a_2 \cup \ldots \cup a_{1n} \cup \ldots = A \cup p = \overline{A}$$

and since $$(a_0 a_1 - a_0) \cup a_1 a_2 \cup \ldots \cup a_{1n} \cup \ldots \subseteq (a_0 a_1 - a_0) \cup a_1 a_2 \cup \ldots \cup a_{1n} \cup \ldots \cup p \subseteq A \cup p$$ we see that $$(a_0 a_1 - a_0) \cup a_1 a_2 \cup \ldots \cup a_{1n} \cup \ldots \cup p$$ is connected. Thus, $\overline{A} - a_0$ is connected, i.e., $a_0$ is a non-cut point of $\overline{A}$. We must now show that $p$ and $a_0$ are the only non-cut points of $\overline{A}$. Let $v$ be any point of $\overline{A}$ such that $v \neq p$ and $v \neq a_0$. There is some $i$ such that $v \in a_{i-1} a_i$. There are two distinct cases to consider.

(i) Suppose $v$ is an interior point of $a_{i-1} a_i$. Under this assumption the reader may easily verify that

$$(A \cup p) - v = \bigcup_{j \leq i} a_{j-1} a_j \cup (a_{i-1} v - v) / \bigcup_{j \leq i} a_{j-1} a_j \cup (v a_i - v)$$

i.e., $(A \cup p) - v$ is not connected.

(ii) Suppose $v = a_{i-1}$. Again it is easily seen that

$$(A \cup p) - v = \bigcup_{j \leq i} a_{j-1} a_j \cup (a_{i-2} a_{i-1} a_i) / \bigcup_{j \leq i} a_{j-1} a_j \cup (a_{i-1} a_i - a_{i-1})$$
hence, \((A \cup p)-v\) is disconnected.

Similarly, if \(v = a_i\) we see that \((A \cup p)-v\) is disconnected, thus, in every case an arbitrary point \(v\) of \(A\) such that \(v \notin p\) and \(v \notin a_o\) is found to be a cut point of \(\overline{A}\). Similarly, \(p\) and \(b_o\) are the only non-cut points of \(\overline{B}\). We now see that \(\overline{A}\) and \(\overline{B}\) are each simple arcs with end points \(p\) and \(a_o\) of \(\overline{A}\) and \(p\) and \(b_o\) of \(\overline{B}\). Furthermore, \(\overline{A}\) and \(\overline{B}\) meet in exactly one point \(p\) which is an end point of each set. Thus \(\overline{A} \cup \overline{B}\) is itself a simple arc with end points \(a_o\) and \(b_o\). But \(p \in \overline{A} \cup \overline{B}\), \(p \neq a_o\) and \(p \neq b_o\), hence \(p\) is an interior point of an arc in \(T\). This contradicts our first assumption, hence the theorem is proved. Q.E.D.

**Theorem 5.34 (Cyclic Connectivity Theorem)** Suppose \(T\) is a Peano space. Then \(T\) is cyclic if and only for every two points \(a\) and \(b\) of \(T\) there is a simple closed curve in \(T\) containing \(a\) and \(b\).

**Proof.** First suppose given any two points \(a\) and \(b\) of \(T\) there is a simple closed curve \(C\) in \(T\) containing \(a\) and \(b\). Let \(d\) be any point of \(T\) different from \(a\) and \(b\), thus \(a, b \in T-d\). There are two cases to consider.

(i) Suppose \(d \in C\). By theorem 4.4, \(C\) may be expressed as two independent arcs \((ab)_1\) and \((ab)_2\). Suppose the notation is chosen so that \(d \in (ab)_1\), thus \(d \in (ab)_2\). Since \((ab)_2\) is a connected set it belongs to one component of \(T-d\). But \(a \in (ab)_2\) and \(b \in (ab)_2\), hence \(a\) and \(b\) are in the same component of \(T-d\). Since \(d\) was arbitrary this implies no point of \(T\) cuts between \(a\) and \(b\). Hence \(a \cap b\) for any two points \(a\) and \(b\) of \(T\). Therefore, \(T\) is cyclic.

(ii) Suppose \(d \notin C\). Since \(C\) is a connected set it is
contained in a single component of $T - d$. Thus $a$ and $b$ lie in a single component of $T - d$, hence $a \cup b$. Again then since $a$ and $b$ were arbitrary points of $T$ we see that $T$ is cyclic.

To prove the converse, suppose that $T$ is a cyclic Peano space and that $a$ and $b$ are any two distinct points of $T$. We must show there a simple closed curve in $T$ containing $a$ and $b$. By theorem 5.33, $a$ is an interior point of some arc $xy$ in $T$. Since $T$ is cyclic and the single point $a$ is closed, we readily see that $T - a$ is a region in $T$. Now, $T - a$ is arcwise connected and $x, y \notin a$ hence there is an arc $\hat{xy}$ in $T - a$ joining the points $x$ and $y$.

Clearly $a \notin \hat{xy}$, hence $xy \notin \hat{xy}$. Consider the subarc $ax$ of $xy$. Since $ax$ is closed, it is compact. Furthermore, $x \in ax$ and $x \in \hat{xy}$, hence $ax \cap \hat{xy} \neq \emptyset$. Let $u$ be the first point of $ax$ on $\hat{xy}$. Thus, $u \in ax \cap \hat{xy}$ and we clearly have $au \cap \hat{xy} = t$ where $au$ is a subarc of $ax$. Also, $u \notin a$ since $u \in \hat{xy} \subset T - a$. By similar reasoning, the subarc $ay$ of $xy$ is compact and meets $\hat{xy}$ hence we may let $w$ be the first point of $ay$ on $\hat{xy}$. Thus, $aw \cap \hat{xy} = w$ where $a \notin w$. Now then $u \in \hat{xy}$, $w \in \hat{xy}$ and $u \notin w$ since $au \cap aw \subset ax \cap ay = a$.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

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Consider the subarc $\hat{uv}$ of $xy$. Clearly $\hat{uw} \cap aw \cap aw = w$ and since $w \in uw$ and $w \in aw$ we have $\hat{uw} \cap aw = w$. Similarly, since $\hat{uw} \cap au \cap au = u$ and $u \in \hat{uw}$ and $u \in au$ we have $\hat{uw} \cap au = u$. It is also clear that $au \cap aw = a$. We see that $au$ and $aw$ are simple arcs meeting at one point; namely $a$. But $a$ is an end point of both $au$ and $aw$, hence $au \cup aw$ is a simple arc with end points $u$ and $w$ and clearly

$$(au \cup aw) \cap \hat{uw} = (au \cap \hat{uw}) \cup (aw \cap \hat{uw}) = u \cup w.$$  

Thus $(au \cup aw)$ and $\hat{uw}$ are independent arcs from $u$ to $w$, hence by the corollary to theorem 4.4 we see that $(au \cup aw) \cup \hat{uw}$ is a simple closed curve. Let $C_a = (au \cup aw) \cup \hat{uw}$. Now $a \in C_a$, hence $a$ lies on a simple closed curve in $T$. By similar reasoning there is a simple closed curve $C_b$ containing $b$.

It is immediate that if either $a \in C_b$ or $b \in C_a$ then $a$ and $b$ lie on the same simple closed curve and the theorem is proved. Thus excluding this possibility there are three cases to consider.

(i) Suppose $C_a \cap C_b = \emptyset$. Then $C_a$ and $C_b$ are non-degenerate, disjoint, compact (hence closed), connected subsets of the cyclic Peano space $T$ (4.5). In view of theorem 5.29, there exists disjoint arcs $pq$ and $rs$ each having one end point in $C_a$ and one end point in $C_b$. That is, $p \in C_a$, $q \in C_b$ and $pq \cap (C_a \cup C_b) = p \cup q$. Similarly, $r \in C_a$, $s \in C_b$ and $rs \cap (C_a \cup C_b) = r \cup s$.

![Diagram](https://via.placeholder.com/150)

Figure 3

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It is easily seen that we now have arcs \( pr \) and \( pr' \) such that 
\( pr \cup pr' = C_a \) and also we have arcs \( qs \) and \( qs' \) such that 
\( qs \cup qs' = C_b \). 
Suppose the notation is chosen so that \( a \in pr \) and \( b \in qs \) (a similar discussion would follow if \( a \in pr' \) and \( b \in qs' \) or else \( a \in pr' \) and \( b \in qs' \)).

We know that \( pq \cap rs = \emptyset \) and clearly \( pq \cap pr = p \) and \( pq \cap qs = q \).

Similarly, it is readily seen that \( rs \cap pr = r \) and \( rs \cap qs = s \).

Furthermore, \( pr \cap qs \subseteq C_a \cap C_b = \emptyset \), i.e., \( pr \cap qs = \emptyset \). It follows that \( pr \cup pq \) is the union of two simple arcs which intersect in exactly one point; namely \( p \), which is an end point of each. Hence, \( pr \cup pq \) is itself a simple arc. Similarly, \( (pr \cup pq) \cup qs \) is a simple arc. Now then we have two independent arcs \( (pr \cup pq \cup qs) \) and \( rs \) from \( r \) to \( s \), hence by the corollary to theorem 4.4,

\[
(pr \cup pq \cup qs \cup rs) = A
\]
is a simple closed curve. Furthermore, \( a \in pr \) and \( b \in qs \), hence \( a \) and \( b \) lie on \( A \).

(ii) Suppose \( C_a \cap C_b = z \), i.e., \( C_a \) and \( C_b \) meet in a single point.

\[\text{Figure 4}\]

Clearly \( z \neq a \) and \( z \neq b \). Since \( T \) is cyclic and \( z \) is a closed set, 
\( T - z \) is a region in \( T \) and \( a, b \in T - z \). Thus, there is an arc \( ab \) in 
\( T - z \) joining \( a \) and \( b \). Now \( C_b \) is compact and since \( b \in ab \cap C_b \) we have 
\( ab \cap C_b \neq \emptyset \). Therefore, there is a first point \( y \) of \( C_b \) on \( ab \).
Hence, \( y \in ab, y \in C_b \) and \( ay \cap C_b = y \). Furthermore, \( y \neq z \) since \( ab \subseteq T - z \), hence \( y \notin C_a \). By similar reasoning, there is a first point \( x \) of \( C_a \) on the arc \( ay \) going in order from \( y \) to \( a \), such that \( x \in ab, x \in C_a \), \( x \neq z \) and \( x \notin C_b \). Thus we have a subarc \( xy \) of \( ab \) such that \( xy \cap C_a = x \) and \( xy \cap C_b = y \). Since \( xy \subseteq ab \cap T - z \) we have \( xy \subseteq T - z \). It is clear (see the figure in (ii)) that we now have subarcs \( zx \), \( \widehat{zx} \) of \( C_a \) and \( zy \), \( \widehat{zy} \) of \( C_b \). Suppose the notation is chosen so that \( a \in \widehat{az} \) and \( b \in \widehat{zy} \) (a similar discussion would follow if \( a \in \widehat{zx} \) and \( b \in \widehat{zy} \) or \( a \in \widehat{az} \) and \( b \in \widehat{zy} \)). Clearly, \( \widehat{zx} \cap \widehat{xy} = x \) and since \( x \) is an end point of both \( \widehat{zx} \) and \( \widehat{xy} \) we see that \( \widehat{zx} \cup \widehat{xy} \) is a simple arc.

Furthermore, \((\widehat{zx} \cup \widehat{xy}) \cap \widehat{zy} = z \cup y\), i.e., the two simple arcs \((\widehat{zx} \cup \widehat{xy}) \) and \( \widehat{zy} \) meet in two points; namely, the end points of each. Hence \( \widehat{zx} \cup \widehat{xy} \cup \widehat{zy} = B \) is a simple closed curve. But \( a \in \widehat{az} \) and \( b \in \widehat{zy} \), hence \( a \) and \( b \) both lie on \( B \).

(iii) Suppose \( C_a \cap C_b \) contains more than one point.

Thus, there is a point \( w \) of \( C_a \cap C_b \) such that \( w \neq a, w \neq b \) and \( w \neq z \) where \( z \in C_a \cap C_b \). Now then we have subarcs \( aw \) and \( \hat{aw} \) of \( C_a \). Let \( z \in \hat{aw} \) (since a similar discussion would follow if \( z \in aw \)). Thus \( z \notin aw \) since \( aw \cap \hat{aw} = a \cup w \) and \( z \neq a \) and \( z \neq w \). Let \( x \) be the
first point of \( aw \) on \( C_b \) going in order from \( a \) to \( w \). Thus we have

a subarc \( ax \) of \( aw \) such that \( x \in aw, x \in C_b, x \neq a \) and \( ax \cap C_b = x \).

Now also there is a subarc \( az \) of \( C_a \) such that \( az \subseteq aw \). Let \( y \) be the first point of \( az \) belonging to \( C_b \). Thus \( y \neq a \), \( y \in C_a \) and \( y \in C_b \). Consider the arc \( ay \). Clearly, \( ay \subseteq az \subseteq aw \) and \( ay \cap C_b = y \).

We now have

\[
ax \cap ay \subseteq aw \cap az \subseteq aw \cap aw = a \cup w,
\]

i.e., \( ax \cap ay \subseteq a \cup w \). But \( w \notin az \), hence \( ax \cap ay = a \), where the point

\( a \) is an end point for each arc \( ax \) and \( ay \). Thus, \( ax \cup ay \) is a simple arc joining \( x \) and \( y \) and clearly \( (ax \cup ay) \cap C_b = x \cup y \).

Furthermore, there exists subarcs \( xy \) and \( \tilde{xy} \) of \( C_b \) and we choose the notation so \( b \in \tilde{xy} \). Now since arcs \( (ax \cup ay) \) and \( \tilde{xy} \) meets in exactly two points, \( x \) and \( y \), which are the end points of each arc

we see that \( ax \cup ay \cup \tilde{xy} = D \) is a simple closed curve. But \( a \in ax \) and \( b \in \tilde{xy} \), hence \( a \) and \( b \) lie on \( D \).

Thus, in every case, there is a simple closed curve containing \( a \) and \( b \). Q.E.D.

For a similar presentation of theorems 5.28 through 5.34, the reader may refer to Hall and Spencer's Elementary Topology.
BIBLIOGRAPHY


